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Extensions of Group Actions

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§1. Introduction

Let G be a finite group, A be a discrete abelian group, and

$$0 \longrightarrow A \xrightarrow{\subset} K \xrightarrow{j} G \longrightarrow 1$$

be an extension with the associated operator $\phi: G \to \mathrm{Aut}(A)$, $\phi(j(k))(a) = kak^{-1}$. Also let X be a connected CW-complex with a G-action $\beta: G \times X \to X$ or $\beta: G \to \mathrm{Homeo}(X)$ satisfying $\phi(\mathrm{Ker}\,\beta) = 1_A$ and $\pi: P \to X$ be a principal A-bundle with an A-action $\alpha: A \times P \to P$.

In this paper, we study the existence and the enumeration of K-actions γ on P such that the following diagram is commutative:

$$\begin{array}{ccccc} A \times P & \xrightarrow{\subset} & K \times P & \xrightarrow{j \times \pi} & G \times X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ P & = & P & \xrightarrow{\pi} & X. \end{array}$$

We call such a K-action γ on P an *extended K-action* on P of α over β. In [3], A. Hattori and T. Yoshida have studied this problem for $K = A \times G$, where A is a product of a torus group and a discrete abelian group, (cf. also [2, p. 23, Remark]).

By considering the G-action (2.8) on $[X, BA]$ of all equivalence classes of principal A-bundles over X, we define the map

$$(2.17) \qquad \Theta: [X, BA]^G \longrightarrow H^2_\phi(G, A)$$

from the set $[X, BA]^G$ of all G-invariant classes to the cohomology $H^2_\phi(G, A)$ of the group G with coefficients in the G-module A by ϕ. Then we have the following existence theorem:

Theorem 3.3. *A principal A-bundle P admits an extended K-action of α over β if and only if*

$$[P] \in [X, BA]^G \quad \textit{and} \quad \Theta([P]) = \omega([K]),$$

where ω: $\mathrm{Opext}(G, A, \phi) \to H^2_\phi(G, A)$ *is the bijection given in* [4, Ch. IV, Th. 4.1].

As an application, we obtain

COROLLARY 3.6. *Assume that a finite CW-complex P has the same mod 2 cohomology as the n-sphere* ($n \geqq 1$) *and has a free cellular involution T. If the orbit space* $X=P/T$ *has a free* Z_m*-action, then there is a free* Z_{2m}*-action on P which is an extension of the given involution.*

Furthermore, by studying the map Θ, we have Theorem 4.18 and the following generalization of a theorem of J. C. Su [5, Th. 3.13]:

COROLLARY 4.19. *Assume that the given G-action* β *on X has a fixed point. Then, a principal A-bundle P satisfying* $[P] \in [X, BA]^G$ *admits an extended* $A \times_\phi G$*-action, where* $A \times_\phi G$ *is the semi-direct product.*

For the enumeration of extended K-actions, we have

THEOREM 5.7. *If a principal A-bundle P admits an extended K-action, then the set of all equivalence classes of extended K-actions on P under the conjugation by elements of A is equivalent to the cohomology* $H^1_\phi(G, A)$ *as a set.*

As a corollary to Theorems 3.3 and 5.7, we have

COROLLARIES 3.5 AND 5.8. *If A is a finite abelian group and the orders of A and G are relatively prime, and* $[P] \in [X, BA]^G$, *then P admits a unique extended* $A \times_\phi G$*-action up to the conjugation by elements of A.*

§2. Preliminaries

Let G be a finite group and X be a connected CW-complex having a G-action $\beta: G \times X \to X$, i.e., a homomorphism

$$\beta: G \longrightarrow \mathrm{Homeo}(X). \tag{2.1}$$

For a given discrete abelian group A, consider a group extension

$$K: 0 \longrightarrow A \xrightarrow[\subset]{} K \xrightarrow{j} G \longrightarrow 1 \tag{2.2}$$

over G with kernel A. Throughout this paper, we consider only such an extension K of (2.2) that *the associated operator*

$$\phi: G \longrightarrow \mathrm{Aut}(A), \quad \phi(j(k))(a) = kak^{-1} \quad (k \in K,\ a \in A), \tag{2.3}$$

satisfies the condition

$$\phi(\mathrm{Ker}\,\beta) = 1_A, \tag{2.4}$$

where β is the given G-action (2.1) on X. We notice that *this condition is satisfied for any extension K if* β *is an effective action.*

Also, we consider a principal A-bundle

$$(2.5)\qquad \pi: P \longrightarrow X, \quad \text{with an } A\text{-action} \quad \alpha: A \times P \longrightarrow P$$

which is an inclusion $\alpha: A \subset \mathrm{Isom}(P)$, and denote by

$$\mathrm{Isom}_{G,\phi}(P)$$

the set of all fibre preserving homeomorphisms $f: P \to P$ of π such that

(2.6) there is some $g \in G$ and the diagram

$$\begin{array}{ccccc} P & \xrightarrow{a} & P & \xrightarrow{\pi} & X \\ \downarrow f & & \downarrow f & & \downarrow \beta(g) \\ P & \xrightarrow{\phi(g)(a)} & P & \xrightarrow{\pi} & X \end{array}$$

is commutative for any $a \in A$.

Then, $\mathrm{Isom}_{G,\phi}(P)$ is a group by the composition and the sequence

$$(2.7)\qquad 0 \longrightarrow A \xrightarrow[\subset]{\alpha} \mathrm{Isom}_{G,\phi}(P) \xrightarrow{\rho} \beta(G)$$

is exact by (2.4), where ρ is a homomorphism defined by $\rho(f) = \beta(g)$.

Now, we define the G-action on the set $[X, BA]$ of all equivalence classes of principal A-bundles over X by

$$(2.8)\qquad g \cdot u = B(\phi(g)) \circ u \circ \beta(g^{-1}): X \xrightarrow{\beta(g^{-1})} X \xrightarrow{u} BA \xrightarrow{B(\phi(g))} BA$$

for $g \in G$ and $u \in [X, BA]$, where $B(\phi(g)) \in \mathrm{Homeo}(BA)$ is the map induced by $\phi(g) \in \mathrm{Aut}(A)$. Set

$$(2.9)\qquad [X, BA]^G = \{u \in [X, BA] \mid g \cdot u = u \quad \text{for any} \quad g \in G\}.$$

We notice that we can identify $[X, BA]$ with $H^1(X, A)$ and then $[X, BA]^G$ is a subgroup of $H^1(X, A)$.

LEMMA 2.10. (i) *ρ in (2.7) is surjective if and only if the equivalence class $[P] \in [X, BA]$ of P belongs to $[X, BA]^G$.*

(ii) *If ρ in (2.7) is surjective, i.e., the sequence*

$$0 \longrightarrow A \xrightarrow[\subset]{} \mathrm{Isom}_{G,\phi}(P) \xrightarrow{\rho} \beta(G) \longrightarrow 1$$

is exact, then the associated operator $\phi': \beta(G) \to \mathrm{Aut}(A)$ with this extension satisfies $\phi' \circ \beta = \phi$.

PROOF. (i) Let $u: X \to BA$ be the classifying map of $\pi: P \to X$. For any $g \in G$, we consider the A-action

$$\alpha_g = \alpha \circ (\phi(g) \times 1): A \times P \xrightarrow{\phi(g) \times 1} A \times P \xrightarrow{\alpha} P$$

on P. Then we see easily that $\pi_g=\pi: P\to X$ with this A-action $\alpha_g: A\times P\to P$ is the principal A-bundle with the classifying map $B(\phi(g))^{-1}\circ u: X\to BA$.

Also, the commutativity of the diagram in (2.6) means that f is a bundle map of π to π_g over $\beta(g)$ by the above definition. Thus $f\in \mathrm{Isom}_{G,\phi}(P)$ if and only if $[u]=[B(\phi(g))^{-1}\circ u\circ\beta(g)]=g^{-1}\cdot[u]$ for $\beta(g)=\rho(f)\in G$ by the above facts and (2.8). Thus we have (i).

(ii) By the definition of ϕ' and the commutative diagram in (2.6), we see that

$$\phi'(\beta(g))(a)=f\circ a\circ f^{-1}=\phi(g)(a) \qquad \text{for} \quad \rho(f)=\beta(g).$$

q.e.d.

Here we recall the cohomology group $H^n_\phi(G, A)$ of a group G with coefficients in a G-module A by an operator $\phi: G\to \mathrm{Aut}(A)$, (cf. [4, Ch. IV, § 5]). Let $B_n(G)$ be the free $Z(G)$-module with basis $G^n=G\times\cdots\times G$ (n-times) and set

$$B^n(G, A)=\mathrm{Hom}_{Z(G)}(B_n(G), A),$$

where $Z(G)$ is the group ring of G. Then $H^n_\phi(G, A)$ is the cohomology group of the cochain comples $\{B^n(G, A), \delta\}$, whose coboundary $\delta: B^n(G, A)\to B^{n+1}(G, A)$ is defined by

$$(2.11)\qquad (\delta c)(g_1,\dots, g_{n+1})=(-1)^{n+1}\{\phi(g_1)(c(g_2,\dots, g_{n+1})) + \textstyle\sum_{i=1}^n(-1)^i c(g_1,\dots, g_i g_{i+1},\dots, g_{n+1})+(-1)^{n+1}c(g_1,\dots, g_n)\}.$$

Let $\phi_i: G_i\to\mathrm{Aut}(A)$ ($i=1, 2$) be operators and $h: G_1\to G_2$ be a homomorphism satisfying $\phi_1=\phi_2\circ h$. Then the homomorphism

$$h^*: H^n_{\phi_2}(G_2, A)\longrightarrow H^n_{\phi_1}(G_1, A)$$

is induced from the cochain map $h^\#: B^n(G_2, A)\to B^n(G_1, A)$ given by

$$h^\#(c)=(g_1,\dots, g_n)=c(h(g_1),\dots, h(g_n)).$$

We say that two extensions $0\to A\to K\to G\to 1$ and $0\to A\to K'\to G\to 1$ are *congruent* if there exists an isomorphism $\lambda: K\to K'$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & G & \longrightarrow & 1\\ & & \| & & \cong\downarrow\lambda & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & K' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

is commutative. This relation is an equivalence relation, and two congruent extensions have the same operator. We denote by

$$\mathrm{Opext}(G, A, \phi)$$

the set of all congruence classes of extensions with an operator $\phi: G \to \mathrm{Aut}(A)$.

Let $0 \to A \xrightarrow[\subset]{} K \xrightarrow{j} G \to 1$ be an extension with an operator $\phi: G \to \mathrm{Aut}(A)$. We choose a section $s: G \to K$, i.e., a map $s: G \to K$ such that $j \circ s = 1_G$, and define a cochain $c \in B^2(G, A)$ by

$$(2.12) \qquad c(g_1, g_2) = s(g_1 g_2) s(g_2)^{-1} s(g_1)^{-1} \qquad \text{for} \quad (g_1, g_2) \in G^2.$$

Then by using the equality $\phi(g)(a) = s(g) a s(g)^{-1}$ $(g \in G, a \in A)$ of (2.3), we see easily from the definition (2.11) of δ that $\delta c = 0$ and the cohomology class

$$\omega(K) = [c] \in H^2_\phi(G, A)$$

does not depend on the choice of s. Also it is clear that $\omega(K) = \omega(K')$ if K is congruent with K'. Thus we have a map

$$\omega: \mathrm{Opext}(G, A, \phi) \longrightarrow H^2_\phi(G, A).$$

THEOREM 2.13 ([4, Ch. IV, Th. 4.1]). *This map ω is bijective, and the ω-image of the semi-direct product $A \times_\phi G$ is zero in $H^2_\phi(G, A)$.*

Here, the semi-direct product $0 \to A \to A \times_\phi G \to G \to 1$ is an extension which has a homomorphism $s: G \to A \times_\phi G$ as a section.

For an operator $\phi: G \to \mathrm{Aut}(A)$ satisfying (2.4), let P_1 and P_2 be two principal A-bundles over X such that $[P_1] = [P_2] \in [X, BA]^G$, and $F: P_1 \to P_2$ be an isomorphism. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow[\subset]{\alpha} & \mathrm{Isom}_{G,\phi}(P_1) & \xrightarrow{\rho} & \beta(G) & \longrightarrow & 1 \\ & & \| & & \cong \downarrow F_* & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow[\subset]{\alpha'} & \mathrm{Isom}_{G,\phi}(P_2) & \xrightarrow{\rho'} & \beta(G) & \longrightarrow & 1 \end{array}$$

of the extensions in Lemma 2.10, where F_* is an isomorphism defined by $F_*(f) = F \circ f \circ F^{-1}$. Thus the two extensions $\mathrm{Isom}_{G,\phi}(P_i)$ $(i = 1, 2)$ are congruent, and we have the map

$$(2.14) \quad \theta': [X, BA]^G \longrightarrow \mathrm{Opext}(\beta(G), A, \phi'), \quad \theta'(P) = \mathrm{Isom}_{G,\phi}(P),$$

where $\phi': \beta(G) \to \mathrm{Aut}(A)$ is the associated operator and satisfies $\phi' \circ \beta = \phi$ by Lemma 2.10 (ii).

By the last equality $\phi' \circ \beta = \phi$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Opext}(\beta(G), A, \phi') & \xrightarrow{\omega'} & H^2_{\phi'}(\beta(G), A) \\ \downarrow \beta^* & & \downarrow \beta^* \\ \mathrm{Opext}(G, A, \phi) & \xrightarrow{\omega} & H^2_\phi(G, A), \end{array}$$

where the left β^* is induced by taking the pull backs of extensions by β, the right β^* is the induced homomorphism of β, and ω and ω' are the bijections in Theorem 2.13. We consider the following maps:

$$\theta = \beta^* \circ \theta' : [X, BA]^G \longrightarrow \mathrm{Opext}(G, A, \phi), \tag{2.15}$$

$$\Theta' = \omega' \circ \theta' : [X, BA]^G \longrightarrow H^2_\phi(\beta(G), A), \tag{2.16}$$

$$\Theta = \omega \circ \theta \ : [X, BA]^G \longrightarrow H^2_\phi(G, A). \tag{2.17}$$

Then the above commutative diagram shows the equality

$$\Theta = \beta^* \circ \Theta'. \tag{2.18}$$

§3. A characterization

Let G be a finite group, A be a discrete abelian group, and

$$0 \longrightarrow A \xrightarrow[\subset]{} K \xrightarrow{j} G \longrightarrow 1$$

be an extension with an operator $\phi: G \to \mathrm{Aut}(A)$ satisfying (2.4). Also, let X be a connected CW-complex with a G-action $\beta: G \times X \to X$, and $\pi: P \to X$ be a principal A-bundle with an A-action $\alpha: A \times P \to P$.

In this section, we study a K-action $\gamma: K \times P \to P$ such that the following diagram is commutative:

$$\begin{array}{ccccc} A \times P & \xrightarrow[\subset]{} & K \times P & \xrightarrow{j \times \pi} & G \times X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ P & = & P & \xrightarrow{\pi} & X. \end{array} \tag{3.1}$$

We call such a K-action γ on P an *extended K-action* of α over β.

REMARK 3.2. We see easily that γ *is free* (*resp. effective*) *if and only if* β *is so.*

For the existence of an extended K-action, we have the following

THEOREM 3.3. *A principal A-bundle P admits an extended K-action of α over β if and only if the class $[P]$ belongs to $[X, BA]^G$ of* (2.9) *and the equality $\Theta([P]) = \omega([K])$ holds, where Θ is the map of* (2.17) *and ω is the bijection of Theorem* 2.13.

PROOF. Suppose that there exists a K-action $\gamma: K \times P \to P$ and the diagram (3.1) is commutative. Then this K-action γ induces a homomorphism $\gamma: K \to \mathrm{Isom}_{G,\phi}(P)$ and we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow[\subset]{} & K & \xrightarrow{j} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & A & \xrightarrow[\subset]{\alpha} & \mathrm{Isom}_{G,\phi}(P) & \xrightarrow{\rho} & \beta(G) & \longrightarrow & 1, \end{array} \tag{3.4}$$

(cf. (2.6) and (2.7)). Therefore, ρ is surjective and so $[P]$ belongs to $[X, BA]^G$ by Lemma 2.10 (i). Moreover, the above commutative diagram of the extensions shows that $\theta([P])=\beta^*(\theta'([P]))=[K]$ by the definition (2.15) of θ. This implies $\Theta([P])=\omega([K])$ by the definition (2.17) of Θ.

Conversely we assume that $[P]\in[X, BA]^G$ and $\Theta([P])=\omega([K])$. Then ρ in (3.4) is surjective by Lemma 2.10 (i), and the above diagram (3.4) is commutative for some homomorphism $\gamma\colon K\to\mathrm{Isom}_{G,\phi}(P)$ by the definition of Θ. Therefore, the diagram (3.1) is commutative for this K-action γ. q.e.d.

COROLLARY 3.5. *Assume that A is a finite abelian group, the orders of the group A and $\beta(G)$ are relatively prime and $[P]\in[X, BA]^G$. Then P admits an extended $A\times_\phi G$-action, where $A\times_\phi G$ is the semi-direct product.*

PROOF. Since $H^2_{\phi'}(\beta(G), A)=0$ by the assumption and [4, Ch. IV, Prop. 5.3], we see that $\Theta([P])=0$ by (2.16) and (2.18). Thus the result follows from the above theorem and the last half of Theorem 2.13. q.e.d.

As a special case, we have the following

COROLLARY 3.6. *Assume that a finite CW-complex P has the same mod 2 cohomology as the n-sphere ($n\geqq 1$) and has a free cellular involution T. If the orbit space $X=P/T$ has a free Z_m-action, then there is a free Z_{2m}-action on P which is an extension of the given involution.*

PROOF. By the assumption, we see easily that $[X, BZ_2]=H^1(X, Z_2)=Z_2$ and so $[X, BZ_2]^{Z_m}=[X, BZ_2]$. If m is odd, then P admits a Z_{2m}-action by the above corollary.

If m is even, then $\omega([Z_{2m}])$ of the extension $0\to Z_2\to Z_{2m}\to Z_m\to 1$ is not zero in $H^2_\phi(Z_m, Z_2)=H^2(Z_m, Z_2)=Z_2$, where $\phi\colon Z_m\to\mathrm{Aut}(Z_2)=1$. If $\Theta([P])=0$ in $H^2(Z_m, Z_2)$, then P has a free $Z_2\times Z_m$-action by Theorem 3.3 and Remark 3.2. Therefore, by the assumption on P and the Gysin sequence of the fibering $P\to P/(Z_2\times Z_m)\to B(Z_2\times Z_m)$, this fact implies that the mod 2 cohomology of $Z_2\times Z_m$ is periodic in higher degrees (cf. [1, Ch. XVI, § 9, Appl. 4]), which is clearly a contradiction. Hence, we see that $\Theta([P])=\omega([Z_{2m}])$ in $H^2(Z_m, Z_2)$, and so P admits an extended Z_{2m}-action by Theorem 3.3.

The required Z_{2m}-action is free by Remark 3.2. q.e.d.

§4. Observations of the map Θ

In this section, we assume that there are given a finite group G, a discrete abelian group A, an operator $\phi: G \to \mathrm{Aut}(A)$, and a connected CW-complex X with a G-action $\beta: G \times X \to X$, and we study the map Θ of (2.17).

Let $p_G: EG \to BG$ be the universal G-bundle and $X_G = (EG \times X)/G$ be the orbit space by the diagonal G-action, and consider the fibre bundle

(4.1) $$X \xrightarrow{i} X_G \xrightarrow{p} BG, \quad i(x) = [y_0, x], \quad p([y, x]) = p_G(y),$$

where y_0 is a point of EG. Then we have the exact sequence

(4.2) $$1 \longrightarrow \pi_1(X, x_0) \xrightarrow{i=i_*} \pi_1(X_G, w_0) \xrightarrow{\chi} G \longrightarrow 1.$$

In this section, we use the following notations for the simplicity:

(4.2)′ $$\Gamma = \pi_1(X, x_0), \quad \Pi = \pi_1(X_G, w_0).$$

By considering a G-module A by ϕ as a Π-module by $\phi \circ \chi$ and as a trivial Γ-module, we have the following

LEMMA 4.3 ([4, Ch. XI, §§9–10]). *The sequence* (4.2) *induces the cohomology exact sequence*

$$0 \longrightarrow H^1(G, A) \xrightarrow{\chi^*} H^1(\Pi, A) \xrightarrow{i^*} H^1(\Gamma, A)^G \xrightarrow{\tau} H^2(G, A) \xrightarrow{\chi^*} H^2(\Pi, A).$$

Here $H^1(\Gamma, A)^G$ is the G-invariant subgroup of $H^1(\Gamma, A)$ which is a G-module by the action

(4.4) $$(g \cdot c)(y) = \phi(g)(c(z^{-1}yz))$$

$$\text{for} \quad c \in B^1(\Gamma, A),\ g = \chi(z) \in G,\ z \in \Pi,\ y \in \Gamma,$$

and τ is the transgression.

By (4.2)′, (4.4) and (2.8), we see easily the following

LEMMA 4.5. *As a G-module, we can consider naturally*

$$H^1(\Gamma, A) = H^1(X, A) = [X, BA],$$

where $[X, BA]$ is the G-module by the action (2.8).

For a given principal A-bundle $\pi: P \to X$, we can take a system $\{V_i | i \in I\}$ of coordinate neighborhoods satisfying the following properties:

(4.6) V_i is connected, and there exists only one $j = gi \in I$ such that $V_j = gV_i$ ($= \beta(g,$

V_i)) for each $i \in I$ and $g \in G$.

Let ψ_i: $A \times V_i \to \pi^{-1}(V_i)$ be the coordinate function and $f_{i,j}$: $V_i \cap V_j \to A$ be the transition function given by

$$(\psi_i^{-1} \circ \psi_j)(a, x) = (f_{i,j}(x) + a, x) \quad (x \in V_i \cap V_j,\ a \in A).$$

In the rest of this paper, we assume that

(4.7) *the equivalence class* $[P]$ *belongs to* $[X, BA]^G$.

Then by Lemma 2.10, we have the exact sequence

$$(4.8) \qquad 0 \longrightarrow A \xrightarrow[\subset]{} \mathrm{Isom}_{G,\phi}(P) \xrightarrow{\rho} \beta(G) \longrightarrow 1\,.$$

We choose a section s': $\beta(G) \to \mathrm{Isom}_{G,\phi}(P)$, $\rho \circ s' = 1_{\beta(G)}$, with $s'(1_X) = 1_P$, and set

$$(4.9) \qquad s = s' \circ \beta \colon G \longrightarrow \mathrm{Isom}_{G,\phi}(P),\ \rho \circ s = \beta,\ s(1) = 1_P.$$

LEMMA 4.10. *Under the assumption* (4.7), *there exist maps* h_i: $G \to A$ ($i \in I$) *satisfying* $h_i(1) = 0$ *and*

$$(4.11) \qquad s(g)(\psi_i(a, x)) = \psi_{gi}(\phi(g)(a) + h_i(g), gx) \quad (g \in G,\ x \in V_i),$$

$$(4.12) \qquad f_{gi,gj}(gx) = h_i(g) - h_j(g) + \phi(g)(f_{i,j}(x)) \quad (g \in G,\ x \in V_i \cap V_j),$$

where s *is the map of* (4.9) *and* $gx = \beta(g, x)$.

PROOF. The map h_i can be defined by (4.11) for $a = 0$, which does not depend on $x \in V_i$ since V_i is connected and A is discrete. Then the equality $h_i(1) = 0$ is clear and (4.11) follows from the definition of $\mathrm{Isom}_{G,\phi}(P)$ (cf. (2.6)). The equality (4.12) is obtained from (4.11) and the definition of the transition functions. q.e.d.

Now, we consider the map

$$\Theta = \omega \circ \beta^* \circ \theta' \colon [X, BA]^G \longrightarrow H^2_\phi(G, A)$$

of (2.17). Then, by (2.14) and the definition of ω in Theorem 2.13, we see immediately that

(4.13) *the image* $\Theta([P]) = [c]$ *of* $[P] \in [X, BA]^G$ *is represented by the cocycle* $c \in B^2(G, A)$ *defined by*

$$c(g_1, g_2) = s(g_1 g_2) s(g_2)^{-1} s(g_1)^{-1} \qquad (g_1, g_2 \in G),$$

where s *is the map in* (4.9).

LEMMA 4.14. *The cocycle* c *in* (4.13) *is given by*

$$c(g_1, g_2) = -h_{g_2 i}(g_1) + h_i(g_1 g_2) - \phi(g_1)(h_i(g_2)) \qquad \textit{for any} \quad i \in I,$$

where h_i is the map in the above lemma.

PROOF. By (4.11), we see that

$$s(g_1 g_2)(\psi_i(0, x)) = \psi_{g_1 g_2 i}(h_i(g_1 g_2), g_1 g_2 x),$$

$$s(g_1)s(g_2)(\psi_i(0, x)) = \psi_{g_1 g_2 i}(\phi(g_1)(h_i(g_2)) + h_{g_2 i}(g_1), g_1 g_2 x).$$

These imply the desired equality by the definition of c. q.e.d.

For an element $z=[l] \in \Pi = \pi_1(X_G, w_0)$, we take a path $\tilde{l}$ of $EG \times X$ with the initial point (y_0, x_0) such that $q \circ \tilde{l} = l$ ($q\colon EG \times X \to X_G$ is the projection). Then $\bar{l} = p_2 \circ \tilde{l}$ ($p_2\colon EG \times X \to X$ is the projection) is a path of X from x_0 to $\chi(z)x_0$, where χ is the homomorphism in (4.2). Now, we fix a coordinate neighborhood $V_{i_0} \ni x_0$, and consider a map

$$(4.15) \qquad h\colon \Pi \longrightarrow A, \ \psi_{i_0}(h(z), x_0) = \bar{l}^{\#}(\psi_{\chi(x)i_0}(0, \chi(z)x_0)),$$

where $\bar{l}^{\#}$ is the translation on P back along the path $\bar{l}$. It is clear that this definition is independent of the choice of l.

Since A is a trivial Γ-module, the coboundary $\delta\colon B^0(\Gamma, A) \to B^1(\Gamma, A)$ is zero by (2.11) and hence

$$B^1(\Gamma, A) \supset H^1(\Gamma, A) = [X, BA].$$

LEMMA 4.16. *By the induced cochain map $i^{\#}\colon B^1(\Pi, A) \to B^1(\Gamma, A)$ of i: $\Gamma \to \Pi$ in* (4.2), *the cochain $h \in B^1(\Pi, A)$ of* (4.15) *is mapped to $-[P] \in [X, BA]^G$.*

PROOF. If $z=[l] \in \Gamma = i(\Gamma) \subset \Pi$, then $\bar{l} = p_2 \circ \tilde{l} = l$. Thus we see that $i^{\#}(h) = -[P]$ by (4.15) and the definition of the characteristic map of P. q.e.d.

Furthermore, we have the following

LEMMA 4.17. *For the coboundary $\delta\colon B^1(\Pi, A) \to B^2(\Pi, A)$, we have*

$$\delta(h)(z_1, z_2) = h_{\chi(z_2)i_0}(\chi(z_1)) - h_{i_0}(\chi(z_1)) \qquad (z_1, z_2 \in \Pi).$$

PROOF. For $z_1 = [l] \in \Pi$, we take a path $\bar{l}$ of X as in the definition of (4.15) and covering $\{V_{i_j} | 0 \leqq j \leqq n\}$ of $\bar{l}([0, 1])$ with $V_{i_n} = V_{\chi(z_1)i_0}$. Also, we take real numbers $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ such that $\bar{l}([t_j, t_{j+1}]) \subset V_{i_j}$ and set $x_j = \bar{l}(t_j)$. Then by (4.15) and the definition of $\bar{l}^{\#}$, we see immediately that

$$(*) \qquad h(z_1) = \textstyle\sum_{j=1}^{n} f_{i_{j-1}, i_j}(x_j).$$

By the same way, for $z_2 = [l'] \in \Pi$, we take a path $\bar{l}'$, a covering $\{V_{k_s} | 0 \leqq s \leqq$

$n'\}$ of $\bar{l}'([0, 1])$ and x'_s $(0 \leqq s \leqq n')$. For the element $z_1 z_2 = [m]$, $m = ll'$, we see that $\bar{m} = \bar{l}(\chi(z_1) \circ \bar{l}')$. Then $\{V_{i_j} | 0 \leqq j \leqq n\} \cup \{V_{\chi(z_1)k_s} | 0 \leqq s \leqq n'\}$ is a covering of $\bar{m}([0, 1])$ and we see that

$$(**) \qquad h(z_1 z_2) = \sum_{j=1}^{n} f_{i_{j-1}, i_j}(x_j) + \sum_{s=1}^{n'} f_{\chi(z_1)k_{s-1}, \chi(z_1)k_s}(\chi(z_1)x'_s)$$

by the same way as $(*)$.

Then the desired equality is shown easily from the definition (2.11) of δ, $(*)$, $(**)$ and (4.12). q.e.d.

Now, we can prove the following

THEOREM 4.18. *The map*

$$\Theta\colon [X, BA]^G \longrightarrow H^2_\phi(G, A)$$

of (2.17) *coincides with the transgression*

$$\tau\colon H^1(\Gamma, A)^G \longrightarrow H^2_\phi(G, A)$$

in Lemma 4.3, *under the identification in Lemma* 4.5.

PROOF. Consider a cochain $\bar{h}\colon \Pi \to A$ given by

$$\bar{h}(z) = - h_{i_0}(\chi(z)) - h(z) \qquad (z \in \Pi).$$

Then, by (4.2) and Lemma 4.16, we see that

$$j^*(\bar{h}) = - i^*(h) = [P].$$

Also, the definition (2.11) of $\delta\colon B^1(\Pi, A) \to B^2(\Pi, A)$ and the above lemma show that $(\delta\bar{h})(z_1, z_2)$ is equal to

$$- \phi(\chi(z_1))(h_{i_0}(\chi(z_2))) + h_{i_0}(\chi(z_1 z_2)) - h_{\chi(z_2)i_0}(\chi(z_1)),$$

which is equal to $(\chi^* c)(z_1, z_2)$ by Lemma 4.14. Thus we have

$$\delta\bar{h} = \chi^* c.$$

These show that $\Theta([P]) = [c] = \tau([P])$ by (4.13) and the definition of the transgression τ. q.e.d.

As an application of the above theorem and Theorem 3.3, we have the following

COROLLARY 4.19 (cf. J. C. Su [5, Th. 3.13]). *Assume that the given G-action β on X has a fixed point. Then, a principal A-bundle P satisfying* (4.7) *admits an extended $A \times_\phi G$-action, where $A \times_\phi G$ is the semi-direct product.*

PROOF. The fibre bundle $X \to X_G \to BG$ of (4.1) has a cross-section

$$s: BG \longrightarrow X_G, \quad s([y]) = [y, x_0] \qquad (y \in EG),$$

where $x_0 \in X$ is a fixed point of β. Then the induced homomorphism $s_*: G \to \pi_1(X_G, w_0) = \Pi$ is a right inverse of χ in (4.2), and hence $\tau = 0$ by Lemma 4.3. Therefore, we have desired result by Theorems 4.18, 3.3 and 2.13. q.e.d.

§5. The enumeration of extended K-actions on P

In the proof of Theorem 3.3, we observe that an extended K-action $\gamma: K \times P \to P$ on a principal A-bundle P over X is a homomorphism $\gamma: K \to \mathrm{Isom}_{G,\phi}(P)$ such that

(5.1) the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{i}{\underset{\subset}{\longrightarrow}} & K & \overset{j}{\longrightarrow} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & A & \overset{\alpha}{\underset{\subset}{\longrightarrow}} & \mathrm{Isom}_{G,\phi}(P) & \overset{\rho}{\longrightarrow} & \beta(G) & \longrightarrow & 1. \end{array}$$

In this section, we enumerate the number of extended K-actions on P.

Suppose that $\gamma_i: K \to \mathrm{Isom}_{G,\phi}(P)$ (i=1, 2) are two homomorphism satisfying (5.1), and define a 1-cochain $d: K \to A$ by

$$d(x) = \gamma_1(x) \circ \gamma_2(x)^{-1} \qquad (x \in K). \tag{5.2}$$

Here A is a K-module by $\phi \circ j$, i.e.,

$$x \cdot a = \gamma(x) \circ a \circ \gamma(x)^{-1} \qquad (x \in K,\ a \in A)$$

for any homomorphism $\gamma: K \to \mathrm{Isom}_{G,\phi}(P)$ satisfying (5.1), by (2.3).

Then, we see easily that $d \in B^1(K, A)$ is a cocycle, and we denote its cohomology class by

$$d(\gamma_1, \gamma_2) = [d] \in H^1(K, A).$$

The upper exact sequence in (5.1) induces the cohomology exact sequence

$$0 \longrightarrow H^1(G, A) \overset{j^*}{\longrightarrow} H^1(K, A) \overset{i^*}{\longrightarrow} H^1(A, A)^G \tag{5.3}$$

by the same way as Lemma 4.3. Therefore, by (5.2) and the commutativity of the left square in (5.1) for $\gamma = \gamma_i$, we have

LEMMA 5.4. $d(\gamma_1, \gamma_2) \in \mathrm{Im}\, j^*$.

LEMMA 5.5. *If $d(\gamma_1, \gamma_2) = 0$, then there is some element $a \in A$ such that*

$$\gamma_2(x) = a^{-1} \circ \gamma_1(x) \circ a \qquad (x \in K).$$

PROOF. By the assumption, we see that

$$\gamma_1(x) \circ \gamma_2(x)^{-1} = d(x) = (\delta a)(x) = a - x \cdot a = a \circ \gamma_2(x) \circ a^{-1} \circ \gamma_2(x)^{-1}$$

for some $a \in A = B^0(K, A)$. q.e.d.

LEMMA 5.6. *Let* $\gamma_1 : K \to \mathrm{Isom}_{G,\phi}(P)$ *be a homomorphism satisfying* (5.1) *and* $[d]$ *be any element of* $\mathrm{Im}\, j^*$. *Then, there exists a homomorphism* $\gamma_2 : K \to \mathrm{Isom}_{G,\phi}(P)$ *satisfying* (5.1) *and* $d(\gamma_1, \gamma_2) = [d]$.

PROOF. We see easily the lemma, by defining

$$\gamma_2(x) = d(x)^{-1} \circ \gamma_1(x) \qquad (x \in K). \qquad \text{q.e.d.}$$

Now, we have immediately the following theorem by the above lemmas.

THEOREM 5.7. *For a given G-action on X and an extension* $0 \to A \to K \to G \to 1$, *suppose that a principal A-bundle P over X admits an extended K-action. Then the set*

$$EA(P, K)$$

of all equivalence classes of extended K-actions on P under the conjugation by elements of A is equivalent to $H^1_\phi(G, A) \cong \mathrm{Im}\, j^*$ *as a set.*

COROLLARY 5.8. *If A is a finite abelian group and the orders of A and G are relatively prime in addition, then EA(P, K) consists of one element.*

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