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On Nonoscillatory Solutions of Functional Differential Equations with a General Deviating Argument

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1. Introduction

The equation to be studied in this paper is

$$x^{(n)}(t) + \sigma f(t, x(g(t))) = 0, \tag{A}$$

where the following conditions are always assumed to hold:

(a) $n \geqq 2$, $\sigma = \pm 1$;

(b) $g(t)$ is continuous on $[a, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(c) $f(t, x)$ is continuous on $[a, \infty)\times(-\infty, \infty)$ and $xf(t, x) \geqq 0$.

It is to be noted that $g(t)$ is a general deviating argument, that is, it is allowed to be *retarded* ($g(t)\leqq t$) or *advanced* ($g(t)\geqq t$) or otherwise.

Equation (A) is called *superlinear* if, for each t,

$$|f(t, x_1)|/|x_1| \geqq |f(t, x_2)|/|x_2| \quad \text{for} \quad |x_1| > |x_2|,\ x_1x_2 > 0,$$

and *strongly superlinear* if there is a number $\alpha > 1$ such that, for each t,

$$|f(t, x_1)|/|x_1|^\alpha \geqq |f(t, x_2)|/|x_2|^\alpha \quad \text{for} \quad |x_1| > |x_2|,\ x_1x_2 > 0.$$

Dually, equation (A) is called *sublinear* if, for each t,

$$|f(t, x_1)|/|x_1| \leqq |f(t, x_2)|/|x_2| \quad \text{for} \quad |x_1| > |x_2|,\ x_1x_2 > 0,$$

and *strongly sublinear* if there is a positive number $\beta < 1$ such that, for each t,

$$|f(t, x_1)|/|x_1|^\beta \leqq |f(t, x_2)|/|x_2|^\beta \quad \text{for} \quad |x_1| > |x_2|,\ x_1x_2 > 0.$$

In this paper we are primarily interested in the nonoscillatory solutions of equation (A) which is either strongly superlinear or strongly sublinear. Of particular interest is the effect that $g(t)$ can have on the nonoscillation properties of (A). Hereafter, the term "solution" will be used to mean a solution $x(t)$ of (A) which is defined on some half-line $[T_x, \infty)$ and is nontrivial on any infinite subinterval of $[T_x, \infty)$. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros; otherwise it is said to be *nonoscillatory*.

Following Kiguradze [1] and Lovelady [9], we classify the nonoscillatory solutions of (A) into $(n+1)$ classes (some or all of which may be empty) in such a way that all solutions $x(t)$ with all of $x(t)$, $x'(t),\dots, x^{(n)}(t)$ of the same eventual sign fall into the same classes. We then give, for some of these classes, necessary and/or sufficient conditions for them to be nonempty. As a consequence we are able to indicate certain classes of equations of the form (A) for which necessary and sufficient conditions for oscillation of all solutions are established. Our results extend considerably some of the fundamental results of Kiguradze [1, 2], Lovelady [9] and Onose [11].

2. Classification of nonoscillatory solutions

A classification of nonoscillatory solutions of (A) will be given according to the following lemma.

LEMMA 1. *If $x(t)$ is a nonoscillatory solution of* (A), *then either*

$$x(t)x^{(i)}(t) > 0 \quad (0 \leqq i \leqq n-1), \quad \sup_{s \geqq t}[x(s)x^{(n)}(s)] > 0 \tag{1}$$

for all sufficiently large t, or there exists an integer k, $0 \leqq k \leqq n-1$, such that

$$x(t)x^{(i)}(t) > 0 \ (0 \leqq i \leqq k), \quad x^{(i)}(t)x^{(i+1)}(t) \leqq 0 \ (k \leqq i \leqq n-1) \tag{2}$$

for all sufficiently large t.

PROOF. Let $x(t)$ be a nonoscillatory solution of (A). From (A), $\operatorname{sgn}[x(t)x^{(n)}(t)] = -\operatorname{sgn}\sigma$ for all large t, so that there is a $T_x \geqq a$ such that each of $x'(t), x''(t),\dots, x^{(n-1)}(t)$ is either nonnegative or nonpositive on $[T_x, \infty)$. Let k be the largest integer such that $x(t)x^{(i)}(t) > 0$ on $[T_x, \infty)$ if $0 \leqq i \leqq k$. When $\sigma = -1$, it may happen that $k = n$. Suppose that $k = n-1$. We cannot exclude the possibility that $\sup_{s \geqq t}[x(s)x^{(n)}(s)] > 0$, $t \geqq T_x$, which takes place only when $\sigma = -1$. Otherwise, we have $x(t)x^{(n)}(t) \leqq 0$ or $x^{(n-1)}(t)x^{(n)}(t) \leqq 0$ on $[T_x, \infty)$. Suppose next that $0 \leqq k \leqq n-2$. The claim is that, for every i with $k \leqq i \leqq n-2$, $x^{(i)}(t)x^{(i+1)}(t) \leqq 0$, $t \geqq T_x$, implies $x^{(i+1)}(t)x^{(i+2)}(t) \leqq 0$, $t \geqq T_x$. Assume to the contrary that $x^{(i+1)}(t)x^{(i+2)}(t) \geqq 0$, $t \geqq T_x$, and there is a $\tau \geqq T_x$ such that $x^{(i+1)}(\tau)x^{(i+2)}(\tau) > 0$. Then, in view of the monotonicity of $x^{(i+1)}(t)$ we see that

$$\begin{aligned} |x^{(i)}(t)| &= |x^{(i)}(\tau)| - \int_\tau^t |x^{(i+1)}(s)|ds \\ &\leqq |x^{(i)}(\tau)| - |x^{(i+1)}(\tau)|(t-\tau), \qquad t \geqq \tau, \end{aligned}$$

which implies that $|x^{(i)}(t)| \to -\infty$ as $t \to \infty$, a contradiction. Hence (2) holds if $0 \leqq k \leqq n-2$.

DEFINITION. A nonoscillatory solution of (A) which satisfies (1) is called a solution of class $\mathcal{N}_n$. A nonoscillatory solution of (A) which satisfies (2) for some k, $0 \leqq k \leqq n-1$, is called a solution of class $\mathcal{N}_k$.

REMARK 1. Every solution of class $\mathcal{N}_0$ is bounded. If $x(t)$ is a solution of class $\mathcal{N}_n$, then there is a constant $c>0$ such that $|x(t)| \geqq ct^{n-1}$ for all large t. If $x(t)$ is a solution of class $\mathcal{N}_k$ with $1 \leqq k \leqq n-1$, then there are constants $c_1>0$, $c_2>0$ such that $c_1 t^{k-1} \leqq |x(t)| \leqq c_2 t^k$ for all large t.

LEMMA 2. *Let $x(t)$ be a solution of* (A) *of class $\mathcal{N}_k$ for some k, $0 \leqq k \leqq n-1$. Then the limit $x^{(k)}(\infty) = \lim_{t\to\infty} x^{(k)}(t)$ is finite and there is a T_x such that*

$$(3) \qquad x^{(k)}(t) = x^{(k)}(\infty) + (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, x(g(s)))ds$$

for $t \geqq T_x$.

PROOF. If $k=n-1$, then an integration of (A) yields (3). Let $k \leqq n-2$ and $k+1 \leqq i \leqq n-1$. If we suppose $\lim_{t\to\infty} |x^{(i)}(t)| > 0$, then from the inequality

$$\begin{aligned} |x^{(i-1)}(t)| &= |x^{(i-1)}(T_x)| - \int_{T_x}^t |x^{(i)}(s)|ds \\ &\leqq |x^{(i-1)}(T_x)| - |x^{(i)}(t)|(t-T_x), \qquad t \geqq T_x, \end{aligned}$$

where $T_x \geqq a$ is chosen sufficiently large, we have that $|x^{(i-1)}(t)| \to -\infty$ as $t\to\infty$, which is impossible. Therefore, we must have $x^{(i)}(t) \to 0$ as $t \to \infty$. Taking this fact into account and integratig (A) repeatedly form t to ∞, we find

$$(4) \qquad x^{(i)}(t) = (-1)^{n-i-1}\sigma \int_t^\infty \frac{(s-t)^{n-i-1}}{(n-i-1)!} f(s, x(g(s)))ds$$

if $k+1 \leqq i \leqq n-1$. Integrating (4) with $i=k+1$ once more and noting that $x^{(k)}(\infty)$ exists and is finite, we arrive at the desired inequality (3).

3. Necessary conditions

In this section we wish to find, for each $k=1,\ldots, n-1$, a necessary condition for (A) to have a solution of class $\mathcal{N}_k$. A discussion of the extreme classes $\mathcal{N}_0$ and $\mathcal{N}_n$ seems to be difficult. We begin with a preliminary observation.

PROPOSITION. *Let* (A) *be either superlinear or sublinear. Suppose that* (A) *has a solution $x(t)$ of class $\mathcal{N}_k$ for some k with $0 \leqq k \leqq n-1$.*

If either $\sigma=1$ and $k \equiv n$ (mod 2) or $\sigma=-1$ and $k \not\equiv n$ (mod 2), then $x(t)$ is eventually a polynomial of degree k and there are numbers $T \geqq a$ and $c \neq 0$ such *that* $f(t, cg^k(t))=0$ *for* $t\geqq T$.

PROOF. By Lemma 2 there is a T_x such that (3) holds for $t\geqq T_x$. Hence, using the hypotheses, we have

$$\int_t^\infty (s-t)^{n-k-1} f(s, x(g(s)))ds = (n-k-1)!\{x^{(k)}(\infty) - x^{(k)}(t)\} \tag{5}$$

for $t\geqq T_x$. Suppose that $x(t)$ is positive. Then, the left member of (5) is nonnegative, while the right member is nonpositive for $t\geqq T_x$. It follows therefore that $x^{(k)}(t)=x^{(k)}(\infty)$ and $f(t, x(g(t)))=0$ for $t\geqq T_x$. Thus $x(t)$ is a polynomial of degree k on $[T_x, \infty)$, and so there are positive numbers c_1, c_2 such that

$$c_1 g^k(t) \leqq x(g(t)) \leqq c_2 g^k(t) \quad \text{for} \quad t \geqq T, \tag{6}$$

where T is taken so large that $g(t)\geqq T_x$ for $t\geqq T$. Combining (6) with $f(t, x(g(t)))=0$, we conclude that $f(t, c_i g^k(t))=0$ for $t\geqq T$, where $i=1$ or 2 according as (A) is superlinear or sublinear. A similar argument holds if we suppose that $x(t)$ is negative.

REMARK 2. Suppose that

$$\sup_{t\geqq T} |f(t, c)| > 0 \qquad \text{for all} \quad T\geqq a \quad \text{and} \quad c\neq 0 \tag{7}$$

if (A) is superlinear, and

$$\sup_{t\geqq T} |f(t, cg^{n-1}(t))| > 0 \qquad \text{for all} \quad T\geqq a \quad \text{and} \quad c\neq 0 \tag{8}$$

if (A) is sublinear. Then, from the above proposition we see that equation (A) with $\sigma=1$ [resp. with $\sigma=-1$] cannot possess solutions of class $\mathcal{N}_k$ for k with $k\equiv n \pmod 2$ [resp. with $k\not\equiv n \pmod 2$].

Motivated by this observation, we shall assume without further mention that condition (7) or (8) holds throughout the sequel. The following notation will be used: $g_*(t)=\min\{g(t), t\}$.

We state and prove the main result of this section.

THEOREM 1. *Let* (A) *be strongly superlinear. Suppose that* (A) *has a solution of class* $\mathcal{N}_k$, *where* k *is such that* $1\leqq k\leqq n-1$ *and* $k\not\equiv n \pmod 2$ *if* $\sigma=1$ *and* $k\equiv n \pmod 2$ *if* $\sigma=-1$.

Then, for every differentiable function $h_*(t)$ *satisfying*

$$h_*(t) \leqq g_*(t), \ h'_*(t) \geqq 0, \quad \textit{and} \quad \lim_{t\to\infty} h_*(t) = \infty\,, \tag{9}$$

we have

$$(10) \qquad \int^{\infty} h_*^{n-k}(t)\,|f(t, cg^{k-1}(t))|dt < \infty \qquad \textit{for some} \quad c \neq 0\,.$$

THEOREM 2. *Let* (A) *be strongly sublinear. Suppose that* (A) *has a solution of class* $\mathcal{N}_k$, *where* k *is such that* $1 \leqq k \leqq n-1$ *and* $k \not\equiv n \pmod 2$ *if* $\sigma = 1$ *and* $k \equiv n \pmod 2$ *if* $\sigma = -1$.

Then we have

$$(11) \qquad \int^{\infty}\left(\frac{g_*(t)}{g(t)}\right)^{\beta k} t^{n-k-1}\,|f(t, cg^{k}(t))|\,dt < \infty \qquad \textit{for some} \quad c \neq 0\,,$$

where $\beta < 1$ *is the sublinearity constant for* (A).

In proving these theorems we use an integral inequality which is stated below without proof.

LEMMA 3. *If* $p < q \leqq r$ *and* $\mu,\ \nu \geqq 0$, *then*

$$\int_p^q (q-t)^{\mu}(r-t)^{\nu} \geqq \frac{1}{\mu+\nu+1}(q-p)^{\mu+1}(r-p)^{\nu}\,.$$

PROOF OF THEOREM 1. Let $x(t)$ be a solution of class $\mathcal{N}_k$ as stated above. Without loss of generality we may suppose that $x(t)$ is positive. Noting that $x^{(k-1)}(t)$ is positive and increasing, we have

$$(12) \qquad \begin{aligned} x(s) &= \sum_{i=0}^{k-2} \frac{(s-t)^i}{i!} x^{(i)}(t) + \int_t^s \frac{(s-u)^{k-2}}{(k-2)!} x^{(k-1)}(u)du \\ &\geqq \frac{(s-t)^{k-1}}{(k-1)!} x^{(k-1)}(t) \quad \text{for} \quad s \geqq t \geqq T_x\,, \end{aligned}$$

where $T_x \geqq a$ is sufficiently large. Take an $h_*(t)$ satisfying (9) and let $T \geqq T_x$ be so large that $h_*(t) \geqq T_x$ for $t \geqq T$. From (12) we have

$$(13) \qquad x(g(s)) \geqq \frac{[g(s)-h_*(t)]^{k-1}}{(k-1)!} x^{(k-1)}(h_*(t)), \quad s \geqq t \geqq T\,.$$

On the other hand, there is a constant $c > 0$ such that

$$(14) \qquad x(g(s)) \geqq cg^{k-1}(s) \qquad \text{for} \quad s \geqq T\,.$$

Using (3), (13) and (14) we have

$$\begin{aligned} x^{(k)}(h_*(t)) &\geqq \int_{h_*(t)}^{\infty} \frac{[s-h_*(t)]^{n-k-1}}{(n-k-1)!} f(s, x(g(s)))ds \\ &\geqq \int_t^{\infty} \frac{[h_*(s)-h_*(t)]^{n-k-1}}{(n-k-1)!} f(s, cg^{k-1}(s))\left(\frac{x(g(s))}{cg^{k-1}(s)}\right)^{\alpha} ds \end{aligned}$$

$$\geqq \frac{[x^{(k-1)}(h_*(t))]^\alpha}{[c(k-1)!]^\alpha(n-k-1)!}\int_t^\infty [h_*(s)-h_*(t)]^{n-k-1}\cdot$$

$$\cdot f(s, cg^{k-1}(s))\left(\frac{g(s)-h_*(t)}{g(s)}\right)^{\alpha(k-1)} ds,$$

where $\alpha>1$ is the superlinearity constant for (A). Setting $M=[c(k-1)!]^\alpha\cdot(n-k-1)!$, we compute as follows:

$$\frac{M}{\alpha-1}[x^{(k-1)}(h_*(T))]^{1-\alpha}$$

$$\geqq M\int_T^\infty h_*'(t)x^{(k)}(h_*(t))\,[x^{(k-1)}(h_*(t))]^{-\alpha}dt$$

$$\geqq \int_T^\infty h_*'(t)\int_t^\infty [h_*(s)-h_*(t)]^{n-k-1}[g(s)-h_*(t)]^{\alpha(k-1)}\frac{f(s, cg^{k-1}(s))}{g^{\alpha(k-1)}(s)}dsdt$$

$$\geqq \int_T^\infty \frac{f(s, cg^{k-1}(s))}{g^{\alpha(k-1)}(s)}\int_T^s h_*'(t)\,[h_*(s)-h_*(t)]^{n-k-1}[g(s)-h_*(t)]^{\alpha(k-1)}dtds$$

$$=\int_T^\infty \frac{f(s, cg^{k-1}(s))}{g^{\alpha(k-1)}(s)}\int_{h_*(T)}^{h_*(s)} [h_*(s)-t]^{n-k-1}[g(s)-t]^{\alpha(k-1)}dtds.$$

Since by Lemma 3

$$\int_{h_*(T)}^{h_*(s)} [h_*(s)-t]^{n-k-1}[g(s)-t]^{\alpha(k-1)}dt$$

$$\geqq \frac{[h_*(s)-h_*(T)]^{n-k}[g(s)-h_*(T)]^{\alpha(k-1)}}{n-k+\alpha(k-1)},$$

we conclude that

$$\int_T^\infty\left(1-\frac{h_*(T)}{g(s)}\right)^{\alpha(k-1)}[h_*(s)-h_*(T)]^{n-k}f(s, cg^{k-1}(s))ds<\infty,$$

from which (10) follows immediately. This completes the proof.

PROOF OF THEOREM 2. Let $x(t)$ be a solution of class $\mathcal{N}_k$ as stated in Theorem 2. We may suppose that $x(t)$ is positive. For a sufficiently large T we have

$$\begin{aligned} x(t) &= \sum_{i=0}^{k-1}\frac{(t-T)^i}{i!}x^{(i)}(T)+\int_T^t\frac{(t-s)^{k-1}}{(k-1)!}x^{(k)}(s)ds \\ &\geqq \int_T^t \frac{(t-s)^{k-1}}{(k-1)!}x^{(k)}(s)ds \qquad \text{for} \quad t\geqq T. \end{aligned} \tag{15}$$

From the proof of Lemma 2 we easily see that

$$
\begin{aligned}
x^{(k)}(s) &= x^{(k)}(\infty) + \int_s^\infty \frac{(u-s)^{i-k-1}}{(i-k-1)!}\,|x^{(i)}(u)|\,du \\
&\geqq \int_s^t \frac{(u-s)^{i-k-1}}{(i-k-1)!}\,|x^{(i)}(u)|\,du \\
&\geqq \frac{(t-s)^{i-k}}{(i-k)!}\,|x^{(i)}(t)|
\end{aligned} \tag{16}
$$

for each i with $k \leqq i \leqq n-1$ and for $t \geqq s \geqq T$. Here the nonincreasing nature of $|x^{(i)}(t)|$ has been used. Let $T_0 \geqq T$ be such that $g_*(t) \geqq T$ for $t \geqq T_0$. Combining (15) with (16) and then applying Lemma 3, we find

$$
\begin{aligned}
x(g(t)) \geqq x(g_*(t)) &\geqq \int_T^{g_*(t)} \frac{[g_*(t)-s]^{k-1}}{(k-1)!}\,x^{(k)}(s)\,ds \\
&\geqq \frac{|x^{(i)}(t)|}{(k-1)!(i-k)!}\int_T^{g_*(t)} [g_*(t)-s]^{k-1}(t-s)^{i-k}\,ds \\
&\geqq \frac{[g_*(t)-T]^k(t-T)^{i-k}}{i(k-1)!(i-k)!}\,|x^{(i)}(t)|
\end{aligned} \tag{17}
$$

for $k \leqq i \leqq n-1$ and $t \geqq T_0$.

Let us define

$$
w(t) = \sum_{i=k}^{n-1} \frac{(T-t)^{i-k}}{(i-k)!}\,x^{(i)}(t), \quad t \geqq T. \tag{18}
$$

It is clear that $w(t) \geqq 0$ on $[T, \infty)$ and

$$
w'(t) = -\frac{(t-T)^{n-k-1}}{(n-k-1)!}f(t, x(g(t))), \quad t \geqq T_0. \tag{19}
$$

Moreover, from (17) and (18) it follows that

$$
x(g(t)) \geqq \frac{[g_*(t)-T]^k}{N}\,w(t), \quad t \geqq T_0, \tag{20}
$$

where $N=(k-1)!\sum_{i=k}^{n-1} i$. Noting that $x(g(t)) \leqq cg^k(t)$ on $[T_0, \infty)$ for some $c>0$ and using (19) and (20), we get for $t \geqq T_0$

$$
\begin{aligned}
-w'(t) &= \frac{(t-T)^{n-k-1}}{(n-k-1)!}f(t, x(g(t))) \\
&\geqq \frac{(t-T)^{n-k-1}}{(n-k-1)!}\,f(t, cg^k(t))\left(\frac{x(g(t))}{cg^k(t)}\right)^\beta \\
&\geqq \frac{(t-T)^{n-k-1}}{(cN)^\beta(n-k-1)!}\left(\frac{g_*(t)-T}{g(t)}\right)^{\beta k} f(t, cg^k(t))w^\beta(t).
\end{aligned}
$$

From the above relation we readily deduce that

$$\int_{T_0}^{\infty}\left(\frac{g_*(t)-T}{g(t)}\right)^{\beta k}(t-T)^{n-k-1}f(t,\ cg^k(t))dt$$

$$\leqq -L\int_{T_0}^{\infty} w^{-\beta}(t)w'(t)\,dt \leqq \frac{L}{1-\beta}\,w^{1-\beta}(T_0)\,,$$

where $L=(cN)^{\beta}(n-k-1)!$. This clearly is equivalent to (11), and the proof is complete.

4. Sufficient conditions

The purpose of this section is to obtain a condition under which equation (A) possesses a solution of class $\mathcal{N}_k$ for each $k=1,\dots, n-1$. Our discussion is based on the following existence theorem.

THEOREM 3. *Let* $0\leqq k\leqq n-1$ *and suppose that*

$$\int^{\infty} t^{n-k-1}|f(t,\ cg^k(t))|dt<\infty \qquad \text{for some} \quad c\neq 0\,. \tag{21}$$

(I) *If* (A) *is superlinear, then* (A) *has a solution* $x(t)$ *such that*

$$\lim_{t\to\infty}\frac{x(t)}{t^k}=\frac{c}{2}\,. \tag{22}$$

(II) *If* (A) *is sublinear, then* (A) *has a solution* $x(t)$ *such that*

$$\lim_{t\to\infty}\frac{x(t)}{t^k}=2c\,. \tag{23}$$

PROOF. Without loss of generality we may suppose that the constant c appearing in (21) is positive.

(I) Let (A) be superlinear. Choose T so large that $T_0=\inf_{t\geqq T} g_*(t)>a$ and

$$\int_T^{\infty} t^{n-k-1}f(t,\ cg^k(t))dt\leqq \frac{c}{2}\,k!(n-k-1)!\,.$$

Let C denote the vector space of all continuous functions on $[T_0,\infty)$ with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$. Set

$$X=\{x\in C\colon 0\leqq x(t)\leqq ct^k \qquad \text{for} \quad t\geqq T_0\}\,.$$

Clearly, X is a convex and closed subset of C. We define the operator Φ acting on X by the following formulas:

$$(\Phi x)(t) = \begin{cases} \dfrac{c}{2} + \sigma \displaystyle\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(g(s)))ds, & t \geqq T, \\ \dfrac{c}{2} + \sigma \displaystyle\int_T^\infty \frac{(T-s)^{n-1}}{(n-1)!} f(s, x(g(s)))ds, & T_0 \leqq t \leqq T, \end{cases}$$

if $k=0$, and

$$(\Phi x)(t) = \begin{cases} \dfrac{c}{2} t^k + \sigma \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(s-u)^{n-k-1}}{(n-k-1)!} f(u, x(g(u)))duds, \\ \hfill t \geqq T, \\ \dfrac{c}{2} t^k, \ T_0 \leqq t \leqq T, \end{cases}$$

if $1 \leqq k \leqq n-1$.

i) *Φ maps X into X.* Using the superlinearity, we have

$$\begin{aligned} \left| (\Phi x)(t) - \frac{c}{2} t^k \right| &\leqq \frac{(t-T)^k}{k!} \int_T^\infty \frac{(u-T)^{n-k-1}}{(n-k-1)!} f(u, x(g(u)))du \\ &\leqq \frac{t^k}{k!(n-k-1)!} \int_T^\infty u^{n-k-1} f(u, x(g(u)))\, du \\ &\leqq \frac{c}{2} t^k \quad \text{for} \quad t \geqq T \quad \text{and} \quad 0 \leqq k \leqq n-1 . \end{aligned}$$

Thus $0 \leqq (\Phi x)(t) \leqq ct^k$ on $[T, \infty)$. The validity of these inequalities on $[T_0, T]$ is trivial.

ii) *Φ is continuous on X.* Let $\{x_m\}$ be a sequence in X converging to an $x \in X$ as $m \to \infty$ in the topology of C. This means that $x_m(t) \to x(t)$ as $m \to \infty$ uniformly on any compact subinterval of $[T_0, \infty)$. By the definition of Φ we have

$$|(\Phi x_m)(t) - (\Phi x)(t)| \leqq \frac{1}{(n-1)!} \int_T^\infty s^{n-1} G_m(s)ds, \quad t \geqq T_0,$$

if $k=0$, and

$$|(\Phi x_m)(t) - (\Phi x)(t)| \begin{cases} \leqq \dfrac{t^k}{k!(n-k-1)!} \displaystyle\int_T^\infty s^{n-k-1} G_m(s)ds, & t \geqq T, \\ = 0, \quad T_0 \leqq t \leqq T, \end{cases}$$

if $1 \leqq k \leqq n-1$, where $G_m(s) = |f(s, x_m(g(s))) - f(s, x(g(s)))|$. Since $G_m(s) \leqq 2f(s, cg^k(s))$ and $G_m(s) \to 0$ as $m \to \infty$ for $s \geqq T$, applying the Lebesgue dominated convergence theorem, we conclude that $(\Phi x_m)(t) \to (\Phi x)(t)$ as $m \to \infty$ uniformly on any compact subinterval of $[T_0, \infty)$, that is, $\Phi x_m \to \Phi x$ as $m \to \infty$ in the topology of C.

iii) *ΦX is precompact.* Differentiating $(\Phi x)(t)$, we find that

$$|(\Phi x)'(t)| \leqq \frac{1}{(n-2)!}\int_T^\infty s^{n-2} f(s, c)ds, \quad t \geqq T,$$

if $k=0$, and

$$|(\Phi x)'(t)| \leqq \begin{cases} \dfrac{ck}{2} t^{k-1} + \dfrac{t^{k-1}}{(k-1)!(n-k-1)!}\displaystyle\int_T^\infty s^{n-k-1} f(s, cg^k(s))ds, \\ \hfill t \geqq T, \\ \dfrac{ck}{2} t^{k-1}, \quad T_0 \leqq t \leqq T, \end{cases}$$

if $1 \leqq k \leqq n-1$. It follows that the family $\{(\Phi x)(t)\}$ is uniformly bounded and equicontinuous at every point of $[T_0, \infty)$. This shows that ΦX is precompact.

From the foregoing observations we are able to apply the strong version of Tychonoff's fixed-point theorem (see e.g. Morris and Noussair [10]) to the operator Φ and conclude that Φ has a fixed point x in X. It is a simple matter to check that this fixed point $x=x(t)$ is a solution of equation (A) having the asymptotic property (22).

(II) Let (A) be sublinear. Take T so large that $T_0 = \inf_{t \geqq T} g_*(t) > a$ and

$$\int_T^\infty t^{n-k-1} f(t, cg^k(t))dt \leqq \frac{c}{3} k!\,(n-k-1)!\,.$$

Put $Y=\{x \in C: ct^k \leqq x(t) \leqq 3ct^k \text{ for } t \geqq T_0\}$. We denote by Ψ the operator Ψ which has the same expressions as Φ studied above except that the constant $c/2$ is replaced by $2c$ wherever it appears. Then it can be shown as in the proof of the first part that Ψ is a continuous operator which maps Y into a compact subset of Y. A solution of (A) with the property (23) is provided by a fixed point of Ψ in Y.

This completes the proof of Theorem 3.

A sufficient condition for a class $\mathcal{N}_k$, $1 \leqq k \leqq n-1$, to have a member is given in the following theorems.

THEOREM 4. *Let k be an integer such that $1 \leqq k \leqq n-1$, and $k \not\equiv n$ (mod 2) if $\sigma=1$ and $k \equiv n$ (mod 2) if $\sigma=-1$. Equation (A) which is superlinear has a solution of class $\mathcal{N}_k$ if*

$$\text{(24)} \qquad \int^\infty t^{n-k}|f(t, cg^{k-1}(t))|dt < \infty \qquad \textit{for some} \quad c \neq 0\,.$$

THEOREM 5. *Let k be as in Theorem 4. Equation (A) which is sublinear* *has a solution of class* $\mathcal{N}_k$ *if* (21) *holds for some* $c \neq 0$.

PROOF OF THEOREM 4. On account of Theorem 3 equation (A) has a solution $x(t)$ with the property $\lim_{t\to\infty} x(t)/t^{k-1} = c/2$. By Lemma 1, $x(t)$ is of class $\mathcal{N}_j$ for some j, $0 \leqq j \leqq n$, and from Remark 2 it follows that $j \equiv k \pmod 2$. If $j=0$, then $x(t)$ is bounded, and so $k=1$, which is a contradiction. If $j=n$, then $\liminf_{t\to\infty} |x(t)|/t^{n-1} > 0$. But this is not consistent with the property $\lim_{t\to\infty} x(t)/t^{k-1} = c/2$, since $k \leqq n-1$. Consequently, we must have $1 \leqq j \leqq n-1$, and there are positive constants c_1, c_2 such that $c_1 t^{j-1} \leqq |x(t)| \leqq c_2 t^j$ for all sufficiently large t. It follows that $j \leqq k \leqq j+1$, which implies $j=k$.

Theorem 5 can be proved similarly.

Now, we suppose that there is a differentiable function $h_*(t)$ which satisfies condition (9) and

$$\liminf_{t\to\infty} \frac{h_*(t)}{t} > 0\,. \tag{25}$$

Then, obviously, condition (10) is equivalent to (24), so that by combining Theorem 1 with Theorem 4 we obtain the following result.

THEOREM 6. *Let* k *be an integer as described in Theorem* 4. *If there exists a function* $h_*(t)$ *satisfying* (9) *and* (25), *then* (24) *is a necessary and sufficient condition for* (A) *which is strongly superlinear to have a solution of class* $\mathcal{N}_k$.

Likewise, from Theorem 2 and Theorem 5 we have the following

THEOREM 7. *Let* k *be as above. Assume that*

$$\liminf_{t\to\infty} \frac{g_*(t)}{g(t)} > 0\,. \tag{26}$$

Then, (21) *is a necessary and sufficient condition for* (A) *which is strongly sublinear to have a solution of class* $\mathcal{N}_k$.

REMARK 3. It is easy to see that (25) holds if $g(t)$ is advanced ($g(t) \geqq t$) and (26) holds if $g(t)$ is retarded ($g(t) \leqq t$).

REMARK 4. Theorem 7 includes a recent result of Lovelady [9] as a special case.

5. Oscillation criteria

Finally we shall show that the above results are used to obtain necessary and sufficient conditions for equation (A) to be "almost oscillatory" in the sense defined below.

DEFINITION. Equation (A) is said to be *almost oscillatory* if one of the following cases holds:

(i) $\sigma=1$, n is even, and every solution of (A) is oscillatory;

(ii) $\sigma=1$, n is odd, and every solution of (A) is oscillatory or tends monotonically to zero as $t\to\infty$ together with $x'(t),\dots, x^{(n-1)}(t)$;

(iii) $\sigma=-1$, n is even, and every solution of (A) is oscillatory or tends monotonically to zero or infinity as $t\to\infty$ together with $x'(t),\dots, x^{(n-1)}(t)$;

(iv) $\sigma=-1$, n is odd, and every solution of (A) is oscillatory or tends monotonically to infinity as $t\to\infty$ together with $x'(t),\dots, x^{(n-1)}(t)$.

THEOREM 8. *Let* (A) *be strongly superlinear. Suppose that there exists a function $h_*(t)$ satisfying* (9) *and* (25). *Then,* (A) *is almost oscillatory if and only if*

$$\int^{\infty} t^{n-1}|f(t, c)|dt = \infty \qquad \textit{for all} \quad c \neq 0. \tag{27}$$

THEOREM 9. *Let* (A) *be strongly sublinear. Suppose that $g_*(t)$ satisfies* (26). *Then,* (A) *is almost oscillatory if and only if*

$$\int^{\infty} |f(t, cg^{n-1}(t))|dt = \infty \qquad \textit{for all} \quad c \neq 0. \tag{28}$$

PROOF OF THEOREM 8. The "only if" part is immediate. In fact, if (27) were false, then (A) would have a solution $x(t)$ such that $\lim_{t\to\infty} x(t)=\text{const.} \neq 0$ by Theorem 3, contradicting the hypothesis that (A) is almost oscillatory. We shall prove the "if" part. Because of (25) and the superlinearity, (27) implies that

$$\int^{\infty} t^{n-k}|f(t, cg^{k-1}(t))|dt = \infty \qquad \text{for all} \quad c \neq 0,$$

if $1\leqq k\leqq n-1$, and so from Theorem 6 it follows that the classes $\mathcal{N}_1,\dots, \mathcal{N}_{n-1}$ are all empty. Hence, a nonoscillatory solution of (A), if any, is of class $\mathcal{N}_0$ or $\mathcal{N}_n$.

Case (i): $\sigma=1$ and n is even. If $x(t)$ is a nonoscillatory solution of (A), then $x(t)x^{(n)}(t)\leqq 0$, so that $x(t)$ must be of class $\mathcal{N}_0$. From the proof of Lemma 2 we see that

$$x'(t) = (-1)^{n-2}\int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} f(s, x(g(s)))ds$$

for all sufficiently large t. But this shows that $x(t)x'(t)>0$, a contradiction.

Case (ii): $\sigma=1$ and n is odd. If $x(t)$ is a nonoscillatory solution of (A), then as in Case (i) $x(t)$ is of class $\mathcal{N}_0$, and from Lemma 2 it follows that $x^{(i)}(\infty)=0$ for $1\leqq i\leqq n-1$ and

$$x(T)=x(\infty)+(-1)^{n-1}\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!}f(s,\, x(g(s)))ds \tag{29}$$

for a suitably large T. We claim that $x(\infty)=0$. Suppose the contrary: $x(\infty)=c\neq 0$. If $c>0$, then by (29) and (27) we obtain

$$x(T)\geqq \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!}f(s,\, c)ds=\infty\,,$$

which is a contradiction. Similarly, a contradiction is obtained if we suppose $c<0$. Therefore, we have $x(\infty)=0$.

Case (iii): $\sigma=-1$ and n is even. If $x(t)$ is of class $\mathcal{N}_0$, then the same argument as above shows that $x^{(i)}(\infty)=0$ for $0\leqq i\leqq n-1$. Let $x(t)$ be of class $\mathcal{N}_n$. We may suppose that $x(t)>0$, since a paralle argument holds if $x(t)<0$. There are positive constants c and T such that $x(g(t))\geqq cg^{n-1}(t)$ for $t\geqq T$. Integrating (A) from T to t, we get

$$\begin{aligned} x^{(n-1)}(t) &= x^{(n-1)}(T)+\int_T^t f(s,\, x(g(s)))ds \\ &\geqq \int_T^t f(s,\, cg^{n-1}(s))ds \qquad \text{for} \quad t\geqq T. \end{aligned} \tag{30}$$

In view of (27), (25) and the superlinearity, it is easy to see that the last integral in (30) tends to infinity as $t\to\infty$. Consequently, we have $x^{(n-1)}(\infty)=\infty$. Since by l'Hospital's rule $\lim_{t\to\infty} x^{(i)}(t)/t^{n-i-1}=\lim_{t\to\infty} x^{(n-1)}(t)=\infty$ for $0\leqq i\leqq n-2$, we conclude that $x^{(i)}(\infty)=\infty$ for $0\leqq i\leqq n-2$.

Case (iv): $\sigma=-1$ and n is odd. If $x(t)$ is of class $\mathcal{N}_n$, then as in Case (iii) $x^{(i)}(\infty)=\infty$ for $0\leqq i\leqq n-1$. In this case the class $\mathcal{N}_0$ is empty by Remark 2. Thus the proof is complete.

Theorem 9 is proved in a similar fashion.

REMARK 5. Theorems 8 and 9 extend some of the main results of Kiguradze [1, Theorems 3 and 4], Ličko and Švec [8, Theorems 1 and 2] and Onose [11, Theorems 3.1–3.5]. Theorem 9 might be regarded as a generalization of Theorems 1 and 2 of Kiguradze [2]. For other related results the reader is referred to the articles [3, 4, 5, 6, 7].

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