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*Oscillatory and Asymptotic Behavior of the Bounded Solutions of Differential Equations with Deviating Arguments**)

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1. Introduction

Let r_i $(i=0, 1,\dots, n)$ be positive continuous functions on the interval $[t_0, \infty)$. For a real-valued function h on $[T, \infty)$, $T \geqq t_0$, and any $\mu=0, 1,\dots, n$ we define the *μ-th r-derivative* of h by the formula

$$D_r^{(\mu)}h = r_\mu(r_{\mu-1}(\cdots(r_1(r_0h)')'\cdots)')'.$$

Then we obviously have

$$D_r^{(0)}h = r_0h \quad \text{and} \quad D_r^{(i)}h = r_i(D_r^{(i-1)}h)' \qquad (i = 1, 2,\dots, n).$$

Moreover, if $D_r^{(n)}h$ is defined on the interval $[T, \infty)$, then the function h is said to be *n-times r-differentiable* and if, in addition, $D_r^{(n)}h$ is continuous, h is said to be *n-times continuously r-differentiable.* If $r_i=1$ $(i=0, 1,\dots, n)$, this notion specializes to the one of the usual differentiability.

Now, we consider the n-th order $(n>1)$ differential equation with deviating arguments of the form

$$(E, \delta) \qquad (D_r^{(n)}x)(t) + \delta F(t; x < g(t) >) = b(t),$$

where $r_n=1$, $\delta=\pm 1$ and

$$x<g(t)> = (x[g_1(t)], x[g_2(t)],\dots, x[g_m(t)]), \quad g = (g_1, g_2,\dots, g_m).$$

The continuity of the real-valued functions F on $[t_0, \infty)\times \mathbf{R}^m$ and g_i $(i=1, 2,\dots, m)$, b on $[t_0, \infty)$ as well as sufficient smoothness for the existence of solutions of (E, δ) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E, δ) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called *oscillatory* if it has no last zero, and otherwise

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it is called *nonoscillatory*.

Furthermore the conditions (i) and (ii) below are assumed to hold throughout the paper:

(i) *For every* $i=1, 2,\dots, m$

$$\lim_{t\to\infty} g_i(t) = \infty .$$

(ii) *For every* $t \geqq t_0$,

$$F(t; 0, 0,\dots, 0) = 0$$

and, moreover, $F(t; y)$ *is increasing with respect to* y *in* $\mathbf{R}^m$.

Note. The order in $\mathbf{R}^m$ is considered in the usual sense, i.e.

$$y \leqq z \Longleftrightarrow (\forall i = 1, 2,\dots, m) y_i \leqq z_i .$$

2. Main results

In order to obtain our first result (Theorem 1) we shall apply the fixed point technique, by using the following Schauder's theorem (Schauder [3]).

THE SCHAUDER THEOREM. *Let E be a Banach space and X any nonempty convex and closed subset of E. If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point* (*i.e. there exists an* $x \in X$ *with* $x=Sx$).

This method patterns after that of Staikos [4].

A set $\mathscr{F}$ of real-valued functions defined on the interval $[T_0, \infty)$ is said to be *equiconvergent at* ∞ if all functions in $\mathscr{F}$ are convergent in $\mathbf{R}$ at the point ∞ and, moreover, for every $\varepsilon>0$ there exists a $T_0' \geqq T_0$ such that, for all functions f in $\mathscr{F}$,

$$t \geqq T_0' \Longrightarrow |f(t) - \lim_{S\to\infty} f(s)| < \varepsilon .$$

Let now $B_T([T_0, \infty))$, $T \geqq T_0$, be the Banach space of all continuous and bounded real-valued functions on the interval $[T_0, \infty)$ which are constant on $[T_0, T]$, endowed with the usual sup-norm $\| \ \|$. We need the following compactness criterion for subsets of $B_T([T_0, \infty))$, which is a corollary of the Arzelà-Ascoli theorem. For a proof of this criterion we refer to Staikos [4].

COMPACTNESS CRITERION. *Let* $\mathscr{F}$ *be an equicontinuous and uniformly bounded subset of the Banach space* $B_T([T_0, \infty))$. *If* $\mathscr{F}$ *is equiconvergent at* ∞, *it is also relatively compact.*

THEOREM 1. *Consider the differential equation* (E, δ) *subject to the con-*

ditions (i), (ii) *and*:

(C_1) *There exists an n-times continuously r-differentiable function w on* $[t_0, \infty)$ *with* $D_r^{(n)}w = b$ *and such that*

$$A = \limsup_{t \to \infty} |(D_r^{(0)}w)(t)| < \infty .$$

(C_2) *For some constant c with* $|c| > 2A$,

$$\int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots$$
$$\cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} \left| F\left(s; c\frac{1}{r_0} < g(s) > \right)\right| ds ds_{n-1} \cdots ds_2 ds_1 < \infty .$$

Then for every real number L with $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c| - A$ *there exists a (nonoscillatory) solution x of the equation* (E, δ) *with*

$$\lim_{t \to \infty} [(D_r^{(0)}x)(t) - (D_r^{(0)}w)(t)] = L$$

and

$$\lim_{t \to \infty} [(D_r^{(j)}x)(t) - (D_r^{(j)}w)(t)] = 0 \qquad (j = 1, 2, \ldots, n-1).$$

Proof. Let L be a real number with $Lc > 0$ and $\frac{|c|}{2} < |L| < |c| - A$. Without loss of generality, we suppose that c is positive, since the substitution $z = -x$ transforms (E, δ) into an equation of the same form satisfying the assumptions of the theorem with $-c$ in place of c. Moreover, we assume that

$$\frac{c}{2} < L < c - B,$$

where

$$B = \sup_{t \geqq T_0} |(D_r^{(0)}w)(t)|$$

for some $T_0 \geqq t_0$. Furthermore, by conditions (i) and (C_2), we choose a $T \geqq T_0$ so that

$$g_i(t) \geqq T_0 \qquad \text{for every} \quad t \geqq T \qquad (i = 1, 2, \ldots, m)$$

and

$$(1) \qquad \int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots$$
$$\cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F\left(s; c\frac{1}{r_0} < g(s) > \right)$$
$$ds ds_{n-1} \cdots ds_2 ds_1 \leqq c - L - B .$$

Let now the Banach space $E=\{h\colon r_0 h \in B_T([T_0, \infty))\}$, endowed with the norm $\| \ \|_{r_0}$,

$$\|h\|_{r_0} = \|r_0 h\|.$$

Moreover, let the subset X of E consisting of all functions x in E with

$$|(D_r^{(0)}x)(t) - L| \leqq c - L \quad \text{for every} \quad t \geqq T_0.$$

The set X is obviously *nonempty* and as it is easy to see, it is *convex* and *closed*.

If $x \in X$, by the definition of the set X, we obviously have

$$0 < x(t) \leqq c\frac{1}{r_0(t)} \quad \text{for every} \quad t \geqq T_0$$

and consequently, in view of (ii), we obtain

$$0 \leqq F(t; x < g(t) >) \leqq F\left(t; c\frac{1}{r_0} < g(t) >\right) \tag{2}$$

for all $t \geqq T$. Thus, because of condition (C_2), for any $x \in X$ and every $t \geqq T$

$$\int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F(s; x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1 < \infty$$

and hence the formula

$(Sx)(t) =$

$$= \begin{cases} \dfrac{L}{r_0(t)} + w(t) + \dfrac{\delta(-1)^{n-1}}{r_0(t)} \displaystyle\int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots \\ \quad \cdots \displaystyle\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F(s; x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1, \text{ if } t \geqq T \\ \dfrac{L}{r_0(t)} + \dfrac{(D_r^{(0)}w)(T)}{r_0(t)} + \dfrac{\delta(-1)^{n-1}}{r_0(t)} \displaystyle\int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots \\ \quad \cdots \displaystyle\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F(s; x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1, \\ \hfill \text{if} \quad T_0 \leqq t \leqq T \end{cases}$$

defines a mapping $S\colon X \to E$, which satisfies the assumptions of the Schauder theorem. Namely, it satisfies the following:

α) $SX \subseteq X$.

In fact, taking into account (2) and (1), we obtain that for any $x \in X$ and every $t \geqq T_0$

$$|(D_r^{(0)}(Sx))(t) - L|$$

$$\leqq \sup_{t\geqq T} |(D_r^{(0)}w)(t)| + \int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F(s; x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1$$

$$\leqq B + \int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F\left(s; c\frac{1}{r_0} < g(s) >\right) ds ds_{n-1} \cdots ds_2 ds_1$$

$$\leqq B + (c - L - B) = c - L.$$

β) *SX is relatively compact.*

Let

$$w^*(t) = \begin{cases} w(t), & \text{if } t \geqq T \\ \dfrac{(D_r^{(0)}w)(T)}{r_0(t)}, & \text{if } T_0 \leqq t \leqq T. \end{cases}$$

Then, for any $x \in X$, the function $f = D_r^{(0)}(Sx) - D_r^{(0)}w^*$ is continuous on $[T_0, \infty)$ and constant on $[T_0, T]$. Moreover, since $SX \subseteq X$, by the definition of the set X, for every $t \geqq T_0$ we obtain

$$|f(t)| \leqq |(D_r^{(0)}(Sx))(t)| + |(D_r^{(0)}w^*)(t)| \leqq c + B,$$

namely $\|f\| \leqq c + B$. Therefore the set

$$\mathscr{F} = \{D_r^{(0)}(Sx) - D_r^{(0)}w^* : x \in X\}$$

is an uniformly bounded subset of the Banach space $B_T([T_0, \infty))$. Furthermore, for any $f \in \mathscr{F}$ and every $t \geqq T$ we have

(3) $$|f(t) - L|$$

$$\leqq \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F\left(s; c\frac{1}{r_0} < g(s) >\right) ds ds_{n-1} \cdots ds_2 ds_1.$$

Indeed, if $f = D_r^{(0)}(Sx) - D_r^{(0)}w^*$, $x \in X$, for every $t \geqq T$

$$\begin{aligned}|f(t)-L| &= |[(D_r^{(0)}(Sx))(t)-(D_r^{(0)}w)(t)]-L|\\ &= \int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\\ &\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty F(s;x<g(s)>)ds\,ds_{n-1}\cdots ds_2 ds_1\end{aligned}$$

and hence, by virtue of (2), (3) follows immediately. Thus, from (3) and condition (C_2) we conclude that the set $\mathscr{F}$ is equiconvergent at ∞. Now, in view of (2), for any f in $\mathscr{F}$,

$$f = D_r^{(0)}(Sx) - D_r^{(0)}w^*, \qquad x \in X,$$

and every t_1, t_2 with $T \leqq t_1 \leqq t_2$ we get

$$|f(t_2)-f(t_1)|$$

$$= \begin{cases} \displaystyle\int_{t_1}^{t_2} \frac{1}{r_1(s_1)}\int_{s_1}^\infty F(s;x<g(s)>)ds\,ds_1, & \text{if } n=2\\ \displaystyle\int_{t_1}^{t_2} \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\\ \displaystyle\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty F(s;x<g(s)>)ds\,ds_{n-1}\cdots ds_2 ds_1, & \text{if } n>2\end{cases}$$

$$\leqq \begin{cases} \displaystyle\int_{t_1}^{t_2} \frac{1}{r_1(s_1)}\int_{s_1}^\infty F\left(s;c\frac{1}{r_0}<g(s)>\right)ds\,ds_1, & \text{if } n=2\\ \displaystyle\int_{t_1}^{t_2} \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\\ \displaystyle\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty F\left(s;c\frac{1}{r_0}<g(s)>\right)ds\,ds_{n-1}\cdots ds_2 ds_1,\\ & \text{if } n>2\end{cases}$$

From this it follows that $\mathscr{F}$ is equicontinuous. Finally, by the given compactness criterion, we conclude that the set $\mathscr{F}$ is relatively compact in $B_T([T_0, \infty))$. This, by the definition of the norm $\|\ \|_{r_0}$ in E and because of the boundedness of the function $D_r^{(0)}w^*$, implies that SX is relatively compact.

γ) *The mapping S is continuous.*

Let $x \in X$ and (x_ν) be an arbitrary sequence in X with

$$\|\ \|_{r_0} - \lim x_\nu = x\,.$$

By (2), for all ν and every $t \geqq T$

$$F(t;x_\nu<g(t)>) \leqq F\left(t;c\frac{1}{r_0}<g(t)>\right)$$

and hence, because of condition (C_2), we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\nu} \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s\,;\, x_\nu < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1$$

$$= \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s\,;\, x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1$$

for all $t \geqq T$. So, for every $t \geqq T_0$ we have the pointwise convergence

$$\lim_{\nu} (Sx_\nu)(t) = (Sx)(t)\,.$$

It remains to prove that the above convergence is also r_0-uniform, i.e.

$$\| \ \|_{r_0} - \lim Sx_\nu = Sx\,. \tag{4}$$

To this end, we consider any subsequence (u_μ) of (Sx_ν). Because of the relative compactness of SX, there exist a subsequence (v_λ) of (u_μ) and $y \in E$ so that

$$\| \ \|_{r_0} - \lim v_\lambda = y\,.$$

Since $\| \ \|_{r_0}$-convergence implies the pointwise one to the same limit function, we must always have $y = Sx$, which proves (4).

Finally, by the Schauder theorem, there exists an $x \in X$ with $x = Sx$, i.e. for every $t \geqq T$

$$(D_r^{(0)}x)(t) = L + (D_r^{(0)}w)(t) + \delta(-1)^{n-1} \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s\,;\, x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1$$

and consequently, by virtue of condition (C_1),

$$(D_r^{(n)}x)(t) = (D_r^{(n)}w)(t) - \delta F(t;\, x{<}g(t){>}) = b(t) - \delta F(t;\, x{<}g(t){>})\,.$$

Thus, the fixed point x of the mapping S is a solution on $[T, \infty)$ of the equation (E, δ) and moreover the required one, since

$$|[(D_r^{(0)}x)(t) - (D_r^{(0)}w)(t)] - L|$$

$$= \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F(s; x < g(s) >) ds ds_{n-1} \cdots ds_2 ds_1 \to 0 \quad \text{as} \quad t \to \infty$$

and for $j = 1, 2, \ldots, n-1$

$$|(D_r^{(j)}x)(t) - (D_r^{(j)}w)(t)|$$

$$= \begin{cases} \displaystyle\int_t^\infty F(s; x < g(s) >) ds, & \text{if} \quad j = n-1 \\ \displaystyle\int_t^\infty \frac{1}{r_{j+1}(s_{j+1})} \cdots \\ \quad \displaystyle\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty F(s; x < g(s) >) ds ds_{n-1} \cdots ds_{j+1}, \\ & \text{if} \quad j < n-1 \end{cases}$$

tends to 0 as $t \to \infty$.

REMARK 1. The Tychonoff's fixed point theorem (see Tychonoff [6]) can also be used to prove Theorem 1, but the proof is then somewhat longer. Tychonoff's theorem has been used in numerous papers for obtaining related results concerning differential equations with deviating arguments or functional differential equations. We choose to refer to [1], [2] and [5].

COROLLARY 1. *Consider the differential equation* (E, δ) *subject to the conditions* (i), (ii) *and*:

(C_1') *There exists an n-times continuously r-differentiable function w on* $[t_0, \infty)$ *with* $D_r^{(n)}w = b$ *and such that*

$$\lim_{t \to \infty} (D_r^{(0)}w)(t) = 0.$$

(C_2') *For some nonzero constant c,*

$$\int^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \left| F\left(s; c\frac{1}{r_0} < g(s) >\right) \right| ds ds_{n-1} \cdots ds_2 ds_1 < \infty.$$

Then for every real number L with $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c|$ *there exists a (nonoscillatory) solution x of the equation* (E, δ) *with*

$$\lim_{t \to \infty} (D_r^{(0)}x)(t) = L.$$

PROOF. It follows immediately from Theorem 1 for $A=0$.

In particular, for $b=0$ and $w=0$ from Theorem 1 we immediately obtain the following result concerning the differential equation

$$(E_0, \delta) \qquad (D_r^{(n)}x)(t) + \delta F(t; x<g(t)>) = 0 .$$

COROLLARY II. *Consider the differential equation* (E_0, δ) *subject to the conditions* (i), (ii) *and* (C_2'). *Then for every real number* L *with* $Lc>0$ *and* $\frac{|c|}{2}<|L|<|c|$ *there exists a (nonoscillatory) solution* x *of the equation* (E_0, δ) *with*

$$\lim_{t\to\infty}(D_r^{(0)}x)(t) = L$$

and

$$\lim_{t\to\infty}(D_r^{(j)}x)(t) = 0 \qquad (j = 1, 2,\dots, n-1).$$

Now, in order to obtain our second result (Theorem 2) it is convenient to give the following two lemmas.

LEMMA 1. *Let* h *be an* n*-times* r*-differentiable function on* $[T, \infty)$, $T \geqq t_0$, *such that* $D_r^{(n)}h$ *is of constant sign on* $[T, \infty)$. *Moreover, let* λ, $0\leqq\lambda \leqq n-2$, *be an integer so that*

$$\int^{\infty} \frac{dt}{r_{\lambda+1}(t)} = \infty ,$$

If the $\lim_{t\to\infty}(D_r^{(\lambda)}h)(t)$ *is finite, then*

$$\lim_{t\to\infty}(D_r^{(\lambda+1)}h)(t) = 0 .$$

PROOF. Since $D_r^{(n)}h$ is of constant sign on $[T, \infty)$, it is easy to see that the functions $D_r^{(i)}h$ $(i=1, 2,\dots, n-1)$ are also eventually of constant sign. So, for every $i=0, 1,\dots, n-1$, $\lim_{t\to\infty}(D_r^{(i)}h)(t)$ exists in the extended real line $\mathbf{R}^* =\mathbf{R}\cup\{-\infty, \infty\}$.

Let now $\alpha=\lim_{t\to\infty}(D_r^{(\lambda+1)}h)(t)$, $\alpha\in\mathbf{R}^*$. If $\alpha>0$, then there exist a positive constant M and a $T'\geqq T$ so that for every $t\geqq T'$

$$(D_r^{(\lambda+1)}h)(t) \geqq M, \quad \text{i.e.} \quad (D_r^{(\lambda)}h)'(t) \geqq M\frac{1}{r_{\lambda+1}(t)} .$$

Therefore

$$(D_r^{(\lambda)}h)(t) \geqq (D_r^{(\lambda)}h)(T') + M\int_{T'}^{t} \frac{ds}{r_{\lambda+1}(s)} \qquad \text{for every} \quad t \geqq T'$$

and hence we have

$$\lim_{t\to\infty}(D_r^{(\lambda)}h)(t) = \infty\,,$$

a contradiction. Similarly, if $\alpha<0$, then we obtain the contradiction

$$\lim_{t\to\infty}(D_r^{(\lambda)}h)(t) = -\infty\,.$$

REMARK 2. Let h be as in Lemma 1 and let ν, $1\leqq\nu\leqq n-1$, be an integer so that

$$\int^{\infty}\frac{dt}{r_i(t)} = \infty \qquad (i = 1, 2,\dots, \nu)\,.$$

If $\lim_{t\to\infty}(D_r^{(\nu)}h)(t)\neq 0$, then, applying consecutively Lemma 1 for $\lambda=\nu-1,\dots, 0$, we conclude that $\lim_{t\to\infty}(D_r^{(0)}h)(t)$ is infinite.

LEMMA 2. *Suppose that:*
(C_3) *For every* $i=1, 2,\dots, n-1$

$$\int^{\infty}\frac{dt}{r_i(t)} = \infty\,.$$

Let h be a positive and n-times r-differentiable function on an interval $[T, \infty)$, $T\geqq t_0$, such that $D_r^{(0)}h$ is bounded on $[T, \infty)$ and

$$\delta(D_r^{(n)}h)(t) \leqq 0 \qquad \textit{for every} \quad t\geqq T.$$

Then

$$\delta(-1)^{n+1+j}(D_r^{(j)}h)(t) \geqq 0 \qquad \textit{for every} \quad t\geqq T \qquad (j = 1, 2,\dots, n-1)$$

and, provided that $D_r^{(n)}h$ is continuous on $[T, \infty)$,

$$\int_T^{\infty}\frac{1}{r_1(s_1)}\int_{s_1}^{\infty}\frac{1}{r_2(s_2)}\cdots$$
$$\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|(D_r^{(n)}h)(s)|\,ds\,ds_{n-1}\cdots ds_2\,ds_1 < \infty$$

PROOF. If $\delta(D_r^{(n-1)}h)(\tau)<0$ for some $\tau\geqq T$, then, since the function $\delta D_r^{(n-1)}h$ is decreasing on $[T, \infty)$, we have $\lim_{t\to\infty}(D_r^{(n-1)}h)(t)\neq 0$ and so, taking into account Remark 2, we conclude that $\lim_{t\to\infty}(D_r^{(0)}h)(t)$ is infinite. This is a contradiction, because of the positivity and the boundedness of the function $D_r^{(0)}h$. Therefore,

$$\delta(D_r^{(n-1)}h)(t) \geqq 0 \quad \text{for every} \quad t \geqq T.$$

Let, now, μ be the smallest integer with $1 \leqq \mu \leqq n-1$ and

$$\delta(-1)^{n+1+j}(D_r^{(j)}h)(t) \geqq 0 \quad \text{for every} \quad t \geqq T \qquad (j = \mu, \mu+1, \dots, n-1).$$

Suppose that $\mu > 1$. Obviously, $\delta(-1)^{n+\mu}(D_r^{(\mu-1)}h)(\tau') < 0$ for some $\tau' \geqq T$. Moreover, the function $\delta(-1)^{n+\mu}D_r^{(\mu-1)}h$ is decreasing on $[T, \infty)$. So, $\lim_{t\to\infty}(D_r^{(\mu-1)} h)(t) \neq 0$ and hence, by Remark 2, the contradiction $\lim_{t\to\infty}(D_r^{(0)}h)(t) = \pm\infty$ can again be derived. Thus, we must always have $\mu = 1$. This, by the definition of μ, proves the first part of the conclusion of the lemma.

Next, we assume that $D_r^{(n)}h$ is continuous on $[T, \infty)$. Then

$$\delta(-1)^{n+1+j}(D_r^{(j)}h)(t) \geqq \delta(-1)^{n+j}\int_t^\infty \frac{1}{r_{j+1}(s)}(D_r^{(j+1)}h)(s)ds \tag{5}$$

for $j = 1, 2, \dots, n-1$ and every $t \geqq T$. Indeed, for every $u \geqq t$ we have

$$\begin{aligned}\delta(-1)^{n+1+j}(D_r^{(j)}h)(t) &= \delta(-1)^{n+1+j}(D_r^{(j)}h)(u) \\ &\quad + \delta(-1)^{n+j}\int_t^u \frac{1}{r_{j+1}(s)}(D_r^{(j+1)}h)(s)ds \\ &\geqq \delta(-1)^{n+j}\int_t^u \frac{1}{r_{j+1}(s)}(D_r^{(j+1)}h)(s)ds,\end{aligned}$$

from which (5) follows immediately. Furthermore,

$$\delta(-1)^n[(D_r^{(0)}h)(t) - (D_r^{(0)}h)(T)] = \delta(-1)^n\int_T^t \frac{1}{r_1(s)}(D_r^{(1)}h)(s)ds$$

for all $t \geqq T$. So, since $\lim_{t\to\infty}(D_r^{(0)}h)(t)$ is finite, we obtain

$$\delta(-1)^n\int_T^\infty \frac{1}{r_1(s)}(D_r^{(1)}h)(s)ds < \infty. \tag{6}$$

Finally, using (5) and (6), we can easily derive

$$\begin{aligned}&\int_T^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots \\ &\qquad \cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty [-\delta(D_r^{(n)}h)(s)]ds ds_{n-1}\cdots ds_2 ds_1 < \infty,\end{aligned}$$

which completes the proof of the lemma.

THEOREM 2. *Consider the differential equation* (E_0, δ) *subject to the conditions* (i), (ii), (C_3) *and*:

(C$_4$) *For every nonzero constant c there exists an integer* k, $0 \leqq k \leqq n-1$, *such that*

$$\begin{cases} \displaystyle\int^{\infty} \left| F\Big(t; c\frac{1}{r_0} < g(t) >\Big)\right| dt = \infty, & \text{if } k = n-1 \\ \displaystyle\int^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \cdots \\ \displaystyle\cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} \left| F\Big(s; c\frac{1}{r_0} < g(s) >\Big)\right| ds ds_{n-1} \cdots ds_{k+1} = \infty, \\ & \text{if } k < n-1. \end{cases}$$

Then every solution x of the equation $(E_0, +1)$ [*respectively*, $(E_0, -1)$] *with* $x(t)=O(1/r_0(t))$ *as* $t\to\infty$ *for n even* [*resp. odd*] *is oscillatory, while for n odd* [*resp. even*] *is either oscillatory or such that*

$$\lim_{t\to\infty}(D_r^{(j)}x)(t) = 0 \quad \textit{monotonically} \quad (j = 0, 1, \ldots, n-1).$$

Proof. Let x be a nonoscillatory solution on an interval $[T_0, \infty)$, $T_0 \geqq t_0$, of the equation (E_0, δ) with $x(t)=O(1/r_0(t))$ as $t\to\infty$. Without loss of generality, we suppose that $x(t)\neq 0$ for every $t \geqq T_0$. Furthermore, we can assume that x is positive, since the substitution $z=-x$ transforms (E_0, δ) into an equation of the same form satisfying the assumptions of the theorem.

Next, by (i), we choose a $T \geqq T_0$ so that

$$g_i(t) \geqq T_0 \qquad \text{for every} \quad t \geqq T \qquad (i = 1, 2, \ldots, m).$$

Then, in view of (ii), from equation (E_0, δ) we obtain

$$-\delta(D_r^{(n)}x)(t) = F(t; x<g(t)>) \geqq F(t; 0, 0, \ldots, 0) = 0$$

for all $t \geqq T$. Namely,

$$\delta(D_r^{(n)}x)(t) \leqq 0 \qquad \text{for every} \quad t \geqq T.$$

So, by applying Lemma 2, we have

$$\delta(-1)^{n+1+j}(D_r^{(j)}x)(t) \geqq 0 \qquad \text{for every} \quad t \geqq T \qquad (j = 1, 2, \ldots, n-1) \tag{7}$$

and

$$\cdots\int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |(D_r^{(n)}x)(s)| ds ds_{n-1} \cdots ds_2 ds_1 < \infty. \tag{8}$$

Now,

$$\lim_{t\to\infty}(D_r^{(0)}x)(t) = 0\,. \tag{9}$$

Indeed, in the opposite case there exists a positive constant c such that for every $t \geqq T_0$

$$(D_r^{(0)}x)(t) \geqq c, \quad \text{i.e.} \quad x(t) \geqq c\frac{1}{r_0(t)}$$

and consequently, taking into account (ii), we get

$$\begin{aligned} |(D_r^{(n)}x)(t)| &= -\delta(D_r^{(n)}x)(t) \\ &= F(t;\, x < g(t) >) \geqq F\left(t;\, c\frac{1}{r_0} < g(t) >\right) \geqq 0 \end{aligned}$$

for all $t \geqq T$. By this, (8) gives

$$\int_T^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots$$

$$\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty F\left(s;\, c\frac{1}{r_0} < g(s) >\right) ds ds_{n-1}\cdots ds_2 ds_1 < \infty\,,$$

which contradicts (C_4).

Finally, by (7) and (9), we conclude that $\delta(-1)^n = -1$, namely n is odd (resp. even) for $\delta = +1$ (resp. $\delta = -1$). Moreover, because of (9), from Lemma 1 it follows that

$$\lim_{t\to\infty}(D_r^{(j)}x)(t) = 0 \qquad (j = 0, 1,\ldots, n-1)\,.$$

COROLLARY III. *Consider the differential equation* (E_0, δ) *subject to the conditions* (i), (ii) *and* (C_3). *Then the condition* (C_2') *is a necessary and sufficient condition in order that the equation* (E_0, δ) *have at least one (nonoscillatory) solution* x *so that the* $\lim_{t\to\infty}(D_r^{(0)}x)(t)$ *exists in* $\mathbf{R} - \{0\}$.

PROOF. The necessity of the condition (C_2') follows immediately from Theorem 2, while its sufficiency is contained in Corollary II.

3. Applications

We shall apply now the results of section 2 in the particular case where for some integer N, $1 \leqq N \leqq n-1$, we have

$$r_i = 1 \quad (i = 0, 1, \dots, n-1;\ i \neq n-N) \quad \text{and} \quad r_{n-N} = r.$$

More precisely, we shall derive some interesting corollaries concerning the differential equation

$$(\hat{E}, \delta) \qquad [r(t)x^{(n-N)}(t)]^{(N)} + \delta F(t;\ x\langle g(t)\rangle) = b(t).$$

All corollaries are new.

COROLLARY 1. *Consider the differential equation* $(\hat{E}, \delta)$ *subject to the conditions* (i), (ii) *and*:

$(\hat{C}_1)$ *There exists a bounded function* $w \in C^{n-N}([t_0, \infty))$ *with* $rw^{(n-N)} \in C^N([t_0, \infty))$ *and* $[rw^{(n-N)}]^{(N)} = b$.

$(\hat{C}_2)$ *For some constant c with* $|c| > 2 \limsup_{t\to\infty} |w(t)|$,

$$\int^{\infty} \frac{t^{n-1-N}}{r(t)} \int_t^{\infty} (s-t)^{N-1} |F(s;\ c, c, \dots, c)|\, ds\, dt < \infty.$$

Then for every real number L with $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c| - \limsup_{t\to\infty} |w(t)|$ *there exists a (bounded nonoscillatory) solution x of the equation* $(\hat{E}, \delta)$ *with*

$$\lim_{t\to\infty} [x(t) - w(t)] = L$$

and

$$\begin{cases} \lim_{t\to\infty} [x^{(i)}(t) - w^{(i)}(t)] = 0 \quad (i = 1, 2, \dots, n-N-1), \text{ if } N < n-1 \\ \lim_{t\to\infty} ([r(t)x^{(n-N)}(t)]^{(j)} - [r(t)w^{(n-N)}(t)]^{(j)}) = 0 \quad (j = 0, 1, \dots, N-1). \end{cases}$$

COROLLARY 2. *Consider the differential equation* $(\hat{E}, \delta)$ *subject to the conditions* (i), (ii) *and*:

$(\hat{C}_1')$ *There exists a function* $w \in C^{n-N}([t_0, \infty))$ *with* $rw^{(n-N)} \in C^N([t_0, \infty))$, $[rw^{(n-N)}]^{(N)} = b$ *and* $\lim_{t\to\infty} w(t) = 0$.

$(\hat{C}_2')$ *For some nonzero constant c*,

$$\int^{\infty} \frac{t^{n-1-N}}{r(t)} \int_t^{\infty} (s-t)^{N-1} |F(s;\ c, c, \dots, c)|\, ds\, dt < \infty.$$

Then for every real number L with $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c|$ *there exists a (bounded nonoscillatory) solution x of the equation* $(\hat{E}, \delta)$ *with*

$$\lim_{t\to\infty} x(t) = L.$$

COROLLARY 3. *Consider the differential equation* $(\hat{E}_0, \delta)$,

$$(\hat{E}_0, \delta) \qquad [r(t)x^{(n-N)}(t)]^{(N)} + \delta F(t; x<g(t)>) = 0,$$

subject to the conditions (i), (ii) *and* $(\hat{C}_2')$. *Then for every real number* L *with* $Lc>0$ *and* $\frac{|c|}{2}<|L|<|c|$ *there exists a (bounded nonoscillatory) solution* x *of the equation* $(\hat{E}_0, \delta)$ *with*

$$\lim_{t\to\infty} x(t) = L$$

and

$$\begin{cases} \lim\limits_{t\to\infty} x^{(i)}(t) = 0 \quad (i = 1, 2,\dots, n-N-1), \text{ if } N < n-1 \\ \lim\limits_{t\to\infty} [r(t)x^{(n-N)}(t)]^{(j)} = 0 \quad (j = 0, 1,\dots, N-1). \end{cases}$$

COROLLARY 4. *Consider the differential equation* $(\hat{E}_0, \delta)$ *subject to the* conditions (i), (ii) *and*:

$$(\hat{C}_3) \qquad \int^\infty \frac{dt}{r(t)} = \infty.$$

$(\hat{C}_4)$ *For every nonzero constant c either*

$$\int^\infty t^{N-1}|F(t; c, c,\dots, c)|dt = \infty$$

or

$$\int^\infty \frac{t^{n-1-N}}{r(t)} \int_t^\infty (s-t)^{N-1}|F(s; c, c,\dots, c)|dsdt = \infty.$$

Then every bounded solution x *of the equation* $(\hat{E}_0, +1)$ [*resp.* $(\hat{E}_0, -1)$] *for n even* [*resp. odd*] *is oscillatory, while for n odd* [*resp. even*] *is either oscillatory or such that*

$$\begin{cases} \lim\limits_{t\to\infty} x^{(i)}(t) = 0 \quad \textit{monotonically} \quad (i = 0, 1,\dots, n-N-1) \\ \lim\limits_{t\to\infty} [r(t)x^{(n-N)}(t)]^{(j)} = 0 \quad \textit{monotonically} \quad (j = 0, 1,\dots, N-1). \end{cases}$$

COROLLARY 5. *Consider the differential equation* $(\hat{E}_0, \delta)$ *subject to the conditions* (i), (ii) *and* $(\hat{C}_3)$. *Then the condition* $(\hat{C}_2')$ *is a necessary and sufficient condition in order that the equation* $(\hat{E}_0, \delta)$ *have at least one (bounded nonoscillatory) solution* x *so that the* $\lim_{t\to\infty} x(t)$ *exists in* $\mathbf{R}-\{0\}$.

In the considered case the conditions (C_1) and (C_1') follow from $(\hat{C}_1)$ and $(\hat{C}_1')$ respectively. Also, the condition (C_3) becomes $(\hat{C}_3)$. Moreover, we have the formula

$$\int_u^\infty \int_v^\infty (s-v)^\mu\, p(s)ds\,dv = \int_u^\infty \frac{(s-u)^{\mu+1}}{\mu+1} p(s)ds\,,$$

where p is a continuous nonnegative function on $[u, \infty)$ and μ is a nonnegative integer. By this formula, it is a matter of elementary calculus to see that in the considered case the conditions (C_2), (C_2') and (C_4) follow respectively from $(\hat{C}_2)$, $(\hat{C}_2')$ and $(\hat{C}_4)$. So, Corollaries 1, 2, 3, 4 and 5 follow from Theorem 1, Corollary I, Corollary II, Theorem 2 and Corollary III respectively.

Now, we restrict ourselves in the usual case where $r_0 = r_1 = \cdots = r_{n-1} = 1$. In this case the equation (E, δ) takes the form

$$(\tilde{E}, \delta) \qquad x^{(n)}(t) + \delta F(t; x<g(t)>) = b(t).$$

We shall formulate the results of section 2 for the differential equation $(\tilde{E}, \delta)$. For this purpose, we observe that this equation is obtained from equation $(\hat{E}, \delta)$ by setting $r=1$. So, from Corollaries 1, 2, 3, 4 and 5 we can respectively derive the following ones concerning the equation $(\tilde{E}, \delta)$.

COROLLARY 1$'$. *Consider the differential equation* $(\tilde{E}, \delta)$ *subject to the conditions* (i), (ii) *and*:

$(\tilde{C}_1)$ *There exists a bounded and n-times continuously differentiable function* w *on* $[t_0, \infty)$ *with* $w^{(n)} = b$.

$(\tilde{C}_2)$ *For some constant* c *with* $|c| > 2 \limsup_{t\to\infty} |w(t)|$,

$$\int^\infty t^{n-1} |F(t; c, c,\dots, c)| dt < \infty\,.$$

Then for every real number L *with* $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c| - \limsup_{t\to\infty} |w(t)|$ *there exists a (bounded nonoscillatory) solution* x *of the equation* $(\tilde{E}, \delta)$ *with*

$$\lim_{t\to\infty} [x(t) - w(t)] = L$$

and

$$\lim_{t\to\infty} [x^{(j)}(t) - w^{(j)}(t)] = 0 \qquad (j = 1, 2,\dots, n-1)\,.$$

COROLLARY 2$'$. *Consider the differential equation* $(\tilde{E}, \delta)$ *subject to the conditions* (i), (ii) *and*:

($\tilde{C}_1'$) *There exists an n-times continuously differentiable function w on* $[t_0, \infty)$ *with* $w^{(n)} = b$ *and* $\lim_{t\to\infty} w(t) = 0$.

($\tilde{C}_2'$) *For some nonzero constant c,*

$$\int^{\infty} t^{n-1}|F(t; c, c, \ldots, c)|dt < \infty .$$

Then for every real number L with $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c|$ *there exists a (bounded nonoscillatory) solution x of the equation* $(\tilde{E}, \delta)$ *with*

$$\lim_{t\to\infty} x(t) = L.$$

COROLLARY 3′. *Consider the differential equation* $(\tilde{E}_0, \delta)$,

$$(\tilde{E}_0, \delta) \qquad x^{(n)}(t) + \delta F(t; x<g(t)>) = 0,$$

subject to the conditions (i), (ii) *and* ($\tilde{C}_2'$). *Then for every real number L with* $Lc > 0$ *and* $\frac{|c|}{2} < |L| < |c|$ *there exists a (bounded nonoscillatory) solution x of the equation* $(\tilde{E}_0, \delta)$ *with*

$$\lim_{t\to\infty} x(t) = L \quad \text{and} \quad \lim_{t\to\infty} x^{(j)}(t) = 0 \qquad (j = 1, 2, \ldots, n-1).$$

COROLLARY 4′. *Consider the differential equation* $(\tilde{E}_0, \delta)$ *subject to the conditions* (i), (ii) *and*:

($\tilde{C}_4$) *For every nonzero constant c,*

$$\int^{\infty} t^{n-1}|F(t; c, c, \ldots, c)|dt = \infty .$$

Then for n even [resp. odd] all bounded solutions of the equation $(\tilde{E}_0, +1)$ *[resp.* $(\tilde{E}_0, -1)$*] are oscillatory, while for n odd [resp. even] all bounded solutions of the equation* $(\tilde{E}_0, +1)$ *[resp.* $(\tilde{E}_0, -1)$*] are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-1$ *derivatives.*

COROLLARY 5′. *Consider the differential equation* $(\tilde{E}_0, \delta)$ *subject to the conditions* (i) *and* (ii). *Then the condition* ($\tilde{C}_2'$) *is a necessary and sufficient condition in order that the equation* $(\tilde{E}_0, \delta)$ *have at least one (bounded nonoscillatory) solution x so that the* $\lim_{t\to\infty} x(t)$ *exists in* $\mathbf{R} - \{0\}$.

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