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Oscillation and Nonoscillation for Perturbed Differential Equations

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1. Introduction

Consider the equation

$$\text{(I)} \qquad x^{(n)} + H(t, x) = Q(t, x), \quad n \text{ even},$$

where H, Q are real continuous functions defined on $[0, \infty) \times (-\infty, \infty)$. The following theorem was given by the author in [4]:

THEOREM A. *Let $H(t, u)$ be increasing in u, $uH(t, u) > 0$ for $u \neq 0$ and such that all bounded solutions of*

$$\text{(II)} \qquad x^{(n)} + H(t, x) = 0$$

oscillate. Moreover, let $|Q(t, x)| \leq Q_0(t)|x|^r$, where $r \geq 1$, Q_0: $[0, \infty) \to [0, \infty)$, continuous and such that

$$\int_0^\infty t^{n-1} Q_0(t) dt < +\infty.$$

Then every bounded solution of (I) *oscillates.*

As it was shown in [4], this theorem does not necessarily hold for $r < 1$, or for functions Q_0 with

$$\int_0^\infty t^{n-1} Q_0(t) dt = +\infty,$$

or for all solutions of (I), provided of course that all solutions of (I) oscillate. In this paper we provide conditions under which an nth order functional differential equation of the form

$$\text{(III)} \qquad x^{(n)} + H(t, x(g_1(t))) = Q(t, x(g_2(t)))$$

has all of its bounded solutions oscillatory. In the particular case $g_1(t) \equiv t$, $g_2(t) \equiv t$, this result does not necessarily demand that the perturbation Q be superlinear or small as in Theorem A. Next, we provide some results under which all solutions of (III) with $g_1(t) \equiv g_2(t) \equiv g(t)$ either oscillate, or are such that the function $H(t, x(g(t))) - Q(t, x(g(t)))$ oscillates. This last property might imply desirable properties for the solution and implies oscillation under conditions of "smallness" of $Q(t, x)$ with respect to $H(t, x)$. For other results related to oscillation of perturbed or forced oscillations, the reader is referred to the paper [4] of the author, which contains an almost complete bibliography on the subject, as well as the papers [1]–[3], [7].

In what follows, $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, and the functions g_1, g_2 in (III) will be assumed defined and continuous on R_+ with

$$\lim_{t\to\infty} g_i(t) = +\infty .$$

A function $f(t)$, $t \in [a, \infty)$, $a \geq 0$ is said to be "oscillatory" if it has an unbounded set of zeros on $[a, \infty)$. By a solution of an equation of the form (III) we shall mean any function which is defined for all large t and satisfies (III) on an infinite subinterval of $[0, \infty)$.

2. Oscillation of bounded solutions

In the following theorem conditions are given on H, Q so that all bounded solutions of (III) oscillate.

THEOREM 1. *Let $H: R_+ \times R \to R$, $Q: R_+ \times R \to R$ be continuous and such that*

(i) *$H(t, u)$ is increasing in u, $uH(t, u) \geq 0$ for $u \neq 0$;*

(ii) *for every $\alpha > 0$ there exists a function $Q_\alpha: R_+ \to R_+$ such that $|Q(t, u)| \leq Q_\alpha(t)$ for every $u \in R$ with $|u| \leq \alpha$;*

(iii) *for every $k > 0$, $\alpha > 0$,*

$$\int_0^\infty t^{n-1}[H(t, \pm k) \mp Q_\alpha(t)]dt = \pm \infty .$$

Then if $x(t)$ is a bounded eventually positive (negative) nonoscillatory solution of (III), there exists a sequence $\{\bar{t}_n\}$, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} \bar{t}_n = +\infty$, and $H(\bar{t}_n, x(g_1(\bar{t}_n))) \leq Q(\bar{t}_n, x(g_2(\bar{t}_n)))$ $(H(\bar{t}_n, x(g_1(\bar{t}_n))) \geq Q(\bar{t}_n, x(g_2(\bar{t}_n))))$. Now, in addition to the above assume that $g_1(t) \equiv g_2(t)$ and that the inequality

$$H(t_n, x_n) \leq Q(t_n, x_n) \quad (H(t_n, x_n) \geq Q(t_n, x_n))$$

for a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = +\infty$ and a sequence $\{x_n\}$ which is positive (negative) and bounded, is impossible; then every bounded solution of (III) is oscillatory.

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of (III). Then there exists a $t_1 \geq 0$ such that $x(t) \neq 0$ for $t \geq t_1$. Assume $x(t) > 0$ for $t \geq t_1$. Now let t_2 be such that $g_1(t) \geq t_1$, $g_2(t) \geq t_1$ for every $t \geq t_2$. It follows that $x(g_1(t)) > 0$ and $x(g_2(t)) > 0$ for $t \geq t_2$. Since $x(t)$ is bounded, there exists a number $\alpha > 0$ such that $|x(t)| \leq \alpha$ for every $t \geq t_1$. Thus, $|Q(t, x(g_2(t)))| \leq Q_\alpha(t)$, $t \geq t_2$. Now assume that

$$H(t, x(g_1(t))) - Q(t, x(g_2(t))) > 0, \qquad t \geq T,$$

where $T \geq t_2$ is a fixed number. Then from (III) we obtain $x^{(n)}(t) < 0$, $t \geq T$. This implies that all the derivatives $x^{(j)}(t)$, $j = 0, 1, \ldots, n-1$ are of fixed sign for all large t (say for $t \geq T_1 \geq T$), and no two consecutive derivatives are of the same sign, because this would imply $\lim_{t\to\infty} x(t) = \pm\infty$, a contradiction to the boundedness of $x(t)$. Now, we may assume that T_1 is sufficiently large so that we also have

$$(1) \qquad (-1)^\mu x^{(\mu)}(g_i(t)) > 0, \qquad \mu = 1, 2, \ldots, n-1, \quad i = 1, 2$$

for every $t \geq T_1$. By differentiation of the function $F(t) \equiv t^{n-1} x^{(n-1)}(t)$, $t \geq T_1$ we have

$$(2) \qquad \begin{aligned} F'(t) &= (n-1)t^{n-2}x^{(n-1)}(t) - t^{n-1}[H(t, x(g_1(t))) - Q(t, x(g_2(t)))] \\ &\leq (n-1)t^{n-2}x^{(n-1)}(t) - t^{n-1}[H(t, \lambda) - Q_\alpha(t)], \end{aligned}$$

for every $t \geq T_1$, where $x(g_1(t)) \geq x(g_1(T_1)) = \lambda > 0$, $t \geq T_1$. This follows from the fact that $x'(g_1(t)) > 0$ for $t \geq T_1$. Now, integrating (2) we get

$$(3) \qquad \begin{aligned} t^{n-1}x^{(n-1)}(t) &- (n-1)\int_{T_1}^t s^{n-2}x^{(n-1)}(s)ds \\ &\leq F(T_1) - \int_{T_1}^t s^{n-1}[H(s, \lambda) - Q_\alpha(s)]ds. \end{aligned}$$

Since the integral in the right-hand side of (3) tends to $+\infty$ as $t \to +\infty$ and $x^{(n-1)}(t) > 0$, $t \geq T_1$, it follows that

$$(4) \qquad \lim_{t\to\infty} \int_{T_1}^t s^{n-2}x^{(n-1)}(s)ds = +\infty.$$

From this point on we can follow the proof of Kartsatos [1, Corollary], to obtain the contradiction $\lim_{t\to\infty} x(t) = +\infty$. Consequently, there is at least one number $\bar{t}_1 \geq T$ such that

$$H(\bar{t}_1, x(g_1(\bar{t}_1))) - Q(t_1, x(g_2(\bar{t}_1))) \leq 0.$$

Since T was arbitrary, it follows that there exists a sequence $\{\bar{t}_n\}$, $n = 1, 2, \ldots$, such that $\bar{t}_n \geq t_2$ and

(5) $$H(\bar{t}_n, x(g_1(\bar{t}_n))) - Q(\bar{t}_n, x(g_2(\bar{t}_n))) \leq 0, \qquad n = 1, 2, \dots.$$

A similar proof holds in the case of an eventually negative solution $x(t)$, and this concludes the first part of the theorem. As far as the second part is concerned, it is obvious from (5) with $g_1(t_n)=g_2(t_n)$ and the corresponding inequality for a negative $x(t)$. This concludes the proof of the theorem.

EXAMPLES. An equation that satisfies all the hypotheses of the first part of the above theorem is the following:

(6) $$x^{(6)} + t^{-5}x^3(\sqrt{t}) = t^{-(5+\varepsilon)}|x(t^{3-2\varepsilon})|^{1/2}\operatorname{sgn} x + 720t^{-7},$$

where $0<\varepsilon<1$. Here $g_1(t)\equiv\sqrt{t}$, $g_2(t)\equiv t^{3-2\varepsilon}$, $|Q(t, u)|\leq\alpha^{1/2}t^{-(5+\varepsilon)}+720t^{-7}$. The last term in (6) can be written as $720(t^{3-2\varepsilon})^{-7/(3-2\varepsilon)}$. Of course, H, Q are defined here for $t>0$ but this does not affect the above theorem since solutions are defined for all large t. In (6) we have the solution $x(t)\equiv t^{-1}$, $t\geq 1$, which is nonoscillatory. An equation satisfying the assumptions of the second part of Theorem 1 is the following:

(7) $$x'' + (1 + t^{-\varepsilon}\sin^3 t)x = t^{-\varepsilon}(\sin^2 t)x^2, \qquad 0 < \varepsilon < 1.$$

Here $g_1(t)=g_2(t)\equiv t$, $Q_\alpha(t)\equiv\alpha^2t^{-\varepsilon}$. Now assume that

$$(1 + t_n^{-\varepsilon}\sin^3 t_n)x_n \leq t_n^{-\varepsilon}(\sin^2 t_n)x_n^2$$

with $x_n>0$, bounded, and $t_n\to+\infty$ as $n\to\infty$.

Then we have

(8) $$1 \leq t_n^{-\varepsilon}(\sin^2 t_n)x_n/(1 + t_n^{-\varepsilon}\sin^3 t_n).$$

Since the right-hand side tends to zero as $n\to\infty$, we obtain a contradiction. An analogous argument applies to the case $x_n<0$. Thus, all bounded solutions of (7) oscillate. One such solution is $x(t)=\sin t$, $t\geq 1$. This example suggests an interesting corollary which we now state.

COROLLARY 1. *Assume that* (i)–(iii) *of Theorem* 1 *are satisfied. Moreover, let* $g_1(t)\equiv g_2(t)$ *and, for every* $k>0$,

$$\lim_{t\to\infty}\sup_{\substack{|u|\leq k\\ u\neq 0}}|Q(t, u)/H(t, u)| < 1.$$

Then every bounded solution of (III) *oscillates.*

It suffices to observe that the first conclusion of Theorem 1 is now impossible. Thus, there are no bounded nonoscillatory solutions.

A large class of equations satisfying the assumptions of this corollary is the following:

$$x^{(n)} + t^{-n}|x|^\alpha\operatorname{sgn} x = t^{-(n+\varepsilon)}|x|^\beta\operatorname{sgn} x, \quad n \text{ even}, \qquad t\geq 1,\ \beta>\alpha>0,\ \varepsilon>0.$$

3. General oscillation results

Before we state the next theorem we state a lemma given by the author in [4, Theorem 2].

LEMMA 1. *For the equation* (I) *assume the following*:

(i) *$H: R_+ \times R \to R$, $Q: R_+ \times R \to R$ are continuous, $H(t, u)$ is increasing in u, $uH(t, u) > 0$ for $u \neq 0$;*

(ii) *$|Q(t, u)| \leq Q_0(t, |u|)$, $(t, u) \in R_+ \times R$, where $Q_0(t, s)$ is continuous and increasing in s;*

(iii) *the homogeneous equation has all of its solutions oscillatory.*

Then if $x(t)$ is a positive solution of (I), *there exist $L > 0$ and $t_0 \geq 0$ such that $x(t) < y(t)$, $t \in [t_0, \infty)$, where $y(t) \equiv y(t, L)$ is any positive solution of the equation $y^{(n)}(t) = Q_0(t, y(t))$ with $y(t_0) = L$ and $y^{(j)}(t_0) = 0$, $j = 1, 2, \ldots, n-1$.*

Actually this lemma was proven in [4] for a more special case of a function Q, but the proof there carries over to the present case without any actual modifications. In the following theorems we shall assume, without further mention, that $H(t, -u) - Q(t, -u) = -H(t, u) + Q(t, u)$ in order to ease the exposition because in this case $-x(t)$ is a solution if $x(t)$ is one. We also set $C(t_0, L) = \{u \in C[t_0, \infty); 0 < u(t) \leq y(t, L), t \in [t_0, \infty)\}$, where $t_0 \geq 0$, $L > 0$ and $y(t, L)$ is as in the Lemma.

THEOREM 2. *For the equation* (I) *assume the following*:

(i) *the hypotheses* (i)–(iii) *of Lemma* 1 *are satisfied*;

(ii) *there exists a function $h: R \to R$ such that h is continuous, increasing, $h(-u) = -h(u)$, $uh(u) > 0$ for every $u \neq 0$, and for every $\varepsilon > 0$,*

$$\int_{\varepsilon}^{\infty} \frac{ds}{h(s)} < +\infty ;$$

(iii) *for any $t_0 \geq 0$, $L > 0$, $u \in C(t_0, L)$,*

$$\int_{t_0}^{\infty} t^{n-1}[[H(t, u(t)) - Q(t, u(t))]/h(u(t))]dt = +\infty .$$

Then

a) *if $x(t)$ is an eventually positive* (*negative*) *solution of* (I), *there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = +\infty$ and, for $n = 1, 2, \ldots$,*

$$H(t_n, x(t_n)) \leq Q(t_n, x(t_n)) \quad (H(t_n, x(t_n)) \geq Q(t_n, x(t_n)) .$$

b) *If, moreover,*

$$\lim_{t\to\infty} [Q(t, u(t))/H(t, u(t))] = 0$$

for every $t_0 \geq 0$, $L>0$ *and* $u \in C(t_0, L)$, *then every solution of* (I) *is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (I). Assume that $x(t)$ is bounded and positive for all $t \geq \bar{t} \geq 0$. Let t_0, L, $t_0 \geq \bar{t}$ be such that the conclusion of Lemma 1 holds and, moreover, $H(t, x(t)) > Q(t, x(t))$, $(-1)^j x^{(j)}(t) < 0$, $j=1, 2,\dots, n-1$, $t \geq t_0$. Then $0 < h(x(t)) \leq h(K)$ for $t \geq t_0$, where $x(t) \leq K$ for $t \geq t_0$. The integral condition in (iii) in conjunction with Lemma 1 show that the conditions of Theorem 1 are now satisfied for $x(t)$, $t \geq t_0$. In fact,

$$[1/h(K)]\int_{t_0}^{\infty} t^{n-1}[H(t, x(t_0)) - Q_K(t)]dt$$

$$\geq \int_{t_0}^{\infty} t^{n-1}[[H(t, x(t)) - Q(t, x(t))]/h(x(t))]dt = +\infty,$$

where $Q_K(t) = Q_0(t, K)$. Thus, the first conclusion of the theorem is true for $x(t)$ bounded. Now assume that $x(t)$ is eventually positive and unbounded. Furthermore, assume that $H(t, x(t)) - Q(t, x(t)) > 0$ for all large t. Then $x^{(n)}(t) < 0$ for all large t. Thus, there exist numbers $t_0 \geq 0$, $L>0$ such that the conclusion of the Lemma holds, and, moreover,

$$H(t, x(t)) - Q(t, x(t)) > 0,$$

$$x^{(n-1)}(t) > 0,\ x'(t) > 0$$

for all $t \geq t_0$. Now consider the function $F(t) \equiv t^{n-1}x^{(n-1)}(t)/h(x(t))$, $t \geq t_0$. Then we have

$$F(t) \leq F(t_0) - \int_{t_0}^{t} s^{n-1}[[H(s, x(s)) - Q(s, x(s))]/h(x(s))]ds$$

$$+ (n-1)\int_{t_0}^{t} [s^{n-2}x^{(n-1)}(s)/h(x(s))]ds + \int_{t_0}^{t} s^{n-1}x^{(n-1)}(s)d[1/h(x(s))]$$

with the last integral considered in the Riemann-Stieltjes sense. Since $1/h(x(t))$ is decreasing ($x'(t) > 0$), the above inequality implies

$$\lim_{t\to+\infty}\left[F(t) - \int_{t_0}^{t} [s^{n-2}x^{(n-1)}(s)/h(x(s))]ds\right] = -\infty.$$

The proof now can continue as in Theorem 3 of Kartsatos [1], and we omit the rest of it which contradicts the assumption $H - Q > 0$ as $t \geq t_0$. Thus, the first conclusion of the theorem is true, and the second holds as in Corollary 1.

Before we provide an example illustrating the above theorem, we should mention that the equation $y^{(n)}=Q_0(t, y)$ was used in Theorem 2 for an appraisal of any positive solution of (III). However, we could have used instead of $C(t_0, L)$ any other class of functions to which a particular positive solution (assumed to exist in the proof) belongs.

Now consider the following equation:

$$x^{(n)} + p(t)|x|^{\alpha}\operatorname{sgn} x = q(t)|x|^{\beta}\operatorname{sgn} x, \quad n \text{ even}, \tag{9}$$

where p, q are continuous and positive for $t \geq 0$, $\alpha > \beta \geq 1$, and

$$\int_1^{\infty} t^{n-1}q(t)dt < +\infty, \int_1^{\infty} t^{n-1}p(t)dt = +\infty, \lim_{t\to\infty}[q(t)/p(t)] = 0.$$

Then every bounded solution of (9) oscillates, and every unbounded nonoscillatory solution $x(t)$ satisfies $\liminf_{t\to\infty}|x(t)|=0$. In fact, the bounded solutions of (9) oscillate by Theorem A because all solutions of the homogeneous equation oscillate (cf., for example, Kartsatos [1, Corollary]). Now let $x(t)$ be positive and unbounded for all large t, and assume that $p(t)x^{\alpha}(t)-q(t)x^{\beta}(t)>0$ for all large t. Then there exists $t_1 \geq 1$ such that $x(t)>0$, $x'(t)>0$, $x^{(n-1)}(t)>0$ for all $t \geq t_1$. Since $x(t) \geq x(t_1)$, taking $h(t) \equiv x^{\alpha}(t)$, we obtain

$$\int_{t_1}^{\infty} t^{n-1}[[p(t)x^{\alpha}(t) - q(t)x^{\beta}(t)]/h(t)]dt$$

$$\geq \int_{t_1}^{\infty} t^{n-1}[p(t) - [x(t_1)]^{\beta-\alpha}q(t)]dt = +\infty.$$

Continuing as in the proof of Theorem 2, we get a contradiction. Thus, for some sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = +\infty$, $p(t_n)x^{\alpha}(t_n) \leq q(t_n)x^{\beta}(t_n)$, which implies,

$$0 = \lim_{n\to\infty} x^{\alpha-\beta}(t_n) \leq \lim_{n\to\infty}[q(t_n)/p(t_n)].$$

Thus, $\liminf_{t\to\infty} x(t)=0$. Since for any solution $x(t)$, $-x(t)$ is also a solution, the assertion is true.

The above theorem corresponds to the case of a "superlinear" homogeneous equation because $1/h(t)$ is integrable on (ε, ∞). We establish below a theorem which covers the sublinear case.

THEOREM 3. *Assume that H, Q satisfy* (i)–(iii) *of Lemma* 1. *Moreover, assume that for every $L>0$, $t_0 \geq 0$ and every $u \in C(t_0, L)$,*

$$\int_t^{\infty} t^{\alpha(n-1)}[[H(t, u(t)) - Q(t, u(t))]/u^{\alpha}(t)]dt = +\infty, \tag{10}$$

where α is a fixed constant with $0<\alpha<1$.

Then the conclusion a) *of Theorem* 2 *holds. If moreover,*

$$\lim_{t\to\infty}[Q(t, u(t))/H(t, u(t))] = 0$$

for every $t_0\geq 0$, $L>0$ and $u\in C(t_0, L)$, then every solution of (I) *oscillates.*

PROOF. Let $x(t)$ be a positive solution of (I). Then since the homogeneous equation oscillates, it follows that there exists $t_0\geq 0$ and $L>0$ such that $0<x(t)\leq y(t)$, $t\in[t_0, \infty)$, where $y(t)$ is any (but fixed) solution of $y^{(n)}(t)=Q(t, y(t))$ with $y(t_0)=L$ and $y^{(j)}(t_0)=0$, $j=1, 2,\dots, n-1$. Now assume that $H(t, x(t)) > Q(t, x(t))$, $t\in[t_1, \infty)$, $t_1\geq t_0$. Then Equation (I) can be written as follows:

$$x^{(n)} + P(t)x^{\alpha} = 0, \tag{11}$$

where

$$P(t) \equiv H(t, x(t)) - Q(t, x(t))/x^{\alpha}(t) > 0 \tag{12}$$

for every $t\geq t_1$. Since, however,

$$\int_{t_1}^{\infty} t^{\alpha(n-1)}P(t)dt = +\infty, \tag{13}$$

it follows from Theorem 1 in Ličko and Švec [6] that (11) cannot have eventually positive solutions, a contradiction. Thus,

$$H(t_n, x(t_n)) \leq Q(t_n, x(t_n)), \tag{14}$$

where $\{t_n\}$ is some sequence with $\lim_{n\to\infty} t_n=+\infty$. The rest of the proof follows as in Theorem 2 and is omitted.

The above theorem has an interesting corollary for second order equations which we now state:

COROLLARY 2. *Consider the equation*

$$x'' + p(t)|x|^{\alpha}\operatorname{sgn} x = q(t)|x|^{\beta}\operatorname{sgn} x, \qquad t\geq 0, \tag{15}$$

where p, q are positive and continuous for $t\geq 0$, $0<\alpha<\beta<1$; furthermore,

$$\int_{t_0}^{\infty} t^{\alpha}[p(t) - q(t)u^{\beta-\alpha}(t)]dt = +\infty, \tag{16}$$

$$\lim_{t\to\infty}[q(t)u^{\beta-\alpha}(t)/p(t)]dt = 0$$

for every $t_0\geq 0$, $L>0$, where $u(t)$ is any function with

$$0 < u(t) \leq [L^{1-\beta} + (1-\beta)\int_{t_0}^{t} (t-s)q(s)ds]^{\frac{1}{1-\beta}}, \qquad t \geq t_0 .$$

Then every solution of (15) *oscillates.*

Let us first remark that all solutions of the problem $y'' = q(t)|y|^{\beta}$, $y(t_0) = L > 0$, $y'(t_0) = 0$, are extendable on $[t_0, \infty)$. Now let $y(t)$ be a solution of this problem for which $y(t) > 0$, $t \geq t_0$. Then we have

$$[y'(t)/y^{\beta}(t)]' = q(t) - \beta[y'(t)]^2 y^{\beta-1}(t) \leq q(t), \tag{17}$$

which, by double integration from t_0 to $t \geq t_0$, gives

$$y(t) \leq [L^{1-\beta} + (1-\beta)\int_{t_0}^{t} (t-s)q(s)ds]^{\frac{1}{1-\beta}}. \tag{18}$$

From this follows the conclusion of the corollary as in Theorem 3. If we take $\alpha = 1/5$, $\beta = 1/3$, $p(t) = (t+1)^{-1}$, $q(t) = (t+1)^{-3}$, then $u(t) \leq \lambda t^{1/2}$ and the second of (16) becomes

$$\int_{t_0}^{\infty} t^{1/5}[(t+1)^{-1} - \lambda(t+1)^{-3}t^{1/2}]dt = +\infty \tag{17}$$

which is true for any $\lambda > 0$, $t_0 \geq 0$.

4. Discussion

We first note that the function $Q(t, u)$ in all the above results can be replaced by a function $Q(t, u, u', \ldots, u^{(n-1)})$, which is bounded above by $Q_0(t, |u|)$. We also note that in all the above results it was assumed that the equation $x^{(n)} + H(t, x) = 0$ has all of its solutions, or its bounded solutions, oscillatory. If such assumptions are not made we need stronger conditions on $H - Q$ which would guarantee oscillation. We are planning to examine such cases in a forthcoming paper. Theorems 2, 3 were given for the ordinary case $g_1(t) = g_2(t) \equiv t$. However, results can obviously be formulated without this assumption at the expense of extra calculations. In view of the importance of the class $C(t_0, L)$, the author thinks that it would be very interesting to establish the extendability of the solutions of

$$x^{(n)} = Q_0(t, x),\ x(t_0) = L,\ x^{(j)}(t_0) = 0, \qquad j = 1, 2, \ldots, n-1$$

on $[t_0, \infty)$ in the case of a superlinear $Q_0(t, u)$ for any t_0 sufficiently large, any $L > 0$, and any even $n > 2$. Instead of $u^{\alpha}(t)$ in (10) one could consider more general functions like, for example, a function $f(u(t))$, where f is as in Theorem 4 of Kusano and Onose [5].

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