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On the Radial Limits of Potentials and Angular Limits of Harmonic Functions

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1. Introduction

In the n-dimensional Euclidean space R^n $(n \geqq 2)$, the Riesz potential of order α of a non-negative function f in $L^p(R^n)$ is defined by

$$U_\alpha^f(x) = \int |x-y|^{\alpha-n} f(y)dy, \qquad x \in R^n,$$

where $0<\alpha<n$ and $1<p<\infty$. Our first aim is to discuss the existence of radial limits of U_α^f at a point of R^n, which can be assumed to be the origin O of R^n without loss of generality. For this purpose we shall use the capacity $C_{\alpha,p}$, which is a special case of the capacity $C_{k;\mu;p}$ introduced by N. G. Meyers [4] and is defined by

$$C_{\alpha,p}(E) = \inf \|g\|_p^p, \qquad E \subset R^n,$$

the infimum being taken over all non-negative functions $g \in L^p(R^n)$ such that $U_\alpha^g(x) \geqq 1$ for all $x \in E$; in case $\alpha p \geqq n$, we assume further that g vanishes outside the open ball with center at O and radius 2. In § 3, setting $S=\{x \in R^n; |x|=1\}$, we shall show that for a non-negative function $f \in L^p(R^n)$ satisfying $\int |y|^{\alpha p-n} f(y)^p dy < \infty$,

$$\text{(i)} \qquad \lim_{r \downarrow 0} U_\alpha^f(r\xi) = U_\alpha^f(O)$$

holds for $\xi \in S$ except those in a Borel set with $C_{\alpha,p}$-capacity zero. In case $U_\alpha^f(O) = \infty$, $\lim_{x \to O} U_\alpha^f(x) = U_\alpha^f(O)$ by the lower semi-continuity of U_α^f, and hence (i) holds for all $\xi \in S$. In this case, we shall investigate the order of infinity; in fact, we shall show that if $\alpha p \leqq n$ and f is a non-negative function in $L^p(R^n)$ with $U_\alpha^f \not\equiv \infty$, then we have

$$\begin{cases} \lim_{r \downarrow 0} r^{(n-\alpha p)/p} U_\alpha^f(r\xi) = 0 & \text{in case } \alpha p < n, \\ \lim_{r \downarrow 0} \left(\log \frac{1}{r}\right)^{1/p-1} U_\alpha^f(r\xi) = 0 & \text{in case } \alpha p = n \end{cases}$$

for $\xi \in S$ except those in a Borel set with $C_{\alpha,p}$-capacity zero. These results can be considered as an improvement of the following fact (cf. [2; Theorem IX, 7]): Let $U_\alpha^\mu \not\equiv \infty$ be the Riesz potential of order α of a non-negative (Radon) measure μ. Then there is a Borel set $E \subset S$ such that $C_\alpha(E)=0$,

$$\lim_{r \downarrow 0} U_\alpha^\mu(r\xi) = U_\alpha^\mu(O)$$

and

$$\lim_{r \downarrow 0} r^{n-\alpha} U_\alpha^\mu(r\xi) = \mu(\{O\})$$

for all $\xi \in S \backslash E$, where C_α denotes the Riesz capacity of order α.

As an application of the results obtained above, we shall study the existence of radial limits of p-precise functions (see [9]) defined on a neighborhood G of the origin. Since all p-precise functions on G are continuous if $p>n$, we are interested in the case $p \leqq n$. We shall show that if u is a p-precise function on G satisfying

$$\int_G |\mathrm{grad}\, u| \cdot |x|^{1-n} dx < \infty \quad \text{and} \quad \int_G |\mathrm{grad}\, u|^p |x|^{p-n} dx < \infty,$$

then $\lim_{r \downarrow 0} u(r\xi)$ exists for $\xi \in S$ except those in a Borel set with $C_{1,p}$-capacity zero. It will also be shown that for a p-precise function u on G, we have

$$\begin{cases} \lim_{r \downarrow 0} r^{(n-p)/p} u(r\xi) = 0 & \text{in case } p < n, \\ \lim_{r \downarrow 0} \left(\log \dfrac{1}{r}\right)^{1/p-1} u(r\xi) = 0 & \text{in case } p = n \end{cases}$$

for $\xi \in S$ except those in a Borel set with $C_{1,p}$-capacity zero.

In the final section we shall be concerned with harmonic functions on a cone of the form $\Gamma(a)=\{(x', x_n) \in R^{n-1} \times R^1;\ |x'|<ax_n,\ |x'|^2+x_n^2<1\}$, where $a>0$. Our aim is to prove that if h is a harmonic function on $\Gamma(a)$ satisfying

$$\text{(ii)} \qquad \int_{\Gamma(a)} |\mathrm{grad}\, h|^p g(|x|) |x|^{p-n} dx < \infty,$$

then $\lim_{x \to O, x \in \Gamma(a')} h(x)$ exists and is finite for any a' with $0<a'<a$, where g is a positive and non-increasing function on the interval (0, 1) such that

$$\text{(iii)} \qquad \int_0^1 \frac{dt}{t g(t)^{1/(p-1)}} < \infty.$$

Moreover we shall show that (iii) is necessary in the following sense: If g is a positive and non-increasing function on (0, 1) such that $t^{-\delta} g(t)^{-1}$ is non-increasing on (0, 1) for some δ with $0<\delta<p/2$ and $\int_0^1 g(t)^{-1/(p-1)} t^{-1} dt = \infty$, then we

can find a harmonic function h on $\Gamma(a)$ satisfying (ii) such that $\lim_{x_n \downarrow 0} h(0,\ldots, 0, x_n)$ does not exist. These are an extension of a result obtained by T. Murai [7].

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2. Preliminaries

Throughout this paper, let $0<\alpha<n$ and $1<p<\infty$. We define the capacity $C_{\alpha,p}$ as follows:

$$C_{\alpha,p}(E) = \inf \|f\|_p^p, \qquad E \subset R^n,$$

where the infimum is taken over all non-negative functions f in $L^p(R^n)$ such that $U_\alpha^f(x) \geqq 1$ for all $x \in E$ and $\|f\|_p$ denotes the L^p-norm. We need another capacity: Let G be a bounded open set in R^n and define

$$C_{\alpha,p}(E;\, G) = \inf \|f\|_p^p, \qquad E \subset R^n,$$

where the infimum is taken over all non-negative functions f in $L^p(R^n)$ such that $f=0$ outside G and $U_\alpha^f \geqq 1$ on E.

Let us begin with

LEMMA 1. *Assume $\alpha p<n$. Let F be a compact set in a bounded open set $G \subset R^n$. Then there is a positive constant M such that $C_{\alpha,p}(E;\, G) \leqq M C_{\alpha,p}(E)$ whenever $E \subset F$.*

PROOF. Let f be a non-negative function in $L^p(R^n)$ such that $U_\alpha^f \geqq 1$ on E. By Hölder's inequality, we have

$$\int_{R^n \setminus G} |x-y|^{\alpha-n} f(y)\,dy \leqq \left\{\int_{R^n \setminus G} |x-y|^{p'(\alpha-n)}\,dy\right\}^{1/p'} \|f\|_p,$$

where $1/p+1/p'=1$. Hence there is $\varepsilon>0$ such that $\|f\|_p^p<\varepsilon$ implies

$$\sup_{x \in F} \int_{R^n \setminus G} |x-y|^{\alpha-n} f(y)\,dy \leqq 1/2,$$

so that $\int_G |x-y|^{\alpha-n} f(y)\,dy \geqq 1/2$ for $x \in E$. From this it follows that $C_{\alpha,p}(E;\, G) \leqq 2^p C_{\alpha,p}(E)$ whenever $E \subset F$ and $C_{\alpha,p}(E)<\varepsilon$. On the other hand, considering the potential $U(x)=\int_G |x-y|^{\alpha-n}\,dy$, we easily see that $C_{\alpha,p}(F;\, G)<\infty$. Thus the inequality of our lemma is satisfied with $M=\max\{2^p,\, \varepsilon^{-1} C_{\alpha,p}(F;\, G)\}$.

COROLLARY. *Let E be a bounded set in R^n. Then $C_{\alpha,p}(E)=0$ implies $C_{\alpha,p}(E;\, G)=0$ for any bounded open set G which contains $\bar{E}$ (the closure of E).*

Conversely, if $C_{\alpha,p}(E; G)=0$ for some bounded open set G such that $\bar{E}\subset G$, then $C_{\alpha,p}(E)=0$.

In the general case we have the following lemma, which can be proved in a way similar to the above proof.

LEMMA 2. *Let G and G' be bounded open sets in R^n. Let F be a compact subset of $G\cap G'$. Then there is a positive constant M such that $C_{\alpha,p}(E; G)\leqq MC_{\alpha,p}(E; G')$ for any $E\subset F$.*

COROLLARY. *If $\bar{E}\subset G\cap G'$, then $C_{\alpha,p}(E; G)=0$ is equivalent to $C_{\alpha,p}(E; G')=0$.*

Let G and G' be open sets in R^n. A mapping T: $G\to G'$ is said to be Lipschitzian if there exists a positive constant M such that

$$M^{-1}|x-y| \leqq |Tx-Ty| \leqq M|x-y|$$

for all x and y in G; one refers to M as a Lipschitz constant for T.

We shall show

LEMMA 3. *Let G be a bounded open set in R^n and T: $G\to TG$ be a Lipschitzian mapping with Lipschitz constant $M>0$. Then for $E\subset G$,*

$$N^{-1}C_{\alpha,p}(E; G) \leqq C_{\alpha,p}(TE; TG) \leqq NC_{\alpha,p}(E; G)$$

with $N=M^{n+p(2n-\alpha)}$.

PROOF. Let f be a non-negative function in $L^p(R^n)$ such that f vanishes outside G and $U_\alpha^f\geqq 1$ on E. Define the function

$$g(z) = \begin{cases} f(T^{-1}z) & \text{for } z\in TG, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for $x\in E$,

$$\int|Tx-z|^{\alpha-n}g(z)dz \geqq M^{\alpha-2n}\int|x-y|^{\alpha-n}f(y)dy \geqq M^{\alpha-2n}.$$

This gives

$$C_{\alpha,p}(TE; TG) \leqq M^{p(2n-\alpha)}\int g(z)^p dz \leqq M^{p(2n-\alpha)}M^n\int f(y)^p dy,$$

which implies that $C_{\alpha,p}(TE; TG)\leqq M^{n+p(2n-\alpha)}C_{\alpha,p}(E; G)$. Thus the inequalities in our lemma are satisfied with $N=M^{n+p(2n-\alpha)}$.

For $r>0$ and $E\subset R^n$, we set $rE=\{rx; x\in E\}$. When T is the Lipschitzian mapping defined by $Tx = rx$ for $x \in R^n$, we obtain

LEMMA 4. *For $r > 0$ and $E \subset R^n$, we have*

$$C_{\alpha,p}(rE) = r^{n-\alpha p} C_{\alpha,p}(E) .$$

This follows with a slight modification of the above proof.

For a set $E \subset R^n$, we denote by $\tilde{E}$ the set of all points $\xi \in S = \{x \in R^n;\ |x| = 1\}$ such that $r\xi \in E$ for some $r > 0$. For $a > 0$ and $x \in R^n$, we denote by $B(x, a)$ the open ball with center at x and radius a. We shall write simply $B(a)$ for $B(O, a)$.

We are now ready to show our main lemma.

LEMMA 5. *There exists a positive constant M such that for $E \subset B(2) \setminus B(1)$,*

$$C_{\alpha,p}(\tilde{E};\ B(3)) \leqq M C_{\alpha,p}(E;\ B(3)) .$$

Especially, in case $\alpha p < n$, $C_{\alpha,p}(\tilde{E}) \leqq M C_{\alpha,p}(E)$ for $E \subset B(2) \setminus B(1)$.

PROOF. Set

$$G = \{x = (x', x_n) \in R^{n-1} \times R^1;\ |x'| < x_n,\ 1/2 < |x| < 3\},$$

$$F = \{x = (x', x_n) \in R^{n-1} \times R^1;\ |x'| \leqq x_n/2,\ 1 \leqq |x| \leqq 2\} .$$

On account of the subadditivity of $C_{\alpha,p}(\cdot;\ G)$ (cf. [4]) and Lemmas 1, 2, it suffices to show that

(1) $\quad C_{\alpha,p}(\tilde{E} \cap F;\ G) \leqq M C_{\alpha,p}(E \cap F;\ G) \quad$ with some constant $M > 0$.

Consider the mapping $T\colon G \to TG$ defined by

$$Tx = \left(\frac{x_1}{|x|}, \ldots, \frac{x_{n-1}}{|x|}, |x|\right), \qquad x = (x_1, \ldots, x_n).$$

Note that $TG = \{(y', y_n) \in R^{n-1} \times R^1;\ |y'| < 1/\sqrt{2},\ 1/2 < y_n < 3\}$ and that T is Lipschitzian. By Lemma 3, there is a constant $M' > 0$ such that $C_{\alpha,p}(T(E \cap F);\ TG) \leqq M' C_{\alpha,p}(E \cap F;\ G)$. In the same way as in the proof of Lemma 1 in [6], we can show that

$$C_{\alpha,p}(T(E \cap F)^*;\ C) \leqq C_{\alpha,p}(T(E \cap F);\ C),$$

where $T(E \cap F)^*$ is the projection of $T(E \cap F)$ to the hyperplane $\{(x', x_n) \in R^{n-1} \times R^1;\ x_n = 1\}$ and $C = \{(x', x_n);\ |x'| < \sqrt{2},\ -1 < x_n < 3\}$. Noting that $T(E \cap F)^* = T(\tilde{E} \cap F)$ and using Lemmas 2 and 3, we have the required inequality (1).

COROLLARY. *Let $r > 1$. If $C_{\alpha,p}(E;\ B(r)) = 0$ for $E \subset B(r/2)$, then $C_{\alpha,p}(\tilde{E};\ B(r)) = 0$.*

PROOF. Set $E_n = E \cap (B(2^{-n}r) \setminus B(2^{-n-1}r))$. Evidently $C_{\alpha,p}(E_n; B(r)) = 0$. On account of the subadditivity it suffices to show $C_{\alpha,p}(\tilde{E}_n; B(r)) = 0$ for each n. Fixing n we have $C_{\alpha,p}((2^{n+1}/r)E_n; B(2^{n+1})) = 0$ by Lemma 3 and hence $C_{\alpha,p}((2^{n+1}/r)E_n; B(3)) = 0$ by Lemma 2. Lemma 5 yields $C_{\alpha,p}(\tilde{E}_n; B(3)) = 0$ and $C_{\alpha,p}(\tilde{E}_n; B(r)) = 0$ follows from Lemma 2.

REMARK 1. (i) If $\alpha p \geqq n$, then $C_{\alpha,p}(R^n) = 0$.
(ii) If $\alpha p > n$ and $x^0 \in R^n$, then $C_{\alpha,p}(\{x^0\}; B(2)) > 0$.
(iii) If $C_{\alpha,p}(E; B(2)) = 0$, then E is of (n-dimensional) measure zero.
(iv) If $\alpha p \leqq n$, then $C_{\alpha,p}(E; B(2)) = 0$ implies that

$$\begin{cases} C_{\alpha p}(E) = 0 & \text{in case } p \leqq 2, \\ C_{\alpha p - \varepsilon}(E) = 0 & \text{for any } \varepsilon \text{ with } 0 < \varepsilon < \alpha p \text{ in case } p > 2. \end{cases}$$

Here C_β denotes the Riesz capacity of order β.

For (i) we have only to show $C_{\alpha,p}(B(1)) = 0$ on account of Lemma 4. For $a > 1$, define the function

$$f_a(y) = \begin{cases} |y|^{-n/p}(\log |y|)^{-1} & \text{if } a < |y| < a^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can find a positive constant M independent of a such that

$$\int f_a(y)^p dy \leqq M \int_a^\infty \frac{dr}{r(\log r)^p}$$

and

$$\int |x - y|^{\alpha - n} f_a(y) dy \geqq \int_{a < |y| < a^2} (2|y|)^{\alpha - n} f_a(y) dy \geqq M^{-1}$$

for all $x \in B(1)$. These imply $C_{\alpha,p}(B(1)) = 0$.

To show (ii), we take $f \in L^p(R^n)$ such that f vanishes outside $B(2)$ and $U_\alpha^f(x^0) \geqq 1$. Hölder's inequality gives

$$1 \leqq \int |x^0 - y|^{\alpha - n} f(y) dy \leqq \left\{ \int_{|y| < 2} |y|^{p'(\alpha - n)} dy \right\}^{1/p'} \|f\|_p,$$

where $1/p + 1/p' = 1$. Since $p'(\alpha - n) + n = p'(\alpha - n/p) > 0$, $C_{\alpha,p}(\{x^0\}; B(2)) > 0$.

The assertion (iv) is a consequence of a result of B. Fuglede [3]. The assertion (iii) follows immediately from (ii) and (iv).

3. Radial limits of potentials at the origin

We first show

LEMMA 6. *Let f be a non-negative measurable function such that $\int |y|^{\beta} f(y)^p dy < \infty$ for a number β. Then there is a Borel set $E \subset S$ such that $C_{\alpha,p}(E; B(2)) = 0$ and*

$$\lim_{r \downarrow 0} r^{(n-\alpha p+\beta)/p} \int_{|r\xi - y| \leqq r/2} |r\xi - y|^{\alpha-n} f(y) dy = 0$$

for every $\xi \in S \backslash E$. If, in addition, $\alpha p > n$, then

$$\lim_{x \to O} |x|^{(n-\alpha p+\beta)/p} \int_{|x-y| \leqq |x|/2} |x - y|^{\alpha-n} f(y) dy = 0.$$

PROOF. Set

$$U(x) = \int_{|x-y| \leqq |x|/2} |x - y|^{\alpha-n} f(y) dy.$$

Set also $a_k = \int_{2^{-k-1} \leqq |y| < 2^{-k+2}} |y|^{\beta} f(y)^p dy$ for each positive integer k, and choose a sequence $\{b_k\}$ of positive numbers so that $\lim_{k \to \infty} b_k = \infty$ and $\sum_{k=1}^{\infty} a_k b_k < \infty$. Further we set

$$E_k = \{x \in R^n;\ 2^{-k} \leqq |x| < 2^{-k+1},\ U(x) \geqq b_k^{-1/p} 2^{k(n-\alpha p+\beta)/p}\}$$

for each k. Define the function

$$g_k(z) = \begin{cases} f(2^{-k} z) & \text{if } 1/2 \leqq |z| < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for $x \in E_k$

$$U(x) \leqq 2^{k(n-\alpha)} \int_{2^{-k-1} \leqq |y| < 2^{-k+2}} |2^k x - 2^k y|^{\alpha-n} f(y) dy$$

$$= 2^{-k\alpha} \int |2^k x - z|^{\alpha-n} g_k(z) dz,$$

so that

$$C_{\alpha,p}(2^k E_k; B(4)) \leqq 2^{-k\alpha p} b_k 2^{-k(n-\alpha p+\beta)} \int g_k(z)^p dz$$

$$\leqq 2^{-k(n+\beta)} b_k \left\{ \int_{2^{-k-1} \leqq |y| < 2^{-k+2}} |y|^{\beta} f(y)^p dy \right\}$$

$$\times \max\{2^{(k+1)\beta}, 2^{(k-2)\beta}\} 2^{kn}$$

$$\leqq 4^{|\beta|} a_k b_k.$$

This together with Lemmas 2 and 5 gives

$$C_{\alpha,p}(\tilde{E}_k;\ B(2)) = C_{\alpha,p}(\widetilde{2^k E_k};\ B(2)) \leqq M a_k b_k,$$

where M is a positive constant independent of k. Setting $E=\cap_{j=1}^{\infty}\cup_{k=j}^{\infty}\tilde{E}_k$, we see that $C_{\alpha,p}(E;\ B(2))=0$ and $\lim_{r\downarrow 0} U(r\xi)=0$ for $\xi\in S\backslash E$. If $\alpha p>n$, then E_k is empty for k sufficiently large on account of Remark 1, (ii). Thus our lemma is proved.

For a non-negative locally integrable function f on R^n, we also define

$$U_\alpha^f(x) = \int |x-y|^{\alpha-n} f(y)dy, \qquad x\in R^n.$$

THEOREM 1. *Let f be a non-negative measurable function such that $\int |y|^{\alpha p-n} f(y)^p dy<\infty$. Then there is a Borel set $E\subset S$ such that $C_{\alpha,p}(E;\ B(2))=0$ and*

$$\lim_{r\downarrow 0} U_\alpha^f(r\xi) = U_\alpha^f(O) \qquad \textit{for every } \xi\in S\backslash E.$$

If, in addition, $\alpha p>n$, then $\lim_{x\to O} U_\alpha^f(x)=U_\alpha^f(O)$.

Proof. We decompose U_α^f as $F+U$, where

$$F(x) = \int_{|x-y|>|x|/2} |x-y|^{\alpha-n} f(y)dy,$$

$$U(x) = \int_{|x-y|\leqq|x|/2} |x-y|^{\alpha-n} f(y)dy.$$

If $U_\alpha^f(O)=\infty$, then $\lim_{x\to O} U_\alpha^f(x)=\infty$ by the lower semicontinuity of U_α^f. Hence it suffices to be concerned with the case $U_\alpha^f(O)<\infty$. In this case we have by Lebesgue's dominated convergence theorem

$$\lim_{x\to O} F(x) = U_\alpha^f(O)$$

since $|x-y|>|x|/2$ implies $|x-y|>|y|/3$. By the aid of Lemma 6 we conclude our theorem.

COROLLARY. *Let f be a non-negative function in $L^p(R^n)$ and set*

$$A = \left\{x^0\in R^n;\ \int_{|x^0-y|<1} |x^0-y|^{\alpha p-n} f(y)^p dy = \infty\right\}.$$

Then to each $x^0\in R^n\backslash A$, there corresponds a Borel set $E_{x^0}\subset S$ such that $C_{\alpha,p}(E_{x^0};\ B(2))=0$ and

$$\lim_{r\downarrow 0} U_\alpha^f(x^0+r\xi) = U_\alpha^f(x^0) \qquad \textit{for every } \xi\in S\backslash E_{x^0}.$$

We remark here that A is empty in case $\alpha p \geqq n$ and $C_{\alpha p}(A)=0$ in case $\alpha p<n$; if $\alpha p<n$ and $p \geqq 2$, then $C_{\alpha p}(A)=0$ implies $C_{\alpha,p}(A)=0$ in view of [3; Theorem 4.2].

REMARK 2. If we set $A=\cup_{k=1}^{\infty} E_k$ in the proof of Lemma 6, then

$$\lim_{\substack{x \to O \\ x \in R^n \setminus A}} U_\alpha^f(x) = U_\alpha^f(O)$$

and

$$\sum_{k=1}^{\infty} 2^{k(n-\alpha p)} C_{\alpha,p}(A_k;\, B(2)) < \infty, \quad A_k = E_k = A \cap B(2^{-k+1}) \setminus B(2^{-k}).$$

From this condition we can derive that $\lim_{r \downarrow 0} C_{\alpha,p}(\widetilde{A \cap B(r)};\, B(2))=0$. From this the conclusion in Theorem 1 follows immediately.

REMARK 3. In case $\alpha<1$ and $\alpha p<n$, there is a non-negative function f in $L^p(R^n)$ such that $U_\alpha^f(O)<\infty$ but $\limsup_{r \downarrow 0} U_\alpha^f(r\xi)=\infty$ for every $\xi \in S$.

To construct a function f with these properties, we set $r_j=2^{-j}$ and

$$s_j = \begin{cases} 2^{-am} & \text{if } j = 2^m \text{ and } m \text{ is a positive integer,} \\ 0 & \text{otherwise} \end{cases}$$

for each positive integer j, where $1<a<1/\alpha$. Define the function

$$f(y) = \begin{cases} j r_j^{-\alpha} & \text{if } (1 - s_j) r_j < |y| < (1 + s_j) r_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that f has the required properties.

In case $U_\alpha^f(O)=\infty$, we shall investigate the order of infinity.

THEOREM 2. *Let $n-\alpha p+\beta \geqq 0$ and let f be a non-negative locally integrable function such that $U_\alpha^f \not\equiv \infty$ and $\int |y|^\beta f(y)^p dy<\infty$. Then there is a Borel set $E \subset S$ such that $C_{\alpha,p}(E;\, B(2))=0$ and*

$$\begin{cases} \lim_{r \downarrow 0} r^{(n-\alpha p+\beta)/p} U_\alpha^f(r\xi) = 0 & \text{in case } n - \alpha p + \beta > 0, \\ \lim_{r \downarrow 0} \left(\log \frac{1}{r}\right)^{1/p-1} U_\alpha^f(r\xi) = 0 & \text{in case } n - \alpha p + \beta = 0 \end{cases}$$

for every $\xi \in S \setminus E$. If, in addition, $\alpha p>n$, then

$$\begin{cases} \lim_{x \to O} |x|^{(n-\alpha p+\beta)/p} U_\alpha^f(x) = 0 & \text{in case } n - \alpha p + \beta > 0, \\ \lim_{x \to O} \left(\log \frac{1}{|x|}\right)^{1/p-1} U_\alpha^f(x) = 0 & \text{in case } n - \alpha p + \beta = 0. \end{cases}$$

REMARK 4. In the theorem we assumed $U_\alpha^f \not\equiv \infty$. This is equivalent to $\int (1+|y|)^{\alpha-n} f(y)dy < \infty$.

PROOF OF THEOREM 2. We decompose U_α^f as in the proof of Theorem 1. In view of Lemma 6, we have only to show

$$\begin{cases} \lim_{x\to O} |x|^{(n-\alpha p+\beta)/p} F(x) = 0 & \text{in case } n-\alpha p+\beta > 0, \\ \lim_{x\to O} \left(\log \frac{1}{|x|}\right)^{1/p-1} F(x) = 0 & \text{in case } n-\alpha p+\beta = 0. \end{cases}$$

Case 1: $\alpha p - n < \beta < n(p-1)$. Choosing γ so that $\alpha p - \beta < \gamma < n$, we have by Hölder's inequality

$$F(x) \leqq \left\{\int_{|x-y|>|x|/2} |x-y|^{\gamma-n}|y|^\beta f(y)^p dy\right\}^{1/p}$$

$$\times \left\{\int_{|x-y|>|x|/2} |x-y|^{p'(\alpha-\gamma/p)-n}|y|^{-\beta p'/p} dy\right\}^{1/p'},$$

where $1/p+1/p'=1$. We can easily verify

$$\int_{|x-y|>|x|/2} |x-y|^{p'(\alpha-\gamma/p)-n}|y|^{-\beta p'/p}\, dy \leqq \text{const.}\ |x|^{p'(\alpha-\gamma/p-\beta/p)},$$

dividing the domain of integration into two parts, that is,

(i) $|x-y| > |x|/2,\ |y| \leqq |x|/2,$ (ii) $|x-y| > |x|/2,\ |y| > |x|/2.$

Hence we obtain

$$|x|^{(n-\alpha p+\beta)/p} F(x) \leqq \text{const.}\ |x|^{(n-\gamma)/p} \left\{\int_{|x-y|>|x|/2} |x-y|^{\gamma-n}|y|^\beta f(y)^p dy\right\}^{1/p}$$

$$\leqq \text{const.} \left\{\int_{|x-y|>|x|/2} \left(\frac{|x|}{|x-y|}\right)^{n-\gamma} |y|^\beta f(y)^p dy\right\}^{1/p},$$

which tends to zero as $x \to O$ by Lebesgue's dominated convergence theorem.

Case 2: $\beta \geqq n(p-1)$. In this case $(n-\alpha p+\beta)/p \geqq n-\alpha$. We have

$$|x|^{(n-\alpha p+\beta)/p} F(x) = |x|^{-n/p'+\beta/p} \int_{|x-y|>|x|/2} \left(\frac{|x|}{|x-y|}\right)^{n-\alpha} f(y)dy.$$

If $|x|<1$ and $|x-y|>|x|/2$, then $|x|\cdot|x-y|^{-1} < 5(1+|y|)^{-1}$, so that $|x|^{(n-\alpha p+\beta)/p} \times F(x) \to 0$ as $x \to O$ on account of Remark 3 and Lebesgue's dominated convergence theorem.

Case 3: $n-\alpha p+\beta=0$. Given ε such that $0<\varepsilon<1$, we see that $\int_{|x-y|>|x|/2,|y|>\varepsilon}|x-y|^{\alpha-n}f(y)dy$ tends to a finite number $\int_{|y|>\varepsilon}|y|^{\alpha-n}f(y)dy$ as $x\to O$. On the other hand Hölder's inequality gives

$$\int_{|x-y|>|x|/2,|y|\leqq\varepsilon}|x-y|^{\alpha-n}f(y)\,dy \leqq \left\{\int_{|y|\leqq\varepsilon}|y|^{\beta}f(y)^{p}dy\right\}^{1/p}$$

$$\times\left\{\int_{|x-y|>|x|/2,|y|\leqq\varepsilon}|x-y|^{(\alpha-n)p'}|y|^{-\beta p'/p}dy\right\}^{1/p'}.$$

It is easy to show

$$\int_{|x-y|>|x|/2,|y|\leqq\varepsilon}|x-y|^{(\alpha-n)p'}|y|^{-\beta p'/p}dy \leqq \text{const.}\log\frac{1}{|x|}$$

for any $x\in R^n$ with $|x|<1/2$, if we divide the domain of integration into two parts, that is, (iii) $|x-y|>|x|/2$, $|y|\leqq\varepsilon$, $|y|<|x|/2$, (iv) $|x-y|>|x|/2$, $|y|\leqq\varepsilon$, $|y|\geqq|x|/2$. Hence

$$\limsup_{x\to O}\left(\log\frac{1}{|x|}\right)^{-1/p'}\int_{|x-y|>|x|/2,|y|\leqq\varepsilon}|x-y|^{\alpha-n}f(y)dy$$

$$\leqq \text{const.}\left\{\int_{|y|\leqq\varepsilon}|y|^{\beta}f(y)^{p}dy\right\}^{1/p},$$

so that

$$\limsup_{x\to O}\left(\log\frac{1}{|x|}\right)^{1/p'}F(x) \leqq \text{const.}\left\{\int_{|y|\leqq\varepsilon}|y|^{\beta}f(y)^{p}dy\right\}^{1/p},$$

which implies $\lim_{x\to O}(\log 1/|x|)^{-1/p'}F(x)=0$. Thus the proof is now complete.

REMARK 5. Let $a(r)$ be a non-increasing function on the interval $(0,\infty)$ such that $\lim_{r\downarrow 0}a(r)=\infty$. Then there is a non-negative measurable function f such that $\int|y|^{\beta}f(y)^{p}dy<\infty$, $f=0$ on $R^n\backslash B(2)$ and

$$\begin{cases}\limsup_{r\downarrow 0}a(r)r^{(n-\alpha p+\beta)/p}U_\alpha^f(r\xi)=\infty & \text{in case } n-\alpha p+\beta>0,\\ \limsup_{r\downarrow 0}a(r)\left(\log\frac{1}{r}\right)^{1/p-1}U_\alpha^f(r\xi)=\infty & \text{in case } n-\alpha p+\beta=0\end{cases}$$

for every $\xi\in S$. In case $\beta<n(p-1)$, f is locally integrable because of $\int|y|^{\beta}f(y)^{p}dy<\infty$, so that Remark 4 gives $U_\alpha^f\not\equiv\infty$.

In case $n-\alpha p+\beta>0$ we choose $\{k_j\}$ so that $2k_j<k_{j+1}$ and $\sum_{j=1}^{\infty}1/a(2^{-k_j})<\infty$. We set $b_j=2^{(n+\beta)k_j/p}a(2^{-k_j})^{-1/p}$ and define $f(y)$ by b_j in $2^{-k_j-1}<|y|<2^{-k_j+1}$ and by 0 elsewhere. In case $n-\alpha p+\beta=0$ we choose $\{k_j\}$ so that $2k_j<k_{j+1}$ and $\sum_{j=1}^{\infty}1/a(2^{-2k_j+1})<\infty$. We set $c_j=a(2^{-2k_j+1})$ and define

$$f(y)=\begin{cases} c_j^{-1/p}|y|^{-\alpha}\left(\log\dfrac{1}{|y|}\right)^{-1/p} & \text{if } 2^{-2k_j}<|y|<2^{-k_j},\\ 0 & \text{elsewhere.}\end{cases}$$

In both cases it is easy to check that $\int |y|^{\beta}f(y)^p dy<\infty$. In case $n-\alpha p+\beta>0$

$$U_\alpha^f(2^{-k_j}\xi)\leqq\int_{|2^{-k_j}\xi-y|<2^{-k_j-1}}|2^{-k_j}\xi-y|^{\alpha-n}f(y)dy$$

$$=b_j2^{-(k_j+1)\alpha}\int_{|y|<1}|y|^{\alpha-n}dy.$$

It is immediate to see that $a(2^{-k_j})2^{-k_j(n-\alpha p+\beta)/p}U_\alpha^f(2^{-k_j}\xi)\to\infty$ as $j\to\infty$. In case $n-\alpha p+\beta=0$ write r_j for 2^{-k_j}. For y with $r_j^2<|y|<r_j$ we observe that $|2r_j^2\xi-y|/|y|\leqq 3$. Hence

$$U_\alpha^f(2^{-2k_j+1}\xi)=U_\alpha^f(2r_j^2\xi)\geqq 3^{\alpha-n}\int_{r_j^2<|y|<r_j}|y|^{\alpha-n}f(y)dy$$

$$=\text{const.}\ c_j^{-1/p}\int_{r_j^2}^{r_j}\frac{dr}{r\left(\log\dfrac{1}{r}\right)^{1/p}}=\text{const.}\ c_j^{-1/p}k_j^{1-1/p}.$$

It is easy to see that $a(2^{-2k_j+1})(\log 2^{2k_j-1})^{1/p-1}U_\alpha^f(2^{-2k_j+1}\xi)\to\infty$ as $j\to\infty$.

REMARK 6. Theorems 1 and 2 are the best possible as to the size of the exceptional set.

In order to prove this fact, we let E be a set in S with $C_{\alpha,p}(E;B(2))=0$. If we set $E_k=\{k^{-1}x;\ x\in E\}$ for each positive integer k, then $C_{\alpha,p}(E_k;B(2))=0$ for each k. By Lemma 2,

$$C_{\alpha,p}(E_k;G_k)=0,$$

where $G_k=\{x\in R^n;\ 1/(k+1)<|x|<1/(k-1)\}$. Hence there is a non-negative function $f_k\in L^p(R^n)$ such that $f_k=0$ outside G_k, $U_\alpha^{f_k}(O)<2^{-k}$, $\int|y|^{\alpha p-n}f_k(y)^pdy<2^{-k}$ and $U_\alpha^{f_k}(x)=\infty$ for all $x\in E_k$. Set $f=\sum_{k=1}^{\infty}f_k$. Clearly, $\int|y|^{\alpha p-n}f(y)^pdy<\infty$ and

$$\limsup_{r\downarrow 0}r^{\beta}U_\alpha^f(r\xi)=\limsup_{r\downarrow 0}\left(\log\frac{1}{r}\right)^{\beta}U_\alpha^f(r\xi)=\infty$$

for any $\xi \in E$ and any number β.

4. Radial limits of functions defined on a punctured ball

We say that a function u on an open set $G \subset R^n$ is locally p-precise if u is p-precise on any relatively compact open subset of G; for p-precise functions, see [9]. Note that for a locally p-precise function u on G, grad u is defined a.e. on G and $\int_\omega |\operatorname{grad} u|^p dx < \infty$ for any relatively compact open subset ω of G. For the details of p-precise or locally p-precise functions, see [8; Chap. IV].

In this section we are concerned with locally p-precise functions u on the punctured ball $D = B(1) \setminus \{O\}$, and discuss the existence of radial limits of u at the origin.

THEOREM 3. *Let $D = \{x \in R^n; 0 < |x| < 1\}$ and let u be a locally p-precise function on D satisfying*

$$\int_D |\operatorname{grad} u|^p |x|^{p-n} dx < \infty, \tag{2}$$

$$\int_D |\operatorname{grad} u| \cdot |x|^{1-n} dx < \infty. \tag{3}$$

Then there are a (finite) constant c and a Borel set $E \subset S$ such that $C_{1,p}(E; B(2)) = 0$ and $\lim_{r \downarrow 0} u(r\xi) = c$ for all $\xi \in S \setminus E$. In case $p > n$, u has a finite limit at the origin.

PROOF. First we consider the case $p \leqq n$. In this case, u is p-precise on $B(1)$. Let φ be a function in $C_0^\infty(B(1))$ which equals one on a neighborhood of the origin. Then φu is p-precise on R^n and satisfies conditions (2) and (3). Hence we may assume that u is p-precise on R^n and has compact support in $B(1)$. By [8; Theorem 9.11] or [5; Theorem 3.1], we have the following integral representation of u: For some constants a_i, $i = 1, 2, \ldots, n$,

$$u(x) = \sum_{i=1}^n a_i \int \frac{x_i - y_i}{|x - y|^n} \frac{\partial u}{\partial y_i} dy \tag{4}$$

holds except on a Borel set $E_1 \subset B(1)$ with $C_{1,p}(E_1; B(2)) = 0$. Set

$$c = -\sum_{i=1}^n a_i \int \frac{y_i}{|y|^n} \frac{\partial u}{\partial y_i} dy.$$

Then c is finite by (3). By Lebesgue's dominated convergence theorem, we see that

$$\sum_{i=1}^n a_i \int_{|x-y| > |x|/2} \frac{x_i - y_i}{|x - y|^n} \frac{\partial u}{\partial y_i} dy \longrightarrow c \qquad \text{as } x \longrightarrow O,$$

and by Lemma 6 that

$$\left|\sum_{i=1}^{n} a_i \int_{|r\xi-y|\leq r/2} \frac{r\xi_i - y_i}{|r\xi - y|^n} \frac{\partial u}{\partial y_i} dy\right|$$

$$\leq \sum_{i=1}^{n} |a_i| \int_{|r\xi-y|\leq r/2} |r\xi - y|^{1-n} |\operatorname{grad} u| dy \longrightarrow 0 \qquad \text{as } r \downarrow 0$$

for $\xi \in S$ except those in a Borel set E_2 with $C_{1,p}(E_2; B(2))=0$. Since $C_{1,p}(\tilde{E}_1; B(2))=0$ by the Corollary to Lemma 5, our theorem for $p \leq n$ is shown.

We next consider the case $p>n$. In this case u is continuous on D. Let $1<q<n$. Then we have by Hölder's inequality

$$\int_D |\operatorname{grad} u|^q dx \leq \left\{\int_D |\operatorname{grad} u|^p |x|^{p-n} dx\right\}^{q/p} \left\{\int_D |x|^{-\frac{q(p-n)}{p-q}} dx\right\}^{1-q/p} < \infty,$$

which implies that u can be considered to be q-precise on $B(1)$. As above we may assume that u is q-precise on R^n and (4) holds on $B(1)$ except for a set E_3 with $C_{1,q}(E_3)=0$. Set for $i=1, 2,\dots, n$,

$$v_i(x) = \int \frac{x_i - y_i}{|x - y|^n} \frac{\partial u}{\partial y_i} dy.$$

Let $x^0 \in D$ and consider $\overline{B(x^0, r_0)} \subset D$. We note that $|\operatorname{grad} u|^p$ is locally integrable, and hence that $\int_{B(x^0,r_0)} |x^0-y|^{1-n} |\operatorname{grad} u| dy$ is finite by Hölder's inequality. Thus v_i is finite-valued in D. To see that v_i is continuous in D, denote by $g(x, y)$ the integrand of the integral for v_i. Set $I_1(x)=\int_{|x-y|<|x^0-x|/2} g(x, y)dy$ and $I_2(x)=\int_{|x-y|>|x^0-x|/2} g(x, y)dy$. Since $\{y \in R^n;\ |x-y|<|x^0-x|/2\} \subset B(x^0, 2|x-x^0|)$,

$$I_1 \leq \int_{B(x^0,2|x-x^0|)} \frac{1}{|x^0-y|^{n-1}} |\operatorname{grad} u| dy \longrightarrow 0$$

as $x \to x^0$. We see also that $I_2(x) \to v_i(x^0)$ by Lebesgue's dominated convergence theorem. Thus $v_i(x)=I_1(x)+I_2(x) \to v_i(x^0)$ as $x \to x^0$. Hence (4) holds on D with no exceptional set. Since $\int_{|x-y|\leq|x|/2} |x-y|^{1-n} |\operatorname{grad} u| dy \to 0$ as $x \to O$ by Lemma 6, v_i is continuous at the origin. These imply that u has a finite limit at the origin.

THEOREM 4. *Let $p-n \leq \beta < n(p-1)$ and let u be a locally p-precise function on D such that $\int_D |\operatorname{grad} u|^p |x|^\beta dx < \infty$. Then there is a Borel set $E \subset S$ such that $C_{1,p}(E; B(2))=0$ and*

$$\begin{cases} \lim_{r\downarrow 0} r^{(n-p+\beta)/p}u(r\xi)=0 & \textit{in case } n-p+\beta>0, \\ \lim_{r\downarrow 0}\left(\log\frac{1}{r}\right)^{1/p-1}u(r\xi)=0 & \textit{in case } n-p+\beta=0 \end{cases}$$

for every $\xi \in S\backslash E$.

PROOF. Choose q such that $1<q<\min\{p, np/(n+\beta)\}$. Then $\beta q/(p-q)<n$. By Hölder's inequality we have $\int_D |\operatorname{grad} u|^q dx<\infty$. As in the previous proof, we may suppose that u is a q-precise function on R^n with compact support, and hence (4) holds a.e. on R^n. Since $|\operatorname{grad} u|^p$ is locally integrable on D, (4) holds on D except for E' with $C_{1,p}(E'; B(2))=0$ (cf. [8; Theorem 9.10]). We can now apply Theorem 2 to obtain the desired result.

REMARK 7. Condition (3) is necessary in Theorem 3. In fact, the function $u(x)=(\log(2/|x|))^{\varepsilon}$ satisfies (2) if ε is chosen so that $0<\varepsilon<1-1/p$, but $u(x)\to\infty$ as $x\to O$. We shall show below, however, that if u is a harmonic function on D satisfying (2), then u has a finite limit at the origin.

THEOREM 5. *Let h be a function harmonic on D. Then h can be extended to a harmonic function on $B(1)$ if one of the following conditions is fulfilled*:

$$(2)' \qquad \int_D |\operatorname{grad} h|^p|x|^{p(n/p'-1)}dx<\infty,$$

$$(3)' \qquad \int_D |\operatorname{grad} h|\cdot|x|^{-1}dx<\infty,$$

where $1/p+1/p'=1$.

PROOF. We shall prove only that h can be extended to a harmonic function on $B(1)$ if (2)′ is satisfied; the case when (3)′ is satisfied can be proved similarly. Assume that (2)′ is satisfied. Since $\partial h/\partial x_i$, $i=1,\dots, n$, are harmonic on D,

$$\frac{\partial h}{\partial x_i}(x)=M_1|x|^{-n}\int_{B(x,|x|/2)}\frac{\partial h}{\partial x_i}(y)dy,$$

where M_1 is a constant independent of $x\in B(1/2)\backslash\{O\}$. From Hölder's inequality it follows that

$$\left|\frac{\partial h}{\partial x_i}(x)\right| \leqq M_2|x|^{1-n}A(h; x),$$

where M_2 is a positive constant independent of x and

$$A(h; x)=\left\{\int_{0<|y|<2|x|}|\operatorname{grad} h|^p|y|^{p(n/p'-1)}dy\right\}^{1/p}.$$

Setting $K(x)=\log(1/|x|)$ in case $n=2$ and $=|x|^{2-n}$ in case $n\geqq 3$, we note that for ε, $0<\varepsilon<1/2$,

$$\limsup_{x\to O} K(x)^{-1}|h(x)| \leqq \limsup_{x\to O} K(x)^{-1}\left\{\left|h\left(\varepsilon\frac{x}{|x|}\right)\right| + \int_{|x|}^{\varepsilon}\left|\operatorname{grad} h\left(r\frac{x}{|x|}\right)\right| dr\right\}$$
$$\leqq 3M_2 A(h;\varepsilon),$$

which implies that $\lim_{x\to O} K(x)^{-1}h(x)=0$. Now our theorem follows from a result in [1; p. 204].

5. Radial limits of functions defined on a cone

For positive numbers a and b, we set

$$\Gamma(a, b) = \{(x', x_n)\in R^{n-1}\times R^1;\ |x'| < ax_n,\ |x'|^2 + x_n^2 < b^2\}.$$

We shall write simply $\Gamma(a)$ for $\Gamma(a, 1)$.

LEMMA 7. *Let g be a positive and non-increasing function on the interval* $(0, 1)$ *such that*

$$\int_0^1 \frac{dt}{tg(t)^{p'/p}} < \infty, \tag{5}$$

where $1/p+1/p'=1$. *If f is a non-negative measurable function on $B(1)$ satisfying*

$$\int_{B(1)} f(x)^p g(|x|)|x|^{p-n}dx < \infty,$$

then $\int_{B(1)} f(x)|x|^{1-n}dx<\infty$.

This follows immediately from (5) and Hölder's inequality.

THEOREM 6. *Let g be as in Lemma 7. Let u be a locally p-precise function on $\Gamma(a)$ such that*

$$\int_{\Gamma(a)} |\operatorname{grad} u|^p g(|x|)|x|^{p-n}dx < \infty. \tag{6}$$

Then there are a constant c and a Borel set $E\subset S$ such that $C_{1,p}(E; B(2))=0$ and $\lim_{r\downarrow 0} u(r\xi)=c$ for all $\xi\in S\cap\Gamma(a, 2)\setminus E$.

PROOF. It is convenient to adopt the polar coordinates $(r, \Theta)=(r, \theta_1,\dots, \theta_{n-1})$ so that $r\geqq 0$, $0\leqq\theta_1\leqq\pi,\dots$, $0\leqq\theta_{n-2}\leqq\pi$, $0\leqq\theta_{n-1}\leqq 2\pi$ and

$$\begin{aligned} x_1 &= r\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \\ x_2 &= r\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ &\vdots \\ \bar{x}_{n-1} &= r\sin\theta_1\cos\theta_2, \\ x_n &= r\cos\theta_1. \end{aligned}$$

Regard u as the function of $(r, \theta_1,\ldots, \theta_{n-1})$ and note

$$\int\cdots\int_{r<1,\ \tan\theta_1<a}\left[\left\{\frac{\partial u}{\partial r}(r, \theta)\right\}^2 + \frac{1}{r^2}\left\{\frac{\partial u}{\partial\theta_1}(r, \theta)\right\}^2 + \cdots \right.$$

$$\left. + \left(\frac{1}{r\sin\theta_1\cdots\sin\theta_{n-2}}\right)^2\left\{\frac{\partial u}{\partial\theta_{n-1}}(r, \theta)\right\}^2\right]^{p/2}$$

$$\times\ g(r)r^{p-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}drd\theta_1\cdots d\theta_{n-1} < \infty.$$

Let $a' = \pi/(2\tan^{-1}a)$ and define the function

$$v(x) = v(r, \theta_1, \theta_2,\ldots, \theta_{n-1}) = u(r, \theta_1/a', \theta_2,\ldots, \theta_{n-1})$$

for $0<r<1$, $0<\theta_1<\pi/2$, $0\leqq\theta_2\leqq\pi,\ldots$, $0\leqq\theta_{n-2}\leqq\pi$ and $0\leqq\theta_{n-1}\leqq 2\pi$. Since $M^{-1}\sin(\theta_1/a')\leqq\sin\theta_1\leqq M\sin(\theta_1/a')$ if $0<\theta_1<\pi/2$ for some positive constant M, $\int_{B(1)^+}|\operatorname{grad} v|^p g(|x|)|x|^{p-n}dx<\infty$, where $B(1)^+=\{x=(x', x_n)\in B(1);\ x_n>0\}$. This and Hölder's inequality give $\int_{B(b)^+}|\operatorname{grad} v|^q dx<\infty$ for $1<q<\min\{p, n\}$ and $0<b<1$. According to [8; Theorem 5.6], the function

$$\tilde{v}(x) = \begin{cases} v(x) & \text{for } x\in B(b)^+ \\ v(x', -x_n) & \text{for } (x', x_n)\in B(b)^- \end{cases}$$

can be extended to a q-precise function on $B(b)$, where $B(b)^-=\{x=(x', x_n)\in B(b);\ x_n<0\}$. The resulting function satisfies condition (6) with $\Gamma(a)$ replaced by $B(b)$. Therefore, by Lemma 7 and Theorem 3 we can find a constant c and a Borel set $E\subset S$ such that $C_{1,p}(E; B(2))=0$ and $\lim_{r\downarrow 0} v(r\xi)=c$ for all $\xi\in S\cap B(2)^+\backslash E$. Denoting by E' the set of all points x such that $(r, a'\theta_1, \theta_2,\ldots, \theta_{n-1})$ is the polar coordinates of a point in E if $(r, \theta_1,\ldots, \theta_{n-1})$ is the polar coordinates of x, we see by Lemma 3 that $C_{1,p}(E'; B(2))=0$. Thus our theorem is proved.

6. Angular limits of harmonic functions

In this section we shall study the existence of angular limits at the origin of harmonic functions defined on the cone $\Gamma(a)$.

THEOREM 7. *Let g be as in Lemma 7. Let h be a harmonic function on*

$\Gamma(a)$ with $a>0$ satisfying (6). Then for any a' with $0<a'<a$, $\lim_{x\to O, x\in\Gamma(a')} h(x)$ exists and is finite.

PROOF. By Theorem 6, there is $\sigma^* \in S \cap \Gamma(a; 2)$ such that $\lim_{r\downarrow 0} h(r\sigma^*)$ exists and is finite. We denote the limit by c. For a number a' such that $0<a'<a$ and $\{r\sigma^*;\ r>0\} \cap \Gamma(a') \neq \emptyset$, we shall show that $\lim_{x\to O, x\in\Gamma(a')} h(x)$ exists and equals c. Choose $d>0$ such that $B(x, d|x|) \subset \Gamma(a)$ for $x \in \Gamma(a', 1/2)$. Then for $x \in \Gamma(a', 1/2)$

$$\left|\frac{\partial h}{\partial x_j}(x)\right| = M_1(d|x|)^{-n}\left|\int_{B(x,d|x|)} \frac{\partial h}{\partial y_j} dy\right|$$
$$\leq M_1 d^{-n}|x|^{-n}\left\{\int_{B(x,d|x|)} |\operatorname{grad} h|^p |y|^{p-n} dy\right\}^{1/p}$$
$$\times \left\{\int_{B(x,d|x|)} |y|^{p'(n-p)/p} dy\right\}^{1/p'},$$

where M_1 is a constant independent of x. Note that

$$\int_{B(x,d|x|)} |y|^{p'(n-p)/p} dy \leq M_2 |x|^{p'(n-p)/p} \int_{B(x,d|x|)} dx = M_3 |x|^{p'(n-1)}$$

for some constants M_2 and M_3 independent of x. For $x \in \Gamma(a')$, set $x^* = |x|\sigma^*$ and denote by L_x the line segment between x and x^*. If $x \in \Gamma(a', 1/2)$, then

$$|h(x) - h(x^*)| \leq |x - x^*| \sup_{L_x} |\operatorname{grad} h| \leq 2|x| \sup_{L_x} |\operatorname{grad} h|$$
$$\leq 2M_1 M_3^{1/p'} d^{-n} \sqrt{n} \left\{\int_{\Gamma(a,(1+d)|x|)} |\operatorname{grad} h|^p |y|^{p-n} dy\right\}^{1/p}$$
$$\longrightarrow 0 \qquad \text{as } x \longrightarrow O.$$

Therefore $\lim_{x\to O, x\in\Gamma(a')} h(x) = c$. Thus the theorem is proved.

Finally we shall discuss the sharpness of Theorem 7. For simplicity we shall write Γ instead of $\Gamma(a)$.

THEOREM 8. *Let g be a positive and non-increasing function on the interval $(0, 1)$ such that*

$$\int_0^1 \frac{dt}{t g(t)^{p'/p}} = \infty \tag{7}$$

and that $t^{-\delta} g(t)^{-1}$ is non-increasing on $(0, 1)$ for some δ with $0<\delta<p/2$, where $1/p + 1/p' = 1$. Then there is a harmonic function h on Γ such that h satisfies (6) but $\lim_{x_n\downarrow 0} h(0,\ldots, 0, x_n) = \infty$.

PROOF. First we deal with the case $n=2$. Define the functions a, b and f as follows:

$$a(r) = \int_r^1 \frac{dt}{tg(t)^{p'/p}} + 1,$$

$$b(r) = \log a(r),$$

$$f(r) = \frac{b(r)^{\varepsilon-1}}{r^2 g(r)^{p'/p} a(r)\left(\log \frac{1}{r} + 1\right)}, \quad 0 < \varepsilon < 1/p'.$$

We see that $a(0)=\infty$ by (7). We set $\hat{\Gamma}=\{-x;\, x\in\Gamma\}$ and consider the function

$$h(x) = \int_{\hat{\Gamma}} \log \frac{1}{|x-y|} f(|y|)dy, \qquad x\in\Gamma.$$

Note that h is harmonic on Γ and $\lim_{x_2 \downarrow 0} h(0, x_2)=\infty$. We shall show that h satisfies (6).

Since $\inf\{|x-y|(|x|+|y|)^{-1};\, x\in\Gamma,\, y\in\hat{\Gamma}\}>0$, there is a positive constant M_1 such that for $x\in\Gamma$,

$$|\operatorname{grad} h(x)| \leqq M_1 \int_0^1 \frac{f(r)}{|x| + r} r dr.$$

Letting $I_1(s)=\int_0^s f(r)(s+r)^{-1}rdr$ and $I_2(s)=\int_s^1 f(r)(s+r)^{-1}rdr$, we estimate them separately. Hereafter $M_2, M_3,\ldots,$ will stand for constants. By Hölder's inequality, we have for $1/p'<\beta<1$

$$I_1(s) \leqq s^{-1}\left\{\int_0^s \frac{dr}{r\left(\log \frac{1}{r} + 1\right)^{\beta p'}}\right\}^{1/p'} \left\{\int_0^s \frac{b(r)^{p(\varepsilon-1)}dr}{rg(r)^{p'}a(r)^p\left(\log \frac{1}{r} + 1\right)^{(1-\beta)p}}\right\}^{1/p}$$

$$\leqq M_2 s^{-1}\left(\log \frac{1}{s} + 1\right)^{(1-\beta p')/p'} \left\{\int_0^s \frac{b(r)^{p(\varepsilon-1)}dr}{rg(r)^{p'}a(r)^p\left(\log \frac{1}{r} + 1\right)^{(1-\beta)p}}\right\}^{1/p}.$$

Hence

$$\int_\Gamma I_1(|x|)^p g(|x|)|x|^{p-2}dx$$

$$\leqq M_3 \int_0^1 \left\{\int_0^s \frac{b(r)^{p(\varepsilon-1)}dr}{rg(r)^{p'}a(r)^p\left(\log \frac{1}{r} + 1\right)^{(1-\beta)p}}\right\}$$

$$\times\ g(s)s^{-1}\left(\log\frac{1}{s}+1\right)^{p(1-\beta p')/p'}ds$$

$$= M_3\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{rg(r)^{p'}a(r)^p\left(\log\frac{1}{r}+1\right)^{(1-\beta)p}}\left\{\int_r^1 \frac{g(s)ds}{s\left(\log\frac{1}{s}+1\right)^{p(\beta-1/p')}}\right\}dr$$

$$\leqq M_3\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{rg(r)^{p'}a(r)^p\left(\log\frac{1}{r}+1\right)^{(1-\beta)p}}g(r)\left\{\frac{\left(\log\frac{1}{r}+1\right)^{(1-\beta)p}}{(1-\beta)p}\right\}dr$$

$$\leqq M_4\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{rg(r)^{p'/p}a(r)}dr = M_5 b(r)^{p(\varepsilon-1)+1}\Big|_0^1$$

$$= M_5 b(1)^{p(\varepsilon-1)+1} < \infty.$$

For I_2, we have

$$I_2(s) \leqq \left\{\int_s^1 \frac{dr}{r^{(2-1/p-2\delta/p)p'}}\right\}^{1/p'}\left\{\int_s^1 \frac{b(r)^{p(\varepsilon-1)}dr}{r^{1+2\delta}g(r)^{p'}a(r)^p}\right\}^{1/p}$$

$$\leqq M_6 s^{-1+2\delta/p}\left\{\frac{1}{s^\delta g(s)}\int_s^1 \frac{b(r)^{p(\varepsilon-1)}dr}{r^{1+\delta}g(r)^{p'/p}a(r)^p}\right\}^{1/p}.$$

Consequently,

$$\int_\Gamma I_2(|x|)^p g(|x|)|x|^{p-2}dx \leqq M_7\int_0^1\left\{\int_s^1 \frac{b(r)^{p(\varepsilon-1)}dr}{r^{1+\delta}g(r)^{p'/p}a(r)^p}\right\}s^{\delta-1}ds$$

$$= M_7\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{r^{1+\delta}g(r)^{p'/p}a(r)^p}\left\{\int_0^r s^{\delta-1}ds\right\}dr$$

$$\leqq M_7\delta^{-1}\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{rg(r)^{p'/p}a(r)}dr < \infty.$$

Thus we obtain the theorem for $n=2$.

Next we are concerned with the case $n\geqq 3$. Let a and b be as above. Set

$$f(r) = \frac{b(r)^{\varepsilon-1}}{r^2 g(r)^{p'/p}a(r)},\quad 0<\varepsilon<1/p',$$

and consider the function

$$h(x) = \int_{\hat{\Gamma}} |x-y|^{2-n}f(|y|)dy,\quad x\in\Gamma.$$

Note that h is harmonic on Γ and $\lim_{x_n\downarrow 0} h(0,\dots,0,x_n)=\infty$. For $x\in\Gamma$ and $j=1,\dots,n$,

$$\left|\frac{\partial h}{\partial x_j}(x)\right| \leqq (n-2)\int_{\hat{\Gamma}} |x-y|^{1-n} f(|y|)dy$$

$$\leqq M_8 \int_0^1 (|x|+r)^{1-n} f(r) r^{n-1} dr.$$

As above we write

$$I_1(s) = \int_0^s f(r)(s+r)^{1-n} r^{n-1} dr,$$

$$I_2(s) = \int_s^1 f(r)(s+r)^{1-n} r^{n-1} dr,$$

and estimate them separately. Take a number β such that $1<\beta p'<n-1$. Then

$$I_1(s) \leqq s^{1-n}\left\{\int_0^s r^{(n-2-\beta)p'} dr\right\}^{1/p'} \left\{\int_0^s \frac{b(r)^{p(\varepsilon-1)} dr}{r^{(1-\beta)p} g(r)^{p'} a(r)^p}\right\}^{1/p}$$

$$\leqq M_9 s^{-\beta-1/p}\left\{\int_0^s \frac{b(r)^{p(\varepsilon-1)} dr}{r^{(1-\beta)p} g(r)^{p'} a(r)^p}\right\}^{1/p}.$$

Hence

$$\int_\Gamma I_1(|x|)^p g(|x|)|x|^{p-n} dx$$

$$\leqq M_{10}\int_0^1 \left\{\int_0^s \frac{b(r)^{p(\varepsilon-1)} dr}{r^{(1-\beta)p} g(r)^{p'} a(r)^p}\right\} g(s) s^{p-\beta p-2} ds$$

$$= M_{10}\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{r^{(1-\beta)p} g(r)^{p'} a(r)^p}\left\{\int_r^1 g(s) s^{p-\beta p-2} ds\right\} dr$$

$$\leqq M_{11}\int_0^1 \frac{b(r)^{p(\varepsilon-1)}}{r g(r)^{p'/p} a(r)^p} dr < \infty.$$

In a way similar to the case $n=2$, we also obtain

$$\int_\Gamma I_2(|x|)^p g(|x|)|x|^{p-n} dx < \infty.$$

The proof is now complete.

Remark 8. Let g be as in the theorem. Then by modifying the harmonic function h in the proof of the theorem, we can construct a harmonic function $\tilde{h}$ on Γ such that $\tilde{h}$ satisfies (6) but $\lim_{x_n \downarrow 0} \tilde{h}(0,\dots, 0, x_n)$ does not exist.

For this purpose, let a, b, f be as in the proof and set

$$K(x) = \begin{cases} \log \dfrac{2}{|x|} & \text{in case } n = 2, \\ |x|^{2-n} & \text{in case } n \neq 2. \end{cases}$$

We write $\ell^+ = \{(x', x_n) \in R^{n-1} \times R^1;\ x' = O \text{ and } x_n > 0\}$. Let $x^{(1)} \in \ell^+$ and $0 < \alpha_1'' < 1$ be arbitrary. We can find $x^{(2)} \in \ell^+ \cap B(1/2)$ and $\alpha_2', \alpha_2'' > 0$ such that $\alpha_2'' < \alpha_2' < \alpha_1''$,

$$\int_{\hat{\Gamma} \cap B(\alpha_2')} K(x^{(1)} - y) f(|y|) dy \leqq 1$$

and

$$\int_{\hat{\Gamma} \cap B(\alpha_2') \setminus B(\alpha_2'')} K(x^{(2)} - y) f(|y|) dy \geqq \int_{\hat{\Gamma} \setminus B(\alpha_1'')} K(x^{(2)} - y) f(|y|) dy + 2.$$

We proceed inductively and obtain $\{x^{(i)}\}$, $\{\alpha_i'\}$ and $\{\alpha_i''\}$ such that

$$x^{(i)} \in \ell^+ \cap B(1/i),\ 0 < \alpha_i'' < \alpha_i' < \alpha_{i-1}'',$$

$$\int_{\hat{\Gamma} \cap B(\alpha_i')} K(x^{(j)} - y) f(|y|) dy \leqq 1$$

and

$$\int_{\hat{\Gamma} \cap B(\alpha_i') \setminus B(\alpha_i'')} K(x^{(i)} - y) f(|y|) dy \geqq \int_{\hat{\Gamma} \setminus B(\alpha_{i-1}'')} K(x^{(i)} - y) f(|y|) dy + 2$$

for any i and j such that $i \geqq 2$ and $1 \leqq j < i$.

Define the function

$$\tilde{f}(y) = \begin{cases} (-1)^i f(|y|) & \text{if } y \in \hat{\Gamma} \text{ and } \alpha_i'' < |y| < \alpha_i', \\ 0 & \text{otherwise} \end{cases}$$

and set $\tilde{h}(x) = \int_{\hat{\Gamma}} K(x-y) \tilde{f}(y) dy$ for $x \in \Gamma$. It is clear that $\tilde{h}$ satisfies (6). Furthermore $\tilde{h}(x^{(2j)}) \geqq 1$ and $\tilde{h}(x^{(2j-1)}) \leqq -1$ for each j. This implies that $\tilde{h}$ has the desired properties.

REMARK 9. In case $p=2$, Theorem 7 has been shown by T. Murai [7]. He has also obtained a harmonic function as in Remark 8 in case $p=n=2$.

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