Asymptotic Properties of Solutions of Two-dimensional Differential Systems with Deviating Argument

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Introduction

In this paper we consider the two-dimensional differential system with deviating argument

(A)
$$\begin{cases} x'(t) = p(t)y(t) \\ y'(t) = f(t, x(g(t))) \end{cases}$$

which, in the particular case where $p(t) > 0$, is equivalent to the second order scalar differential equation

(B)
$$\left(\frac{1}{p(t)} x'(t) \right)' = f(t, x(g(t))).$$

The conditions we always assume for p, g, f are as follows:

(a) $p(t)$ is continuous and nonnegative on $[a, \infty)$; $p(t) \not\equiv 0$ on any infinite subinterval of $[a, \infty)$.
(b) $g(t)$ is continuous on $[a, \infty)$ and $\lim_{t \to \infty} g(t) = \infty$.
(c) $f(t, x)$ is continuous on $[a, \infty) \times (-\infty, \infty)$ and $|f(t, x)| \leq \omega(t, |x|)$ for $(t, x) \in [a, \infty) \times (-\infty, \infty)$ where $\omega(t, r)$ is continuous on $[a, \infty) \times [0, \infty)$ and nondecreasing in r.

We note that $g(t)$ is a general deviating argument, that is, it is allowed to be *retarded* ($g(t) \leq t$) or *advanced* ($g(t) \geq t$) or otherwise. System (A) is called *superlinear* or *sublinear* according to whether $\omega(t, r)/r$ is nondecreasing or nonincreasing in r for $r > 0$.

The purpose of this paper is to study the asymptotic behavior of solutions of system (A) which is either superlinear or sublinear. We are particularly interested in obtaining information about the growth or decay of oscillatory solutions as well as of nonoscillatory solutions. Hereafter the term "solution" will be understood to mean a solution $\{x(t), y(t)\}$ of (A) which exists on some half-line $[\tau, \infty)$, $\tau > a$, and satisfies

$$\sup \{|x(t)| + |y(t)| : t \geq \tau'\} > 0 \quad \text{for any} \quad \tau' \geq \tau.$$

Such a solution is said to be *oscillatory* [resp. *weakly oscillatory*] if each of its components [resp. at least one component] has arbitrarily large zeros. A solution is said to be *nonoscillatory* [resp. *weakly nonoscillatory*] if each of its components [resp. at least one component] is eventually of constant sign.

We distinguish the two cases

$$\int_a^\infty p(t)dt = \infty, \quad \int_a^\infty p(t)dt < \infty$$

and examine them separately in § 1 and § 2. The theorems in § 1 and § 2 are formulated in terms of $P(t) = \int_a^t p(s)ds$ and $\pi(t) = \int_t^\infty p(s)ds$, respectively, and exhibit a kind of duality. Our results include as special cases some of the main results of the papers [1], [2], [6], [7], [8] and [9]. For other related problems regarding systems of the form (A) we refer to [3], [4] and [10].

1. The case where $\int_a^\infty p(t)dt = \infty$

We begin by considering system (A) in which $p(t)$ satisfies the condition $\int_a^\infty p(t)dt = \infty$. The results are formulated in terms of the function $P(t) = \int_a^t p(s)ds$. The following notation will be used throughout the paper:

$$g^*(t) = \max \{g(t), t\}, \qquad g_*(t) = \min \{g(t), t\},$$

$$h^*(t) = \sup_{a \leq s \leq t} g^*(s), \qquad h_*(t) = \inf_{s \geq t} g_*(s).$$

A) We first prove a theorem which enables us to classify all the solutions of (A) according to the behavior as $t \to \infty$.

THEOREM 1.1. *Assume that either (A) is superlinear and*

(1) $$\int^\infty \frac{P(g^*(t))}{P(g(t))} \omega(t, cP(g(t)))dt < \infty \qquad for\ all\ \ c > 0$$

or (A) is sublinear and

(2) $$\int^\infty P(g^*(t))\omega(t, c)dt < \infty \qquad for\ all\ \ c > 0.$$

If $\{x(t), y(t)\}$ is a solution of (A), then exactly one of the following cases holds:

(I) $$\limsup_{t \to \infty} \frac{|x(t)|}{P(t)} = \infty, \quad \limsup_{t \to \infty} |y(t)| = \infty.$$

(II) *There exists a nonzero constant α such that*

$$\lim_{t \to \infty} \frac{x(t)}{P(t)} = \alpha, \quad \lim_{t \to \infty} y(t) = \alpha.$$

(III) *There exists a constant β such that*

$$\lim_{t \to \infty} x(t) = \beta, \quad \lim_{t \to \infty} P(t)y(t) = 0.$$

PROOF. Let $\{x(t), y(t)\}$ be a solution of (A) defined on $[\tau, \infty)$ and let $T \geqq \tau$ be such that $h_*(T) \geqq \tau$.

First we assume that (A) is superlinear and (1) holds. We note that (1) implies that the functions

$$\omega(t, cP(g(t))), \ [P(t)/P(g(t))]\omega(t, cP(g(t))), \ P(t)\omega(t, c)$$

are integrable at ∞ for all $c > 0$. Suppose $\limsup_{t \to \infty} |x(t)|/P(t) = \infty$. Then it holds necessarily that $\limsup_{t \to \infty} |y(t)| = \infty$, since otherwise from the relation

(3) $$\qquad\qquad x(t) = x(T) + \int_T^t p(s)y(s)ds, \quad t \geqq T,$$

we would have $\limsup_{t \to \infty} |x(t)|/P(t) < \infty$, a contradiction. Suppose $\limsup_{t \to \infty} |x(t)|/P(t) < \infty$, that is, $x(t) = O(P(t))$ as $t \to \infty$. Then it is clear that $f(t, x(g(t))) \in L^1[T, \infty)$, and so from the equation

(4) $$\qquad\qquad y(t) = y(T) + \int_T^t f(s, x(g(s)))ds, \quad t \geqq T,$$

we obtain

(5) $$\qquad\qquad y(t) = \alpha - \int_t^\infty f(s, x(g(s)))ds, \quad t \geqq T,$$

where

$$\alpha = y(T) + \int_T^\infty f(s, x(g(s)))ds.$$

As a result we have $\lim_{t \to \infty} y(t) = \alpha$. Using this fact in (3) we easily see that $\lim_{t \to \infty} x(t)/P(t) = \alpha$. Thus Case (II) holds if $\alpha \neq 0$. Supposing that $\alpha = 0$, we show that Case (III) occurs. Choose a $T_1 \geqq T$ such that

$$T_0 = h_*(T_1) \geqq T, \quad |x(g(t))| \leqq P(g(t)) \text{ for } t \geqq T_1,$$

and

$$\int_{T_1}^\infty \omega(s, P(g(s)))ds \leqq \frac{1}{3}, \ \int_{T_1}^\infty \frac{P(s)}{P(g(s))} \omega(s, P(g(s)))ds \leqq \frac{1}{3}.$$

Combining (3) and (5) (with $\alpha = 0$) yields

(6)
$$|x(t)| \leq |x(T)| + \int_T^t P(s)\omega(s, |x(g(s))|)ds$$
$$+ P(t)\int_t^\infty \omega(s, |x(g(s))|)ds, \quad t \geq T,$$

whence it follows that

(7)
$$\frac{|x(t)|}{P(t)} \leq \frac{k}{P(t)} + \frac{1}{P(t)}\int_{T_1}^t P(s)\omega(s, |x(g(s))|)ds$$
$$+ \int_t^\infty \omega(s, |x(g(s))|)ds, \quad t \geq T_1,$$

where k is a positive constant.   We now define

$$u(t) = \sup_{s \geq t} \frac{|x(s)|}{P(s)}, \quad t \geq T_0.$$

Observing that the right-hand side of (7) is decreasing in t and using the inequality

(8)
$$\omega(s, mn) \leq m\omega(s, n), \quad 0 < m \leq 1, \quad n > 0,$$

which is a consequence of the superlinearity of (A), we can derive the following inequality from (7):

(9)
$$P(t)u(t) \leq k + \int_{T_1}^t u(g(s))P(s)\omega(s, P(g(s)))ds$$
$$+ P(t)\int_t^\infty u(g(s))\omega(s, P(g(s)))ds, \quad t \geq T_1.$$

For each $t \geq T_1$ we let I_t, J_t denote the sets

(10) $I_t = \{s \in [T_1, \infty): g(s) \leq t\}, \quad J_t = \{s \in [T_1, \infty): g(s) > t\}.$

We then have

$$P(g(s))u(g(s)) \leq \sup_{T_0 \leq \sigma \leq t} [P(\sigma)u(\sigma)] \text{ for } s \in I_t,$$

$$u(g(s)) \leq u(t) \text{ for } s \in J_t.$$

In view of this fact, the right-hand side of (9) is bounded from above by

$$k + \sup_{T_0 \leq s \leq t} [P(s)u(s)] \cdot \int_{I_t \cap [T_1, t)} \frac{P(s)}{P(g(s))} \omega(s, P(g(s)))ds$$

$$+ u(t)\int_{J_t \cap [T_1, t)} P(s)\omega(s, P(g(s)))ds$$

$$+ P(t) \sup_{T_0 \le s \le t} [P(s)u(s)] \cdot \int_{I_t \cap [t, \infty)} \frac{1}{P(g(s))} \omega(s, P(g(s))) ds$$

$$+ P(t)u(t) \int_{J_t \cap [t, \infty)} \omega(s, P(g(s))) ds$$

$$\le k + \sup_{T_0 \le s \le t} [P(s)u(s)] \cdot \int_{T_1}^{\infty} \frac{P(s)}{P(g(s))} \omega(s, P(g(s))) ds$$

$$+ P(t)u(t) \int_{T_1}^{\infty} \omega(s, P(g(s))) ds$$

$$\le k_1 + \frac{1}{3} \sup_{T_1 \le s \le t} [P(s)u(s)] + \frac{1}{3} P(t)u(t)$$

for $t \ge T_1$, where $k_1 = k + \frac{1}{3} \sup_{T_0 \le s \le T_1} [P(s)u(s)]$. It follows that

$$P(t)u(t) \le \frac{3}{2} k_1 + \frac{1}{2} \sup_{T_1 \le s \le t} [P(s)u(s)], \quad t \ge T_1,$$

which implies

(11) $$|x(t)| \le \sup_{T_1 \le s \le t} [P(s)u(s)] \le 3k_1, \quad t \ge T_1.$$

From (5) (with $\alpha = 0$) and (11) we obtain

(12) $$\int_T^{\infty} p(s)|y(s)| ds \le \int_T^{\infty} P(s)\omega(s, 3k_1) ds,$$

and

(13) $$P(t)|y(t)| \le \int_t^{\infty} P(s)\omega(s, 3k_1) ds.$$

Since $p(t)y(t) \in L^1[T, \infty)$ by (12), rewriting (3) as

$$x(t) = x(T) + \int_T^{\infty} p(s)y(s) ds - \int_t^{\infty} p(s)y(s) ds,$$

we see that

$$\lim_{t \to \infty} x(t) = \beta = x(T) + \int_T^{\infty} p(s)y(s) ds.$$

That $\lim_{t \to \infty} P(t)y(t) = 0$ follows from (13).

Next we assume that (A) is sublinear and (2) holds. It is clear that (2) implies the integrability at ∞ of the functions

$$P(t)\omega(t, c), \quad P(g(t))\omega(t, c), \quad \omega(t, cP(g(t)))$$

for all $c > 0$. Of course, Case (I) may occur. Let $x(t) = O(P(t))$ as $t \to \infty$. Then, $f(t, x(g(t))) \in L^1[T, \infty)$, so that (5) holds. If $\alpha \neq 0$, then we have Case (II). It remains to examine the case where $\alpha = 0$. We show that $x(t)$ is bounded on $[T, \infty)$ in this case. Suppose the contrary. We are able to choose T_1, T_2 and T_3 so that

$$T < T_1 < T_2 < T_3, \quad T_0 = h_*(T_1) \geqq T, \quad |x(T_0)| \geqq 1,$$

$$\sup_{T_0 \leqq s \leqq t} |x(s)| = \sup_{T_2 \leqq s \leqq t} |x(s)| \text{ for } t \geqq T_2,$$

$$\int_{T_2}^{\infty} P(s)\omega(s, 1)ds \leqq \frac{1}{4}, \quad \int_{T_2}^{\infty} P(g(s))\omega(s, 1)ds \leqq \frac{1}{4},$$

and

$$|x(T)| + \int_{T}^{T_2} P(s)\omega(s, |x(g(s))|)ds \leqq \frac{1}{4} |x(T_3)|.$$

Let us define

$$v(t) = \sup_{T_0 \leqq s \leqq t} |x(s)|, \quad t \geqq T_0.$$

Noting the increasing nature of the right-hand side of (6) and using the inequality

(14) $$\omega(s, mn) \leqq m\omega(s, n), \quad m \geqq 1, \quad n > 0,$$

which follows from the sublinearity of (A), we have from (6) that

$$v(t) \leqq |x(T)| + \int_{T}^{T_2} P(s)\omega(s, |x(g(s))|)ds$$

$$+ \int_{T_2}^{t} P(s)\omega(s, |x(g(s))|)ds + P(t)\int_{t}^{\infty} \omega(s, |x(g(s))|)ds$$

$$\leqq \frac{1}{4} v(t) + \int_{T_2}^{t} v(g(s))P(s)\omega(s, 1)ds + P(t)\int_{t}^{\infty} v(g(s))\omega(s, 1)ds,$$

and consequently

(15) $$\frac{3}{4}\frac{v(t)}{P(t)} \leqq \frac{1}{P(t)}\int_{T_2}^{t} v(g(s))P(s)\omega(s, 1)ds + \int_{t}^{\infty} v(g(s))\omega(s, 1)ds$$

for $t \geqq T_3$. Since

$$v(g(s)) \leqq v(t) \text{ for } s \in I_t,$$

$$\frac{v(g(s))}{P(g(s))} \leqq \sup_{\sigma \geqq t} \frac{v(\sigma)}{P(\sigma)} \text{ for } s \in J_t,$$

where I_t and J_t are defined in (10), the right-hand side of (15) is bounded from above by

$$\frac{v(t)}{P(t)}\left(\int_{I_t \cap [T_2, t)} P(s)\omega(s, 1)ds + P(t)\int_{I_t \cap [t, \infty)} \omega(s, 1)ds \right)$$

$$+ \sup_{s \geq t} \frac{v(s)}{P(s)} \cdot \left(\frac{1}{P(t)}\int_{J_t \cap [T_2, t)} P(g(s))P(s)\omega(s, 1)ds \right.$$

$$\left. + \int_{J_t \cap [t, \infty)} P(g(s))\omega(s, 1)ds \right)$$

$$\leqq \frac{v(t)}{P(t)}\int_{T_2}^{\infty} P(s)\omega(s, 1)ds + \sup_{s \geq t} \frac{v(s)}{P(s)} \cdot \int_{T_2}^{\infty} P(g(s))\omega(s, 1)ds$$

$$\leqq \frac{1}{4}\left(\frac{v(t)}{P(t)} + \sup_{s \geq t} \frac{v(s)}{P(s)} \right), \quad t \geqq T_3.$$

Therefore we obtain

$$\frac{1}{2}\frac{v(t)}{P(t)} \leqq \frac{1}{4}\sup_{s \geq t} \frac{v(s)}{P(s)}, \text{ and so } 0 < \sup_{s \geq t} \frac{v(s)}{P(s)} \leqq \frac{1}{2}\sup_{s \geq t} \frac{v(s)}{P(s)}.$$

This contradiction shows that $x(t)$ is bounded on $[T, \infty)$. We now proceed exactly as in the superlinear case to conclude that $\{x(t), y(t)\}$ is subject to Case (III). Thus the proof of Theorem 1.1 is complete.

B) On the basis of Theorem 1.1 we wish to determine the growth or decay of all nonoscillatory solutions of (A) for which the following sign assumption is made:

(16) $x f(t, x) \leqq 0$ for $(t, x) \in [a, \infty) \times (-\infty, \infty)$.

In addition it is assumed that

(17) $\sup_{t \geqq T} |f(t, x)| > 0$ for any $T \geqq a$ and $x \neq 0$.

We remark that under assumption (16) a solution of (A) is oscillatory [resp. nonoscillatory] if and only if it is weakly oscillatory [resp. weakly nonoscillatory].

THEOREM 1.2. *Assume that (16), (17) and the hypotheses of Theorem 1.1 are satisfied. If $\{x(t), y(t)\}$ is a nonoscillatory solution of (A), then either*

(18) $$\lim_{t \to \infty} \frac{x(t)}{P(t)} = \alpha, \quad \lim_{t \to \infty} y(t) = \alpha$$

for some constant $\alpha \neq 0$, or else

(19) $$\lim_{t\to\infty} x(t) = \beta, \quad \lim_{t\to\infty} P(t)y(t) = 0$$

for some constant $\beta \neq 0$.

PROOF. Let $\{x(t), y(t)\}$ be a solution of (A) such that $x(t) \neq 0$ and $y(t) \neq 0$ on $[\tau, \infty)$. Take a $T \geq \tau$ such that $h_*(T) \geq \tau$. Suppose $x(t) > 0$ for $t \geq \tau$. The second equation of (A) then implies that $y(t)$ is decreasing on $[T, \infty)$. If $y(t_0) < 0$ for some $t_0 > T$, then $y(t) \leq y(t_0)$ for $t \geq t_0$. Taking this into account and integrating the first equation of (A), we obtain

$$x(t) \leq x(t_0) + y(t_0) \int_{t_0}^{t} p(s)ds, \quad t \geq t_0,$$

which implies $x(t) \to -\infty$ as $t \to \infty$, a contradiction. Therefore, we must have $y(t) > 0$ for $t \geq T$.

Now, from the first equation of (A) we see that $x(t)$ is increasing, and so $x(t)$ is bounded away from below by a positive constant on $[T, \infty)$. Again integrating the first equation of (A) and using the decreasing nature of $y(t)$, we have

$$x(t) \leq x(T) + y(T) \int_{T}^{t} p(s)ds, \quad t \geq T.$$

This shows that $x(t)/P(t)$ is bounded from above by a positive constant on $[T, \infty)$. A similar argument holds if we assume that $x(t) < 0$ on $[T, \infty)$. It follows that

$$\liminf_{t\to\infty} |x(t)| > 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{|x(t)|}{P(t)} < \infty,$$

and hence Case (I) and Case (III) (with $\beta = 0$) are excluded from the possibilities listed in Theorem 1.1. This completes the proof.

REMARK 1.1. When specialized to the scalar equation (B), Theorem 1.2 extends previous results of Belohorec [1, Theorem 3], Moore and Nehari [7, Theorem IV] and Odaric and Ševelo [8, Theorem 3].

REMARK 1.2. Under the hypotheses of Theorem 1.2 system (A) actually possesses nonoscillatory solutions of the type (18) for all $\alpha \neq 0$ as well as those of the type (19) for all $\beta \neq 0$. This follows from the existence theory developed by the present authors in [3] and [4].

EXAMPLE 1.1. Consider the sublinear system

(20)
$$\begin{cases} x'(t) = \dfrac{1 + \cos(t + \pi/4)}{\sqrt{2} - \sin t} e^{2t} y(t) \\[3mm] y'(t) = -\dfrac{2[1 + \cos(t + \pi/4)]}{(\sqrt{2} + \cos 3t)^{1/3}} e^{-2t} x^{1/3}(3t) \end{cases}$$

Here we can take $P(t) = e^{2t}$, $g(t) = g^*(t) = 3t$, and

$$\omega(t, r) = \frac{2[1 + \cos(t + \pi/4)]}{(\sqrt{2} + \cos 3t)^{1/3}} e^{-2t} r^{1/3}.$$

As easily verified, condition (2) is violated and system (20) has a nonoscillatory solution

$$x(t) = e^t(\sqrt{2} + \cos t), \quad y(t) = \sqrt{2} e^{-t}(\sqrt{2} - \sin t)$$

which has the properties:

$$\lim_{t \to \infty} \frac{x(t)}{P(t)} = 0, \quad \lim_{t \to \infty} y(t) = 0,$$

$$\lim_{t \to \infty} x(t) = \infty, \quad \lim_{t \to \infty} P(t) y(t) = \infty.$$

This example shows that violation of the integral condition of Theorem 1.2 may give rise to nonoscillatory solutions with asymptotic nature different from (18) and (19). For a related result concerning (B) we refer to the paper [5].

C) We now turn to investigating the behavior of oscillatory solutions of system (A). No sign condition is placed on $f(t, x)$ but the following conditions on $g(t)$ are needed.

Condition (G*): There exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$

as $n \to \infty$ and $h^*(t_n) = t_n$ for $n = 1, 2, \dots$.

Condition (G*): There exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$

as $n \to \infty$ and $h_*(t_n) = t_n$ for $n = 1, 2, \dots$.

We observe that condition (G*) [resp. (G*)] is satisfied if $g(t) \leq t$ [resp. $g(t) \geq t$]. As an example of functions satisfying both (G*) and (G*) [resp. neither (G*) nor (G*)] we give

$$g(t) = t\left[1 + \frac{1}{4} \sin(\log t)\right] \quad [\text{resp. } g(t) = t + 2\pi \sin t].$$

THEOREM 1.3. (i) *Assume that (A) is superlinear and condition (G*) is satisfied. If (1) holds, then every oscillatory solution $\{x(t), y(t)\}$ of (A) has the property*

(21) $$\limsup_{t \to \infty} \frac{|x(t)|}{P(t)} = \infty, \quad \limsup_{t \to \infty} |y(t)| = \infty.$$

(ii) *Assume that (A) is sublinear and condition (G*) is satisfied. If (2)*

holds, then every oscillatory solution $\{x(t), y(t)\}$ of (A) has the property

(22) $$\lim_{t \to \infty} x(t) = 0, \quad \lim_{t \to \infty} P(t)y(t) = 0.$$

PROOF. Let $\{x(t), y(t)\}$ be an oscillatory solution of (A) defined on $[\tau, \infty)$. Choose a $T \geq \tau$ such that $h_*(T) \geq \tau$. Since the solution is oscillatory by hypothesis, Case (II) and Case (III) (with $\beta \neq 0$) of Theorem 1.1 can never occur, so that it must satisfy either (21) or (22).

(i) Consider the case where (A) is superlinear and (G_*) holds. Suppose (22) is true. From the proof of Theorem 1.1 we then have

$$x(t) = -\int_t^\infty p(s)y(s)ds, \quad y(t) = -\int_t^\infty f(s, x(g(s)))ds,$$

and combining these we get

(23) $$|x(t)| \leq \int_t^\infty P(s)\omega(s, |x(g(s))|)ds, \quad t \geq T.$$

Let us put $u(t) = \sup\limits_{s \geq t} |x(s)|$ and choose T_1 and T_2 such that $T < T_1 < T_2$, $|x(t)| \leq 1$ for $t \geq T_1$ and $h_*(T_2) \geq T_1$. With the aid of (8) we derive from (23)

$$u(t) \leq \int_t^\infty P(s)u(g(s))\omega(s, 1)ds \leq u(h_*(t))\int_t^\infty P(s)\omega(s, 1)ds$$

for $t \geq T_2$, which implies

(24) $$\frac{u(t)}{u(h_*(t))} \leq \int_t^\infty P(s)\omega(s, 1)ds, \quad t \geq T_2.$$

But this is a contradiction, because the right-hand side of (24) tends to zero as $t \to \infty$, while the left-hand side equals 1 along a sequence diverging to infinity by condition (G_*). It follows that (21) is the only possibility.

(ii) Consider the case where (A) is sublinear and (G^*) holds. Let (21) hold. We can select T_1, T_2 and T_3 in the following manner:

$$T < T_1 < T_2 < T_3, \ T_0 = h_*(T_1) \geq T, \quad |x(T_0)| \geq P(T_0),$$

$$\sup_{T_0 \leq s \leq t} \frac{|x(s)|}{P(s)} = \sup_{T_2 \leq s \leq t} \frac{|x(s)|}{P(s)} \ \text{ for } t \geq T_2,$$

$$\int_{T_2}^\infty \omega(s, P(g(s)))ds \leq \frac{1}{4},$$

$$\frac{|x(T)|}{P(T)} + |y(T)| + \int_T^{T_2} \omega(s, |x(g(s))|)ds \leq \frac{1}{2}\frac{|x(T_3)|}{P(T_3)}.$$

We define

$$v(t) = \sup_{T_0 \leqq s \leqq t} \frac{|x(s)|}{P(s)}.$$

Using $v(t)$ and (14) in the inequality

$$\frac{|x(t)|}{P(t)} \leqq \frac{|x(T)|}{P(T)} + |y(T)| + \int_T^{T_2} \omega(s, |x(g(s))|)ds$$

$$+ \int_{T_2}^t \omega(s, |x(g(s))|)ds, \ t \geqq T_2,$$

which follows from (3) and (4), we find

$$v(t) \leqq \frac{1}{2} v(t) + v(h^*(t)) \int_{T_2}^t \omega(s, P(g(s)))ds, \quad t \geqq T_3,$$

and hence

$$\frac{v(t)}{v(h^*(t))} \leqq \frac{1}{2} \ \text{ for } t \geqq T_3.$$

Because of (G*) this is a contradiction, and so the solution $\{x(t), y(t)\}$ has to satisfy (22). This completes the proof of Theorem 1.3.

REMARK 1.3. The second part of Theorem 1.3 includes recent results of [2, Theorem 2] for the retarded sublinear system (A) and of [6, First half of Theorem 5] for the retarded sublinear equation (B).

EXAMPLE 1.2. Consider the superlinear system

(25)
$$\begin{cases} x'(t) = \exp\left(\frac{2t}{\sqrt{3}}\right)y(t) \\ y'(t) = -\frac{4}{\sqrt{3}} \exp\left(\frac{t}{\sqrt{3}} - 3\sqrt{3}g(t)\right)x^3(g(t)) \end{cases}$$

where $g(t) = \arccos(\sin^{1/3}t)$ with branches taken as follows:

$$(n + 1)\pi \leqq g(t) < (n + 2)\pi \text{ if } n\pi - \frac{\pi}{2} \leqq t < n\pi + \frac{\pi}{2}, n = 0, 1, 2, \dots.$$

Since $g(t) > t$, condition (G_*) is satisfied and clearly the integral condition (1) holds. By the first part of Theorem 1.3 every oscillatory solution of (25) enjoys the property (21). One such solution is

$$x(t) = \exp(\sqrt{3}t)\cos t, \ y(t) = 2\exp\left(\frac{t}{\sqrt{3}}\right)\cos\left(t + \frac{\pi}{6}\right).$$

As a corollary of Theorem 1.3 we have the following nonoscillation result

for *almost linear* systems of the form (A).

THEOREM 1.4. *Assume that*

$$(26) \qquad |f(t, x)| \leqq q(t)|x| \quad for \ (t, x) \in [a, \infty) \times (-\infty, \infty),$$

where $q(t)$ *is continuous and nonnegative on* $[a, \infty)$. *Assume moreover that conditions* (G^*) *and* (G_*) *hold. If*

$$(27) \qquad \int^\infty P(g^*(t))q(t)dt < \infty,$$

then all solutions of (A) are weakly nonoscillatory.

PROOF. Suppose to the contrary that there exists an oscillatory solution $\{x(t), y(t)\}$ of (A). Since by (26) system (A) is both superlinear and sublinear, and since (27) is equivalent to (1) or (2), we can apply Theorem 1.3 to conclude that $\{x(t), y(t)\}$ satisfies both (21) and (22). But this is impossible, and so (A) has no oscillatory solutions.

REMARK 1.4. Let $\{x(t), y(t)\}$ be a solution of (A). If $x(t)$ has arbitrarily large zeros, then so does $y(t)$, since otherwise the first equation of (A) would imply that $x(t)$ is a monotone function, a contradiction. It follows that the first component of a weakly nonoscillatory solution is always eventually positive or negative.

EXAMPLE 1.3. Consider the linear system of ordinary differential equations

$$(28) \qquad \begin{cases} x' = e^t\left[1 + \sin\left(t + \dfrac{\pi}{4}\right)\right][1 - e^{-t}(\sqrt{2} + \cos t)]^2 y \\[2mm] y' = -\dfrac{2\sqrt{2}e^{-2t}(1 + e^{-t}\sin t)}{[1 - e^{-t}(\sqrt{2} + \cos t)]^4}\, x. \end{cases}$$

Since the hypotheses of Theorem 1.4 are satisfied, every nontrivial solution of (28) is weakly nonoscillatory (in fact, it is nonoscillatory). This can be seen directly, as the general solution of (28) is given explicitly by

$$\begin{cases} x(t) = [c_1 e^t(\sqrt{2} + \sin t) + c_2][1 - e^{-t}(\sqrt{2} + \cos t)] \\[2mm] y(t) = \dfrac{\sqrt{2}\,[c_1 - \sqrt{2}c_1 e^{-t}\cos(t + \pi/4) + c_2 e^{-2t}]}{[1 - e^{-t}(\sqrt{2} + \cos t)]^2} \end{cases}$$

where c_1 and c_2 are arbitrary constants.

2. The case where $\int_a^\infty p(t)dt < \infty$

Let us now consider the system (A) in which $p(t)$ is subject to the condition $\int_a^\infty p(t)dt < \infty$.   We are able to obtain results which are parallel to the theorems proved in §1.   Use is made of the function $\pi(t) = \int_t^\infty p(s)ds$.

A)   A classification of solutions according to the behavior as $t \to \infty$ is described in the following theorem.

THEOREM 2.1.   *Assume that either (A) is superlinear and*

$$(29) \qquad \int^\infty \pi(g_*(t))\omega(t, c)dt < \infty \quad \textit{for all } c > 0$$

or (A) is sublinear and

$$(30) \qquad \int^\infty \frac{\pi(g_*(t))}{\pi(g(t))}\omega(t, c\pi(g(t)))dt < \infty \textit{ for all } c > 0.$$

If $\{x(t), y(t)\}$ is a solution of (A), then exactly one of the following cases holds:

(I) $\displaystyle\limsup_{t \to \infty} |x(t)| = \infty, \quad \limsup_{t \to \infty} |y(t)| = \infty.$

(II)   *There exists a nonzero constant α such that*

$\displaystyle\lim_{t \to \infty} x(t) = \alpha, \quad \lim_{t \to \infty} \pi(t)y(t) = 0.$

(III)   *There exists a constant β such that*

$\displaystyle\lim_{t \to \infty} \frac{x(t)}{\pi(t)} = \beta, \quad \lim_{t \to \infty} y(t) = -\beta.$

PROOF.   Let $\{x(t), y(t)\}$ be a solution of (A) defined on $[\tau, \infty)$ and let $T \geqq \tau$ be such that $h_*(T) \geqq \tau$.

Suppose (A) is superlinear and (29) holds.   Because of (29)

$$\pi(t)\omega(t, c), \quad \pi(g(t))\omega(t, c), \quad \omega(t, c\pi(g(t)))$$

are integrable at ∞ for all $c > 0$.   If $\limsup_{t \to \infty} |x(t)| = \infty$, then $\limsup_{t \to \infty} |y(t)| = \infty$, since otherwise a contradiction follows from (3).   Let $\limsup_{t \to \infty} |x(t)| < \infty$.   Then $p(t)y(t) \in L^1[T, \infty)$, because from (3) and (4) we have

$$\int_T^\infty p(s)|y(s)|ds \leqq \pi(T)|y(T)| + \int_T^\infty \pi(s)\omega(s, |x(g(s))|)ds$$

and (3) can be written as

$$(31) \qquad\qquad x(t) = \alpha - \int_t^\infty p(s)y(s)ds$$

for $t \geq T$, where

$$\alpha = x(T) + \int_T^\infty p(s)y(s)ds.$$

Consequently, $\lim_{t \to \infty} x(t) = \alpha$. On the other hand, we have

$$\pi(t)|y(t)| \leq \pi(t)|y(T)| + \pi(t)\int_T^t \omega(s, k)ds$$

$$\leq \pi(t)|y(T)| + \pi(t)\int_T^{T_1} \omega(s, k)ds + \int_{T_1}^t \pi(s)\omega(s, k)ds,$$

where k is a constant such that $|x(t)| \leq k$ for $t \geq \tau$. From this we see that $\pi(t)|y(t)|$ can be made arbitrarily small by taking T_1 sufficiently large and then letting t increase without bound. Thus, $\lim_{t \to \infty} \pi(t)y(t) = 0$, and we arrive at Case (II) if $\alpha \neq 0$. Now we suppose that $\alpha = 0$. Let $T_1 \geq T$ be such that

$$T_0 = h_*(T_1) \geq T, \quad |x(g(t))| \leq 1 \text{ for } t \geq T_1,$$

$$\int_{T_1}^\infty \pi(s)\omega(s, 1)ds \leq \frac{1}{3}, \quad \int_{T_1}^\infty \pi(g(s))\omega(s, 1)ds \leq \frac{1}{3}.$$

From (4) and (31) (with $\alpha = 0$) we get

$$(32) \qquad \begin{aligned} |x(t)| &\leq \pi(t)|y(T)| + \pi(t)\int_T^t \omega(s, |x(g(s))|)ds \\ &\quad + \int_t^\infty \pi(s)\omega(s, |x(g(s))|)ds, \quad t \geq T. \end{aligned}$$

Putting $u(t) = \sup_{s \geq t} |x(s)|$ and using the decreasing nature of the right-hand side of (32), we obtain

$$(33) \qquad \begin{aligned} \frac{u(t)}{\pi(t)} &\leq k_0 + \int_{T_1}^t \omega(s, u(g(s)))ds \\ &\quad + \frac{1}{\pi(t)}\int_t^\infty \pi(s)\omega(s, u(g(s)))ds, \quad t \geq T_1, \end{aligned}$$

where k_0 is a positive constant. Since

$$\frac{u(g(s))}{\pi(g(s))} \leq \sup_{T_0 \leq \sigma \leq t} \frac{u(\sigma)}{\pi(\sigma)} \text{ for } s \in I_t,$$

$$u(g(s)) \leqq u(t) \quad \text{for } s \in J_t,$$

where I_t and J_t are as in (10), taking (8) into account, we find

$$\frac{u(t)}{\pi(t)} \leqq k_0 + \sup_{T_0 \leqq s \leqq t} \frac{u(s)}{\pi(s)} \cdot \left(\int_{I_t \cap [T_1, t)} \pi(g(s)) \omega(s, 1) ds \right.$$

$$+ \frac{1}{\pi(t)} \int_{I_t \cap [t, \infty)} \pi(g(s)) \pi(s) \omega(s, 1) ds \bigg)$$

$$+ \frac{u(t)}{\pi(t)} \left(\pi(t) \int_{J_t \cap [T_1, t)} \omega(s, 1) ds + \int_{J_t \cap [t, \infty)} \pi(s) \omega(s, 1) ds \right)$$

$$\leqq k_0 + \sup_{T_0 \leqq s \leqq t} \frac{u(s)}{\pi(s)} \cdot \int_{T_1}^{\infty} \pi(g(s)) \omega(s, 1) ds + \frac{u(t)}{\pi(t)} \int_{T_1}^{\infty} \pi(s) \omega(s, 1) ds$$

$$\leqq k_0 + \frac{1}{3} \sup_{T_0 \leqq s \leqq t} \frac{u(s)}{\pi(s)} + \frac{1}{3} \frac{u(t)}{\pi(t)}, \quad t \geqq T_1,$$

and consequently

$$\frac{u(t)}{\pi(t)} \leqq k_1 + \frac{1}{2} \sup_{T_1 \leqq s \leqq t} \frac{u(s)}{\pi(s)}, \quad t \geqq T_1,$$

where k_1 is a positive constant. It follows that

$$\frac{|x(t)|}{\pi(t)} \leqq \sup_{T_1 \leqq s \leqq t} \frac{u(s)}{\pi(s)} \leqq 2k_1 \quad \text{for } t \geqq T_1,$$

that is, $x(t) = O(\pi(t))$ as $t \to \infty$. Now, this fact implies that $f(t, x(g(t))) \in L^1[T, \infty)$, and so from (4) we have

$$(34) \qquad\qquad y(t) = -\beta - \int_t^{\infty} f(s, x(g(s))) ds$$

for $t \geqq T$, where

$$\beta = -y(T) - \int_T^{\infty} f(s, x(g(s))) ds.$$

Therefore, $\lim_{t \to \infty} y(t) = -\beta$. Coupling (31) (with $\alpha = 0$) and (34) yields

$$x(t) = \beta \pi(t) + \int_t^{\infty} [\pi(t) - \pi(s)] f(s, x(g(s))) ds, \quad t \geqq T,$$

from which we easily see that $\lim_{t \to \infty} x(t)/\pi(t) = \beta$. Thus we are led to Case (III) when $\alpha = 0$.

Next suppose (A) is sublinear and (30) holds. Notice that

$$[\pi(t)/\pi(g(t))]\omega(t,\, c\pi(g(t))),\quad \omega(t,\, c\pi(g(t))),\quad \pi(t)\omega(t,\, c)$$

are integrable at ∞ for all $c>0$. It suffices to discuss the case where $\limsup\limits_{t\to\infty} |x(t)|$ $<\infty$ and (31) holds with $\alpha=0$. We shall show that $\limsup\limits_{t\to\infty} |x(t)|/\pi(t)<\infty$. Suppose the contrary. Then it is possible to select T_1, T_2 and T_3 in such a way that

$$T < T_1 < T_2 < T_3,\quad T_0 = h_*(T_1) \geqq T,\quad |x(T_0)| \geqq \pi(T_0),$$

$$\sup_{T_0 \leqq s \leqq t} \frac{|x(s)|}{\pi(s)} = \sup_{T_2 \leqq s \leqq t} \frac{|x(s)|}{\pi(s)} \quad \text{for } t \geqq T_2,$$

$$\int_{T_2}^{\infty} \frac{\pi(s)}{\pi(g(s))}\,\omega(s,\, \pi(g(s)))ds \leqq \frac{1}{4},\quad \int_{T_2}^{\infty} \omega(s,\, \pi(g(s)))ds \leqq \frac{1}{4}.$$

and

$$|y(T)| + \int_T^{T_2} \omega(s,\, |x(g(s))|)ds \leqq \frac{1}{4}\,\frac{|x(T_3)|}{\pi(T_3)}\,.$$

We rewrite (32) as follows:

$$(35) \quad \begin{aligned} \frac{|x(t)|}{\pi(t)} &\leqq |y(T)| + \int_T^{T_2} \omega(s,\, |x(g(s))|)ds + \int_{T_2}^{t} \omega(s,\, |x(g(s))|)ds \\ &\quad + \frac{1}{\pi(t)} \int_t^{\infty} \pi(s)\omega(s,\, |x(g(s))|)ds,\ t \geqq T_2. \end{aligned}$$

Define

$$v(t) = \sup_{T_0 \leqq s \leqq t} \frac{|x(s)|}{\pi(s)} \quad \text{for } t \geqq T_0.$$

Noting that the right-hand side of (35) is an increasing function of t and using the sublinearity (14), we obtain

$$\begin{aligned} v(t) &\leqq \frac{1}{4}\,v(t) + \int_{T_2}^{t} v(g(s))\omega(s,\, \pi(g(s)))ds \\ &\quad + \frac{1}{\pi(t)} \int_t^{\infty} v(g(s))\pi(s)\omega(s,\, \pi(g(s)))ds \end{aligned}$$

or

$$(36) \quad \begin{aligned} \frac{3}{4}\,\pi(t)v(t) &\leqq \pi(t) \int_{T_2}^{t} v(g(s))\omega(s,\, \pi(g(s)))ds \\ &\quad + \int_t^{\infty} v(g(s))\pi(s)\omega(s,\, \pi(g(s)))ds \end{aligned}$$

for $t \geqq T_3$. Using the inequalities

$$v(g(s)) \leqq v(t) \text{ for } s \in I_t,$$

$$\pi(g(s))v(g(s)) \leqq \sup_{\sigma \geqq t} [\pi(\sigma)v(\sigma)] \text{ for } s \in J_t,$$

we see from (36) that for $t \geqq T_3$

$$\frac{3}{4} \pi(t)v(t) \leqq \pi(t)v(t)\left(\int_{I_t \cap [T_2, t)} \omega(s, \pi(g(s)))ds\right.$$

$$+ \frac{1}{\pi(t)} \int_{I_t \cap [t, \infty)} \pi(s)\omega(s, \pi(g(s)))ds\Bigg)$$

$$+ \sup_{s \geqq t} [\pi(s)v(s)] \cdot \left(\pi(t) \int_{J_t \cap [T_2, t)} \frac{1}{\pi(g(s))} \omega(s, \pi(g(s)))ds\right.$$

$$+ \int_{J_t \cap [t, \infty)} \frac{\pi(s)}{\pi(g(s))} \omega(s, \pi(g(s)))ds\Bigg)$$

$$\leqq \pi(t)v(t) \int_{T_2}^{\infty} \omega(s, \pi(g(s)))ds$$

$$+ \sup_{s \geqq t} [\pi(s)v(s)] \cdot \int_{T_2}^{\infty} \frac{\pi(s)}{\pi(g(s))} \omega(s, \pi(g(s)))ds$$

$$\leqq \frac{1}{4} \pi(t)v(t) + \frac{1}{4} \sup_{s \geqq t} [\pi(s)v(s)].$$

Thus we arrive at

$$0 < \sup_{s \geqq t} [\pi(s)v(s)] \leqq \frac{1}{2} \sup_{s \geqq t} [\pi(s)v(s)], \quad t \geqq T_3,$$

a contradiction. Therefore we must have $\limsup_{t \to \infty} |x(t)|/\pi(t) < \infty$, and arguing as in the superlinear case, we are led to the relation (34).
This completes the proof.

B) The following theorem describes the asymptotic behavior of nonoscillatory solutions of (A).

THEOREM 2.2. *Suppose that the conditions* (16), (17) *and the hypotheses of Theorem 2.1 are satisfied. If* $\{x(t), y(t)\}$ *is a nonoscillatory solution of* (A), *then either*

(37) $$\lim_{t \to \infty} x(t) = \alpha, \quad \lim_{t \to \infty} \pi(t)y(t) = 0$$

for some constant $\alpha \neq 0$, *or else*

(38)
$$\lim_{t\to\infty} \frac{x(t)}{\pi(t)} = \beta, \quad \lim_{t\to\infty} y(t) = -\beta$$

for some constant $\beta \neq 0$.

PROOF. Let $\{x(t), y(t)\}$ be a nonoscillatory solution of (A) defined on $[\tau, \infty)$. Suppose $x(t) > 0$ for $t \geq \tau$. Let $T \geq \tau$ be such that $h_*(T) \geq \tau$. By the second equation of (A), $y(t)$ is decreasing for $t \geq T$. If $y(t) > 0$ for $t \geq T$, then the first equation of (A) implies that $x(t)$ is increasing, and so from (3) we get

$$x(T) \leq x(t) \leq x(T) + \pi(T)y(T) \quad \text{for } t \geq T.$$

If $y(t) < 0$ for $t \geq T'$ $(> T)$, then again from (3)

$$-\int_{T'}^{t} p(s)y(s)ds = x(T') - x(t) \leq x(T'), \quad t \geq T',$$

which shows that $p(t)y(t) \in L^1[T', \infty)$. Taking this fact into account and noting that $x(t)$ is decreasing, we have

$$x(T') \geq x(t) = x(T') + \int_{T'}^{\infty} p(s)y(s)ds - \int_{t}^{\infty} p(s)y(s)ds$$

$$\geq x(T') + \int_{T'}^{\infty} p(s)y(s)ds + \pi(t)|y(T')| \geq \pi(t)|y(T')|$$

for $t \geq T'$. A parallel argument applies if we assume that $x(t) < 0$ for $t \geq \tau$. Therefore we conclude that

$$\limsup_{t\to\infty} |x(t)| < \infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{|x(t)|}{\pi(t)} > 0.$$

The conclusion of the theorem now follows from Theorem 2.1. This finishes the proof.

REMARK 2.1. From the existence theorems established in [3] and [4] it readily follows that under the hypotheses of Theorem 2.1 system (A) actually possesses nonoscillatory solutions of the type (37) for all $\alpha \neq 0$ as well as those of the type (38) for all $\beta \neq 0$.

EXAMPLE 2.1. Consider the superlinear system

(39)
$$\begin{cases} x'(t) = \dfrac{1 + \sin(t + \pi/4)}{\sqrt{2} + \sin t} e^{-2t}y(t) \\[2mm] y'(t) = -\dfrac{2[1 + \sin(t + \pi/4)]}{[\sqrt{2} + \cos(t/3)]^3} e^{2t}x^3(t/3). \end{cases}$$

As easily checked, the integral condition (29) is not satisfied and system (39) has a nonoscillatory solution

$$x(t) = e^{-t}(\sqrt{2} + \cos t), \quad y(t) = -\sqrt{2}\, e^t(\sqrt{2} + \sin t)$$

whose asymptotic property is different from (37) and (38).

C)   Finally we establish oscillation and nonoscillation theorems corresponding to Theorems 1.3 and 1.4.

THEOREM 2.3.   (i) *Suppose that (A) is superlinear and condition* (G_*) *is satisfied. If (29) holds, then every oscillatory solution* $\{x(t), y(t)\}$ *of (A) has the property*:

$$(40) \qquad\qquad \limsup_{t \to \infty} |x(t)| = \infty, \quad \limsup_{t \to \infty} |y(t)| = \infty.$$

(ii)   *Suppose that (A) is sublinear and condition* (G^*) *is satisfied. If (30) holds, then every oscillatory solution* $\{x(t), y(t)\}$ *of (A) has the property*:

$$(41) \qquad\qquad \lim_{t \to \infty} \frac{x(t)}{\pi(t)} = 0, \quad \lim_{t \to \infty} y(t) = 0.$$

PROOF.   Let $\{x(t), y(t)\}$ be an oscillatory solution of (A) defined on $[\tau, \infty)$. Take $T \geq \tau$ so that $h_*(T) \geq \tau$. From the possibilities listed in Theorem 2.1 Case (II) and Case (III) (with $\beta \neq 0$) are excluded, and hence $\{x(t), y(t)\}$ satisfies either (40) or (41).

(i)   Let (A) be superlinear and (G_*) hold. Suppose to the contrary that (41) holds true. Then, proceeding as in the proof of the first part of Theorem 1.3, we have

$$|x(t)| \leq \pi(t) \int_t^\infty \omega(s, |x(g(s))|)ds, \quad t \geq T,$$

from which using the function

$$u(t) = \sup_{s \geq t} \frac{|x(t)|}{\pi(t)},$$

we can derive

$$\frac{u(t)}{u(h_*(t))} \leq \int_t^\infty \omega(s, \pi(g(s)))ds$$

for all sufficiently large t.   But because of (G_*) this is impossible.

(ii)   Let (A) be sublinear and (G^*) hold.   Suppose we have (40).   Then there are T_1, T_2 and T_3 such that

$$T < T_1 < T_2 < T_3, \quad T_0 = h_*(T_1) \geqq T, \quad |x(T_0)| \geqq 1,$$

$$\sup_{T_0 \leqq s \leqq t} |x(s)| = \sup_{T_2 \leqq s \leqq t} |x(s)| \text{ for } t \geqq T_2,$$

$$\int_{T_2}^{\infty} \pi(s)\omega(s, 1)ds \leqq \frac{1}{4},$$

$$|x(T)| + \pi(T)|y(T)| + \int_T^{T_2} \pi(s)\omega(s, |x(g(s))|)ds \leqq \frac{1}{2}|x(T_3)|.$$

Now defining $v(t) = \sup\limits_{T_0 \leqq s \leqq t} |x(s)|$ and arguing as in the second part of Theorem 1.3, we are able to derive a contradiction that

$$\frac{v(t)}{v(h^*(t))} \leqq \frac{1}{2}, \quad t \geqq T_3,$$

from the inequality

$$|x(t)| \leqq |x(T)| + \pi(T)|y(T)| + \int_T^t \pi(s)\omega(s, |x(g(s))|)ds$$

which is a consequence of (3) and (4). It follows that $\{x(t), y(t)\}$ must satisfy (41). Thus the proof is complete.

REMARK 2.2. The second part of Theorem 2.3 extends and improves recent results obtained in [2, Theorem 1], [6, Second half of Theorem 5] and [9, Theorems 2 and 3].

EXAMPLE 2.2. Consider the sublinear system

$$(42) \qquad \begin{cases} x'(t) = \exp\left(-\dfrac{2t}{\sqrt{3}}\right)y(t) \\[3mm] y'(t) = -\dfrac{4}{\sqrt{3}}\exp\left(\dfrac{g(t) - t}{\sqrt{3}}\right)x^{1/3}(g(t)) \end{cases}$$

where $g(t) = \arccos(-\sin^3 t)$. The branches of $g(t)$ is taken as follows:

$$(n - 2)\pi \leqq g(t) < (n - 1)\pi \text{ if } n\pi - \frac{\pi}{2} \leqq t < n\pi + \frac{\pi}{2}, \, n = 1, 2, 3, \ldots .$$

Since $g(t) < t$, condition (G*) is satisfied. It is easy to verify that (30) is valid. According to (ii) of Theorem 2.3 every oscillatory solution of (42) vanishes asymptotically in the sense (41). In fact,

$$x(t) = \exp(-\sqrt{3}\,t)\cos t, \quad y(t) = -2\exp\left(-\frac{t}{\sqrt{3}}\right)\sin\left(t + \frac{\pi}{3}\right)$$

is an oscillatory solution of (42) having this property.

THEOREM 2.4.   *Suppose that* (26), *(G*) and (G*) hold.   If*

$$(43) \qquad\qquad \int^{\infty} \pi(g_*(t))q(t)dt < \infty,$$

then all solutions of (A) are weakly nonoscillatory.
This is an immediate consequence of Theorem 2.3.

EXAMPLE 2.3.   Consider the linear system

$$(44) \qquad \begin{cases} x'(t) = \exp\left(-\dfrac{2t}{\sqrt{3}}\right)y(t) \\[2ex] y'(t) = -\dfrac{8}{\sqrt{3}}\exp\left(\dfrac{t}{2\sqrt{3}} + \dfrac{\sqrt{3}\pi}{4}\right)\cos\left(\dfrac{t}{2} + \dfrac{\pi}{4}\right)x(g(t)). \end{cases}$$

Let $g(t) = t\left[1 + \dfrac{1}{4}\sin(\log t)\right]$.   All the conditions of Theorem 2.4 are satisfied, so that system (44) has no oscillatory solutions.

Let $g(t) = \dfrac{t}{2} + \dfrac{\pi}{4}$.   Although (43) holds, condition (G*) is violated.   As a result (44) possesses an oscillatory solution

$$x(t) = \exp\left(-\sqrt{3}t\right)\sin t, \quad y(t) = 2\exp\left(-\dfrac{t}{\sqrt{3}}\right)\cos\left(t + \dfrac{\pi}{3}\right).$$

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