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# *Differential Calculus in Linear Ranked Spaces*

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## §0. Introduction

The theory of differential calculus in Banach spaces has been already established (cf. e.g., J. Dieudonné [1, Ch. VIII]), and there have been various attempts to construct differential calculus in more general linear spaces. For example, A. Fröhlicher and W. Bucher [2] have studied in linear spaces with limit structures based on filters, H. H. Keller [3] has studied the notion of $C^p$-mappings in locally convex spaces, and S. Yamamuro [7] has introduced the notion of equicontinuous differentiability in topological linear spaces.

In this paper, we try to develop differential calculus in linear ranked spaces. The notion of ranked spaces was first introduced by K. Kunugi [4]; and M. Yamaguchi [6] considered differentiation in linear ranked spaces. Using a modified formulation of linear ranked spaces given in M. Washihara [5, II], we shall study differentiation further than [6] and show that many important results in differential calculus can be included in our theory. In many respects, our construction of the theory and the methods of proofs are analogous to those in [2] and [7], though the underlying structures of the spaces are different.

We prepare in §1 some notions and results on linear ranked spaces. We define the notion of R-differentiability in §2, and prove the chain rule (Theorem 2.2) and the mean value theorem (Theorem 3.1). Further we study the Gâteaux differentiability in §4, and the invertibility of differentiable mappings in §5 (Theorems 5.2–5). Finally in §6, the higher derivatives are considered.

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## §1. Linear ranked spaces

Let $E$ be a linear space over the real field $\boldsymbol{R}$. Suppose that a sequence $\{\mathfrak{V}_n\}_{n=0}^{\infty}$ of families of subsets in $E$ is given to satisfy the following condition (E.1):

(E.1) $0 \in V$ *for any* $V \in \mathfrak{V} = \bigcup_{n=0}^{\infty} \mathfrak{V}_n$, $E \in \mathfrak{V}_0$; *and for any* $V \in \mathfrak{V}$ *and for any integer* $n \geqq 0$, *there are another integer* $m > n$ *and* $U \in \mathfrak{V}_m$ *such that* $U \subset V$.

Sets in $\mathfrak{V}_n$ are called *preneighborhoods of the origin* 0 *with rank n.*

A sequence $\{V_k\}_{k=0}^{\infty}$ of subsets of $E$ is called a *fundamental sequence at* 0, if

(1) $V_1 \supset V_2 \supset \cdots \supset V_k \supset \cdots$, *and*
(2) $V_k \in \mathfrak{B}_{n_k}$ $(k=1, 2,\ldots)$ *with* $n_1 \leqq n_2 \leqq \cdots \leqq n_k \leqq \cdots \longrightarrow \infty$.

Hereafter, we simply call any fundamental sequence at 0 a *f.s.*. Given a f.s. $\{V_k\}$, let

$$\mathrm{E}(\{V_k\}) = \{x \in E \mid \text{for each } k, \text{ there is } \lambda_k > 0 \text{ such that } x \in \lambda_k V_k\}.$$

If $(E, \{\mathfrak{B}_n\})$ satisfies the following conditions (E. 2–5) in addition to (E. 1), then $E=(E, \{\mathfrak{B}_n\})$ is called a *linear ranked space* (cf. K. Kunugi [4], M. Washihara [5] and M. Yamaguchi [6]):

(E. 2) *For any two f.s.'s* $\{V_k\}$ *and* $\{U_k\}$, *there is another f.s.* $\{W_k\}$ *such that* $V_k+U_k \subset W_k$ *for each* $k$.
(E. 3) *For any f.s.* $\{V_k\}$ *and* $\lambda>0$, *there are integers* $1 \leqq m(1) \leqq m(2) \leqq \cdots \to \infty$ *and* $k_0 \geqq 1$ *such that* $\lambda V_k \subset V_{m(k)}$ *for* $k \geqq k_0$.
(E. 4) *For any* $V \in \mathfrak{B}$ *and* $|\lambda| \leqq 1$, $\lambda V \subset V$.
(E. 5) *For any* $x \in E$, *there is a f.s.* $\{V_k\}$ *such that* $x \in \mathrm{E}(\{V_k\})$.

REMARK 1.1. (E. 4) follows from the condition that $\lambda V \subset V$ for $0<\lambda \leqq 1$ and the *symmetricity* $-V=V$. We assume condition (E. 4), since the symmetricity is essential for the study in this paper.

A linear ranked space $E$ is said to be $T_1^*$, if
($T_1^*$) $\bigcap_{k=1}^{\infty} V_k = \{0\}$ *for any f.s.* $\{V_k\}$.

DEFINITION 1.1. Given a f.s. $\{V_k\}$, a sequence $\{x_n\}$ in $E$ is said to be $\{V_k\}$-*convergent to* $x \in E$, *in symbols* $x_n \to x(\{V_k\})$, if for any $k$ there is $n_k$ such that $n \geqq n_k$ implies $x_n \in x+V_k$. Also, $\{x_n\}$ is said to be R-*convergent to* $x$, in symbols $x_n \to x(\mathrm{R})$, if $x_n \to x(\{V_k\})$ for some f.s. $\{V_k\}$; and then $x$ is called an R-*limit of* $\{x_n\}$.

PROPOSITION 1.1. *If E is a* $T_1^*$ *linear ranked space, then an* R-*limit of a sequence* $\{x_n\}$ *is unique if it exists.*

PROOF. Suppose $x_n \to x(\{V_k\})$ and $x_n \to x'(\{U_k\})$, and choose a f.s. $\{W_k\}$ such that $V_k+U_k \subset W_k$ by (E. 2). Then there is some $n_k$ such that $n \geqq n_k$ implies $x_n \in x+V_k$ and $x_n \in x'+U_k$, and so $x-x' \in V_k+U_k \subset W_k$ since $-V_k=V_k$ by (E. 4). Thus $x-x' \in \bigcap_{k=1}^{\infty} W_k$, and hence $x=x'$ since $E$ is $T_1^*$.

LEMMA 1.1. (a) *Let* $\{V_k\}$ *be a f.s.. Then for any* $\lambda>0$ *and* $k$, *there is* $k'$ *such that* $\lambda V_k \supset V_{k'}$.

(b) *If* $\{V_k\}$ *is a convex f.s., then for each* $k$, *there is* $k'$ *such that* $V_{k'}+V_{k'} \subset V_k$. (Here, we say that a f.s. $\{V_k\}$ is *convex*, if each $V_k$ is convex.)

PROOF. (a) By (E. 3), there are integers $1 \leqq m(1) \leqq m(2) \leqq \cdots \to \infty$ and $k_0 \geqq 1$ such that $\lambda^{-1} V_k \subset V_{m(k)}$ for $k \geqq k_0$. Thus, for each $k$, choosing $k' \geqq k_0$ with $m(k') \geqq k$, we obtain $\lambda V_k \supset \lambda V_{m(k')} \supset V_{k'}$.

(b) If $V_k$ is convex, then $2^{-1} V_k + 2^{-1} V_k \subset V_k$. Thus we see (b) by (a).

LEMMA 1.2. *Let $\{V_k\}$ be a f.s.. Then there are a sequence $\{\lambda_k\}$ of positive numbers and a sequence $\{N_k\}$ of positive integers such that*

$$\lambda_k \downarrow 0 \text{ and } V_j \subset \lambda_j V_k \quad \text{if} \quad j \geqq N_k.$$

PROOF. Put $j_1 = 1$. By induction, using Lemma 1.1 (a) we can choose $\{j_k\}$ such that $k V_{j_k} \subset V_{j_{k-1}}$ and $j_1 < j_2 < \cdots$. Put $\lambda_j = 1/k$ if $j_k \leqq j < j_{k+1}$ ($\lambda_1 = 1$). Then $\lambda_j \downarrow 0$. For each $k$, choose the smallest $m \geqq 2$ such that $k \leqq j_{m-1}$ and put $N_k = j_m$. If $j \geqq N_k$, then $j_l \leqq j < j_{l+1}$ for some $l \geqq m$. Then $k \leqq j_{m-1} \leqq j_{l-1}$, so that $\lambda_j^{-1} V_j = l V_j \subset l V_{j_l} \subset V_{j_{l-1}} \subset V_k$.

LEMMA 1.3. (a) *For any f.s.'s $\{V_k\}$ and $\{U_k\}$, let $\{W_k\}$ be a f.s. as in* (E. 2). *If $x_n \to x(\{V_k\})$ and $y_n \to y(\{U_k\})$, then $x_n + y_n \to x + y(\{W_k\})$.*

(a′) *If $x_n \to x(\mathrm{R})$ and $y_n \to y(\mathrm{R})$, then $x_n + y_n \to x + y(\mathrm{R})$.*

(b) *Let $\{V_k\}$ be a f.s.. If $x_n \to x(\{V_k\})$ and $\lambda > 0$, then $\lambda x_n \to \lambda x(\{V_k\})$.*

(c) *If $x_n \to x(\mathrm{R})$ and $\lambda_n \to \lambda(\lambda_n, \lambda \in \boldsymbol{R})$, then $\lambda_n x_n \to \lambda x(\mathrm{R})$.*

(d) *If $x_n \to 0(\{V_k\})$ and $\{\lambda_n\}$ is a bounded sequence of real numbers, then $\lambda_n x_n \to 0(\{V_k\})$.*

PROOF. We see easily (a) and (a′) by definition, and (b) using Lemma 1.1 (a).

(c) Assume $x_n \to x(\{V_k\})$. Then $\lambda_0 x_n \to \lambda_0 x(\{V_k\})$ by (b), where $\lambda_0 = \sup_n |\lambda_n|$. Hence for each $k$ there is $n_1(k)$ such that $n \geqq n_1(k)$ implies $\lambda_0 x_n \in \lambda_0 x + V_k$ and so $\lambda_n(x_n - x) \in \lambda_n \lambda_0^{-1} V_k \subset V_k$ by (E. 4). Also, choose a f.s. $\{U_k\}$ such that $x \in \mathrm{E}(\{U_k\})$, i.e., $x \in \mu_k U_k$ for some $\mu_k > 0$, by (E. 5). Since $\lambda_n \to \lambda$, there is $n_2(k)$ such that $n \geqq n_2(k)$ implies $|\lambda_n - \lambda| < 1/\mu_k$ and so $(\lambda_n - \lambda)x \in (\lambda_n - \lambda)\mu_k U_k \subset U_k$ by (E. 4). Then $n \geqq \max(n_1(k), n_2(k))$ implies

$$\lambda_n x_n - \lambda x = \lambda_n(x_n - x) + (\lambda_n - \lambda)x \in V_k + U_k \subset W_k,$$

where $\{W_k\}$ is a f.s. in (E. 2). Thus $\lambda_n x_n \to \lambda x(\mathrm{R})$.

(d) follows easily from Lemma 1.1 (a) and (E. 4).

LEMMA 1.4. *Let $x_n \to 0(\{V_k\})$ for some f.s. $\{V_k\}$. Then there is a sequence $\{\mu_k\}$ of positive numbers such that $\mu_k \uparrow \infty$ and $\mu_n x_n \to 0(\{V_k\})$.*

PROOF. By Definition 1.1, there are positive integers $m(1) \leqq m(2) \leqq \cdots$ such that $n \geqq m(k)$ implies $x_n \in V_k$. Choose sequences $\{\lambda_k\}$ and $\{N_k\}$ as in Lemma 1.2, and put $j_k = m(N_k)$ and

$$\mu_n = 1 \quad \text{for} \quad 1 \leqq n < j_1, \quad \mu_n = 1/\lambda_{N_k} \quad \text{for} \quad j_k \leqq n < j_{k+1} \qquad (k = 1, 2, \ldots).$$

Then $\mu_n \uparrow \infty$. Also, we see that $\mu_n x_n \in V_k$ if $j_k \leqq n < j_{k+1}$, which implies $\mu_n x_n \to 0(\{V_k\})$.

DEFINITION 1.2. Given a f.s. $\{V_k\}$, a sequence $\{x_n\}$ in $E$ is called a *Cauchy sequence by* $\{V_k\}$, if for each $k$ there is $n_k$ such that $m > n \geqq n_k$ implies $x_m - x_n \in V_k$. A sequence $\{x_n\}$ in $E$ is called an R-*Cauchy sequence* if it is a Cauchy sequence by some f.s. $\{V_k\}$. Also, $E$ is said to be R-*complete*, if for each Cauchy sequence $\{x_n\}$ by $\{V_k\}$ there is $x \in E$ such that $x_n \to x(\{V_k\})$.

LEMMA 1.5. *If* $x_n \to x(\mathrm{R})$, *then* $\{x_n\}$ *is an* R-*Cauchy sequence.*

PROOF. If $x_n \to x(\{V_k\})$ and $\{U_k\}$ is a f.s. such that $V_k + V_k \subset U_k$, then we see easily that $\{x_n\}$ is a Cauchy sequence by $\{U_k\}$.

Now, we consider the following additional assumptions for a linear ranked space $E = (E, \{\mathfrak{B}_n\})$, which will be assumed frequently:

(A. 1) *For each f.s.* $\{V_k\}$, *there is* $k_0$ *such that* $V_{k_0} \subset \mathrm{E}(\{V_k\})$.
(A. 2) *Let* $\{V_k\}$ *be a f.s.. If* $x_n \to 0(\mathrm{R})$ *and* $\{x_n\} \subset \mathrm{E}(\{V_k\})$, *then* $x_n \to 0(\{V_k\})$.
(A. 3) *Let* $\{V_k\}$ *be a f.s.. Then for each* $k$ *and* $x \in V_k$ *there is* $m$ *such that* $x + V_m \subset V_k$.

The following are some examples of linear ranked spaces which satisfy (A. 1–3) (cf. [4], [5], [6]).

EXAMPLE 1. *Normed linear spaces.* Let $E$ be a normed linear space, and let $V(\varepsilon) = \{x \in E \mid \|x\| < \varepsilon\}$ for $\varepsilon > 0$. Put

$$\mathfrak{B}_0 = \{V(\varepsilon) \mid \varepsilon > 1\} \cup \{E\}, \quad \mathfrak{B}_n = \{V(\varepsilon) \mid 1/(n+1) < \varepsilon \leqq 1/n\} \qquad (n = 1, 2, \ldots).$$

Then $(E, \{\mathfrak{B}_n\})$ is a $T_1^*$ linear ranked space. Hereafter we shall always regard a normed linear space as a linear ranked space with this structure. The R-convergence coincides with the norm-convergence, and any R-Cauchy sequence is a Cauchy sequence with respect to the norm. Hence $E$ is R-complete if and only if $E$ is a Banach space. It is easy to see that $E$ satisfies (A. 1–3).

EXAMPLE 2. *The Schwartz space* $\mathscr{D}$. Let $\boldsymbol{R}^n$ be the $n$-dimensional Euclidean space and put $\Omega_l = \{x \in \boldsymbol{R}^n \mid |x| < l\}$ $(|(x_1, \ldots, x_n)| = (x_1^2 + \cdots + x_n^2)^{1/2})$. For integers $m \geqq 0$, $l > 0$ and a real number $\varepsilon > 0$, let

$$U(m, l, \varepsilon) = \{\varphi \in \mathscr{D} \mid \operatorname{supp} \varphi \subset \Omega_l, \ |D^\alpha \varphi| < \varepsilon \quad \text{for all multi-indices } \alpha \text{ with } |\alpha| \leqq m\}.$$

Put $\mathfrak{B}_0=\{U(0, l, \varepsilon) \mid l=1, 2,\dots; \varepsilon>1\} \cup \{\mathscr{D}\}$ and

$$\mathfrak{B}_n = \{U(n, l, \varepsilon) \mid l = 1, 2,\dots; 1/(n+1) < \varepsilon \leqq 1/n\} \qquad (n = 1, 2,\dots).$$

Then $(\mathscr{D}, \{\mathfrak{B}_n\})$ is a $T_1^*$ linear ranked space. For a sequence $\{\varphi_n\}$ in $\mathscr{D}$, $\varphi_n \to 0(\mathrm{R})$ means that supp $\varphi_n$ is contained in a fixed bounded set and $|D^\alpha \varphi_n| \to 0$ uniformly for each $\alpha$. $\mathscr{D}$ is R-complete and satisfies (A. 1–3).

Example 3. *Inductive limits of metrizable topological vector spaces.* Let $\{(E_n, d_n)\}$ be a sequence of metrizable topological vector spaces such that $d_n$ is an invariant absorbing metric of $E_n$ for each $n$, $E_1 \subsetneqq E_2 \subsetneqq \cdots$ and $d_{n+1}(x, 0) \leqq d_n(x, 0)$ for $x \in E_n$. Consider the inductive limit $E=\bigcup_{n=1}^{\infty} E_n$, and put $\mathfrak{B}_0 =\{V(l; \varepsilon) \mid l=1, 2,\dots; \varepsilon>1\} \cup \{E\}$ and $\mathfrak{B}_n=\{V(l; \varepsilon) \mid l=1, 2,\dots; 1/(n+1)<\varepsilon \leqq 1/n\}$ $(n=1, 2,\dots)$, where $V(l; \varepsilon)=\{x \in E_l \mid d_l(x, 0)<\varepsilon\}$. Then $(E, \{\mathfrak{B}_n\})$ is a $T_1^*$ linear ranked space satisfying (A. 1–3), and $x_n \to 0(\mathrm{R})$ if and only if there is some $k$ such that $\{x_n\} \subset E_k$ and $d_k(x_n, 0) \to 0$ $(n \to \infty)$. Also, if each $(E_n, d_n)$ is complete, then $E$ is R-complete. The space $\mathscr{D}$ of the above example is a special case.

Now, we define several notions for a linear ranked space $E=(E, \{\mathfrak{B}_n\})$.

Definition 1.3. For a subset $S$ of $E$ and a f.s. $\{V_k\}$ in $E$, the *$\{V_k\}$-closure $\bar{S}(\{V_k\})$ of $S$* is the set of all $x \in E$ such that there is $\{x_n\}$ in $S$ with $x_n \to x(\{V_k\})$.

$$\bar{S} = \cup \{\bar{S}(\{V_k\}) \mid \{V_k\} \text{ is a f.s. in } E\}$$

is called the R-*closure of S*. $S$ is said to be $\{V_k\}$- or R-*closed* if $S=\bar{S}(\{V_k\})$ or $S=\bar{S}$. Also, a set $D \subset E$ is said to be R-*open* if $E \setminus D$ is R-closed.

Lemma 1.6. (a) *$x \in \bar{S}(\{V_k\})$ if and only if $(x+V_k) \cap S \neq \phi$ for each $k$.*

(a′) *$x \in \bar{S}$ if and only if there is some f.s. $\{V_k\}$ such that $(x+V_k) \cap S \neq \phi$ for each $k$.*

(b) *$\overline{\lambda S}(\{V_k\})=\lambda \bar{S}(\{V_k\})$, $\overline{\lambda S}=\lambda \bar{S}$, for any $\lambda>0$.*

(c) *$\overline{(x+S)}(\{V_k\})=x+\bar{S}(\{V_k\})$, $\overline{x+S}=x+\bar{S}$, for any $x \in E$.*

(d) *If $\{V_k\}$ is a convex f.s., then $\bar{S}(\{V_k\})$ is $\{V_k\}$-closed.*

(e) *If $\{V_k\}$ is a convex f.s. and $S$ is convex, then $\bar{S}(\{V_k\})$ is also convex.*

(e′) *If $S$ is convex, then so is $\bar{S}$.*

Proof. We see easily (a)–(c) by the above definition and Lemma 1.3 (b).

(d) Let $T=\bar{S}(\{V_k\})$. Then $\bar{T}(\{V_k\}) \supset T$ is obvious. For each $k$, choose $m$ such that $V_m+V_m \subset V_k$ by Lemma 1.1 (b). If $x \in \bar{T}(\{V_k\})$, then (a) implies that there exist $x' \in (x+V_m) \cap T$ and $x'' \in (x'+V_m) \cap S$. Hence $x'' \in x+V_m+V_m \subset x+V_k$ and $x'' \in S$, and so $x \in T$. Thus $\bar{T}(\{V_k\}) \subset T$.

(e) and (e′) are seen easily by (a) and (E. 2).

Lemma 1.7. *If $\{V_k\}$ is a convex f.s., then $\bar{V}_n(\{V_k\}) \subset \lambda V_n$ for any $\lambda>1$ and $n$.*

PROOF By Lemma 1.1 (a), there is an integer $m$ such that $(\lambda-1)V_n \supset V_m$. If $x \in \bar{V}_n(\{V_k\})$, then there exists $x' \in (x+V_m) \cap V_n$ by Lemma 1.6 (a). Thus $x = x - x' + x' \in V_m + V_n \subset (\lambda-1)V_n + V_n = \lambda V_n$ since $V_n$ is symmetric and convex.

DEFINITION 1.4. Let $\{V_k\}$ be a f.s. in $E$. A subset $S \subset E$ is said to be *$\{V_k\}$-bounded* if there is a sequence $\{\lambda_k\}$ of positive numbers such that $S \subset \lambda_k V_k$ for each $k$. $S \subset E$ is said to be R-*bounded* if it is $\{V_k\}$-bounded for some f.s. $\{V_k\}$. A sequence $\{x_n\}$ in $E$ is called a *$\{V_k\}$-quasi bounded sequence* (*$\{V_k\}$-q.b.s.*) if $\lambda_n x_n \to 0(\{V_k\})$ for any sequence $\{\lambda_n\}$ of positive numbers such that $\lambda_n \to 0$ $(n \to \infty)$. $\{x_n\}$ is called an R-*quasi bounded sequence* (R-*q.b.s.*) if it is a $\{V_k\}$-q.b.s. for some f.s. $\{V_k\}$ (cf. [6, II]).

REMARK 1.2. If $E$ is a normed linear space, then $S$ is R-bounded if and only if it is norm-bounded, and a sequence $\{x_n\}$ is an R-q.b.s. if and only if it is norm-bounded (cf. Lemma 1.10 below).

LEMMA 1.8. (a) *If $S_1$ and $S_2$ are* R-*bounded, then so are $S_1 \cup S_2$ and $S_1 + S_2$.*

(b) *Any finite set is* R-*bounded.*

(c) *If $S$ is* R-*bounded and $\lambda > 0$, then $\lambda S$ is* R-*bounded.*

(d) *Let $\{V_k\}$ be a convex f.s.. If $S$ is $\{V_k\}$-bounded, then so is $\bar{S}(\{V_k\})$.*

PROOF. (a) We see easily that if $S_1$ and $S_2$ are R-bounded, then so is $S_1 + S_2$, by using (E. 2). Thus $S_1 \cup S_2$ is R-bounded since $S_1 \cup S_2 \subset (S_1 \cup \{0\}) + (S_2 \cup \{0\})$.

(b) By (E. 5), given $x \in E$ there is a f.s. $\{V_k\}$ such that $x \in \mathrm{E}(\{V_k\})$. Then $\{x\}$ is $\{V_k\}$-bounded and so R-bounded. Thus we see (b) by (a).

(c) is obvious from definition, and (d) is immediate from Lemma 1.7.

LEMMA 1.9. (a) *If $\{x_n\}$ and $\{y_n\}$ are* R-*q.b.s.'s, then so is $\{x_n + y_n\}$.*

(b) *If $\{x_n\}$ is an* R-*q.b.s. and $\{\alpha_n\}$ is a bounded sequence of non-negative numbers, then $\{\alpha_n x_n\}$ is an* R-*q.b.s..*

PROOF. (a) follows from Lemma 1.3 (a) and definition.

(b) If $\lambda_n \to 0$ $(n \to \infty)$, then $\lambda_n \alpha_n \to 0$ $(n \to \infty)$. Thus we have (b).

LEMMA 1.10. (a) *Given a f.s. $\{V_k\}$, any $\{V_k\}$-bounded sequence is a $\{V_k\}$-q.b.s..*

(a′) *Any* R-*bounded sequence is an* R-*q.b.s..*

(b) *Conversely, if $\{x_n\}$ is a $\{V_k\}$-q.b.s. and $\{x_n\} \subset \mathrm{E}(\{V_k\})$, then $\{x_n\}$ is $\{V_k\}$-bounded.*

(b′) *If $E$ satisfies* (A. 1) *and $\{x_n\}$ is a $\{V_k\}$-q.b.s., then $\{x_n\}_{n \geq n_0}$ is $\{V_k\}$-bounded for some $n_0$.*

(b″) *If $E$ satisfies* (A. 1), *then any* R-*q.b.s. is* R-*bounded.*

PROOF. (a), (a′) Suppose $\{x_n\}$ is $\{V_k\}$-bounded. Then there is a sequence $\{\mu_k\}$ of positive numbers such that $x_n \in \mu_k V_k$ for all $n$, $k$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_n \to 0$ $(n \to \infty)$. For each $k$ there is an integer $n(k)$ such that $n \geqq n(k)$ implies $\lambda_n \leqq 1/\mu_k$. Hence $n \geqq n(k)$ implies $\lambda_n x_n \in \lambda_n \mu_k V_k \subset V_k$. Hence $\lambda_n x_n \to 0(\{V_k\})$. Thus $\{x_n\}$ is a $\{V_k\}$-q.b.s..

(b) Assume that $\{x_n\} \subset \mathrm{E}(\{V_k\})$ and $\{x_n\}$ is not $\{V_k\}$-bounded. Then there is $k$ such that $\{x_n\} \not\subset \lambda V_k$ for all $\lambda > 0$, and we can choose $n_j$ such that $x_{n_j} \notin jV_k$ for each $j = 1, 2, \dots$. Since $\{x_n\} \subset \mathrm{E}(\{V_k\})$, we see that $\{j \mid n_j = n\}$ is a finite set. Hence $\{n_j\}$ is unbounded, so that we can choose a subsequence $\{n_{j_l}\}$, $n_{j_1} < n_{j_2} < \cdots \to \infty$. Since $j_l \to \infty$, we can choose $\{\lambda_n\}$ such that $\lambda_{n_j} = 1/j$ for $j = j_l$, $l = 1, 2, \dots$, and $\lambda_n \to 0$ $(n \to \infty)$. Then $\lambda_{n_j} x_{n_j} \notin V_k$ if $j = j_l$ $(l = 1, 2, \dots)$, and hence $\lambda_n x_n \not\to 0$ $(\{V_k\})$. Thus $\{x_n\}$ is not a $\{V_k\}$-q.b.s..

(b′) Assume that $\{x_n\}$ is a $\{V_k\}$-q.b.s.. By (A.1), there is $k_0$ such that $V_{k_0} \subset \mathrm{E}(\{V_k\})$. Since $n^{-1}x_n \to 0$ $(\{V_k\})$, there is $n_0$ such that $n \geqq n_0$ implies $n^{-1}x_n \in V_{k_0}$, i.e., $x_n \in \mathrm{E}(\{V_k\})$. Then $\{x_n\}_{n \geqq n_0}$ is $\{V_k\}$-bounded by (b).

(b″) follows from (b′) and Lemma 1.8 (a), (b).

LEMMA 1.11. (a) *Each* R-*convergent sequence is an* R-*q.b.s..*

(b) *If* $x_n \to 0$ $(\{V_k\})$, *then* $\{x_n\}$ *is a* $\{V_k\}$-*q.b.s..*

PROOF. (a) follows from Lemma 1.3 (c); and (b) from Lemma 1.3 (d).

The continuity of a mapping between two linear ranked spaces is defined as follows.

DEFINITION 1.5. Let $E = (E, \{\mathfrak{B}_n\})$ and $F = (F, \{\mathfrak{W}_n\})$ be two linear ranked spaces and $D$ be a subset of $E$. A mapping $f: D \to F$ is said to be R-*continuous at* $a \in D$ (relative to $D$) if for each f.s. $\{V_k\}$ in $E$, there is a f.s. $\{W_k\}$ in $F$ such that

$$f((a + V_k) \cap D) \subset f(a) + W_k \qquad \text{for each} \quad k.$$

If $f$ is R-continuous at every $a \in D' \subset D$, then we say that $f$ is R-*continuous on* $D'$ (relative to $D$).

Let $L(E, F)$ be the set of all R-continuous linear mappings from $E$ to $F$.

LEMMA 1.12. *Let* $E \supset D$ *and* $F$ *be as in Definition* 1.5.

(a) *If* $f: D \to F$ *is* R-*continuous at* $a \in D$ *and* $x_n \to a(\mathrm{R})$ *with* $x_n \in D$, *then* $f(x_n) \to f(a)(\mathrm{R})$.

(b) *If* $f: D \to F$ *and* $g: D \to F$ *are* R-*continuous at* $a \in D$ *and if* $\lambda > 0$, *then* $f + g$ *and* $\lambda f$ *are* R-*continuous at* $a \in D$.

(c) *Let* $D$ *be a linear subspace of* $E$. *If* $f: D \to F$ *is linear and* R-*continuous at* 0, *then* $f$ *is* R-*continuous on* $D$.

(d) *Let* $f: E \to F$ *be a linear* R-*continuous mapping. If* $S$ *is* R-*bounded in* $E$, *then so is* $f(S)$ *in* $F$. *If* $\{x_n\}$ *is an* R-*q.b.s. in* $E$, *then so is* $\{f(x_n)\}$ *in* $F$.

PROOF. This lemma is seen easily by definition.

REMARK 1.3. If $E$ and $F$ are both normed linear spaces, then the R-continuity coincides with the continuity with respect to norms.

DEFINITION 1.6. Let $E=(E, \{\mathfrak{V}_n\})$ and $F=(F, \{\mathfrak{W}_n\})$ be two linear ranked spaces. For the product linear space $E\times F$, we take

$$\mathfrak{X}_n = \{V\times W \mid V\in\mathfrak{V}_l,\ W\in\mathfrak{W}_m,\ \min(l, m) = n\}$$

as the family of preneighborhoods of rank $n$.

LEMMA 1.13. (a) *If $\{V_k\}$ and $\{W_k\}$ are f.s.'s in $E$ and $F$, respectively, then $\{V_k\times W_k\}$ is a f.s. in $E\times F$.*

(b) *If $\{V_k\times W_k\}$ is a f.s. in $E\times F$, then there exist f.s.'s $\{V_k^*\}$ in $E$ and $\{W_k^*\}$ in $F$ such that $V_k\subset V_k^*\in\{V_k\}$ and $W_k\subset W_k^*\in\{W_k\}$ for each $k$.*

PROOF. (a) is obvious by definition.

(b) Assume $V_k\times W_k\in\mathfrak{X}_{n_k}$, $V_k\in\mathfrak{V}_{l_k}$, $W_k\in\mathfrak{W}_{m_k}$ with $n_k=\min(l_k, m_k)$. Define $k(j)$ inductively as follows: Let $k(1)=1$. Choose $k(j+1)>k(j)$ such that $l_{k(j+1)}>l_{k(j)}$ and $m_{k(j+1)}>m_{k(j)}$. Then $k(j)\geqq j$ and $\{V_{k(j)}\}_j$, $\{W_{k(j)}\}_j$ are f.s.'s in $E$, $F$, respectively. Put $V_k^*=V_{k(j)}$ and $W_k^*=W_{k(j)}$ if $k(j)\leqq k<k(j+1)$. Then $\{V_k^*\}$ and $\{W_k^*\}$ are the desired f.s.'s.

THEOREM 1.1 (cf. [6, I]). *$E\times F=(E\times F, \{\mathfrak{X}_n\})$ is a linear ranked space.*

PROOF. Let $\{V_k\times W_k\}$ and $\{V'_k\times W'_k\}$ be f.s.'s in $E\times F$. Then by Lemma 1.13(b), there are f.s.'s $\{V_k^*\}$, $\{V_k'^*\}$ in $E$ and $\{W_k^*\}$, $\{W_k'^*\}$ in $F$ such that $V_k\subset V_k^*$, $V'_k\subset V_k'^*$ and $W_k\subset W_k^*$, $W'_k\subset W_k'^*$. Also, by (E.2) for $E$ and $F$, there are f.s.'s $\{V''_k\}$ in $E$ and $\{W''_k\}$ in $F$ such that $V_k^*+V_k'^*\subset V''_k$ and $W_k^*+W_k'^*\subset W''_k$ for each $k$. Then $\{V''_k\times W''_k\}$ is a f.s. in $E\times F$ by Lemma 1.13(a), and

$$V_k\times W_k + V'_k\times W'_k = (V_k+V'_k)\times(W_k+W'_k)\subset V''_k\times W''_k \qquad \text{for each}\quad k.$$

Thus $(E\times F, \{\mathfrak{X}_n\})$ satisfies (E.2).

Let $\{V_k\times W_k\}$ be a f.s. in $E\times F$ and $\lambda>0$. Choose f.s.'s $\{V_k^*\}$ in $E$ and $\{W_k^*\}$ in $F$ as in Lemma 1.13(b). Then, by (E.3) for $E$ and $F$, there are integers $1\leqq m(1)\leqq m(2)\leqq\cdots\to\infty$, $1\leqq l(1)\leqq l(2)\leqq\cdots\to\infty$ and $k_0$ such that $\lambda V_k^*\subset V_{m(k)}^*$, $\lambda W_k^*\subset W_{l(k)}^*$ for $k\geqq k_0$. Thus $\lambda(V_k\times W_k)=\lambda V_k\times\lambda W_k\subset V_{n(k)}^*\times W_{n(k)}^*$ for $k\geqq k_0$, where $n(k)=\min(m(k), l(k))$. These show (E.3) for $(E\times F, \{\mathfrak{X}_n\})$.

(E.4) and (E.5) for $(E\times F, \{\mathfrak{X}_n\})$ are verified easily.

LEMMA 1.14. (a) *If $E$ and $F$ are both $T_1^*$, then so is $E\times F$.*

(b) *$(x_n, y_n)\to 0$(R) in $E\times F$ if and only if $x_n\to 0$(R) in $E$ and $y_n\to 0$(R) in $F$.*

(c) *If $E$ and $F$ both satisfy (A.1), (A.2) or (A.3), then so does $E\times F$.*

(d) *If $E$ and $F$ are* R-*complete, then so is* $E\times F$.

(e) $S=S_1\times S_2(S_1\subset E,\ S_2\subset F)$ *is* R-*bounded in* $E\times F$ *if and only if* $S_1$ *and* $S_2$ *are* R-*bounded in $E$ and $F$, respectively.*

(f) *The projections* $p_1: E\times F\to E$ *and* $p_2: E\times F\to F$ *are* R-*continuous.*

PROOF. Let $\{U_k\}=\{V_k\times W_k\}$ be a f.s. in $E\times F$, and choose f.s.'s $\{V_k^*\}$ in $E$ and $\{W_k^*\}$ in $F$ such that $V_k\subset V_k^*\in\{V_k\}$ and $W_k\subset W_k^*\in\{W_k\}$ for each $k$, by Lemma 1.13(b). Then we see easily that $\mathrm{E}(\{U_k\})=\mathrm{E}(\{V_k^*\times W_k^*\})=\mathrm{E}(\{V_k^*\})\times\mathrm{E}(\{W_k^*\})$.

(a) Since $\bigcap_k U_k\subset(\bigcap_k V_k^*)\times(\bigcap_k W_k^*)$, we see (a).

(b) If $(x_n, y_n)\to 0(\{U_k\})$, then $x_n\to 0(\{V_k^*\})$ and $y_n\to 0(\{W_k^*\})$. This shows the 'only if' part. The 'if' part is clear by Lemma 1.13(a).

(c) If $E$ and $F$ satisfy (A.1), then there is $k_0$ such that $V_{k_0}^*\subset\mathrm{E}(\{V_k^*\})$ and $W_{k_0}^*\subset\mathrm{E}(\{W_k^*\})$. Put $k'=\max(k_1, k_2)$, where $V_{k_0}^*=V_{k_1}$ and $W_{k_0}^*=W_{k_2}$. Then $U_{k'}=V_{k'}\times W_{k'}\subset\mathrm{E}(\{V_k^*\})\times\mathrm{E}(\{W_k^*\})=\mathrm{E}(\{U_k\})$. Thus $E\times F$ satisfies (A.1).

Suppose that $E$ and $F$ satisfy (A.2). If $(x_n, y_n)\to 0(\mathrm{R})$ in $E\times F$, then $x_n\to 0(\mathrm{R})$ in $E$ and $y_n\to 0(\mathrm{R})$ in $F$ by (b). If $(x_n, y_n)\in\mathrm{E}(\{U_k\})$ for all $n$ in addition, then $x_n\in\mathrm{E}(\{V_k^*\})$ and $y_n\in\mathrm{E}(\{W_k^*\})$, and so $x_n\to 0(\{V_k^*\})$ and $y_n\to 0(\{W_k^*\})$ by (A.2) for $E$ and $F$. Thus we see that $(x_n, y_n)\to 0(\{U_k\})$.

Finally suppose $E$ and $F$ satisfy (A.3), and $(x, y)\in U_k=V_k\times W_k$. Then, there is $m$ such that $x+V_m^*\subset V_k$ and $y+W_m^*\subset W_k$ by (A.3) for $E$ and $F$. Thus $(x, y)+U_m\subset(x, y)+(V_m^*\times W_m^*)\subset U_k$, and (A.3) holds for $E\times F$.

(d) Suppose $E$ and $F$ are R-complete. If $\{(x_n, y_n)\}$ is a Cauchy sequence by $\{U_k\}$, then we see easily by definition that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences by $\{V_k^*\}$ and $\{W_k^*\}$, respectively. Thus there are $x\in E$ and $y\in F$ such that $x_n\to x(\{V_k^*\})$ and $y_n\to y(\{W_k^*\})$, and hence $(x_n, y_n)\to(x, y)(\{U_k\})$. Hence $E\times F$ is also R-complete.

(e) and (f) are seen easily by definition.

LEMMA 1.15. *Let $E$, $F$ be linear ranked spaces and let $T$ be an* R-*continuous bilinear mapping of the product linear ranked space* $E^2=E\times E$ *into $F$. Then for any f.s.'s $\{V_k\}$ and $\{U_k\}$ in $E$, there is a f.s. $\{W_k\}$ in $F$ such that*

$$T(x_n, y_n)\longrightarrow 0(\{W_k\})$$

*for any sequence $\{x_n\}$ with $x_n\to 0(\{V_k\})$ and any $\{U_k\}$-q.b.s. $\{y_n\}$.*

PROOF. Since $\{V_k\times U_k\}$ is a f.s. in $E^2$ by Lemma 1.13(a), the R-continuity of $T$ at $0=(0, 0)$ implies that there is a f.s. $\{W_k\}$ in $F$ with

$$T(V_k\times U_k)\subset W_k \qquad \text{for each} \quad k.$$

If $x_n\to 0(\{V_k\})$, then there is a sequence $\{\mu_n\}$ such that $\mu_n>0$, $\mu_n\uparrow\infty$ and $\mu_n x_n\to 0(\{V_k\})$ by Lemma 1.4. Thus if $\{y_n\}$ is a $\{U_k\}$-q.b.s., then we see $T(x_n, y_n)$

$=T(\mu_n x_n, \mu_n^{-1} y_n) \to 0(\{W_k\})$ as desired.

## §2. Differentiation

*In the sequel, let E and F be linear ranked spaces and let D be a non-empty* R-*open subset of E* (cf. Definition 1.3).

DEFINITION 2.1 (cf. [6, V]). A mapping $f: D \to F$ is said to be R-*differentiable at* $x \in D$, if there exists an R-continuous linear mapping $l: E \to F$ for which

$$(2.1) \qquad r: D-x \longrightarrow F, \quad r(h)=f(x+h)-f(x)-l(h) \qquad (h \in D-x).$$

satisfies the following condition:

(2.2) For any f.s. $\{V_k\}$ in $E$, there exists a f.s. $\{U_m\}$ in $F$ such that

$$\lambda_n^{-1} r(\lambda_n h_n) \longrightarrow 0(\{U_m\})$$

for each $\{V_k\}$-q.b.s. $\{h_n\}$ in $E$ and each sequence $\{\lambda_n\}$ of positive numbers with $\lambda_n \to 0$ $(n \to \infty)$.

REMARK 2.1. Since $D$ is R-open, for each $x \in D$ and each f.s. $\{V_k\}$, there is $k_0$ such that $k \geqq k_0$ implies $V_k \subset D-x$ by Lemma 1.6(a'). Hence, if $\{h_n\}$ is a $\{V_k\}$-q.b.s. and $\lambda_n \to 0$ $(\lambda_n > 0)$, then $\lambda_n h_n \to 0(\{V_k\})$ and so $\lambda_n h_n \in D-x$ for large $n$.

LEMMA 2.1. *If* (2.2) *holds, then for any sequence* $\{\lambda_k\}$ *of positive numbers with* $\lambda_k \to 0$ *and for any m, there is* $k_0$ *such that*

$$\lambda_k^{-1} r(\lambda_k V_k) \subset U_m \qquad \textit{for each} \quad k \geqq k_0.$$

PROOF. Suppose there are $m_0$ and a sequence $\{\lambda_k\}$ with $\lambda_k > 0$, $\lambda_k \to 0$ such that for each $k$ there is $k' > k$ with $\lambda_{k'}^{-1} r(\lambda_{k'} V_{k'}) \not\subset U_{m_0}$. Then we can choose $k_1 < k_2 < \cdots$ and $h_j \in V_{k_j}$ such that

$$\lambda_{k_j}^{-1} r(\lambda_{k_j} h_j) \notin U_{m_0} \qquad \text{for each} \quad j.$$

Thus $h_j \to 0(\{V_k\})$ and so $\{h_j\}$ is a $\{V_k\}$-q.b.s. by Lemma 1.11(b). Also, $\lambda_{k_j} \to 0$ $(j \to \infty)$, but $\lambda_{k_j}^{-1} r(\lambda_{k_j} h_j) \not\to 0(\{U_m\})$, which contradicts (2.2).

THEOREM 2.1. *If* $f: D \to F$ *is* R-*differentiable at* $x \in D$, *then it is* R-*continuous at* $x$.

PROOF. Let a f.s. $\{V_k\}$ in $E$ be given. By Lemma 1.2, there exist $\{\lambda_k\}$ $(\lambda_1 = 1, 0 < \lambda_k \leqq 1, \lambda_k \downarrow 0)$ and $\{N_k\}$ $(N_1 = 1, N_k \uparrow \infty)$ such that $V_j \subset \lambda_j V_k$ if $j \geqq N_k$. If we put $k(j) = \max\{k | j \geqq N_k\}$ and $V'_j = V_{k(j)}$, then we see that $\{V'_j\}$ is a f.s. in $E$ and $V_j \subset \lambda_j V'_j$ for all $j$. Let $l \in L(E, F)$ and $r$ be as in Definition 2.1. By

(2.1),

$$f((x+V_k)\cap D)\subset f(x)+l(V_k)+r(V_k\cap(D-x)),$$

where $V_k\subset D-x$ for large $k$, by Remark 2.1. By Lemma 2.1, there is a f.s. $\{U_m\}$ in $F$ such that

$$\lambda_j^{-1}r(\lambda_j V'_j)\subset U_m \qquad \text{if}\quad j\geqq j(m),$$

for some sequence $\{j(m)\}$ of integers with $j(m)\uparrow\infty$. Hence

$$r(V_j)\subset r(\lambda_j V'_j)\subset \lambda_j U_m\subset U_m \qquad \text{if}\quad j\geqq j(m).$$

Put $m(j)=\max\{m\mid j\geqq j(m)\}$ ($m(j)=0$ if $j<j(m)$ for all $m$) and $U'_j=U_{m(j)}$ ($U_0=F$). Then $\{U'_j\}$ is a f.s. in $F$. On the other hand, since $l$ is R-continuous at 0, there is a f.s. $\{U''_m\}$ in $F$ such that $l(V_j)\subset U''_j$ for each $j$. Choose a f.s. $\{W_j\}$ in $F$ such that $U'_j+U''_j\subset W_j$ by (E. 2). Then

$$f((x+V_j)\cap D)\subset f(x)+l(V_j)+r(V_j)\subset f(x)+U'_j+U''_j\subset f(x)+W_j$$

for large $j$. Hence $f$ is R-continuous at $x$.

LEMMA 2.2. *If $F$ is $T_1^*$ and if $f: D\to F$ is* R-*differentiable at $x\in D$, then $l\in L(E,F)$ in Definition* 2.1 *is uniquely determined.*

PROOF. Let $l_1, l_2\in L(E,F)$,

$$r_j(h)=f(x+h)-f(x)-l_j(h) \qquad (h\in D-x),\quad j=1,2,$$

and suppose $r_1$ and $r_2$ both satisfy (2.2).

For any $h\in E$, we can find a f.s. $\{V_k\}$ in $E$ such that $h\in \mathrm{E}(\{V_k\})$ by (E. 5). Then $\{h\}$ is $\{V_k\}$-bounded and hence is a $\{V_k\}$-q.b.s. by Lemma 1.10(a). Thus by (2.2), there are f.s.'s $\{U_m(1)\}$ and $\{U_m(2)\}$ in $F$ such that

$$\lambda_n^{-1}r_j(\lambda_n h)\longrightarrow 0 \quad (\{U_m(j)\}) \quad (j=1,2)$$

for any sequence $\{\lambda_n\}$ with $\lambda_n>0$, $\lambda_n\to 0$. Then for any $m$, there is $n$ such that $\lambda_n^{-1}r_j(\lambda_n h)\in U_m(j)$ ($j=1,2$). Let $\{W_m\}$ be a f.s. in $F$ such that $U_m(1)+U_m(2)\subset W_m$. Then

$$\begin{aligned} l_1(h)-l_2(h)&=\lambda_n^{-1}\{l_1(\lambda_n h)-l_2(\lambda_n h)\}\\ &=\lambda_n^{-1}\{r_1(\lambda_n h)-r_2(\lambda_n h)\}\in U_m(1)+U_m(2)\subset W_m. \end{aligned}$$

This implies $l_1(h)=l_2(h)$ as desired, since $F$ is $T_1^*$, i.e., $\bigcap_m W_m=\{0\}$.

DEFINITION 2.2. Suppose $F$ is $T_1^*$ and a mapping $f: D\to F$ is R-differentiable at $x\in D$. Then the unique $l\in L(E,F)$ in Definition 2.1 is called the

R-*derivative of* $f$ *at* $x$ and is denoted by $f'(x)$.

Hereafter, we shall always assume that $F$ *is* $T_1^*$.

REMARK 2.2 (cf. [6, V]). If $E$ and $F$ are normed linear spaces, then R-differentiability of $f: E \to F$ with $E$ and $F$ being regarded as linear ranked spaces coincides with Fréchet differentiability of $f$ and $f'(x)$ is the Fréchet derivative of $f$ at $x$ (see Theorem 2.3 below).

LEMMA 2.3. *Any* R-*continuous linear mapping* $l \in L(E, F)$ *is* R-*differentiable at every* $a \in E$ *and* $l'(a)(x) = l(x)$.

PROOF. Since $l(a+x) - l(a) = l(x)$, we see immediately the lemma by definition.

THEOREM 2.2. *Let E be a linear ranked space, F and G be* $T_1^*$ *linear ranked spaces. Let* $D_1$ *and* $D_2$ *be* R-*open subsets of E and F, respectively. Suppose* $f: D_1 \to F$ *and* $g: D_2 \to G$ *are* R-*differentiable at* $a \in D_1$ *and* $f(a) \in D_2$, *respectively, and* $f(D_1) \subset D_2$. *Then the composed mapping* $g \circ f: D_1 \to G$ *is* R-*differentiable at* $a \in D_1$ *and*

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a).$$

PROOF. Consider the remainders

$$r_1(x) = f(a+x) - f(a) - f'(a)(x) \qquad (x \in D_1 - a),$$
$$r_2(y) = g(f(a)+y) - g(f(a)) - g'(f(a))(y) \qquad (y \in D_2 - f(a)).$$

Then we see easily that

$$\text{(2.3)} \qquad r(x) \equiv (g \circ f)(a+x) - (g \circ f)(a) - (g'(f(a)) \circ f'(a))(x)$$
$$= g'(f(a))(r_1(x)) + r_2(f'(a)(x) + r_1(x)) \qquad (x \in D_1 - a).$$

Let $\{V_k\}$ be a f.s. in $E$. Choose a f.s. $\{U_m\}$ in $F$ such that

$$\lambda_n^{-1} r_1(\lambda_n h_n) \longrightarrow 0\ (\{U_m\})$$

for any $\{V_k\}$-q.b.s. $\{h_n\}$ and any sequence $\{\lambda_n\}$ with $\lambda_n > 0$, $\lambda_n \to 0$. Since $f'(a)$ is R-continuous at 0, there is a f.s. $\{U'_m\}$ in $F$ such that $f'(a)(V_m) \subset U'_m$ for each $m$. Let $\{U''_m\}$ be a f.s. in $F$ such that $U_m + U'_m \subset U''_m$ for each $m$. By the R-differentiability of $g$, there is a f.s. $\{W_l\}$ in $G$ such that

$$\lambda_n^{-1} r_2(\lambda_n k_n) \longrightarrow 0\ (\{W_l\})$$

for any $\{U''_m\}$-q.b.s. $\{k_n\}$ and any sequence $\{\lambda_n\}$ with $\lambda_n > 0$, $\lambda_n \to 0$. Also, since $g'(f(a)) \in L(F, G)$, there is a f.s. $\{W'_l\}$ in $G$ such that $g'(f(a))(U_l) \subset W'_l$ for each $l$. Choose a f.s. $\{W''_l\}$ in $G$ such that $W_l + W'_l \subset W''_l$ for each $l$.

Now, let $\{h_n\}$ be a $\{V_k\}$-q.b.s. in $E$ and $\{\lambda_n\}$ be a sequence such that $\lambda_n > 0$,

$\lambda_n \to 0$. Since $\lambda_n^{-1} r_1(\lambda_n h_n) \to 0\ (\{U_m\})$, we see that

$$\lambda_n^{-1} g'(f(a))(r_1(\lambda_n h_n)) = g'(f(a))(\lambda_n^{-1} r_1(\lambda_n h_n)) \longrightarrow 0\ (\{W_l'\}). \tag{2.4}$$

Next, we shall show that the sequence $\{k_n\}$ given by

$$k_n = f'(a)(h_n) + \lambda_n^{-1} r_1(\lambda_n h_n)$$

is a $\{U_m''\}$-q.b.s.. For any sequence $\{\mu_n\}$ with $\mu_n > 0$ and $\mu_n \to 0$, $\mu_n h_n \to 0(\{V_k\})$ by definition, so that

$$\mu_n f'(a)(h_n) = f'(a)(\mu_n h_n) \longrightarrow 0(\{U_m'\}).$$

On the other hand, $\mu_n \lambda_n^{-1} r_1(\lambda_n h_n) \to 0(\{U_m\})$ by Lemma 1.3 (d). Hence $\mu_n k_n \to 0(\{U_m''\})$ by Lemma 1.3 (a). Thus $\{k_n\}$ is a $\{U_m''\}$-q.b.s.. Therefore,

$$\lambda_n^{-1} r_2(f'(a)(\lambda_n h_n) + r_1(\lambda_n h_n)) = \lambda_n^{-1} r_2(\lambda_n k_n) \longrightarrow 0(\{W_l\}). \tag{2.5}$$

By (2.3–5) and Lemma 1.3 (a), we have

$$\lambda_n^{-1} r(\lambda_n h_n) \longrightarrow 0(\{W_l''\}).$$

Hence we have proved the theorem.

In the case that $E$ is a normed linear space, we have the following

THEOREM 2.3. *Let $E$ be a normed linear space, $D$ be an open subset of $E$ and $F$ be a $T_1^*$ linear ranked space. Then $f: D \to F$ is* R*-differentiable at $x \in D$ with $E$ being regarded as a linear ranked space, if and only if there exists $l \in L(E, F)$ such that*

(2.6) *for any sequence $\{h_n\}$ in $E$ with $h_n \to 0$, $h_n \neq 0$,*

$$\|h_n\|^{-1} r(h_n) \longrightarrow 0(\mathrm{R}) \quad (r(h) = f(x+h) - f(x) - l(h)).$$

PROOF. The necessity follows immediately from Definition 2.1; note that $\{h_n/\|h_n\|\}$ is an R-q.b.s..

Conversely, suppose (2.6) holds. If $\{h_n\}$ is an R-q.b.s. in $E$, then $\{h_n\}$ is bounded by Remark 1.2 (cf. Lemma 1.10 (b″)), and so $\lambda_n h_n \to 0$ for any $\{\lambda_n\}$ with $\lambda_n > 0$, $\lambda_n \to 0$. Thus $\|\lambda_n h_n\|^{-1} r(\lambda_n h_n) \to 0(\mathrm{R})$ by (2.6), which implies $\lambda_n^{-1} r(\lambda_n h_n) \to 0(\mathrm{R})$ by Lemma 1.3 (d). Hence $f$ is R-differentiable at $x$.

COROLLARY 2.4. *If $E = \mathbf{R}$ in the above theorem, then* (2.6) *is the following*:

(2.6′) *For any sequence $\{\delta_n\}$ with $\delta_n \to 0$, $\delta_n \neq 0$,*

$$\delta_n^{-1}(f(x+\delta_n) - f(x)) \longrightarrow l(1)\ (\mathrm{R}).$$

In this case, we denote $l(1)=f'(x)(1)$ by $f^{\cdot}(x)$. Obviously $f'(x)(\lambda)=\lambda f^{\cdot}(x)$ for all $\lambda \in \boldsymbol{R}$.

## §3. The mean value theorem

THEOREM 3.1 (cf. [2, §5.1], [7, (1.3.1)]). *Let $E$ be a $T_1^*$ linear ranked space satisfying* (A.2). *Let $\alpha<\beta$ and let $f:[\alpha, \beta]\to E$ and $\varphi:[\alpha, \beta]\to \boldsymbol{R}$ satisfy the following conditions*:

(a) *$f$ and $\varphi$ are* R*-continuous on* $[\alpha, \beta]$;

(b) *$f$ and $\varphi$ are* R*-differentiable at each point $t\in(\alpha, \beta)\setminus D_1$, where $D_1$ is at most countable*;

(c) *$\varphi$ is monotone non-decreasing.*

*Furthermore, let $\{V_k\}$ be a convex f.s. in $E$ and $B$ be a subset of $E$ satisfying*

(d) $f([\alpha, \beta])\subset f(\alpha)+\mathrm{E}(\{V_k\}), f^{\cdot}((\alpha, \beta)\setminus D_1)\subset \mathrm{E}(\{V_k\})$;

(e) *$B$ is $\{V_k\}$-closed and convex, and $B\cap \mathrm{E}(\{V_k\})\neq\phi$.*

*If $f^{\cdot}(t)\in\varphi^{\cdot}(t)B$ for all $t\in(\alpha, \beta)\setminus D_1$, then*

$$f(\beta)-f(\alpha)\in(\varphi(\beta)-\varphi(\alpha))B.$$

PROOF. First remark that $\mathrm{E}(\{V_k\})$ is a linear subspace, since each $V_k$ is symmetric and convex. Thus, we may assume without loss of generality that $\alpha=0$, $\varphi(0)=0$ and $f(0)=0$. Furthermore, for $x_0\in B\cap \mathrm{E}(\{V_k\})$, consider $f_1(t)=f(t)-\varphi(t)x_0$ and $B_1=B-x_0$. Then $B_1$ is convex and $\{V_k\}$-closed by Lemma 1.6(c), $f_1$ satisfies (a), (b) and (d) and $0\in B_1\cap \mathrm{E}(\{V_k\})$. Therefore, we may assume that $0\in B$. For simplicity, let $S^a=\bar{S}(\{V_k\})$ for each $S(\subset E)$. Note that if $S$ is convex, then so is $S^a$ (Lemma 1.6(e)). Also $(S^a)^a=S^a$ (Lemma 1.6(d)).

Now, to prove $f(\beta)\in\varphi(\beta)B$, it is enough to show

$$(3.1)\qquad \begin{aligned} &f(\beta)\in\varphi(\beta)(V_k+B)^a \qquad \text{for each } k, \text{ if } \varphi(\beta)\neq 0;\\ &f(\beta)\in(V_k)^a \qquad\qquad\quad \text{for each } k, \text{ if } \varphi(\beta)=0.\end{aligned}$$

For, in case $\varphi(\beta)=0$, (3.1), Lemma 1.7, Lemma 1.1(a) and $(T_1^*)$ for $E$ imply that $f(\beta)=0\in\varphi(\beta)B$. In case $\varphi(\beta)\neq 0$, if $f(\beta)\notin\varphi(\beta)B$, then there would exist $k'$ such that $(\varphi(\beta)^{-1}f(\beta)+V_{k'})\cap B=\phi$ by Lemma 1.6(a), since $B$ is $\{V_k\}$-closed. Then there would exist $k$ such that $(\varphi(\beta)^{-1}f(\beta)+V_k)\cap(B+V_k)=\phi$ by Lemma 1.1(b), or $\varphi(\beta)^{-1}f(\beta)\notin(B+V_k)^a$ by Lemma 1.6(a), which contradicts (3.1).

To prove (3.1), fix $k$ and set $V=V_k$. Let $\varepsilon>0$ be arbitrary and fixed for a while. Let $D_1=\{\rho_1, \rho_2,\dots\}$ and consider the function

$$\chi(s)=\varphi(s)+\varepsilon s+\varepsilon\sum_{\rho_n<s}2^{-n}\qquad(0\leqq s\leqq\beta).$$

Then $\chi(0)=0$ and $\chi(s)>0$ if $s>0$. Put

$$A = \{t \in [0, \beta] \mid f(s) \in \chi(s)(V + B)^a \quad \text{for all} \quad s \in [0, t]\}.$$

Obviously, $0 \in A$ and $[0, t] \subset A$ if $t \in A$. We shall show that

$$\gamma = \sup A \in A \quad \text{and} \quad \gamma = \beta, \quad \text{i.e.,} \quad A = [0, \beta]. \tag{3.2}$$

If $\gamma > 0$ and $t_n \uparrow \gamma$, then $\chi(t_n) \uparrow \chi(\gamma) > 0$ by definition, and $f(t_n) \to f(\gamma)(\mathrm{R})$ since $f$ is R-continuous at $\gamma$. Hence $x_n = \chi(t_n)^{-1} f(t_n) - \chi(\gamma)^{-1} f(\gamma) \to 0(\mathrm{R})$ by Lemma 1.3 (c). Since $\mathrm{E}(\{V_k\})$ is a linear subspace, $\{x_n\} \subset \mathrm{E}(\{V_k\})$ by the first condition of (d). Thus

$$x_n \longrightarrow 0(\{V_k\}), \quad \text{i.e.,} \quad \chi(t_n)^{-1} f(t_n) \longrightarrow \chi(\gamma)^{-1} f(\gamma)(\{V_k\})$$

by (A.2). Hence $\chi(\gamma)^{-1} f(\gamma) \in ((V+B)^a)^a = (V+B)^a$, and $\gamma \in A$. If $\gamma = 0$, then $\gamma \in A$ is clear. Thus we see $\gamma = \sup A \in A$. Next we shall prove that $\gamma = \beta$.

Suppose $\gamma < \beta$ and $\gamma \notin D_1$. Set

$$r_1(h) = f(\gamma + h) - f(\gamma) - h f^{\cdot}(\gamma), \quad r_2(h) = \varphi(\gamma + h) - \varphi(\gamma) - h\varphi^{\cdot}(\gamma).$$

If $h_n \to 0$ $(0 < h_n < \beta - \gamma)$, then $h_n^{-1} r_1(h_n) \to 0(\mathrm{R})$ by Theorem 2.3. Thus $h_n^{-1} r_1(h_n) \to 0(\{V_k\})$ by (A.2), since $\{r_1(h_n)\} \subset \mathrm{E}(\{V_k\})$ by (d). Obviously, $h_n^{-1} r_2(h_n) \to 0$. Hence, we can find $h > 0$ $(h < \beta - \gamma)$ such that

$$h^{-1} r_1(h) \in \varepsilon V/2 \quad \text{and} \quad |h^{-1} r_2(h)| < \varepsilon/2.$$

Since $\gamma \in A$, i.e., $f(\gamma) \in \chi(\gamma)(V+B)^a$ and $(V+B)^a$ is convex by (e), by using the assumption $f^{\cdot}(\gamma) \in \varphi^{\cdot}(\gamma)B$, we have

$$\begin{aligned} f(\gamma + h) &= f(\gamma) + h f^{\cdot}(\gamma) + r_1(h) \\ &\in \chi(\gamma)(V + B)^a + h\varphi^{\cdot}(\gamma)B + \varepsilon h V/2 \\ &\subset (\chi(\gamma) + h\varphi^{\cdot}(\gamma) + \varepsilon h/2)(V + B)^a. \end{aligned}$$

By the definition of $\chi$ and (c),

$$\begin{aligned} 0 < \chi(\gamma) + h\varphi^{\cdot}(\gamma) + \varepsilon h/2 &= \varphi(\gamma) + \varepsilon\gamma + \varepsilon \textstyle\sum_{\rho_n < \gamma} 2^{-n} + h\varphi^{\cdot}(\gamma) + \varepsilon h/2 \\ &= \varphi(\gamma + h) + \varepsilon(\gamma + h) + \varepsilon \textstyle\sum_{\rho_n < \gamma} 2^{-n} - r_2(h) - \varepsilon h/2 \leqq \chi(\gamma + h). \end{aligned}$$

Hence $f(\gamma + h) \in \chi(\gamma + h)(V + B)^a$, i.e., $\gamma + h \in A$, which contradicts $\gamma = \sup A$.

Suppose $\gamma < \beta$ and $\gamma = \rho_m \in D_1$. By the R-continuity of $f$ at $\gamma$, the condition (d) and (A.2), we can choose $\delta > 0$ $(\delta < \min(\beta - \gamma, \gamma))$ such that

$$f(\xi) - f(\gamma) \in \varepsilon V/2^m \quad \text{for} \quad |\xi - \gamma| < \delta.$$

Let $\gamma < \xi < \gamma + \delta$. Since $0 < \chi(\gamma) + \varepsilon/2^m < \chi(\xi)$ and $f(\gamma) \in \chi(\gamma)(V+B)^a$, we have

$$f(\xi) = (f(\xi) - f(\gamma)) + f(\gamma) \in \varepsilon V/2^m + \chi(\gamma)(V + B)^a \subset \chi(\xi)(V + B)^a,$$

so that $\xi \in A$, which contradicts $\gamma = \sup A$ again. Thus we have shown (3.2).

Now, we prove (3.1). If $\varphi(\beta)=0$, then $\varphi(t)\equiv 0$ by (c), so that $\varphi^{\cdot}(t)=0$ for all $t\in(0, \beta)$. Hence $f^{\cdot}(t)=0$ for $t\in(0, \beta)\backslash D_1$ by the assumption $f^{\cdot}(t)\in\varphi^{\cdot}(t)B$, so that the above arguments are valid with $B=\{0\}$. Hence by (3.2),

$$f(\beta)\in\chi(\beta)V^a \subset \varepsilon(\beta+1)V^a.$$

Choosing $\varepsilon>0$ such that $\varepsilon(\beta+1)\leqq 1$, we obtain (3.1).

If $\varphi(\beta)>0$, then $0<\chi(\beta)\leqq\varphi(\beta)+\varepsilon(\beta+1)$, so that

$$f(\beta)\in(\varphi(\beta)+\varepsilon(\beta+1))(V+B)^a.$$

Let $\varepsilon_n>0$ and $\varepsilon_n\downarrow 0$. Then, since $f(\beta)\in \mathrm{E}(\{V_k\})$, we see that

$$(1+\varepsilon_n(\beta+1)\varphi(\beta)^{-1})^{-1}\varphi(\beta)^{-1}f(\beta)\longrightarrow\varphi(\beta)^{-1}f(\beta)(\{V_k\})$$

by Lemma 1.3(c) and (A.2). Thus $\varphi(\beta)^{-1}f(\beta)\in((V+B)^a)^a=(V+B)^a$, and we obtain (3.1). Therefore, Theorem 3.1 is proved completely.

## §4. Gâteaux differentiation

DEFINITION 4.1. Let $D$ be an R-open subset of a linear ranked space $E$, and $f: D\to F$ be a mapping into a $T_1^*$ linear ranked space $F$. Then we say that $f$ is *Gâteaux* R-*differentiable at* $x\in D$ if there exists $l\in L(E, F)$ such that

$$r: D-x\longrightarrow F,\ r(h)=f(x+h)-f(x)-l(h)\qquad(h\in D-x),$$

satisfies the following condition:

(4.1) For each $h\neq 0$, there is a f.s. $\{U_m\}$ in $F$ such that

$$\lambda_n^{-1}r(\lambda_n h)\longrightarrow 0(\{U_m\})$$

for each sequence $\{\lambda_n\}$ of positive numbers with $\lambda_n\to 0$.

As in the case of the R-differentiation, if $f$ is Gâteaux R-differentiable, then $l\in L(E, F)$ in the above definition is uniquely determined, and is denoted by $f'_g(x)$.

If $E$ and $F$ are normed linear spaces, then Gâteaux R-differentiability coincides with ordinary Gâteaux differentiability.

Obviously, R-differentiability implies Gâteaux R-differentiability and $f'(x)=f'_g(x)$. In order to state a condition under which the inverse is valid, we introduce

DEFINITION 4.2. Let $E$, $F$ and $G$ be linear ranked spaces and $D\subset E$. A mapping $T: D\to L(F, G)$ is said to be R-*hypo-continuous at* $a\in D$, if for any f.s.'s $\{V_k\}$ in $E$ and $\{U_m\}$ in $F$, there is a f.s. $\{W_l\}$ in $G$ satisfying the following condition:

(4.2) For any $\{U_m\}$-q.b.s. $\{y_j\}$ and any $l$, there is $k_0$ such that

$$x \in (D-a) \cap V_{k_0} \quad \text{implies} \quad T(a+x)(y_j) - T(a)(y_j) \in W_l \quad \text{for each} \quad j.$$

DEFINITION 4.3. We say that a linear ranked space $F$ is *convex*, if each preneighborhood of 0 in $F$ is convex.

THEOREM 4.1 (cf. [7, (1.4.4)]). *Let $E$ be a linear ranked space satisfying* (A.1) *and $F$ be a convex $T_1^*$ linear ranked space satisfying* (A.1–2). *Let $D$ be an* R-*open subset of $E$, and a mapping $f: D \to F$ be Gâteaux* R-*differentiable at every point of $D$. Suppose that for any f.s. $\{V_k\}$ in $E$, there are $k_0$ and a f.s. $\{W_m\}$ in $F$ such that $f(D \cap (a+V_{k_0})) \subset \mathrm{E}(\{W_m\})$ and that $f'_g: D \to L(E, F)$ is* R-*hypo-continuous at $a \in D$. Then $f: D \to F$ is* R-*differentiable at $a \in D$.*

PROOF. Let a f.s. $\{V_k\}$ in $E$ be given. By assumption, there are $k_0$ and a f.s. $\{W_m\}$ in $F$ such that $a+V_{k_0} \subset D$, $f(a+V_{k_0}) \subset \mathrm{E}(\{W_m\})$. Also, by the R-hypo-continuity of $f'_g$ at $a$, there is a f.s. $\{W'_m\}$ in $F$ having the following property:

(4.3) For any $\{V_k\}$-q.b.s. $\{h_j\}$ and any $m$, there is $k_m$ such that

$$x \in (D-a) \cap V_{k_m} \quad \text{implies} \quad [f'_g(a+x) - f'_g(a)](\{h_j\}) \subset W'_m.$$

Since $f'_g(a)$ is R-continuous at 0, there is a f.s. $\{W''_m\}$ in $F$ such that $f'_g(a)(V_k) \subset W''_k$ for each $k$. Choose a f.s. $\{U_m\}$ in $F$ such that $W_m + W'_m + W''_m \subset U_m$ for each $m$ by (E. 2).

Now, let $\{h_j\}$ be a $\{V_k\}$-q.b.s. and $\{\lambda_j\}$ be a sequence such that $\lambda_j > 0$, $\lambda_j \to 0$. Since $\lambda_j h_j \to 0(\{V_k\})$, there is $j_0$ such that

$$0 \leqq \lambda \leqq \lambda_j \quad \text{implies} \quad \lambda h_j \in V_{k_0} \quad \text{and} \quad h_j \in \mathrm{E}(\{V_k\}) \qquad \text{for} \quad j \geqq j_0,$$

by (A. 1) for $E$. For $j \geqq j_0$, put

$$g_j(\lambda) = f(a+\lambda h_j) - f(a) - \lambda f'_g(a)(h_j) \qquad (0 \leqq \lambda \leqq \lambda_j).$$

Then $f(a+\lambda h_j) - f(a) \in \mathrm{E}(\{W_m\})$. Also, $f'_g(a)(h_j) \in \mathrm{E}(\{W''_m\})$, since $f'_g(a)(V_k) \subset W''_k$ and $h_j \in \mathrm{E}(\{V_k\})$. Thus

$$g_j([0, \lambda_j]) \subset \mathrm{E}(\{U_m\}).$$

On the other hand, if $0 \leqq \lambda \leqq \lambda_j$ and $0 \leqq \lambda + \varepsilon \leqq \lambda_j$, then

$$g_j(\lambda+\varepsilon) - g_j(\lambda) = \varepsilon[f'_g(a+\lambda h_j) - f'_g(a)](h_j) + r(\varepsilon h_j),$$

where $r(h) = f(a+\lambda h_j + h) - f(a+\lambda h_j) - f'_g(a+\lambda h_j)(h)$. By Definition 4.1, there is a f.s. $\{U'_m\}$ in $F$ such that if $\varepsilon_n \to 0$ ($\varepsilon_n > 0$), then

$$\varepsilon_n^{-1} r(\varepsilon_n h_j) \longrightarrow 0(\{U'_m\}) \qquad (n \longrightarrow \infty).$$

Thus, we see by Corollary 2.4 that $g_j$: $[0, \lambda_j] \to F$ is R-continuous on $[0, \lambda_j]$, R-differentiable at $\lambda \in (0, \lambda_j)$ and

$$g_j^{\cdot}(\lambda) = [f_g'(a + \lambda h_j) - f_g'(a)](h_j). \tag{4.4}$$

By (A. 1) for $F$, there is $m_0$ such that $m \geqq m_0$ implies $U_m \subset \mathrm{E}(\{U_m\})$. Given $m \geqq m_0$, since $\lambda_j h_j \to 0(\{V_k\})$, there is $j_m \geqq j_0$ such that $j \geqq j_m$ implies $\lambda_j h_j \in V_{k_m}$, so that $\lambda h_j \in V_{k_m}$ for $0 \leqq \lambda \leqq \lambda_j$, where $k_m$ is the one in (4.3). Then by (4.4) and (4.3),

$$g_j^{\cdot}(\lambda) \in W_m' \subset U_m \subset \mathrm{E}(\{U_m\}) \qquad \text{for} \quad 0 \leqq \lambda \leqq \lambda_j,\ j \geqq j_m.$$

Apply Theorem 3.1 with $\varphi(\lambda) = \lambda (0 \leqq \lambda \leqq \lambda_j)$ and $B = \overline{U}_m(\{U_m\})$. Then we obtain

$$g_j(\lambda_j) - g_j(0) \in \lambda_j \overline{U}_m(\{U_m\}) \qquad (j \geqq j_m)$$

or $$\lambda_j^{-1}\{f(a + \lambda_j h_j) - f(a) - f_g'(a)(\lambda_j h_j)\} \in \overline{U}_m(\{U_m\}) \qquad (j \geqq j_m).$$

In view of Lemma 1.7 and Lemma 1.1 (a), this means that $f$ is R-differentiable at $a$.

## §5. Invertible mappings

DEFINITION 5.1. Let $E$ be a linear ranked space and $D \subset E$. $f: D \to E$ is called an R-*contraction* if for any f.s. $\{V_k\}$, there is a sequence $\{L_k\}$ of positive numbers such that $0 < L_k < 1$ and

$$a - b \in V_k \quad \text{implies} \quad f(a) - f(b) \in L_k V_k \qquad \text{for each} \quad k.$$

THEOREM 5.1 (cf. [7, (3.3.4)]). *Let $E$ be convex* R-*complete $T_1^*$ linear ranked space. If $u \in L(E, E)$ is an* R-*contraction, then $I - u$ ($I$ is the identity mapping) has the inverse $(I-u)^{-1}$: $E \to E$,*

$$(I - u)^{-1}(x) = \textstyle\sum_{n=0}^{\infty} u^n(x)(\mathrm{R}) \qquad \textit{for every} \quad x \in E,$$

*where $u^0 = I$, $u^n = u \circ \cdots \circ u$ (n-times, $n \geqq 1$), and $y = \sum_{n=0}^{\infty} u^n(x)(\mathrm{R})$ means $\sum_{n=0}^{l} u^n(x) \to y(\mathrm{R})(l \to \infty)$. If $E$ satisfies* (A. 1) *in addition, then $(I-u)^{-1} \in L(E, E)$.*

PROOF. Let $x \in E$, and choose a f.s. $\{V_k\}$ such that $x \in \mathrm{E}(\{V_k\})$. Then there is $\{\beta_k\}$ such that $\beta_k > 0$, $x \in \beta_k V_k$ for each $k$. Since $u$ is an R-contraction, there is a sequence $\{L_k\}$ such that $0 < L_k < 1$ and $u(V_k) \subset L_k V_k$ for each $k$. Then $u^n(x) \in \beta_k L_k^n V_k$ for each $n$. Put $s_l(x) = \sum_{n=0}^{l} u^n(x)$. Choose $l(1) < l(2) < \cdots$ such that $\beta_k \sum_{n=l(k)+1}^{\infty} L_k^n \leqq 1$. Then, since $V_k$ is convex, $s_{l'}(x) - s_l(x) \in V_k$ for $l' > l \geqq l(k)$. Hence $\{s_l(x)\}$ is a Cauchy sequence by $\{V_k\}$. Since $E$ is R-complete, there is $f(x) \in E$ such that $s_l(x) \to f(x)(\{V_k\})$.

Then $u(s_l(x)) \to u(f(x))(\mathrm{R})$ since $u$ is R-continuous. Also $u^n(x) \to 0(\mathrm{R})$ since $u^n(x) \in \beta_k L_k^n V_k$. Thus, in view of Proposition 1.1, the equalities

$$(I - u)s_l(x) = s_l(x - u(x)) = x - u^{l+1}(x)$$

imply $(I-u)f(x) = f(x-u(x)) = x$, so that

$$\sum_{n=0}^{\infty} u^n(x) = f(x) = (I - u)^{-1}(x) \qquad (x \in E).$$

It is easy to see that $f$ is linear. To show that $f$ is R-continuous, let $\{U_k\}$ be any f.s. in $E$. Since $u$ is an R-contraction, there is a sequence $\{\lambda_k\}$ such that $0<\lambda_k<1$ and $u(U_k) \subset \lambda_k U_k$ for each $k$. By Lemma 1.1 (a) and (A. 1), choose $1 \leqq j(1) < j(2) < \cdots$ such that

$$U_{j(k)} \subset 2^{-1}(1 - \lambda_k)U_k \quad \text{and} \quad U_{j(k)} \subset \mathrm{E}(\{U_k\}) \qquad \text{for each} \quad k.$$

Let $x \in U_{j(k)}$. Then by the above proof, $s_l(x) \to f(x)(\{U_k\})$ since $x \in \mathrm{E}(\{U_k\})$, and also $s_l(x) = \sum_{n=0}^{l} u^n(x) \in 2^{-1}(1-\lambda_k)\sum_{n=0}^{l} \lambda_k^n U_k \subset 2^{-1} U_k$. Thus, by Lemma 1.7, $f(x) \in 2^{-1}\overline{U}_k(\{U_k\}) \subset U_k$. Hence

$$f(U_{j(k)}) \subset U_k \qquad \text{for each} \quad k.$$

If we choose a f.s. $\{W_k\}$ in $E$ so that $W_n = E$ if $1 \leqq n < j(1)$ and $W_n = U_k$ if $j(k) \leqq n < j(k+1)$, then $f(U_k) \subset W_k$ for each $k$. Hence $f = (I-u)^{-1}$ is R-continuous.

Definition 5.2. Let $E$ and $F$ be linear ranked spaces and $D$ be an R-open subset of $E$. Then $f: D \to F$ is called an R-*q.b. preserving mapping at* $a \in D$, if for any f.s. $\{V_k\}$ in $E$ there exists a f.s. $\{U_m\}$ in $F$ satisfying the following condition:

(5.1) If $\{h_n\}$ is a $\{V_k\}$-q.b.s. and $\{\lambda_n\}$ is a sequence such that $\lambda_n \to 0$, $\lambda_n > 0$ and $a + \lambda_n h_n \in D$, then $\{\lambda_n^{-1}(f(a+\lambda_n h_n) - f(a))\}_n$ is a $\{U_m\}$-q.b.s..

We see easily that if $f: D \to F$ is R-differentiable at $a \in D$, then it is R-q.b. preserving at $a \in D$.

Definition 5.3. Two linear ranked spaces $E$ and $F$ are said to be R-*isomorphic* if there exists a bijective linear mapping $T: E \to F$ such that for any f.s. $\{V_k\}$ in $E$, $\{T(V_k)\}$ is a f.s. in $F$, and for any f.s. $\{U_l\}$ in $F$, $\{T^{-1}(U_l)\}$ is a f.s. in $E$. In this case, $T$ is called an R-*isomorphism of E onto F*.

In the rest of this section, *let E and F be two* R-*isomorphic* $T_1^*$ *linear ranked spaces, D be an* R-*open subset of E, and*

$$(5.2) \qquad f: D \longrightarrow F$$

*be a mapping such that f(D) is* R-*open in F*. We shall study a (local) inverse of $f$

under suitable assumptions.

THEOREM 5.2 (cf. [7, (3.2.4)]). *Suppose that $f$ of* (5.2) *is* R-*differentiable at $a\in D$ and $f'(a)$ is an* R-*isomorphism of $E$ onto $F$. If in addition $f$ is injective and $f^{-1}: f(D)\to E$ is* R-*q.b. preserving at $f(a)$, then $f^{-1}$ is* R-*differentiable at $f(a)$ and*

$$(f^{-1})'(f(a))=f'(a)^{-1}.$$

PROOF. Let $b=f(a)$, and put

$$r(h)=f(a+h)-f(a)-f'(a)(h) \qquad (h\in D-a),$$

$$R(k)=f^{-1}(b+k)-f^{-1}(b)-f'(a)^{-1}(k) \qquad (k\in f(D)-b).$$

Given a f.s. $\{U_m\}$ in $F$, since $f^{-1}$ is R-q.b. preserving at $b$, there is a f.s. $\{V_k\}$ in $E$ such that if $\{k_n\}$ is a $\{U_m\}$-q.b.s. and $\lambda_n>0$, $\lambda_n\to 0$, then $\{h_n\}$ given by

$$h_n=\lambda_n^{-1}(f^{-1}(b+\lambda_n k_n)-f^{-1}(b))$$

is a $\{V_k\}$-q.b.s.. Note that the above equality implies $a+\lambda_n h_n=f^{-1}(b+\lambda_n k_n)$, or

$$k_n=\lambda_n^{-1}(f(a+\lambda_n h_n)-f(a))=f'(a)(h_n)+\lambda_n^{-1}r(\lambda_n h_n).$$

Since $f$ is R-differentiable at $a$, there is a f.s. $\{U'_m\}$ such that

$$k_n-f'(a)(h_n)=\lambda_n^{-1}r(\lambda_n h_n)\longrightarrow 0(\{U'_m\}).$$

Let $V'_k=f'(a)^{-1}(U'_k)$. Since $f'(a)^{-1}$ is an R-isomorphism, $\{V'_k\}$ is a f.s. in $E$ and

$$f'(a)^{-1}(k_n)-h_n\longrightarrow 0(\{V'_k\}).$$

Now, $f'(a)^{-1}(k_n)-h_n=-\lambda_n^{-1}R(\lambda_n k_n)$. Hence $\lambda_n^{-1}R(\lambda_n k_n)\to 0(\{V'_k\})$. Thus $f^{-1}$ is R-differentiable at $b$ and $(f^{-1})'(b)=f'(a)^{-1}$.

THEOREM 5.3 (cf. [7, (3.4.4)]). *Let $E$ and $F$ be convex and satisfy* (A. 1–2). *Suppose that $f: D\to F$ of* (5.2) *is* R-*differentiable at every point of $D$ and injective. Let $a\in D$ and suppose in addition that $f^{-1}$ is* R-*continuous at $f(a)$, $f'(a)$ is an* R-*isomorphism of $E$ onto $F$ and $g=f'(a)^{-1}\circ f: D\to E$ satisfies the following condition* (5.3):

(5.3) *For any f.s.'s $\{V_k\}$, $\{U_k\}$ in $E$, there exist $k_0$, a f.s. $\{W_k\}$ in $E$ and a sequence $\{L_n\}$ such that $a+V_{k_0}\subset D$, $U_k\subset W_k$ for each $k$, $0<L_n<1$ and*

$$[g'(a+x)-I](W_n)\subset L_n W_n \qquad \text{for all} \quad x\in V_{k_0},\quad n=1,2,\ldots.$$

*Then $f^{-1}: f(D)\to E$ is* R-*differentiable at $f(a)$.*

REMARK 5.1. In case $E$ and $F$ are normed linear spaces, if $f: D \to F$ is R-differentiable at every point of $D$ and $f': D \to L(E, F)$ is continuous at $a \in D$, then $g$ satisfies (5.3).

PROOF OF THEOREM 5.3. By Lemma 2.3 and Theorem 2.2, $g: D \to E$ is R-differentiable at every $x \in D$ and

$$g'(x) = f'(a)^{-1} \circ f'(x),$$

in particular $g'(a) = I \in L(E, E)$. $g$ is obviously injective and $g(D) = f'(a)^{-1}(f(D))$ is R-open since $f'(a)$ is an R-isomorphism. We shall show that $g^{-1}$ is R-q.b. preserving at $b = g(a)$. Then, by the above theorem, we conclude that $g^{-1}$ is R-differentiable at $b$, and again by Lemma 2.3 and Theorem 2.2, $f^{-1} = g^{-1} \circ f'(a)^{-1}$ is R-differentiable at $f'(a)(b) = f(a)$.

Let $\{V_k\}$ be any f.s. in $E$. Since $f^{-1}$ is R-continuous at $f(a)$ and $f'(a)$ is an R-isomorphism, $g^{-1} = f^{-1} \circ f'(a)$ is R-continuous at $b$. Hence there is a f.s. $\{V'_k\}$ in $E$ such that

$$(5.4) \qquad g^{-1}((b + V_k) \cap g(D)) \subset a + V'_k \qquad \text{for each} \quad k.$$

Also, since $f$, and hence $g$, is R-continuous at $a$, there is another f.s. $\{V''_k\}$ in $E$ such that

$$(5.5) \qquad g((a + V'_k) \cap D) \subset g(a) + V''_k \qquad \text{for each} \quad k.$$

By condition (5.3) and (E. 2), there exist $k_0$, a f.s. $\{W_k\}$ in $E$ and a sequence $\{L_n\}$ such that $a + V'_{k_0} \subset D$, $V_k + V'_k + V''_k \subset W_k$ for each $k$, $0 < L_n < 1$ and

$$(5.6) \quad [g'(a + x) - g'(a)](W_n) \subset L_n W_n \qquad \text{for all} \quad x \in V'_{k_0}, \ n = 1, 2, \dots.$$

Let $\{h_n\}$ be a $\{V_k\}$-q.b.s. and $\{\lambda_n\}$ be a sequence such that $\lambda_n > 0$, $\lambda_n \to 0$ and $\lambda_n h_n \in g(D) - b$, and put

$$y_n = \lambda_n^{-1}\{g^{-1}(b + \lambda_n h_n) - g^{-1}(b)\}, \qquad n = 1, 2, \dots.$$

If we show that $\{y_n\}$ is a $\{W_k\}$-q.b.s., then we can conclude that $g^{-1}$ is R-q.b. preserving at $b$.

The above equality implies

$$(5.7) \qquad \lambda_n h_n = g(a + \lambda_n y_n) - g(a), \qquad n = 1, 2, \dots.$$

Since $\lambda_n h_n \to 0(\{V_k\})$, for each $k$ there is $n(k)$ such that $n \geqq n(k)$ implies $\lambda_n h_n \in V_k$. Thus, by (5.4), if $n \geqq n(k)$, then $g^{-1}(b + \lambda_n h_n) \in a + V'_k$, i.e., $\lambda_n y_n \in V'_k$. Therefore, if $n \geqq n(k_0)$ and $t \in [0, \lambda_n]$, then $a + t y_n \in a + V'_{k_0} \subset D$. Put

$$F_n(t) = t y_n - g(a + t y_n) + g(a), \quad t \in [0, \lambda_n], \qquad n \geqq n(k_0).$$

Each $F_n$ is R-continuous on $[0, \lambda_n]$, R-differentiable at each $t \in (0, \lambda_n)$ and

(5.8) $$F_n^{\boldsymbol{\cdot}}(t) = y_n - g'(a + ty_n)(y_n) = [g'(a) - g'(a + ty_n)](y_n).$$

By (A. 1), there is $k_1 \geqq k_0$ such that $W_{k_1} \subset \mathrm{E}(\{W_k\})$. If $n \geqq n(k_1)$, then $a + ty_n \in (a + V'_{k_1}) \cap D$, so that $g(a + ty_n) - g(a) \in V''_{k_1}$ by (5.5). Hence $F_n(t) \in V'_{k_1} + V''_{k_1} \subset W_{k_1} \subset \mathrm{E}(\{W_k\})$ for $t \in [0, \lambda_n]$, $n \geqq n(k_1)$.

Next, let $\varphi_k(y) = \inf\{\lambda > 0 \mid \lambda^{-1} y \in W_k\}$ be the Minkowski functional for $W_k$. Since $y_n \in \mathrm{E}(\{W_k\})$ for $n \geqq n(k_1)$, $\varphi_k(y_n)$ is finite for each $k$ and $n \geqq n(k_1)$. Thus $y_n \in (\varphi_k(y_n) + \varepsilon) W_k$ for any $\varepsilon > 0$, $n \geqq n(k_1)$. Hence, by (5.6) and (5.8), we see that

$$F_n^{\boldsymbol{\cdot}}(t) \in L_k(\varphi_k(y_n) + \varepsilon) W_k, \quad 0 < t < \lambda_n, \qquad n \geqq n(k_1), \quad k = 1, 2, \ldots,$$

for any $\varepsilon > 0$. In particular, $F_n^{\boldsymbol{\cdot}}(t) \in \mathrm{E}(\{W_k\})$ for $0 < t < \lambda_n$, $n \geqq n(k_1)$. Hence we can apply Theorem 3.1 and obtain

$$F_n(\lambda_n) - F_n(0) \in \lambda_n L_k(\varphi_k(y_n) + \varepsilon) \overline{W}_k(\{W_k\}), \qquad n \geqq n(k_1), \quad k = 1, 2, \ldots,$$

for any $\varepsilon > 0$. Since $F_n(\lambda_n) - F_n(0) = \lambda_n y_n - g(a + \lambda_n y_n) + g(a) = \lambda_n(y_n - h_n)$ by (5.7), this shows

(5.9) $$y_n - h_n \in (L_k \varphi_k(y_n) + \varepsilon) W_k, \qquad n \geqq n(k_1), \quad k = 1, 2, \ldots,$$

for any $\varepsilon > 0$, by Lemma 1.7.

If $\mu_n > 0$ and $\mu_n \to 0$, then $\mu_n h_n \to 0(\{W_k\})$ since $\{h_n\}$ is a $\{V_k\}$-q.b.s. and $V_k \subset W_k$ $(k = 1, 2, \ldots)$. Hence for each $k$ there is $m(k)$ such that $n \geqq m(k)$ implies $\mu_n h_n \in 2^{-1}(1 - L_k) W_k$ by Lemma 1.1 (a). Thus if $n \geqq \max(m(k), n(k_1))$, then by (5.9)

$$\begin{aligned} \mu_n y_n &\in \mu_n h_n + (\mu_n L_k \varphi_k(y_n) + \mu_n \varepsilon) W_k \\ &\subset [2^{-1}(1 - L_k) + L_k \varphi_k(\mu_n y_n) + \mu_n \varepsilon] W_k, \end{aligned}$$

which implies

$$\varphi_k(\mu_n y_n) \leqq 2^{-1} + \varepsilon \mu_n (1 - L_k)^{-1}$$

for any $\varepsilon > 0$. Hence it follows that

$$\mu_n y_n \in W_k \qquad \text{for} \quad n \geqq \max(m(k), n(k_1)),$$

which means that $\{y_n\}$ is a $\{W_k\}$-q.b.s..

DEFINITION 5.4. Let $E$ and $F$ be linear ranked spaces, $F$ be $T_1^*$ and $D$ be an R-open subset of $E$. $f: D \to F$ is called a *$C^1$-mapping at $a \in D$*, if $f$ is R-differentiable at every point of $D$ and further $f': D \to L(E, F)$ is R-hypo-continuous at $a$.

THEOREM 5.4 (cf. [7, (3.4.4)]). *Let $E$ and $F$ be convex,* R*-complete and satisfy* (A. 1–2). *Suppose $f: D \to F$ of* (5.2) *is a $C^1$-mapping at every point of $D$,*

*$f$ is injective, $f^{-1}$ is* R*-continuous on $f(D)$, $f'(a)$ is an* R*-isomorphism of $E$ onto $F$ for every $a \in D$ and $g = f'(a)^{-1} \circ f: D \to E$ satisfies condition* (5.3) *for each $a \in D$. Then $f^{-1}: f(D) \to E$ is a $C^1$-mapping at every $b \in f(D)$.*

PROOF. By the above theorem we have to prove that

$$(f^{-1})': f(D) \longrightarrow L(F, E)$$

is R-hypo-continuous at every $b \in f(D)$. Fix $a \in D$ and let $b = f(a)$. Since $g'(z) = f'(a)^{-1} \circ f'(z)$ for $z \in D$, Theorem 5.2 implies

$$(f^{-1})'(f(z)) = g'(z)^{-1} \circ f'(a)^{-1}, \qquad z \in D.$$

Since $f'(a)$ is an R-isomorphism, in order to prove that $(f^{-1})'$ is R-hypo-continuous at $b$, it is enough to show the following: Given a f.s. $\{V_k\}$ in $E$ and a f.s. $\{U_k\}$ in $F$, there is a f.s. $\{W_k\}$ in $E$ satisfying

(5.10) for any $\{V_k\}$-q.b.s. $\{h_n\}$ and for each $l$, there is $k(l)$ such that $a+x \in D$ and $f(a+x)-f(a) \in U_{k(l)}$ imply

$$[g'(a+x)^{-1} - g'(a)^{-1}](h_n) \in W_l \qquad \text{for all} \quad n.$$

Thus, let a f.s. $\{V_k\}$ in $E$ and a f.s. $\{U_k\}$ in $F$ be given. Since $f^{-1}$ is R-continuous at $f(a)$, there is a f.s. $\{V'_k\}$ in $E$ such that

(5.11) $$f^{-1}((f(a) + U_k) \cap f(D)) \subset a + V'_k \qquad \text{for each} \quad k.$$

Since $f$ is a $C^1$-mapping at $a$, $g': D \to L(E, E)$ is R-hypo-continuous at $a$. Hence there is a f.s. $\{V''_k\}$ in $E$ such that for any $\{V_k\}$-q.b.s. $\{h_n\}$ and for each $l$, there is $k'(l)$ such that

(5.12) $x \in (D-a) \cap V'_{k'(l)}$ implies $[g'(a+x) - g'(a)](h_n) \in V''_l$ for all $n$.

By (5.3), there exist $k_0$, a f.s. $\{W_k\}$ in $E$ and a sequence $\{L_n\}$ such that $a + V'_{k_0} \subset D$, $V''_k \subset W_k$, $0 < L_k < 1$ for each $k$ and

(5.13) $$[g'(a+x) - I](W_k) \subset L_k W_k \qquad \text{for all} \quad x \in V'_{k_0}, \quad k = 1, 2, \dots.$$

By (A. 1), we may assume that $W_1 \subset \mathrm{E}(\{W_k\})$. We shall show that with this $\{W_k\}$, (5.10) is satisfied.

Let $\{h_n\}$ be a $\{V_k\}$-q.b.s.. By (5.12), for each $l$ there is $k''(l) \geqq k'(l)$ such that

(5.14) $x \in (D-a) \cap V'_{k''(l)}$ implies $[g'(a+x) - I](h_n) \subset 2^{-1}(1 - L_l)V''_l$

for all $n$. We may assume that $k''(1) \leqq k''(2) \leqq \cdots$. Let $k(l) = \max(k_0, k''(l))$. If $x \in V'_{k(1)}$, then

$$[g'(a+x) - I](h_n) \in V''_1 \subset W_1 \subset \mathrm{E}(\{W_k\})$$

by (5.12). Hence, in view of (5.13), as in the proof of Theorem 5.1, we see that

$$\sum_{\nu=1}^{m}(-1)^{\nu}[g'(a+x)-I]^{\nu}(h_n)$$
$$\longrightarrow -g'(a+x)^{-1}[g'(a+x)-I](h_n)=[g'(a+x)^{-1}-I](h_n)(\{W_k\})$$

as $m\to\infty$, for each $x\in V'_{k(1)}$ and $n$. Furthermore, by (5.14) and (5.13), if $x\in V'_{k(l)}$, then

$$\sum_{\nu=1}^{m}(-1)^{\nu}[g'(a+x)-I]^{\nu}(h_n)\in 2^{-1}(1-L_l)(1-L_l)^{-1}W_l=2^{-1}W_l$$

for all $m$ and $n$, so that

$$[g'(a+x)^{-1}-I](h_n)\in 2^{-1}\overline{W}_l(\{W_k\})\subset W_l \qquad \text{for all} \quad n,$$

by Lemma 1.7. Since $a+x\in D$ and $f(a+x)-f(a)\in U_{k(l)}$ imply $x\in V'_{k(l)}$ by (5.11), we have shown that (5.10) is satisfied.

LEMMA 5.1. *Let E and F be linear ranked spaces and suppose F is convex and satisfies* (A. 2). *Let D be an* R-*open subset of E and* $f: D\to F$ *be* R-*continuous at* $a\in D$. *If* $\{U_k\}$ *is a f.s. in E and* $\{V_k\}$ *is a f.s. in F, then for each k there is* $m(k)$ *such that*

$$f(a+U_{m(k)})\cap[f(a)+\mathrm{E}(\{V_k\})]\subset f(a)+V_k.$$

PROOF. Suppose the contrary. Then there are $k_0$ and a sequence $\{x_m\}$ in $E$ such that

$$x_m\in U_m,\ a+x_m\in D,\ f(a+x_m)-f(a)\in\mathrm{E}(\{V_k\})\backslash V_{k_0} \qquad \text{for all} \quad m.$$

Since $f$ is R-continuous, $f(a+x_m)\to f(a)(\mathrm{R})$. Since $f(a+x_m)-f(a)\in\mathrm{E}(\{V_k\})$, $f(a+x_m)\to f(a)(\{V_k\})$ by (A. 2), which contradicts $f(a+x_m)-f(a)\notin V_{k_0}$.

THEOREM 5.5 (cf. [7, (3.4.5)]). *Let E and F be convex,* R-*complete and satisfy* (A. 1–2). *Suppose that* $f: D\to F$ *of* (5.2) *is* R-*differentiable at every point of D and* $f'(a): E\to F$ *is an* R-*isomorphism at a given* $a\in D$. *Suppose furthermore that* $g=f'(a)^{-1}\circ f$ *and a f.s.* $\{W_k\}$ *in E satisfy the following conditions* (5.15–16) *for some* $k_0$:

(5.15) $\qquad a+W_{k_0}\subset D \quad$ *and* $\quad g(a+W_{k_0})\subset g(a)+\mathrm{E}(\{W_k\})$.

(5.16) *For each l, there is* $L_l$: $0<L_l<1$ *such that*

$$[g'(a+x)-I](W_l)\subset L_lW_l \qquad \textit{for all} \quad x\in W_{k_0}.$$

*Then, there are a set U with* $W_{k'}\subset U\subset W_{k_0}$ *for some* $k'\geq k_0$ *and a preneighborhood V of* 0 *in F such that the restriction*

$$f_1 = f\,|(a+U)\colon a+U \longrightarrow F$$

*is an injection of* $a+U$ *onto* $f(a)+V$ *and* $f_1^{-1}\colon f(a)+V\to E$ *is* R*-continuous at* $f(a)$. *If, in addition,* $E$ *and* $F$ *satisfy* (A. 3), *then* $f_1^{-1}$ *is* R*-continuous on* $f(a)+V$.

PROOF. By (A. 1), we may assume that $W_{k_0}\subset \mathrm{E}(\{W_k\})$. We divide the proof into several steps.

(a) If $v+tu\in W_{k_0}$ for $0\leqq t\leqq 1$ and $u\in \mu_l W_l$, $\mu_l>0$, then

$$u-g(a+v+u)+g(a+v)\in L'_l\mu_l W_l,$$

where $L_l<L'_l<1$.

Proof of (a): By (5.15), we see that

$$F(t)\equiv v+tu-g(a+v+tu)+g(a)\in \mathrm{E}(\{W_k\}),\qquad 0\leqq t\leqq 1.$$

By (5.16), we have

$$F^{\cdot}(t)=u-g'(a+v+tu)(u)\in L_l\mu_l W_l \qquad \text{for}\quad 0\leqq t\leqq 1,\quad l=1,2,\dots,$$

so that $F^{\cdot}(t)\in \mathrm{E}(\{W_k\})$ for $0\leqq t\leqq 1$. Hence we can apply Theorem 3.1 and obtain

$$u-g(a+v+u)+g(a+v)=F(1)-F(0)\in L_l\mu_l\overline{W}_l(\{W_k\}).$$

Thus, in view of Lemma 1.7, we have (a).

(b) Put $L=L'_{k_0}$ and choose $W\in\{W_k\}$ such that $W\subset 2^{-1}(1-L)W_{k_0}$. Then, for any $y_0\in b+W\,(b=g(a))$, there is $x_0\in W_{k_0}$ such that $y_0=g(a+x_0)$.

Proof of (b): Given $y_0\in b+W$, put

$$T(x)=y_0+x-g(a+x),\qquad x\in W_{k_0}.$$

Define $\{u_m\}$ by $u_0=y_0-b$ and $u_m=T(u_{m-1})$, $m=1,2,\dots$. Since $u_0\in W\subset 2^{-1}(1-L)W_{k_0}\subset \mathrm{E}(\{W_k\})$, there is a sequence $\{\alpha_l\}$ of positive numbers such that $u_0\in\alpha_l W_l$ for all $l$. By induction we shall prove

$$\begin{aligned} &u_m-u_{m-1}\in (L'_l)^m\alpha_l W_l, && m=1,2,\dots;\ l=1,2,\dots,\\ (5.17)\qquad &u_m-u_{m-1}\in 2^{-1}(1-L)L^m W_{k_0}, && m=1,2,\dots,\\ &u_{m-1}+t(u_m-u_{m-1})\in W_{k_0}, && 0\leqq t\leqq 1,\ m=1,2,\dots. \end{aligned}$$

Since $u_0\in\alpha_l W_l$, $u_0\in 2^{-1}(1-L)W_{k_0}\subset W_{k_0}$ and

$$u_1-u_0=u_0-g(a+u_0)+g(a),$$

(a) implies that $u_1-u_0\in L'_l\alpha_l W_l$ and $u_1-u_0\in 2^{-1}(1-L)LW_{k_0}$. Then, $u_0+t(u_1-u_0)\in 2^{-1}(1-L)(1+L)W_{k_0}$. Thus (5.17) holds with $m=1$. Suppose (5.17) holds for $m=1,2,\dots,n$. Since

$$u_{n+1} - u_n = u_n - u_{n-1} - g(a + u_n) + g(a + u_{n-1}),$$

(5.17) for $m \leqq n$ and (a) imply the first two relations in (5.17) with $m=n+1$ and

$$u_n + t(u_{n+1} - u_n) = u_0 + \sum_{m=1}^{n}(u_m - u_{m-1}) + t(u_{n+1} - u_n)$$

$$\in 2^{-1}(1 - L)(1 + \sum_{m=1}^{n+1}L^m)W_{k_0} \subset W_{k_0}$$

for $0 \leqq t \leqq 1$. Thus we obtain (5.17).

From the first relation in (5.17), it follows that $\{u_m\}$ is a Cauchy sequence by $\{W_k\}$. Since $E$ is assumed to be R-complete, there is $x_0 \in E$ such that $u_m \to x_0(\{W_k\})$. By the second relation in (5.17), we see that $u_m \in 2^{-1}W_{k_0}$ and hence $x_0 \in W_{k_0}$. Since $g$, and hence $T$, is R-continuous, from the definition of $T$ and $\{u_m\}$, we derive that

$$x_0 = T(x_0) = y_0 + x_0 - g(a + x_0),$$

i.e., $y_0 = g(a + x_0)$.

(c) For each $l$, let $\varphi_l(y) = \inf\{\lambda > 0 \mid \lambda^{-1}y \in W_l\}$ be the Minkowski functional for $W_l$. Then for any $z_1, z_2 \in W_{k_0}$,

$$\varphi_l(z_1 - z_2 - g(a + z_1) + g(a + z_2)) \leqq L'_l\varphi_l(z_1 - z_2).$$

Proof of (c): For any $\varepsilon > 0$, since $z_1 - z_2 \in (\varphi_l(z_1 - z_2) + \varepsilon)W_l$,

$$z_1 - z_2 - g(a + z_1) + g(a + z_2) \in L'_l(\varphi_l(z_1 - z_2) + \varepsilon)W_l$$

by (a). Thus $\varphi_l(z_1 - z_2 - g(a + z_1) + g(a + z_2)) \leqq L'_l(\varphi_l(z_1 - z_2) + \varepsilon)$ for any $\varepsilon > 0$, and we obtain (c).

(d) Put $U = W_{k_0} \cap \{g^{-1}(b + W) - a\}$. Then $g_1 = g \mid (a + U)$: $a + U \to E$ is injective, $g_1(a + U) = b + W$ and there is $k' \geqq k_0$ such that $W_{k'} \subset U$.

Proof of (d): If $x_0, x_1 \in W_{k_0}$ and $g(a + x_0) = g(a + x_1)$, then by (c)

$$\varphi_l(x_0 - x_1) \leqq L'_l\varphi_l(x_0 - x_1), \qquad l = 1, 2, \dots.$$

Since $L'_l < 1$, this means that $\varphi_l(x_0 - x_1) = 0$ for all $l$, i.e., $x_0 - x_1 \in W_l$ for all $l$. Hence $x_0 = x_1$. Thus $g_1$ is injective. By (b), $g_1(a + U) = b + W$. Applying Lemma 5.1, we find $k' \geqq k_0$ such that

$$g(a + W_{k'}) \cap [b + \mathrm{E}(\{W_k\})] \subset b + W.$$

By (5.15), $g(a + W_{k'}) \subset b + W$, which implies $W_{k'} \subset U$.

(e) $g_1^{-1}$: $b + W \to a + U$ is R-continuous at $b$; if we assume (A. 3) for $E$ and $F$, then $g_1^{-1}$ is R-continuous on $b + W$.

Proof of (e): Let $y_0 \in b + W$ and $y_0 = g(a + x_0)$ with $x_0 \in U$. If $y_0 \neq b$, choose $k^* \geqq k_0$ such that $y_0 + W_{k^*} \subset b + W$ by (A. 3). If $y_0 = b$, then let $W_{k^*} = W$. First we show that

(5.18) for each $l$, there is $k(l) \geqq k^*$ satisfying $g_1^{-1}(y_0 + W_{k(l)}) \subset x_0 + a + W_l$.

Let $u \in b + W$ and $u = g_1(a + z)$ with $z \in U$. By (c)

$$\varphi_l((z - x_0) - (u - y_0)) \leqq L'_l \varphi_l(z - x_0).$$

Since $\varphi_l$ is subadditive, it follows that

$$(1 - L'_l)\varphi_l(z - x_0) \leqq \varphi_l(u - y_0),$$

so that $u \in y_0 + 2^{-1}(1 - L'_l)W_l$ implies $z \in x_0 + W_l$. Hence (5.18) is valid with $k(l) \geqq k^*$ such that $W_{k(l)} \subset 2^{-1}(1 - L'_l)W_l$.

Now, given a f.s. $\{U_k\}$ in $E$, applying Lemma 5.1 with $f = I$, we find $m(l)$ such that

$$U_{m(l)} \cap \mathrm{E}(\{W_k\}) \subset W_{k(l)}$$

for each $l$. Since $b - y_0 + W \subset W + W \subset \mathrm{E}(\{W_k\})$, (5.18) shows that

$$g_1^{-1}[(y_0 + U_{m(l)}) \cap (b + W)] \subset x_0 + a + W_l,$$

which shows that $g_1^{-1}$ is R-continuous at $y_0$.

(f) Since $f'(a)$ is an R-isomorphism, $V = f'(a)(W)$ is a preneighborhood of 0 in $F$. Thus we have the theorem by (d) and (e).

## §6. Higher derivatives

Let $E$ and $F$ be $T_1^*$ linear ranked spaces and $D$ be an R-open subset of $E$. Let $E^2 = E \times E$ be the product linear ranked space of 2-copies of $E$.

DEFINITION 6.1. A mapping $f: D \to F$ is said to be twice *R-differentiable at* $a \in D$, if $f$ is R-differentiable at every point of $D$ and if there is an R-continuous bilinear mapping $f''(a)$ of $E^2$ into $F$ such that $r^1: D - a \to L(E, F)$, given by

$$r^1(h)(x) = (f'(a + h) - f'(a))(x) - f''(a)(h, x) \quad (h \in D - a,\ x \in E),$$

satisfies the following condition:

(6.1) For any f.s.'s $\{V_k\}$ and $\{V'_k\}$ in $E$, there is a f.s. $\{W_k\}$ in $F$ such that

$$\lambda_n^{-1} r^1(\lambda_n h_n)(h'_n) \longrightarrow 0(\{W_k\})$$

for every $\{V_k\}$-q.b.s. $\{h_n\}$, every $\{V'_k\}$-q.b.s. $\{h'_n\}$ and every sequence $\{\lambda_n\}$ with $\lambda_n > 0$, $\lambda_n \to 0$.

We can prove the following as in the proof of Lemma 2.3.

LEMMA 6.1. *$f''(a)$ in the above definition is uniquely determined.*

THEOREM 6.1 (cf. [2, §9.1], [7, (1.8.2)]). *Let $F$ be a convex $T_1^*$ linear ranked space satisfying* (A.1–2) *and $D$ be an* R-*open subset of $E$. If $f: D \to F$ is twice* R-*differentiable at $a \in D$, then*

$$f''(a)(x, y) = f''(a)(y, x) \qquad \textit{for all} \quad x, y \in E.$$

PROOF. By (E.5) and (E.2) for $E$, there is a f.s. $\{V_k\}$ in $E$ such that $x, y \in \mathrm{E}(\{V_k\})$. Since $f$ is R-differentiable at $a \in D$, $f$ is R-continuous at $a \in D$ by Theorem 2.1. Thus there is a f.s. $\{U_k\}$ in $F$ such that

$$f((a + V_k) \cap D) \subset f(a) + U_k \qquad \text{for all} \quad k.$$

Choose $k_0$ such that $U_{k_0} \subset \mathrm{E}(\{U_k\})$ by (A.1). Then

$$f((a + V_n) \cap D) \subset f(a) + \mathrm{E}(\{U_k\}) \qquad \text{if} \quad n \geqq k_0. \tag{6.2}$$

Let $\{V_k^*\}$ be a f.s. in $E$ such that $V_k + V_k \subset V_k^*$ for all $k$. Since $D$ is R-open, there is $k_1 \geqq k_0$ such that $a + V_{k_1}^* \subset D$. Choose $\lambda_0 > 0$ such that $\lambda_0 x \in V_{k_1}$ and $\lambda_0 y \in V_{k_1}$. Then

$$\lambda\xi x + \lambda\xi' y \in \lambda\lambda_0^{-1} V_{k_1}^* \subset D - a \qquad \text{if} \quad \xi, \xi' \in [0, 1] \quad \text{and} \quad \lambda \in [0, \lambda_0].$$

For any $\xi \in [0, 1]$ and $\lambda \in [0, \lambda_0]$, put

$$g(\xi; \lambda) = f(a + \lambda\xi x + \lambda y) - f(a + \lambda\xi x). \tag{6.3}$$

Then, by Theorem 2.2 and Corollary 2.4, the R-derivative $g^{\cdot}(\xi; \lambda)$ of $g(\xi; \lambda)$ with respect to $\xi$ is given by

$$g^{\cdot}(\xi; \lambda) = (f'(a + \lambda\xi x + \lambda y) - f'(a + \lambda\xi x))(\lambda x) \quad (0 < \xi < 1, \quad 0 \leqq \lambda \leqq \lambda_0). \tag{6.4}$$

By the definition of the remainder $r^1$ in Definition 6.1, we see easily that

$$\begin{aligned} &(f'(a + \lambda\xi x + \lambda y) - f'(a + \lambda\xi x))(x) \\ &= f''(a)(\lambda y, x) + r^1(\lambda\xi x + \lambda y)(x) - r^1(\lambda\xi x)(x). \end{aligned} \tag{6.5}$$

Now, by (6.1) choose a f.s. $\{W_k\}$ in $F$ such that

$$\lambda_n^{-1} r^1(\lambda_n h_n)(h_n') \longrightarrow 0(\{W_k\})$$

for every $\{V_k^*\}$-q.b.s. $\{h_n\}$, every $\{V_k\}$-q.b.s. $\{h_n'\}$ and every $\{\lambda_n\}$ with $\lambda_n > 0$, $\lambda_n \to 0$. Then, for any $\{\lambda_n\}$ with $\lambda_0 > \lambda_n > 0$, $\lambda_n \to 0$ and for any $k$, there is $N(k)$ such that

$$\lambda_m^{-1} r^1(\lambda_m \xi x + \lambda_m y)(x) \in W_k \qquad \text{if} \quad m \geqq N(k) \quad \text{and} \quad \xi \in [0, 1]. \tag{6.6}$$

In fact, suppose the contrary. Then there are $k_1$, $m(1) \leqq m(2) \leqq \cdots \to \infty$ and $\{\xi_n\}$ such that $\lambda_{m(n)}^{-1} r^1(\lambda_{m(n)} \xi_n x + \lambda_{m(n)} y)(x) \notin W_{k_1}$ and $\xi_n \in [0, 1]$ for all $n$. Since

$x, y \in E(\{V_k\})$ and $\xi_n \in [0, 1]$, we see that $\{\xi_n x + y\}$ is $\{V_k^*\}$-bounded and hence $\{V_k^*\}$-q.b. by Lemma 1.10(a). Thus the above definition of $\{W_k\}$ implies that $\lambda_{m(n)}^{-1} r^1(\lambda_{m(n)}\xi_n x + \lambda_{m(n)} y)(x) \to 0(\{W_k\})$, which is a contradiction. Hence we see (6.6).

By (6.6) and (A. 1) for $F$, there is an integer $N_1$ such that

$$\text{(6.7)} \qquad \lambda_n^{-1} r^1(\lambda_n \xi x + \lambda_n y)(x) \in E(\{W_k\}) \qquad \text{if} \quad n \geqq N_1 \quad \text{and} \quad \xi \in [0, 1].$$

Similarly, there are $\{N'(k)\}$ and $N_2$ such that

$$\text{(6.8)} \qquad \lambda_m^{-1} r^1(\lambda_m \xi x)(x) \in W_k \qquad \text{if} \quad m \geqq N'(k) \quad \text{and} \quad \xi \in [0, 1];$$

$$\text{(6.9)} \qquad \lambda_n^{-1} r^1(\lambda_n \xi x)(x) \in E(\{W_k\}) \qquad \text{if} \quad n \geqq N_2 \quad \text{and} \quad \xi \in [0, 1].$$

Let $\{W'_k\}$ be a f.s. in $F$ such that $f''(a)(x, y), f''(a)(y, x) \in E(\{W'_k\})$ and $\{W_k^*\}$ be a f.s. in $F$ such that $U_k + 2W_k + W'_k \subset W_k^*$ for each $k$. Then, by (6.2–5, 7, 9) there is an integer $N_0$ such that $n \geqq N_0$ implies

$$g(\xi; \lambda_n) \in E(\{W_k^*\}) \qquad \text{for all} \quad \xi \in [0, 1]; \; g^{\boldsymbol{\cdot}}(\xi; \lambda_n) \in E(\{W_k^*\}) \qquad \text{for all} \quad \xi \in (0, 1).$$

Also by (6.4–6) and (6.8), there is $\{n'(k)\}$ such that $n \geqq n'(k)$ and $\xi \in (0, 1)$ imply

$$g^{\boldsymbol{\cdot}}(\xi; \lambda_n) \in \lambda_n^2(f''(a)(y, x) + 2W_k) \subset \lambda_n^2(f''(a)(y, x) + W_k^*).$$

Thus Theorem 3.1 and Lemma 1.7 show that

$$\text{(6.10)} \qquad g(1; \lambda_n) - g(0; \lambda_n) \in \lambda_n^2(f''(a)(y, x) + \overline{W}_k^*(\{W_k^*\})) \subset \lambda_n^2(f''(a)(y, x) + 2W_k^*) \qquad \text{for large} \quad n.$$

On the other hand, (6.3) shows that $g(1; \lambda_n) - g(0; \lambda_n)$ is symmetric with respect to $x$ and $y$. Thus by repeating the above discussion, we see that

$$\text{(6.11)} \qquad g(1; \lambda_n) - g(0; \lambda_n) \in \lambda_n^2(f''(a)(x, y) + 2W_k^*) \qquad \text{for large} \quad n.$$

(6.10–11) show that $f''(a)(y, x) - f''(a)(x, y) \in 4W_k^*$ for all $k$, and hence

$$f''(a)(y, x) = f''(a)(x, y)$$

by $(T_1^*)$ as desired.

THEOREM 6.2 (cf. [7, (1.8.3)]). *Let $E$, $F$ and $G$ be $T_1^*$linear ranked spaces and $D$, $D_1$ be* R*-open subsets of $E$, $F$, respectively. If $f: D \to F$ and $g: D_1 \to G$ with $f(D) \subset D_1$ are twice* R*-differentiable at $a \in D$ and at $b = f(a) \in D_1$, respectively, then the composed mapping $g \circ f: D \to G$ is twice* R*-differentiable at $a \in D$ and*

$$(g \circ f)''(a)(x, y) = g''(b)(f'(a)(x), f'(a)(y)) + g'(b)(f''(a)(x, y)).$$

PROOF. By Theorem 2.2, $g\circ f$ is R-differentiable and $(g\circ f)'(a+x)=g'(f(a+x))\circ f'(a+x)$ for $x\in D-a$. Put $l=f'(a)$, $L=f''(a)$, $l_1=g'(b)$, $L_1=g''(b)$ and

$$r(x)=f(a+x)-b-l(x),\quad r^1(x)(y)=(f'(a+x)-l)(y)-L(x,y),$$

$$s(z)=g(b+z)-g(b)-l_1(z),$$

$$s^1(z)(w)=(g'(b+z)-l_1)(w)-L_1(z,w),$$

$$R(x)(y)=((g\circ f)'(a+x)-l_1\circ l)(y)-\{L_1(l(x),l(y))+l_1(L(x,y))\}$$

for $x\in D-a$, $y\in E$, $z\in D_1-b$ and $w\in F$. Then for $\varepsilon>0$ with $\varepsilon x\in D-a$, we see easily that

$$\varepsilon^{-1}R(\varepsilon x)(y)=\textstyle\sum_{i=1}^{8}S_i,\quad S_i=S_i(x,y,\varepsilon), \tag{6.12}$$

where

$$S_1=L_1(\varepsilon^{-1}r(\varepsilon x),l(y)),\qquad S_2=\varepsilon^{-1}s^1(f(a+\varepsilon x)-b)(l(y)),$$

$$S_3=l_1(\varepsilon^{-1}r^1(\varepsilon x)(y)),\qquad S_4=L_1(l(\varepsilon^{1/2}x),L(\varepsilon^{1/2}x,y)),$$

$$S_5=L_1(l(\varepsilon x),\varepsilon^{-1}r^1(\varepsilon x)(y)),\qquad S_6=\varepsilon L_1(\varepsilon^{-1}r(\varepsilon x),\varepsilon^{-1}r^1(\varepsilon x)(y)),$$

$$S_7=L_1(\varepsilon^{-1}r(\varepsilon x),L(\varepsilon x,y)),$$

$$S_8=\varepsilon^{-1}s^1(f(a+\varepsilon x)-b)(L(\varepsilon x,y)+\varepsilon(\varepsilon^{-1}r^1(\varepsilon x)(y))).$$

Now, let $\{V_k\}$ and $\{V'_k\}$ be given f.s.'s in $E$. Then by using Lemma 1.15, we can show easily the following:

(6.13) For any $1\leqq i\leqq 8$, there is a f.s. $\{W_k^{(i)}\}$ in $G$ such that

$$S_i(x_n,y_n,\varepsilon_n)\longrightarrow 0(\{W_k^{(i)}\})$$

for any $\{V_k\}$-q.b.s. $\{x_n\}$, any $\{V'_k\}$-q.b.s. $\{y_n\}$ and any $\{\varepsilon_n\}$ with $\varepsilon_n>0$, $\varepsilon_n\to 0$.

Thus if we choose a f.s. $\{W_k\}$ with $\sum_{i=1}^{8}W_k^{(i)}\subset W_k$ for all $k$, then we see by (6.12–13) that $\varepsilon_n^{-1}R(\varepsilon_n x_n)(y_n)\to 0(\{W_k\})$, which shows the theorem.

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