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On a Lemma of Tate-Thompson

Ryoshi HOTTA and Kiyoshi MATSUI
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In his famous account [12], J. Tate stated that the algebraic cycles span all the (l-adic) cohomology groups of the hypersurface defined by the equation:

$$x_0^n + x_1^n + \cdots + x_r^n = 0$$

in the r-dimensional projective space $\mathbf{P}^r$ over an algebraically closed field k of characteristic p, if r is odd and $p^\nu \equiv -1 \bmod n$ for some ν. The statement can easily be reduced to the case that $n = q+1$ ($q = p^\nu$). The crucial point, which is due to Tate and Thompson, is that the middle-dimensional l-adic cohomology group $H^{r-1}(S, \mathbf{Q}_l)$ of the hypersurface S defined by the equation:

$$x_0^{q+1} + x_1^{q+1} + \cdots + x_r^{q+1} = 0 \quad \text{in} \quad \mathbf{P}^r,$$

breaks up into the sum of two irreducible $U_{r+1}(\mathbf{F}_q)$-modules, one of which is the trivial one, where $U_{r+1}(\mathbf{F}_q)$ is the finite unitary group of rank $r+1$ over the finite field $\mathbf{F}_q$ with q elements and $H^*(S, \mathbf{Q}_l)$ has the $U_{r+1}(\mathbf{F}_q)$-module structure given by the natural action of $U_{r+1}(\mathbf{F}_q)$ on S.

In this paper, we shall first, in § 1, give the identification of this non-trivial irreducible piece in $H^{r-1}(S, \mathbf{Q}_l)$ with a certain unipotent representation of $U_{r+1}(\mathbf{F}_q)$ classified by Lusztig-Srinivasan [10]. This argument also gives the proof of the above mentioned Tate-Thompson's statement. Secondly, in § 2, we shall determine the character of this irreducible representation, by a method similar to that of [9]. Since the arguments in § 2 are quite independent of those in § 1, one can immediately obtain an alternative proof of the irreducibility of the Tate-Thompson representation.

We understand that some parts of this paper, especially results in § 1, which are essentially easy exercises of Lusztig's results [8], may be known to experts. However, since Tate-Thompson's result just stated is Mecca of recent developments of the use of l-adic cohomologies in the representation theory of the finite linear groups, and since the original proof of Tate-Thompson does not seem to be highly available to many people, we consider it to be of some meaning that we write up the following account on these subjects. Of course, for various reasons from a historical point of view, one of our proofs of the irreducibility, given in § 1, seems to be different from that of Tate-Thompson.

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us in these subjects.

Added on January 19, 1978: We have been informed that Shioda and Katsura have obtained a direct geometric proof of Tate's result stated in the top of the paper without using the representation theory of the unitary groups.

Notations

$\mathbf{F}_q$ denotes the finite field with q elements and k denotes an algebraic closure of $\mathbf{F}_q$. If X is an algebraic variety defined over $\mathbf{F}_q$, F denotes the Frobenius endomorphism on X. For an endomorphism σ on X, X^σ denotes the set of fixed points of σ; thus X^F is the set of $\mathbf{F}_q$-rational points of X. $H^i(X)=H^i(X, \mathbf{Q}_l)$ (resp. $H^i_c(X)=H^i_c(X, \mathbf{Q}_l)$) denotes the i-th l-adic cohomology group of X (resp. with compact supports) with coefficients in the l-adic sheaf $\mathbf{Q}_l$ for some fixed $l\neq p=\operatorname{char}\mathbf{F}_q$. For an endomorphism σ on X, σ^* is the action on $H^*(X)$ (or $H^*_c(X)$ if σ is proper) given by that of σ. We simply write

$$\operatorname{Tr}(\sigma^*, H^*(X)) = \textstyle\sum_i(-1)^i\operatorname{Tr}(\sigma^*, H^i(X)),$$

or

$$\operatorname{Tr}(\sigma^*, H^*_c(X)) = \textstyle\sum_i(-1)^i\operatorname{Tr}(\sigma^*, H^i_c(X)).$$

For a set S, $|S|$ denotes the cardinality of S. A partition $\lambda=(\lambda_1, \lambda_2,\dots, \lambda_r)=(1^{m_1}2^{m_2}\cdots n^{m_n})$ of degree n means $n=\lambda_1+\lambda_2+\cdots+\lambda_r$ $(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r)$ and that m_i is the number of the parts λ_j equal to i. The set of all partitions of degree n will be denoted by Λ_n.

§1. Unipotent representations of the finite unitary groups and the Tate-Thompson representations

In the classification of the irreducible characters of the finite unitary groups by Lusztig-Srinivasan [10], the basic step is to complete that of the "unipotent" characters.

Let $G=U_n$ be the unitary group of rank n over $\mathbf{F}_q$ and F the Frobenius endomorphism on G. We regard G as the general linear group $GL_n(k)$ over k, an algebraic closure of $\mathbf{F}_q$, and F as the map defined by $F((x_{ij}))=(x_{ji}^q)^{-1}$; thus $G^F=U_n(\mathbf{F}_q)$. Let T_0 be the F-stable maximal torus consisting of diagonal matrices (T_0 is anisotropic!) and $W=N_G(T_0)/T_0$ the Weyl group for T_0. Then the Frobenius F acts trivially on W. The set $\mathscr{T}$ of G^F-conjugacy classes of F-stable maximal tori corresponds bijectively to the set of conjugacy classes of W, via

$$xT_0x^{-1} \longmapsto x^{-1}F(x)\in N_G(T_0) \bmod T_0.$$

Since W is isomorphic to the symmetric group S_n of degree n and since the set of conjugacy classes of S_n is parametrized naturally by Λ_n, the set of all partitions of n, one has the natural correspondence

$$\Lambda_n \ni \rho \longmapsto T(\rho) \in \mathscr{T} \qquad (T((1^n)) = T_0).$$

Let R_T^1 be the Deligne-Lusztig character for $T \in \mathscr{T}$ ([3]). An irreducible character is said to be *unipotent* if it is a constituent of R_T^1 for some $T \in \mathscr{T}$. Lusztig-Srinivasan's classification [10] of unipotent characters is as follows. Let χ^λ be the irreducible character of S_n corresponding to $\lambda \in \Lambda_n$, and χ_ρ^λ the value of χ^λ at the class corresponding to $\rho \in \Lambda_n$. (The notation is the classical one, the same as that in [6; § 2]. Hence, for example, $\chi^{(n)}$ is the trivial character, $\chi^{(n-1,1)}$ is the non-trivial constituent in the permutation representation, and $\chi^{(1^n)}$ is the sign representation.) Denote by z_ρ the order of the centralizer of the class in S_n corresponding to $\rho \in \Lambda_n$; hence $n!/z_\rho$ is the cardinality of the class of type ρ. Define the class function ψ^λ on G^F ($\lambda \in \Lambda_n$) by

$$\psi^\lambda = \sum_{\rho \in \Lambda_n} \frac{\chi_\rho^\lambda}{z_\rho} R_{T(\rho)}^1.$$

Then the results of [10] say that $\{\psi^\lambda\}_{\lambda \in \Lambda_n}$ is the set of all unipotent characters of G^F (up to sign). Note that the only non-trivial point is that ψ^λ is a generalized character. In our later discussion, we shall naturally give a proof of it in a very special case which we shall encounter.

We are now coming back to the problem stated in the introduction. Let S be the hypersurface defined by $\sum_{i=1}^n x_i^{q+1} = 0$ in the $(n-1)$-dimensional projective space $\mathbf{P}^{n-1}$ over k, for $n > 1$. The finite unitary group G^F acts on S and hence on the cohomology group $H^*(S) = H^*(S, \mathbf{Q}_l)$.

THEOREM 1. (i) *If n is even, then $H^i(S) = 0$ for odd i, and $H^i(S)$ is the trivial G^F-module for even i unless $i = n-2$. The character of the G^F-module $H^{n-2}(S)$ equals $1 - \psi^{(n-1,1)}$, where $\psi_n = -\psi^{(n-1,1)}$ is the (proper irreducible) unipotent character corresponding to the partition $(n-1, 1) \in \Lambda_n$.*

(ii) *If n is odd, then $H^i(S) = 0$ for odd $i \neq n-2$, and $H^i(S)$ is the trivial G^F-module for even i. The G^F-module $H^{n-2}(S)$ is irreducible, whose character is $\psi_n = \psi^{(n-1,1)}$.*

PROOF. Let X be the open complement of S in $\mathbf{P}^{n-1}$. Then X is an affine variety; hence by [1] or [2; Arcarta, IV, Th. (6.4)], we have the vanishing:

$$H_c^i(X) = 0 \qquad (i < n-1).$$

Thus by the long exact sequence, $H^i(\mathbf{P}^{n-1}) \xrightarrow{\sim} H^i(S)$ $(i < n-2)$. By the Poincaré duality, we also have $H^i(\mathbf{P}^{n-1}) \xrightarrow{\sim} H^i(S)$ for $n-2 < i \leq 2n-4$. Hence $H_c^i(X) = 0$

for $n-1<i<2n-2$. Thus the non-trivial parts can be read off from the exact sequences:

(i) for even n,

$$0 \longrightarrow H^{n-2}(\mathbf{P}^{n-1}) \longrightarrow H^{n-2}(S) \longrightarrow H_c^{n-1}(X) \longrightarrow 0,$$

(ii) for odd n,

$$0 \longrightarrow H^{n-2}(S) \xrightarrow{\simeq} H_c^{n-1}(X) \longrightarrow 0.$$

By the homotopy theorem [3; Prop. 6.4], the G^F-module action on $H^*(\mathbf{P}^{n-1})$ is trivial. In case (i), $H^{n-2}(\mathbf{P}^{n-1})$ is the trivial module. Thus, in order to prove the theorem, it suffices to show that the character of $H_c^{n-1}(X)$ equals $\psi_n=(-1)^{n-1}\psi^{(n-1,1)}$, which is irreducible. For this, since $H_c^{2n-2}(X) \simeq H^{2n-2}(\mathbf{P}^{n-1})$ is trivial, it suffices to show that the Euler character

$$H_c^{2n-2}(X) + (-1)^{n-1}H_c^{n-1}(X)$$

gives the character $1+\psi^{(n-1,1)}$.

Our claim is that the affine variety X is nothing but a certain variety considered by Lusztig [8] in much more general situation. Let M be the F-stable Levi subgroup of the maximal parabolic subgroup

$$P=\left\{\begin{bmatrix} * & * \cdots * \\ 0 & \\ \vdots & g \\ 0 & \end{bmatrix} \in G \,\middle|\, g \in GL_{n-1}(k)\right\},$$

that is, $M=GL_1(k)\times GL_{n-1}(k)\subset G$ (diagonally imbedded). Let

$$U_P=\left\{\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} \in G\right\}$$

be the unipotent radical of P; hence

$$FU_P=\left\{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & & 0 \\ \vdots & & \ddots & \\ * & 0 & & 1 \end{bmatrix} \in G\right\}.$$

Lusztig [8] considers the variety:

$$\tilde{Y}=\{x\in G \mid x^{-1}F(x)\in FU_P\},$$

which has the $G^F\times M^F$-action (G^F from the left, M^F from the right).

Put $Y=\tilde{Y}/M^F$. We prove that the map

$$Y \in xM^F \longmapsto xP \in G/P$$

gives an isomorphism of Y onto the open complement X of the hypersurface S in $\mathbf{P}^{n-1}$, through $G/P \simeq \mathbf{P}^{n-1}$. Note that if

$$x = \begin{bmatrix} x_1 & \\ \vdots & * \\ x_n & \end{bmatrix} \in \tilde{Y},$$

then

$$\sum_{i=1}^{n} x_i^{q+1} = 1;$$

hence Y maps into X.

Injectivity: Let $x, y \in \tilde{Y}$ such that $xP = yP$; hence $x^{-1}y \in P$. But then $x^{-1}y = (x^{-1}F(x))F(x^{-1}y)(y^{-1}F(y)) \in FP$ by the assumption. Thus $x^{-1}y \in P \cap FP = M$. In order to show $x^{-1}y \in M^F$, it suffices to see $F(x^{-1}y)y^{-1}x = 1$. In fact,

$$F(x^{-1}y)y^{-1}x = (F(x^{-1})x)(x^{-1}y)(y^{-1}F(y))(x^{-1}y)^{-1}$$

belongs to FU_P since $F(x^{-1})x \in FU_P$, $x^{-1}y \in M$, $y^{-1}F(y) \in FU_P$ and M normalizes FU_P. Since $M \cap FU_P = \{1\}$, our assertion has been verified.

Surjectivity: Let $(x_1 : \cdots : x_n) \in X$ in $\mathbf{P}^{n-1}$. Then we may assume

$$\sum_{i=1}^{n} x_i^{q+1} = 1.$$

We have to show that there exists a matrix

$$x = \begin{bmatrix} x_1 & \\ \vdots & * \\ x_n & \end{bmatrix} \in \tilde{Y};$$

that is, ${}^t x^{(q)} x \in FU_P$ (${}^t(x_{ij})^{(q)} = (x_{ji}^q)$). By assumption, it is clear that there exists

$$x = \begin{bmatrix} x_1 & \\ \vdots & * \\ x_n & \end{bmatrix} \in G$$

such that

$${}^t x^{(q)} x = \begin{bmatrix} 1 & 0 \cdots 0 \\ * & \\ \vdots & g \\ * & \end{bmatrix}, \quad \text{where} \quad g \in GL_{n-1}(k).$$

By Lang's theorem, there exists $z \in GL_{n-1}(k)$ such that ${}^t z^{(q)} z = g$. Then

$$x\begin{bmatrix} 1 & 0\cdots 0 \\ 0 & \\ \vdots & z^{-1} \\ 0 & \end{bmatrix} = \begin{bmatrix} x_1 & \\ \vdots & * \\ x_n & \end{bmatrix}$$

and this belongs to $\tilde{Y}$.

Thus we have $Y \cong X$. Hence it suffices to show that the Euler character

$$\Phi = \sum_i (-1)^i H_c^i(Y) = H_c^{2n-2}(Y) + (-1)^{n-1} H_c^{n-1}(Y)$$

gives the character $1+\psi^{(n-1,1)}$. In the notation of [8],

$$\Phi = R_{M\subset P}^G(1).$$

As the characters of $M^F = U_1(\mathbf{F}_q) \times U_{n-1}(\mathbf{F}_q)$, we have the identity

$$\sum_{\rho' \in \Lambda_{n-1}} \frac{1}{z_{\rho'}} R^1_{T(1)\times T(\rho'),M} = 1$$

([3; Cor. 7.14]). Substituting this into Φ, we have

$$\Phi = \sum_{\rho' \in \Lambda_{n-1}} \frac{1}{z_{\rho'}} R_{M\subset P}^G(R^1_{T(1)\times T(\rho'),M}).$$

By [8; 5 Cor.], $R_{M\subset P}^G(R^1_{T(1)\times T(\rho'),M}) = R^1_{T(\rho' 1)}((\rho' 1) \in \Lambda_n)$. Let $\nu = \mathrm{Ind}_{S_{n-1}}^{S_n} 1$ be the character of the permutation representation of S_n ($\nu = 1+\chi^{(n-1,1)}$). Then

$$\begin{aligned}
\Phi &= \sum_{\rho' \in \Lambda_{n-1}} \frac{1}{z_{\rho'}} R^1_{T(\rho' 1)} \\
&= \sum_{\rho \in \Lambda_n} \frac{\nu_\rho}{z_\rho} R^1_{T(\rho)} \\
&= \sum_{\rho \in \Lambda_n} \frac{1}{z_\rho} R^1_{T(\rho)} + \sum_{\rho \in \Lambda_n} \frac{\chi_\rho^{(n-1,1)}}{z_\rho} R^1_{T(\rho)} \\
&= 1 + \psi^{(n-1,1)},
\end{aligned}$$

again by [3; Cor. 7.14] (ν_ρ is the value of ν at the class corresponding to $\rho \in \Lambda_n$). The irreducibility of the character $\pm\psi^{(n-1,1)}$ is an easy consequence of [3; Th. 6.8].

REMARK. In [12], it was guessed that this Tate-Thompson representation ψ_n seems to attain the minimal degree among all irreducible representations with degrees > 1 if n is even. It is, however, known that there exist ones with degrees equal to $\psi_n(1)-1$ ([7]).

§2. The character formula

In this section, we compute the value of the character $\psi_n=(-1)^{n-1}\psi^{(n-1,1)}$ at each conjugacy class. The result is very simple and fits the conjecture of Ennola [4]. By Theorem 1, it suffices to compute the value

$$\mathrm{Tr}(g^*, H^*(S)) = \sum_i(-1)^i \mathrm{Tr}(g^*, H^*(S))$$

for $g \in G^F$. By [3; Th. 3.2], if $g=su$ is the Jordan decomposition, then

$$\mathrm{Tr}(g^*, H^*(S)) = \mathrm{Tr}(u^*, H^*(S^s)). \tag{2.1}$$

From the proof of [3; Prop. 3.3] (cf. (4.1.2) loc. cit.), it follows that

$$\mathrm{Tr}(h^*, H_c^*(Z)) = -\{\textstyle\sum_{m\geq 1} |Z^{F^m h}| t^m\}_{t=\infty} \tag{2.2}$$

for any variety Z defined over $\mathbf{F}_q$ with an automorphism h of finite order ($hF=Fh$). (The right-hand side is the value at $t=\infty$ of the rational function of t expanded as above.) First we compute $|S^{F^{2r}u}|$ for a unipotent $u \in G^F$, $r \geq 1$, and secondly reduce the general case to the first one.

We prepare some more notations in order to simplify our descriptions. Let V be an n-dimensional vector space over k with $\mathbf{F}_{q^2}$-structure. F^2 denotes the Frobenius with respect to this $\mathbf{F}_{q^2}$-structure. Let

$$(\,|\,)\colon V\times V \longrightarrow k$$

be a non-degenerate sesqui-linear form which gives a unitary metric on V^{F^2} over $\mathbf{F}_{q^2}$;

$$(\lambda x|y) = \lambda(x|y),\ (x|\lambda y) = \lambda^q(x|y),\ (x|y)^q = (F^2y|x) \quad (x, y \in V, \lambda \in k).$$

For some $\mathbf{F}_{q^2}$-basis, $(x|y)=\sum_{i=1}^n x_i y_i^q$. Thus our hypersurface is

$$S = S(V) = \{<x> \in \mathbf{P}(V)\,|\,(x|x) = 0\},$$

where $<x>$ denotes the line generated by x, which is identified with a point in the projective space $\mathbf{P}(V)$; our group is

$$G^F = \{g \in GL(V)\,|\,(gx|gy) = (x|y) \quad \text{for all} \quad x, y \in V\}.$$

Let $g \in G^F$ and assume that there exists $v \in V^{F^2}$ such that $v \neq 0$, $(v|v)=0$ and $gv=v$ (in fact, the assumption will be seen to be satisfied for g unipotent). We fix such v and g, once and for all. We make the partition of S so that $S=S_0 \amalg S_1$;

$$S_0 = \{<x> \in S\,|\,(x|v) = 0\},$$

$$S_1 = \{<x> \in S \mid (x|v) \neq 0\}.$$

Note that this partition is stable under the (F^2, g)-action. By assumption, if we put $<v>^{\perp} = \{x \in V \mid (x|v) = 0\}$, then $<v>^{\perp}/<v>$ carries the natural sesqui-linear form induced by $(\,|\,)$ and g acts unitarily on it. Here we have a new hypersurface $S(<v>^{\perp}/<v>)$ of the same kind of dimension $n-4$.

LEMMA 1. *For an integer $r \geq 1$, we have*

$$|S_0^{F^{2r}g}| = q^{2r}|S(<v>^{\perp}/<v>)^{F^{2r}g}| + 1.$$

PROOF. Define the map

$$\phi\colon S_0 - \{<v>\} \ni <x> \longmapsto <x> \bmod <v> \in \mathbf{P}(<v>^{\perp}/<v>)$$

which commutes with the (F^2, g)-action. Then it is easily seen that ϕ is surjective onto $S(<v>^{\perp}/<v>)$ and the fibers are the affine lines. The $F^{2r}g$-action on $\mathbf{P}(V)$ turns out to be a Frobenius action with respect to some $\mathbf{F}_{q^{2r}}$-structure (by Lang's theorem). Hence $|\phi^{-1}(z)^{F^{2r}g}| = q^{2r}$ for $z \in S(<v>^{\perp}/<v>)^{F^{2r}g}$, which implies the lemma.

For S_1, letting $\pi\colon V \to V/<v>$ be the natural projection, we consider the map

$$\psi\colon S_1 \ni <x> \longmapsto \pi(x/(x|v)) \in V/<v>,$$

which is clearly well-defined. Denote by $\tau_{\mu,v} (\mu^q + \mu = 0)$ the linear transformation defined by $\tau_{\mu,v}(x) = x + \mu(x|v)v$ $(x \in V)$. (If $\tau_{\mu,v} \neq 1$, such a $\tau_{\mu,v}$ is said to be a unitary transvection.) Then $\tau_{\mu,v} \in G^F$ and $\psi(<\tau_{\mu,v}(x)>) = \psi(<x>)$. Thus if we consider the abelian group

$$T_v = \{\tau_{\mu,v} \mid \mu^q + \mu = 0\},$$

then ψ factors through

$$\psi\colon S_1 \longrightarrow S_1/T_v \longrightarrow V/<v>.$$

LEMMA 2. *The map $S_1/T_v \to V/<v>$ is injective and the image is*

$$\psi(S_1) = \{\dot{y} \in V/<v> \mid (y|v) = 1\}.$$

(For $y \in V$, $\dot{y} \in V/<v>$ is an element represented by y.)

PROOF. We first see that $S_1/T_v \to V/<v>$ is injective. Let $\pi(x_1/(x_1|v)) = \pi(x_2/(x_2|v))$ for $<x_1>, <x_2> \in S_1$. Then

$$x_1/(x_1|v) = x_2/(x_2|v) + \lambda v$$

for some $\lambda \in k$. But then $(x_1|x_1) = 0$ and $(x_2|x_2) = 0$ imply $\lambda^q + \lambda = 0$. Thus

$$x_1 = \frac{(x_1|v)}{(x_2|v)} \tau_{\lambda,v}(x_2),$$

which implies the injectivity. Secondly, let $y \in V$ such that $(y|v)=1$. Let $\xi \in k$ be a solution of $\xi^q+\xi+(y|y)=0$. Put $x=y+\xi v$. Then $(x|v)=(y|v)=1$, and

$$\begin{aligned}(x|x) &= (y+\xi v|y+\xi v)\\ &= (y|y)+\xi(v|y)+\xi^q(y|v)\\ &= (y|y)+\xi(y|v)^q+\xi^q(y|v)\\ &= 0.\end{aligned}$$

Thus $<x> \in S_1$. It is clear that $\psi(S_1) \subset \{\dot{y} \in V/<v> |(y|v)=1\}$. Hence the lemma.

LEMMA 3. $|\psi(S_1)^{F^{2r}g}| = q^{2r(n-2)}$.

PROOF. By Lang's theorem, the $F^{2r}g$-action on $V/<v>$ turns out to be a Frobenius action with respect to some $\mathbf{F}_{q^{2r}}$-structure. That is, choose $\gamma \in GL(V)$ such that $g=\gamma^{-1}F^{2r}(\gamma)$. Then we have an isomorphism

$$\gamma\colon V/<v> \xrightarrow{\sim} V/<\gamma v>,$$

where $V/<v>$ is an affine space with $F^{2r}g$-action and $V/<\gamma v>$ with the Frobenius F^{2r}-action. Here $\psi(S_1)$ is an $F^{2r}g$-stable affine space of dimension $n-2$, by Lemma 2; hence

$$|\psi(S_1)^{F^{2r}g}| = |\mathbf{A}^{n-2}(\mathbf{F}_{q^{2r}})| = q^{2r(n-2)}.$$

LEMMA 4.

$$\frac{1}{q}\sum_{\mu^q+\mu=0}|S^{F^{2r}g\tau_{\mu,v}}| = q^{2r}|S(<v>^{\perp}/<v>)^{F^{2r}g}| + q^{2r(n-2)} + 1.$$

PROOF. In the partition $S=S_0 \amalg S_1$, T_v acts trivially on S_0. Thus by Lemma 1,

$$|S_0^{F^{2r}g\tau_{\mu,v}}| = |S_0^{F^{2r}g}| = q^{2r}|S(<v>^{\perp}/<v>)^{F^{2r}g}| + 1.$$

In the fibering $S_1 \to S_1/T_v \simeq \psi(S_1)$,

$$|\psi(S_1)^{F^{2r}g}| = \frac{1}{|T_v|}\sum_{\mu^q+\mu=0}|S_1^{F^{2r}g\tau_{\mu,v}}|.$$

Thus Lemma 3 leads to the lemma.

We are now ready to prove:

THEOREM 2. *Let $u \in G^F$ be unipotent. Then*

$$|S^{F^{2r}u}| = \sum_{i=0}^{n-2} q^{2ri} + \frac{q}{q+1}(1-(-q)^{\dim\mathrm{Ker}(u-1)-1})(-q)^{r(n-2)}.$$

PROOF. According to Lusztig [9], we say a unipotent $u \in G^F$ to be *non-exceptional* if there exists $w \in V^{F^{2r}}$ such that $(u-1)w \neq 0$, $(u-1)^2 w = 0$ and $(w|(u-1)w) = 0$; otherwise, *exceptional*. We divide the proof into each case.

(i) Assume u is exceptional. Then by [9; 25. Prop.], u is either 1 or a transvection $\tau_{\mu,v}$ for some $v \in V^F$, $v \neq 0$, $(v|v) = 0$, $\mu^q + \mu = 0$, $\mu \neq 0$. If $u = 1$, the formula follows from [9; 30. Prop.]. If $u = \tau_{\mu,v} = \tau$ ($\mu \neq 0$), then it follows from Lemma 4 that

$$\frac{1}{q}|S^{F^{2r}}| + \frac{q-1}{q}|S^{F^{2r}\tau}|$$

$$= q^{2r}|S(<v>^{\perp}/<v>)^{F^{2r}}| + q^{2r(n-2)} + 1$$

since $\tau_{\mu_1,v}$ is conjugate to $\tau_{\mu_2,v}$ for $\mu_1, \mu_2 \neq 0$. The desired formula immediately follows from that for $u = 1$. (Note that $\dim\mathrm{Ker}(\tau-1) = n-1$.)

(ii) Assume u is non-exceptional. Then by [9; 20. Lem., 22. Lem.], there exists $v \in V^{F^2}$ such that $v \neq 0$, $uv = v$, $(v|v) = 0$, $\dim\mathrm{Ker}(u-1, V) = \dim\mathrm{Ker}(u-1, <v>^{\perp}/<v>)$ and that every $u\tau_{\mu,v}$ ($\mu^q + \mu = 0$) is conjugate to each other. Thus, by Lemma 4, we have

$$|S^{F^{2r}u}| = \frac{1}{q}\sum_{\mu^q+\mu=0}|S^{F^{2r}u\tau_{\mu,v}}|$$

$$= q^{2r}|S(<v>^{\perp}/<v>)^{F^{2r}u}| + q^{2r(n-2)} + 1.$$

Applying the induction on $n = \dim V$, we may assume that

$$|S(<v>^{\perp}/<v>)^{F^{2r}u}| = \sum_{i=0}^{n-4} q^{2ri} + \frac{q}{q+1}(1-(-q)^{\dim\mathrm{Ker}(u-1)-1})(-q)^{r(n-4)}$$

since $\dim\mathrm{Ker}(u-1, V) = \dim\mathrm{Ker}(u-1, <v>^{\perp}/<v>)$. Then the theorem holds.

COROLLARY 1. *For a unipotent $u \in G^F$,*

$$\mathrm{Tr}(u^*, H^*(S)) = n - 1 + \frac{q}{q+1}(1-(-q)^{\dim\mathrm{Ker}(u-1)-1}).$$

PROOF. Immediate from (2.2) and Theorem 2.

We are now going into the character formula for arbitrary $g \in G^F$. For an eigenvalue $\alpha \in k^\times$ of g, let $V_\alpha \subset V$ be the eigenspace of α, and let

$$V = \bigoplus_\alpha V_\alpha$$

be the eigenspace decomposition. If $g=su$ is the Jordan decomposition, then the fixed point subvariety of s in $\mathbf{P}(V)$ is

$$\mathbf{P}(V)^s = \coprod_{\alpha} \mathbf{P}(V_\alpha).$$

LEMMA 5.

$$S^s = \coprod_{\alpha^{q+1}\neq 1} \mathbf{P}(V_\alpha) \amalg \coprod_{\alpha^{q+1}=1} S(V_\alpha).$$

PROOF. If $\alpha^{q+1}=1$, then $(\,|\,)$ defines a non-degenerate sesqui-linear form on V_α, and clearly $\mathbf{P}(V_\alpha)\cap S=S(V_\alpha)$. If $\alpha^{q+1}\neq 1$, then V_α is isotropic for $(\,|\,)$. In fact, if $x\in V_\alpha$, then $(x|x)=(sx|sx)=\alpha^{q+1}(x|x)$. But then since $\alpha^{q+1}\neq 1$, $(x|x)=0$. Thus $\mathbf{P}(V_\alpha)\subset S$ for $\alpha^{q+1}\neq 1$. Thus the lemma.

Let $\psi_n=(-1)^{n-1}\psi^{(n-1,1)}$ be the irreducible character of the G^F-module $H^{n-2}(S)$ for n odd, or of the non-trivial piece of $H^{n-2}(S)$ for n even. Here we have the character formula for ψ_n.

THEOREM 3. *Let $g\in G^F$. Then*

$$\psi_n(g) = (-1)^n\left\{1 - \sum_{\alpha^{q+1}=1} \frac{1-(-q)^{\dim\operatorname{Ker}(g-\alpha 1)}}{1+q}\right\},$$

where the summation runs over the eigenvalues α of g such that $\alpha^{q+1}=1$.

PROOF. Let $g=su$ be the Jordan decomposition. Then by (2.1),

$$\operatorname{Tr}(g^*, H^*(S)) = \operatorname{Tr}(u^*, H^*(S^s)).$$

By definition, the left-hand side equals

$$n-1+(-1)^{n-2}\psi_n(g).$$

But then by Lemma 5, the right-hand side equals

$$\sum_{\alpha^{q+1}\neq 1} \dim V_\alpha + \sum_{\alpha^{q+1}=1} (\operatorname{Tr}(u_\alpha^*, H^*(S(V_\alpha)))),$$

where $u_\alpha=u|V_\alpha$. Hence

$$(-1)^n\psi_n(g) = 1 + \sum_{\alpha^{q+1}=1} (\operatorname{Tr}(u_\alpha^*, H^*(S(V_\alpha))) - \dim V_\alpha).$$

From Corollary 1, it follows that

$$\operatorname{Tr}(u_\alpha^*, H^*(S(V_\alpha))) - \dim V_\alpha$$

$$= -1 + \frac{q}{q+1}(1-(-q)^{\dim\operatorname{Ker}(u_\alpha-1)-1})$$

$$= \frac{1}{q+1}((-q)^{\dim \mathrm{Ker}(g-\alpha 1)} - 1).$$

Thus the theorem.

REMARK. The formula fits the conjecture of Ennola [4], which has not yet been proved in general but for sufficiently large p ([6]).

We now illustrate Ennola's principle by proving the irreducibility of the Tate-Thompson representation ψ_n directly from Theorem 3. Note that we have not used the irreducibility result of § 1 for the proof of Theorem 3.

We are first reminded of the character of $GL_n(\mathbf{F}_q)$ corresponding to ψ_n. Consider the representation of $GL_n(\mathbf{F}_q)$ given by the action on $\mathbf{P}^{n-1}(\mathbf{F}_q)$, the $\mathbf{F}_q$-rational points of the projective space $\mathbf{P}^{n-1}$. It is then well-known that this representation breaks up into two irreducible constituents, one of which is trivial ([5], [11]). Thus if we put

$$\phi_n(g) = |\mathbf{P}^{n-1}(\mathbf{F}_q)^g| - 1 \qquad (g \in GL_n(\mathbf{F}_q)),$$

then ϕ_n is an irreducible character of $GL_n(\mathbf{F}_q)$. We easily have

$$\phi_n(g) = \sum_{\alpha \in \mathbf{F}_q^\times} \frac{1 - q^{\dim \mathrm{Ker}(g-\alpha 1)}}{1-q} - 1. \tag{2.3}$$

We recall the parametrizations of the conjugacy classes of $GL_n(\mathbf{F}_q)$ and $U_n(\mathbf{F}_q)$. Consider the action $\alpha \mapsto \alpha^q$ (resp. $\alpha \mapsto \alpha^{-q}$) in $k^\times$ and let O^0 (resp. O^1) be the set of all orbits under this action. For $a \in O^0$ or $a \in O^1$, put $d(a) = |a|$. Set $\Lambda = \bigcup_{i \geq 0} \Lambda_i$ ($\Lambda_0 = \phi$) where Λ_i is the set of partitions of i, and set $|\lambda| = i$ if $\lambda \in \Lambda_i$. Let

$$C_n^0 = \{f : O^0 \longrightarrow \Lambda \mid \textstyle\sum_{a \in O^0} |f(a)|\, d(a) = n\},$$

$$C_n^1 = \{f : O^1 \longrightarrow \Lambda \mid \textstyle\sum_{a \in O^1} |f(a)|\, d(a) = n\}.$$

Then there is the well-known bijection between C_n^0 and the set of the conjugacy classes of $GL_n(\mathbf{F}_q)$ (resp. C_n^1 and the set of the ones of $U_n(\mathbf{F}_q)$).

Consider the set

$$\Omega_n = \{\delta : \Lambda \longrightarrow \Lambda \mid \textstyle\sum_{\lambda \in \Lambda} |\lambda|\, |\delta(\lambda)| = n\}.$$

Then there is the surjection

$$C_n^i \longrightarrow \Omega_n \qquad (i = 0, 1)$$

such that $f \in C_n^i$ corresponds to $\delta \in \Omega_n$ by $\delta(\lambda) = (d(a))_{f(a)=\lambda}$. Thus we have the surjection

$$\text{(2.4)}\qquad \begin{aligned}\gamma_0: GL_n(\mathbf{F}_q) &\longrightarrow \Omega_n,\\ \gamma_1: U_n(\mathbf{F}_q) &\longrightarrow \Omega_n.\end{aligned}$$

We define the polynomial $\phi_n^\delta(t)$ in t for $\delta \in \Omega_n$ by

$$\phi_n^\delta(t) = \sum_{\lambda\in\Lambda}\delta(\lambda)_1 \frac{1 - t^{[\lambda]}}{1 - t} - 1,$$

where $[\lambda]$ is the number of the parts of $\lambda \in \Lambda$ and $\delta(\lambda)_1$ is the number of the parts 1, i.e., $\delta(\lambda)=(1^{\delta(\lambda)_1}2^{\delta(\lambda)_2}\cdots)$. Then by (2.3), we have the formula

$$\text{(2.5)}\qquad \phi_n(g) = \phi_n^{\gamma_0(g)}(q) \qquad (g \in GL_n(\mathbf{F}_q)),$$

where $\gamma_0(g) \in \Omega_n$ is as in (2.4). On the other hand, for ψ_n, by Theorem 3, we also have

$$\text{(2.6)}\qquad \psi_n(g) = (-1)^{n-1}\phi_n^{\gamma_1(g)}(-q) \qquad (g \in U_n(\mathbf{F}_q)).$$

We want to show

$$<\psi_n, \psi_n>_{U_n} = \frac{1}{|U_n(\mathbf{F}_q)|}\sum_{g\in U_n(\mathbf{F}_q)}\psi_n(g)^2 = 1,$$

or

$$\sum_{g\in U_n(\mathbf{F}_q)}\phi_n^{\gamma_1(g)}(-q)^2 = |U_n(\mathbf{F}_q)|$$

by (2.6). For this it suffices to show

$$\text{(2.7)}\qquad \sum_{\delta\in\Omega_n}|\gamma_1^{-1}(\delta)|\,\phi_n^\delta(-q)^2 = |U_n(\mathbf{F}_q)|.$$

But then since ϕ_n is irreducible, we have

$$\text{(2.8)}\qquad \sum_{\delta\in\Omega_n}|\gamma_0^{-1}(\delta)|\,\phi_n^\delta(q)^2 = |GL_n(\mathbf{F}_q)|.$$

Considering (2.8) as the identity of polynomials in q, we have the identity (2.7) in changing q to $-q$ thanks to the well-known structure theory of the unitary groups [4].

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Department of Mathematics,
Faculty of Science,
Hiroshima University