HIROSHIMA MATH. J.
8 (1978), 499–504

On the Decomposition of Coalgebras

Takefumi SHUDO and Haruo MIYAMOTO

(Received May 17, 1978)

Introduction

It is well known that a cocommutative coalgebra is decomposed into the direct sum of its irreducible components ([3], Theorem 8. 0. 5). This theorem is based on the fact that a commutative artinian ring is uniquely decomposed into a direct product of local rings ([4], p.205). As for the non-commutative case, we know that an artinian ring is the direct sum of its blocks which are uniquely determined and indecomposable as two-sided ideals. A block is the sum of principal indecomposable modules which are linked to each other (e.g., [1], p.171).

The purpose of this paper is to give a criterion for a coalgebra to be indecomposable. Here, a coalgebra is indecomposable provided it cannot be decomposed into a direct sum of two non-zero subcoalgebras.

Since the Brauer's theory stated above is constructed by using one-sided ideals, it seems not to be useful for a theory of coalgebras. Thus we consider the decomposition of rings from a slightly different viewpoint.

Let C be a coalgebra over a field. Then we shall prove the following result: A necessary and sufficient condition for C to be indecomposable is that for any simple subcoalgebras S and S' there exists a sequence

$$S = S_1, S_2,\dots, S_r = S'$$

of simple subcoalgebras such that

$$S_i \wedge S_{i+1} \neq S_{i+1} \wedge S_i \qquad \text{for} \quad i = 1,\dots, r-1.$$

The proof is divided into two parts. First, in Section 2, we reduce this problem to the finite-dimensional case, where coalgebras can be completely turned into algebras. Secondly, in Section 3, we prove the paraphrased assertion.

All notations and terminology we will use are the same as in [3].

1. Notations, Definitions and Main Theorem

Throughout this paper all coalgebras and algebras are over a fixed field k. Let C be a coalgebra. We denote by $\mathfrak{S} = \mathfrak{S}(C)$ the set of all simple subcoalgebras of C.

(1.1) Let $\mathfrak{S}=\bigcup_{\lambda}\mathfrak{S}_\lambda$ be a partition of $\mathfrak{S}$, i.e., $\mathfrak{S}$ is a disjoint union of subsets $\mathfrak{S}_\lambda$, and R_λ the sum of all simple subcoalgebras belonging to $\mathfrak{S}_\lambda$. Then the sum

$$\sum_{\lambda} \wedge^{\infty} R_\lambda$$

is direct, where $\wedge^{\infty} R_\lambda = \bigcup_{n} \wedge^{n} R_\lambda$.

In fact, assume that $(\wedge^{\infty} R_\lambda) \cap (\sum_{\mu \neq \lambda} \wedge^{\infty} R_\mu) \neq 0$. Since this is a subcoalgebra of C, it contains a simple subcoalgebra, say S. Then

$$S \in \mathfrak{S}(\wedge^{\infty} R_\lambda) \cap \mathfrak{S}(\sum_{\mu \neq \lambda} \wedge^{\infty} R_\mu) = \mathfrak{S}_\lambda \cap (\bigcup_{\mu \neq \lambda} \mathfrak{S}_\mu),$$

which contradicts the assumption that the union of $\mathfrak{S}_\lambda$'s is disjoint.

(1.2) A coalgebra is said to be *indecomposable* if it cannot be decomposed into a direct sum of two non-zero subcoalgebras.

(1.3) Let S, S' be simple subcoalgebras of C. We denote $S \sim S'$ if either $S=S'$ or there is a finite sequence

$$S=S_1, S_2, \dots, S_r = S'$$

of simple subcoalgebras of C such that $S_i \wedge S_{i+1} \neq S_{i+1} \wedge S_i$ for $i=1,\dots, r-1$. Then, as easily seen, the relation $\sim$ is an equivalence relation in $\mathfrak{S}(C)$.

Let $\mathfrak{S}_\lambda$ $(\lambda \in \Lambda)$ be the equivalence classes and put $R_\lambda = \sum_{S \in \mathfrak{S}_\lambda} S$.

The main theorem of this paper is

THEOREM. $C = \sum_{\lambda \in \Lambda} \oplus \wedge^{\infty} R_\lambda$. *Moreover each summand* $\wedge^{\infty} R_\lambda$ *is indecomposable.*

2. Reduction to the Finite-Dimensional Case

(2.1) *Let D be a subcoalgebra of C and let S, S' be simple subcoalgebras of D (a priori of C). If $S \sim S'$ in $\mathfrak{S}(D)$, then $S \sim S'$ in $\mathfrak{S}(C)$.*

PROOF. Let S_1, S_2 be in $\mathfrak{S}(D)$. By [2] (2.3.1) we have

$$S_1 \wedge_D S_2 = (S_1 \wedge_C S_2) \cap D.$$

Therefore, for S_1, S_2 in $\mathfrak{S}(D)$ if $S_1 \wedge_C S_2 = S_2 \wedge_C S_1$ then $S_1 \wedge_D S_2 = S_2 \wedge_D S_1$. Thus $S \sim S'$ in $\mathfrak{S}(D)$ implies that $S \sim S'$ in $\mathfrak{S}(C)$.

(2.2) *We can assume that the coalgebra C in* THEOREM *is of finite dimension.*

In fact, assume that THEOREM holds for all finite-dimensional coalgebras. Let C be *any* coalgebra and let c be any element of C. Then the subcoalgebra

D generated by c is of finite dimension. Thus by the assumption D has the decomposition as in THEOREM:

$$D = \sum \oplus D_i.$$

From (2.1) and the definition of D_i it is claer that

$$D_i \subset \wedge^{\infty} R_\lambda$$

for some λ. It follows that

$$c \in D \subset \sum_{\lambda} \oplus \wedge^{\infty} R_\lambda.$$

(2.3) Let C be a finite-dimensional coalgebra and $A = C^*$ the dual algebra of C. The correspondence $W \mapsto W^{\perp}$ gives an inclusion reversing bijection between the set of all linear subspaces of C and that of A. A subcoalgebra corresponds to a two-sided ideal; the sum (resp. intersection, wedge) of two subcoalgebras does to the intersection (resp. sum, product) of corresponding ideals; simple subcoalgebras do to maximal ideals ([3], Propositions 1.4.5, 1.4.6, 9.0.0).

By virtue of (2.2) and (2.3), we have completely reduced the situation to the case of finite-dimensional algebras. In this case we can paraphrase THEOREM as follows:

(2.4) *Let A be a finite-dimensional algebra. We can define an eqivalence relation in the set of all maximal ideals of A as follows: For maximal ideals M, M' of A, we write $M \sim M'$ if either $M = M'$ or there is a sequence of maximal ideals*

$$M = M_1, M_2, \dots, M_r = M'$$

such that $M_i M_{i+1} \neq M_{i+1} M_i$ for $i = 1, \dots, r-1$. Let $\mathfrak{M}_i$ $(i = 1, \dots, s)$ be the equivalence classes and put

$$P_i = \bigcap_{M \in \mathfrak{M}_i} M.$$

Then we have the canonical isomorphism

$$A \cong \prod_{i=1}^{s} A/P_i^{\infty}$$

and each A/P_i^{∞} is indecomposable (as a ring), where $P_i^{\infty} = \bigcap_{n=1}^{\infty} P_i^n$.

3. Finite-Dimensional Algebras

Let A be a finite-dimensional algebra.

(3.1) There exists a positive integer d such that

$$\bigcap_{n=1}^{\infty} I^n = I^d$$

for all (one or two-sided) ideals I of A. We denote I^d by I^∞. In fact, for example, $d = \dim_k A$.

(3.2) $\mathfrak{M}(A)$, the set of all maximal ideals of A, is a finite set and

$$N = \bigcap_{M \in \mathfrak{M}(A)} M$$

is the (Jacobson) radical of A.

(3.3) *Let I, J be two-sided ideals of A such that $I+J=A$. Then*

$$I \cap J = IJ + JI.$$

PROOF. By hypothesis

$$1 = a + b$$

for some a in I and b in J. Let x be any element of $I \cap J$. Then

$$x = ax + bx$$

where $ax \in IJ$ and $bx \in JI$. This implies that

$$I \cap J \subset IJ + JI.$$

The converse inclusion is clear.

(3.4) *Let P_i be as in* (2.4). *Then for $i \neq j$ we have*

$$P_i^\infty + P_j^\infty = A.$$

Indeed, this follows from the fact that only maximal ideals containing P_i are members of the class $\mathfrak{M}_i$, and any (proper) ideal is contained in some maximal one.

This implies, by Chinese Remainder Theorem (e.g., [1], p.46), the canonical homomorphism

$$A \longrightarrow \prod_{i=1}^{s} A/P_i^\infty$$

is surjective. The kernel of this homomorphism is $\bigcap_{i=1}^{s} P_i^\infty$.

(3.5) *Let $M_1, \ldots, M_r$ be distinct maximal ideals and I an ideal of A such that $M_i I = IM_i$ for $i=1, \ldots, r$. Then*

$$I(M_1 \cap \cdots \cap M_r) = (M_1 \cap \cdots \cap M_r)I$$

Proof. We prove this by induction on r. It is trivial for $r=1$. Assume that the assertion holds for $r-1$. Let $K=M_1 \cap \cdots \cap M_{r-1}$. Since all M_i's are distinct maximal ideals we have

$$K + M_r = A.$$

Therefore by (3.3) we have an equality

$$K \cap M_r = KM_r + M_rK,$$

so that this commutes with I.

(3.6) Let P_i be as in (3.4). By (3.5) we have

$$P_i^\infty \cap P_j^\infty = P_i^\infty P_j^\infty.$$

Indeed, two maximal ideals belonging to distinct classes commute. Therefore

$$\begin{aligned} P_1^\infty \cap \cdots \cap P_s^\infty &= P_1^\infty \cdots P_s^\infty \\ &= (P_1 \cdots P_s)^\infty \\ &= N^\infty \\ &= 0. \end{aligned}$$

Consequently the homomorphism

$$A \longrightarrow \prod A/P_i^\infty$$

is an isomorphism. This completes the proof of Theorem.

4. Corollaries and Example

In this section we list some direct consequences of Theorem and an example.

(4.1) *Let C be a coalgebra and $C_1=C_0 \wedge C_0$, where C_0 is the coradical of C. Then C is indecomposable if and only if C_1 is indecomposable.*

In fact, since $S \wedge_C S' \subset C_1$ for simple subcoalgebras S and S', we have

$$S \wedge_C S' = S \wedge_{C1} S'.$$

(4.2) *A necessary and sufficient condition for a coalgebra C to be decomposed into the direct sum of its irreducible components is that*

$$S \wedge S' = S' \wedge S$$

for any simple subcoalgebras S and S′ of C.

Note that when C is cocommutative this condition is trivial.

(4.3) Even if C is indecomposable, the number of simple subcoalgebras need not be finite.

EXAMPLE. Let C be a vector sapce with basis $\{x_n \mid n=1, 2,\dots\} \cup \{y_n \mid n=1, 2,\dots\}$. Define Δ and ε as follows:

$$\Delta(x_n) = x_n \otimes x_n, \quad \varepsilon(x_n) = 1, \qquad \text{for } n = 1, 2,\dots,$$

$$\Delta(y_n) = x_n \otimes y_n + y_n \otimes x_{n+1}, \ \varepsilon(y_n) = 0, \qquad \text{for } n = 1, 2,\dots$$

Then C is actually a coalgebra and $\mathfrak{S}(C)=\{kx_n \mid n=1, 2,\dots\}$. It is easy to see that for each n

$$kx_n \wedge kx_{n+1} \neq kx_{n+1} \wedge kx_n,$$

since y_n belongs to the left but not to the right. Thus by THEOREM C is indecomposable.

References

[1] C. Faith, Algebra II Ring Theory, Springer-Verlag, Berlin · Heiderberg · New York, 1976.

[2] R. G. Heyneman and D. E. Radford, Reflexivity and coalgebras of finite type, J. Algebra **28** (1974), 215–246.

[3] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[4] O. Zarisky and P. Samuel, Commutative Algebra vol. I, Van Nostrand, New York, 1958.

Technical College,
Yamaguchi University
and
Department of Mathematics,
Faculty of Engineering,
Tokushima University