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KO-Groups of Lens Spaces Modulo Powers of Two

Dedicated to Professor A. Komatu on his 70th birthday

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§1. Introduction

The K- and KO-rings of the standard lens space $L^n(m)=S^{2n+1}/Z_m$ mod m are investigated by several authors, and the structures of the reduced K- and KO-rings $\tilde{K}(L^n(m))$ and $\widetilde{KO}(L^n(m))$ are determined by J. F. Adams [1, Th.7.3–4] when $m=2$ ($L^n(2)=RP^{2n+1}$ is the real projective space), and T. Kambe [3] when m is an odd prime. Furthermore, the additive groups $\tilde{K}(L^n(p^r))$ (p: prime) and $\widetilde{KO}(L^n(p^r))$ (p: odd prime) are determined by N. Mahammed [9, Th.3], and an explicit additive base of $\tilde{K}(L^n(p^r))$ (p: odd prime) is given in [5, Th.1.7].

In this note, we shall determine the additive structure of

$$\widetilde{KO}(L^n(2^r)) \qquad \text{for any} \quad r \geqq 2.$$

Let ρ *be the non-trivial real line bundle over* $L^n(2^r)$, and η *be the canonical complex line bundle over* $L^n(2^r)$, i.e., the induced bundle of the canonical complex line bundle over the complex projective space CP^n by the natural projection $\pi: L^n(2^r)\rightarrow CP^n$. Then we can prove the following

PROPOSITION 1.1. *The reduced KO-ring* $\widetilde{KO}(L^n(2^r))$ $(r\geqq 2)$ *is generated by the stable classes*

(1.2) $\qquad \kappa = \rho-1, \quad \bar{\sigma} = r\eta-2$ (*$r\eta$ is the real restriction of η*);

and there hold the following relations:

(1.3) $$\bar{\sigma}^i = 0 \quad \text{for} \quad i > [n/2]+\varepsilon,\ \varepsilon = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 4, \\ 0 & \text{otherwise}; \end{cases}$$

(1.4) $$\bar{\sigma}(r-1) = 2\kappa, \quad \kappa^2 = -2\kappa,$$

(1.5) $$\kappa\bar{\sigma} = -2\kappa + \textstyle\sum_{s=1}^{r-2} \{(2+\bar{\sigma})\bar{\sigma}(s)\prod_{t=s+1}^{r-2}(2+\bar{\sigma}(t))\},$$

where $\bar{\sigma}(s)=\bar{\sigma}^{2^s}+\sum_{j=1}^{2^s-1} y_{sj}\bar{\sigma}^j \in \widetilde{KO}(L^n(2^r))$ *is given inductively by*

(1.6) $$\bar{\sigma}(0) = \bar{\sigma}, \quad \bar{\sigma}(s) = 4\bar{\sigma}(s-1)+\bar{\sigma}(s-1)^2 \qquad (0 < s < r).$$

Hence we see that $\widetilde{KO}(L^n(2^r))$ is generated additively by

(1.7) $$\{\kappa, \bar{\sigma}^i : 1 \leqq i \leqq N'\}, \quad N' = \min\{2^{r-1}-1, [n/2]+\varepsilon\}.$$

Furthermore we have the following main theorem, where $Z_u\langle\alpha\rangle$ means the cyclic group of order u generated by α; a_s and b_s are the integers with

(1.8) $$a_s = [n/2^s], \quad n = 2^s a_s + b_s, \quad 0 \leqq b_s < 2^s \qquad (0 < s < r),$$

ε is the one in (1.3), N' is the one in (1.7); and κ, $\bar{\sigma}$ and $\bar{\sigma}(s)$ are the elements in (1.2) and (1.6).

THEOREM 1.9. *The reduced KO-group of the standard lens space $L^n(2^r)$ mod 2^r ($r \geqq 2$) is given as follows.*

(i) ([6, Th.B]) $\widetilde{KO}(L^0(4)) = Z_2\langle\kappa\rangle$ *and* $\widetilde{KO}(L^n(4))$ *is the direct sum*

$$\widetilde{KO}(L^n(4)) = Z_u\langle\kappa + 2^{a_1}\bar{\sigma}\rangle \oplus Z_{u(1)}\langle\bar{\sigma}\rangle,$$

$$u = 2^{a_1+\varepsilon}, \quad u(1) = 2^{2a_1+1} \qquad (n \geqq 1),$$

where the first generator is able to be replaced by κ if $n \equiv 1 \bmod 4$.

(ii) $\widetilde{KO}(L^n(2^r))$ ($r \geqq 3$) *is the direct sum*

$$\widetilde{KO}(L^n(2^r)) = \textstyle\sum_{i=0}^{N'} Z_{u(i)}\langle\bar{\sigma}_i\rangle, \quad N' = \min\{2^{r-1}-1, a_1 + \varepsilon\},$$

and the order $u(i)$ and the generator $\bar{\sigma}_i$,

$$\bar{\sigma}_0 = \kappa + \textstyle\sum_{j=1}^{N'} z_j \bar{\sigma}^j, \quad \bar{\sigma}_i = \sum_{j=1}^{i} z_{ij}\bar{\sigma}^j \qquad (1 \leqq i \leqq N')$$

where $z_{ii} = 1$ or $1 - 2^{a'(i)-1}$ ($a'(i) \geqq 2$), are given as follows:

(a) *The case $n \not\equiv 1 \bmod 4$: For $i = 0$,*

$$u(0) = 2^{a_{r-1}}, \qquad \bar{\sigma}_0 = \kappa + \textstyle\sum_{t=1}^{r-1} 2^{(2^t-1)(a_{r-1}+1)-1}\bar{\sigma}(r-1-t) \qquad (n \geqq 2^{r-1}),$$

$$u(0) = 2, \qquad \bar{\sigma}_0 = \kappa \qquad (n < 2^{r-1});$$

and for $i = 2^s + d \leqq a_1$ with $0 \leqq s \leqq r-2$ and $0 \leqq d < 2^s$,

$$u(1) = 2^{r-1+2a_1}, \quad \bar{\sigma}_1 = \bar{\sigma} \qquad \text{if } i = 1;$$

$$u(i) = 2^{r-s-2+a_s}, \quad \bar{\sigma}_i = \bar{\sigma}(s) + \textstyle\sum_{t=1}^{s} 2^{(2^t-1)(a_s+1)}\bar{\sigma}(s-t) \qquad \text{if } i = 2^s \geqq 2;$$

$$\begin{cases} u(i) = 2^{r-s-3+a'(i)}, \quad a'(i) = \begin{cases} a_{s+1}+1 & \text{for } 2d \leqq b_{s+1}, \\ a_{s+1} & \text{for } 2d > b_{s+1}, \end{cases} \\ \bar{\sigma}_i = \bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{s-1}(2+\bar{\sigma}(t)) - 2^{a'(i)-1}\bar{\sigma}^d\bar{\sigma}(s) \end{cases}$$

$+ \sum_{t=2}^{s+1} 2^{(2^t-1)a'(i)-1}\bar{\sigma}^d\bar{\sigma}(s+1-t)$ *if* $i = 2^s + d \geqq 3,\ d \geqq 1.$

(b) *The case* $n \equiv 1 \bmod 4$: $u(i)$ *and* $\bar{\sigma}_i$ *are the same as* (a) *if* $i \not\equiv a_1+1 \bmod 2^{r-2}$; *and*

$$u(i) = 2^{a_{r-1}}, \quad \bar{\sigma}_i = \bar{\sigma}^{d-1}\bar{\sigma}(1)\textstyle\prod_{t=0}^{r-3}(2 + \bar{\sigma}(t))$$

$$\text{if } i = a_1 + 1 - 2^{r-2}(a_{r-1}-1) = 2^{r-2} + d,\ 2d = b_{r-1} + 1 \quad (n \geqq 2^{r-1});$$

$$u(i) = 2, \quad \bar{\sigma}_i = \bar{\sigma}^i \qquad \text{if } i = a_1 + 1 \quad (n < 2^{r-1}).$$

As compared with the above theorem, the result for the real projective space $L^n(2) = RP^{2n+1}$ is stated as follows:

(1.10) (J. F. Adams [1, Th.7.4]) $\widetilde{KO}(L^n(2)) = Z_{u'}\langle\kappa\rangle,\ u' = 2^{2a_1+1+\varepsilon}$.

In § 2, we study some relations in the reduced K-ring $\tilde{K}(L^n(2^r))$ which is generated by the stable class $\sigma = \eta - 1$ with the relations $\sigma^{n+1} = 0 = (1+\sigma)^{2^r} - 1$, and give an additive base of $\tilde{K}(L^n(2^r))$ explicitly in Theorem 3.1.

In § 4, we study the induced homomorphism

$$i^*: \widetilde{KO}(L^n(2^r)) \longrightarrow \widetilde{KO}(L^{n-1}(2^r)) \quad (i: L^{n-1}(2^r) \subset L^n(2^r))$$

by using the results of B. J. Sanderson [11, Th. (3.9)] on the KO-ring of the complex projective space, and prove the first half of Proposition 1.1 in Proposition 4.4(ii) and (1.3) in Corollary 4.12 (ii). Furthermore, by using the Bott sequence ([2, (12.2)]), we prove some properties of the complexification

$$c: \widetilde{KO}(L^n(2^r)) \longrightarrow \tilde{K}(L^n(2^r))$$

in Proposition 5.3, which contains the recent result of M. Yasuo [12, (A.13)] that c is monomorphic if $n \equiv 3 \bmod 4$.

By using these results, we study some relations in $\widetilde{KO}(L^n(2^r))$ in §§ 6–7, and obtain the relations (1.4–5) in Proposition 6.3, and prove finally the main theorem in § 7. Moreover, the group $\widetilde{KO}(L_0^n(2^r))$ of the $2n$-skeleton $L_0^n(2^r)$ of $L^n(2^r)$ is given in Theorem 7.5.

§ 2. Some relations in $\tilde{K}(L^n(2^r))$

The group $S^1 = \{z \in C: |z| = 1\}$ acts on the $(2n+1)$-sphere

$$S^{2n+1} = \{(z_0, \dots, z_n) \in C^{n+1}: |z_0|^2 + \cdots + |z_n|^2 = 1\}$$

diagonally (i.e., $z(z_0, \dots, z_n) = (zz_0, \dots, zz_n)$), and the orbit manifold

$$L^n(m) = S^{2n+1}/Z_m, \quad Z_m = \{z \in S^1 : z^m = 1\} \subset S^1, \qquad (m \geqq 2)$$

is the standard lens space mod m. By considering the subspace

$$(2.1) \qquad L_0^n(m) = \{[z_0,\dots, z_n] \in L^n(m) \colon z_n \text{ is real} \geqq 0\} \subset L^n(m),$$

we have the cell-decomposition $L^n(m) = L_0^n(m) \cup e^{2n+1} = \cup_{i=0}^{2n+1} e^i$, and

$$(2.2) \qquad L^n(m)/L_0^n(m) = S^{2n+1}, \quad L_0^n(m)/L_0^{n-1}(m) = S^{2n-1} \cup_m e^{2n},$$

where the attaching map $m \colon S^{2n-1} \to S^{2n-1}$ is the map of degree m.

Let $CP^n = S^{2n+1}/S^1$ be the complex projective space, and

$$(2.3) \qquad \pi\colon L^n(m) \longrightarrow CP^n, \quad \pi\colon L_0^n(m) \longrightarrow CP^n$$

be the natural projection and its restriction.

Let η be the canonical complex line bundle over CP^n. Denote also the canonical one $\pi^*\eta$ over $L^n(m)$ or $L_0^n(m)$ by η, and its stable class by

$$(2.4) \qquad \sigma = \eta - 1 \quad \text{in } \tilde{K}(L^n(m)) \text{ or } \tilde{K}(L_0^n(m)).$$

Since the first Chern class of η^m is 0 in $H^2(L^n(m)) = H^2(L_0^n(m)) = Z_m$,

$$(2.5) \qquad \eta^m - 1 = (1+\sigma)^m - 1 = 0 \quad \text{in } \tilde{K}(L^n(m)) \text{ or } \tilde{K}(L_0^n(m)).$$

Further we have the following by using the Puppe exact sequences of (2.2) and the results of J. F. Adams [1, Th.7.2] on $\tilde{K}(CP^n)$:

PROPOSITION 2.6 ([4, Lemma 2.4, Prop. 2.6]). (i) *The induced homomorphism*

$$i^*\colon \tilde{K}(L^n(m)) \longrightarrow \tilde{K}(L^{n-1}(m)) \qquad (i\colon L^{n-1}(m) \subset L^n(m))$$

is epimorphic, $i^*\sigma = \sigma$ and Ker $i^* = Z_m\langle\sigma^n\rangle$.

(ii) $\tilde{K}(L^n(m)) = \tilde{K}(L_0^n(m))$ *by the induced homomorphism of the inclusion, and this ring contains exactly m^n elements and is generated by σ which satisfies $\sigma^{n+1} = 0$ and* (2.5).

Now let $m = 2^r$, and consider the reduced K-ring

$$\tilde{K}(L^n(2^r)) = \tilde{K}(L_0^n(2^r)) \qquad (r \geqq 1)$$

and its elements

$$(2.7) \qquad \sigma(s) = \eta^{2^s} - 1 = (1+\sigma)^{2^s} - 1 \quad (0 \leqq s \leqq r), \qquad \sigma(0) = \sigma.$$

Then by definition and by $(1+\sigma)^{2^r} - 1 = 0$ in (2.5), we see that

(2.8) $$\sigma(s) = 2\sigma(s-1) + \sigma(s-1)^2 \quad (0 < s \leqq r), \qquad \sigma(r) = 0,$$

and $(1+\sigma(s))^{2^{r-s}}-1=0$. These and $\sigma^{n+1}=0$ in Proposition 2.6 (ii) imply the following

LEMMA 2.9 ([8, Lemmas 2.3–4]). (i) *For any integers* $k_0,\dots, k_{s-1}\geqq 0$ *and* $k_s>0$ $(0\leqq s\leqq r)$,

$$2^h \prod_{t=0}^{s} \sigma(t)^{k_t} = 0 \qquad in \quad \tilde{K}(L^n(2^r))$$

if $h\geqq 0$ *is an integer such that* $2^s(h-r+s) \geqq n-\sum_{t=0}^{s} 2^t k_t$.

(ii) *For any integers* $k_0,\dots, k_{s-1} \geqq 0$ *and* $k_s > l \geqq 0$ $(0 \leqq s < r)$,

$$2^{h'}\alpha\sigma(s)^{k_s} = (-1)^l 2^{h'+l}\alpha\sigma(s)^{k_s-l} \quad (\alpha = \textstyle\prod_{t=0}^{s-1} \sigma(t)^{k_t}) \qquad in \quad \tilde{K}(L^n(2^r))$$

if $h'\geqq 0$ *is an integer such that* $2^{s+1}(h'-r+s+1)\geqq n-\sum_{t=0}^{s} 2^t k_t$.

Furthermore, we see the following

LEMMA 2.10. *If* $0<s<r$, $n<d+2^s k$, $d\geqq 0$ *and* $k\geqq 2$, *then*

$$\sum_{t=0}^{s} 2^{r-s-2+2^t k}\sigma^d\sigma(s-t) = 0 \qquad in \quad \tilde{K}(L^n(2^r)).$$

PROOF. If k is even, the lemma is proved in [8, Lemma 2.5].

Since $n<d+2^s k$, by Lemma 2.9 and $\sigma^{n+1}=0$ in Proposition 2.6 (ii), we see that

(2.11) $$2^{r-s-1}\sigma^d\sigma(s)^k = \pm 2^{r-s-2+k}\sigma^d\sigma(s), \quad 2^{r-s-1}\sigma^{d+2^s k} = 0.$$

Suppose that $k=2k'+1\geqq 2$ is odd. Then we can prove that

(2.12) $$2^{r-s-1}\sigma^d\sigma(s-t+1)^{2^{t-1}k} - 2^{r-s-1}\sigma^d\sigma(s-t)^{2^t k}$$

$$= \begin{cases} \pm 2^{r-s-2+k}\sigma^d\sigma(s-1)\sigma(s) + 2^{r-s-2+2k}\sigma^d\sigma(s-1) & \text{if } t = 1 \leqq s, \\ \pm 2^{r-s-1+2k}\sigma^d\sigma(s-1) + 2^{r-s-2+4k}\sigma^d\sigma(s-2) & \text{if } t = 2 \leqq s, \\ 2^{r-s-2+2^t k}\sigma^d\sigma(s-t) & \text{if } 3 \leqq t \leqq s \end{cases}$$

in $\tilde{K}(L^n(2^r))$, instead of (*) in the proof of [8, Lemma 2.5], as follows.

Set $u=s-t$ for $1\leqq t\leqq s$. Then we see that

(2.13) $$2^{r-s-1}\sigma^d\sigma(u+1)^{2^{t-1}k} = 2^{r-s-1}\sigma^d\{2\sigma(u) + \sigma(u)^2\}^{2^t k'}\sigma(u+1)^{2^{t-1}}$$

(by (2.8))

$$= \sum_{i=0}^{2^t k'} \binom{2^t k'}{i} 2^{r-s-1+i}\sigma^d\sigma(u)^{2^{t+1}k'-i}\sigma(u+1)^{2^{t-1}}$$

$$= 2^{r-s-1}\sigma^d\sigma(u)^{2^{t+1}k'}\sigma(u+1)^{2^{t-1}} \qquad \text{(by Lemma 2.9(i))},$$

by noticing that $\binom{2^t k'}{i} \equiv 0 \bmod 2^{t-v}$ if $i=2^v j \geqq 1$ and j is odd.

If $t=1$, then the left hand side of (2.12) is equal to

$$2^{r-s}\sigma^d\sigma(s-1)^{4k'+1} = 2^{r-s}\sigma^d\sigma(s-1)\{\sigma(s)-2\sigma(s-1)\}^{2k'} \quad \text{(by (2.13) and (2.8))}$$

$$= \textstyle\sum_{i=0}^{2k'}\binom{2k'}{i}(-1)^i 2^{r-s+i}\sigma^d\sigma(s-1)^{i+1}\sigma(s)^{2k'-i}$$

$$= 2^{r-s}\sigma^d\sigma(s-1)\sigma(s)^{k-1} + 2^{r-s-1+k}\sigma^d\sigma(s-1)^k \qquad \text{(by Lemma 2.9(i))}.$$

This is equal to the right hand side of (2.12) by Lemma 2.9.

If $t \geqq 2$, then the left hand side of (2.12) is equal to

$$\textstyle\sum_{i=1}^{2^{t-1}}\binom{2^{t-1}}{i}2^{r-s-1+i}\sigma^d\sigma(u)^{2^t k-i} \qquad \text{(by (2.13) and (2.8))}. \tag{2.14}$$

The i-th term of this sum is equal to

$$(-1)^{i-1}\binom{2^{t-1}}{i}2^{r-s-2+2^t k}\sigma^d\sigma(u) \qquad \text{if} \quad i \neq 1, 2, 4,$$

by Lemma 2.9 (ii). If $i=2^v$ ($v=0$, 1 or 2), then the i-th term in (2.14) is equal to

$$\binom{2^{t-1}}{i}2^{r-s-1+i}\sigma^d\sigma(u)^{4-i}\{\sigma(u+1) - 2\sigma(u)\}^{2^{t-1}k-2}$$

$$= \pm\, 2^{r-u-2+i-v}\sigma^d\sigma(u)^{4-i}\sigma(u+1)^{2^{t-1}k-2}$$

$$+ \binom{2^{t-1}}{i}2^{r-s-3+i+2^{t-1}k}\sigma^d\sigma(u)^{2^{t-1}k+2-i}$$

by Lemma 2.9 (i), since $\binom{2^{t-1}}{2^v} \equiv 2^{t-1-v} \bmod 2^{t-v}$. Further this is equal to

$$\pm\, 2^{r-u-5+i-v+2^{t-1}k}\sigma^d\sigma(u)^{4-i}\sigma(u+1) + (-1)^{i-1}\binom{2^{t-1}}{i}2^{r-s-2+2^t k}\sigma^d\sigma(u)$$

by Lemma 2.9 (ii). Thus the sum (2.14) is equal to the sum of

$$\textstyle\sum_{i=1}^{2^{t-1}}(-1)^{i-1}\binom{2^{t-1}}{i}2^{r-s-2+2^t k}\sigma^d\sigma(u) = 2^{r-s-2+2^t k}\sigma^d\sigma(u),$$

$$\pm\, 2^{r-u-4+2^{t-1}k}\sigma^d\{\sigma(u)^2+\sigma(u)^3\}\sigma(u+1) = \pm\, 2^{r-u-3+2^{t-1}k}\sigma^d\sigma(u+1)$$

(the equality follows from Lemma 2.9 (ii) since $\sigma(u)^2+\sigma(u)^3=2\sigma(u)-\sigma(u+1)+\sigma(u)\sigma(u+1)$ by (2.8)) and

$$\pm\, 2^{r-u-3+2^{t-1}k}\sigma^d\sigma(u+1) \qquad \text{if} \quad t \geqq 3;$$

and the last term does not appear if $t=2$. Hence we see (2.12) also for $t\geqq 2$.

Now, by (2.11–2), we see that

$$(2.15) \qquad \textstyle\sum_{t=0}^{s} 2^{r-s-2+2^t k}\sigma^d\sigma(s-t)$$

$$= \begin{cases} \pm\, 2^{r-3+k}\sigma^{d+1}\sigma(1) & \text{if} \quad s = 1, \\ \pm\, 2^{r-s-2+k}\sigma^d\sigma(s-1)\sigma(s) \pm 2^{r-s-1+2k}\sigma^d\sigma(s-1) & \text{if} \quad s \geqq 2. \end{cases}$$

If $s \geqq 2$, then by (2.15)$\times\sigma(s-1)$ and Lemma 2.9, we see that

$$2^{r-s-2+k}\sigma^d\sigma(s-1)\sigma(s) = \pm\, 2^{r-s-2+2k}\sigma^d\sigma(s-1)^2 = \pm\, 2^{r-s-1+2k}\sigma^d\sigma(s-1).$$

Thus the right hand side of (2.15) is 0 as desired if $s\geqq 2$. The same is also valid if $s=1$ by Lemma 2.9 (i).

Therefore the lemma is proved completely. q.e.d.

§3. The additive structure of $\tilde{K}(L^n(2^r))$

By the results in §2, we have the following theorem, where σ and $\sigma(s)$ are the elements in (2.4) and (2.7), and a_s and b_s are the integers in (1.8), i.e.,

$$n = 2^s a_s + b_s, \quad 0 \leqq b_s < 2^s \qquad (0 \leqq s < r).$$

THEOREM 3.1 (cf. N. Mahammed [9, Th. 3]). *The reduced K-group of the standard lens space $L^n(2^r)$ or its 2n-skeleton $L_0^n(2^r)$ $(r\geqq 1)$ of (2.1) is the direct sum*

$$\tilde{K}(L^n(2^r)) = \tilde{K}(L_0^n(2^r)) = \textstyle\sum_{i=1}^{N} Z_{t(i)}\langle\sigma_i\rangle, \quad N = \min\{2^r-1, n\},$$

and the order $t(i)$ and the generator σ_i of the i-th cyclic factor for

$$1 \leqq i = d + 2^s \leqq N, \quad 0 \leqq s < r \quad \text{and} \quad 0 \leqq d < 2^s,$$

are given as follows:

$$t(i) = 2^{r-s-2+a(i)}, \quad a(i) = \begin{cases} a_s + 1 & \text{if} \quad d \leqq b_s, \\ a_s & \text{if} \quad d > b_s, \end{cases}$$

$$\sigma_i = \begin{cases} \sigma^d\sigma(s) & \text{if} \quad d = b_s, \\ \sum_{t=0}^{s} 2^{(2^t-1)a(i)}\sigma^d\sigma(s-t) & \text{otherwise.} \end{cases}$$

PROOF. By Proposition 2.6 (ii) and $(1+\sigma)^{2^r}-1=0$ in (2.5), we see that $\tilde{K}(L^n(2^r))$ is generated additively by $\{\sigma^i\colon 1\leqq i\leqq N\}$, and hence it is also so by

$$\{\sigma_i\colon 1 \leqq i \leqq N\},$$

since $\sigma_i=\sigma^i+\sum_{j=1}^{i-1} x_{ij}\sigma^j$ by (2.7) and the above definition.

On the other hand, by Lemmas 2.9 (i), 2.10 and the above definition, we see that

$$t(i)\sigma_i = 0 \qquad \text{for} \quad 1 \leqq i \leqq N,$$

since $n=2^s a_s+b_s<d+2^s a(i)$ and $a(i)\geqq 2$ if $i=d+2^s\leqq N$. Further

$$\begin{aligned}&\textstyle\sum_{s=0}^{u-1} \{(r-s-2+a_s)2^s+b_s+1\} + (r-u)(b_u+1)\\ &\quad = \textstyle\sum_{s=0}^{u-1} (r-s-2)2^s+(n+1)u+(r-u)(n-2^u+1) = nr\end{aligned}$$

by (1.8), where $u=r$ if $n\geqq 2^r$ and $a_u=1$ otherwise. Thus

$$\textstyle\prod_{i=1}^{N} t(i) = 2^{rn},$$

which is equal to the order of $\tilde{K}(L^n(2^r))$ by Proposition 2.6 (ii).

Therefore, we have the theorem. q.e.d.

Here, we prepare the following for the purpose of the later sections.

Instead of $\tilde{K}(L^n(2^r))$, we consider the quotient ring

$$\tilde{K}(L^{n+1}(2^r))/\langle 2\sigma^{n+1}\rangle \qquad (r \geqq 2), \tag{3.2}$$

where the ideal $\langle 2\sigma^{n+1}\rangle$ generated by $2\sigma^{n+1}$ is the cyclic subgroup $Z_{2^{r-1}}\langle 2\sigma^{n+1}\rangle$ by Proposition 2.6; and we denote an element in $\tilde{K}(L^{n+1}(2^r))$ and its coset in $\tilde{K}(L^{n+1}(2^r))/\langle 2\sigma^{n+1}\rangle$ by the same letter.

LEMMA 3.3. (i) *$\sigma^{n+1}\neq 0$ in $\tilde{K}(L^{n+1}(2^r))/\langle 2\sigma^{n+1}\rangle$ of* (3.2).

(ii) *Lemma* 2.9 *holds for $\tilde{K}(L^{n+1}(2^r))/\langle 2\sigma^{n+1}\rangle$ instead of $\tilde{K}(L^n(2^r))$.*

(iii) *So does Lemma* 2.10 *under the additional assumption that $s\leqq r-2$ or $n+2\leqq d+2^s k$.*

PROOF. (i) is seen immediately by Proposition 2.6 (i).

(ii) In the proof of Lemma 2.9 ([8, Lemmas 2.3–4]), we use the relation $\sigma^{n+1}=0$ in $\tilde{K}(L^n(2^r))$ only at the first step of the inductive proof of [8, Lemma 2.3] to show that $2^{r+h}\sigma^{n-h}=0$ for $h<0$, which follows from the relations

$$2\sigma^{n+1} = 0 \quad \text{and} \quad \sigma^{n+2} = 0 \qquad \text{in} \quad \tilde{K}(L^{n+1}(2^r))/\langle 2\sigma^{n+1}\rangle, \tag{3.4}$$

since $r\geqq 2$. Hence we can prove (ii) by the same argument as [8,Lemmas 2.3–4].

(iii) In the proof of Lemma 2.10 (and [8, Lemma 2.5]), we use the relation $\sigma^{n+1}=0$ in $\tilde{K}(L^n(2^r))$ only to show that $2^{r-s-1}\sigma^{d+2^s k}=0$ in (2.11) (and [8, p.87]), which follows from (3.4) if $r-s-1\geqq 1$ or $d+2^s k\geqq n+2$. Hence we can prove (iii) by the same argument as Lemma 2.10 (and [8, Lemma 2.5]). q.e.d.

LEMMA 3.5. (i) *If* $0<s<r$, $n<d+2^s k$, $d\geqq 0$ *and* $k\geqq 2$, *then*

$$\pm 2^{r-s-2+k}\sigma^d\sigma(s) = \sum_{t=1}^{s} 2^{r-s-1+2^t(k-1)}\sigma^d\sigma(s-t)^{2^t} \quad \textit{in} \quad \tilde{K}(L^n(2^r)).$$

(ii) *If* $s\leqq r-2$ *or* $n+2\leqq d+2^s k$ *in addition, then the above equality holds also in* $\tilde{K}(L^{n+1}(2^r))/\langle 2\sigma^{n+1}\rangle$ *of* (3.2).

PROOF. We see easily (i) by Lemmas 2.10 and 2.9, and (ii) by (iii) and (ii) of the above lemma. q.e.d.

REMARK 3.6. We notice that the result of [5, Th. 1.7] on $\tilde{K}(L^n(p^r))$ for an odd prime p holds also for $p=2$. In fact, by using Lemma 3.5 (i) instead of Lemma 2.10, we see that *the generator* σ_i *in Theorem* 3.1 *is able to be replaced by*

$$\sum_{t=0}^{s} 2^{(2^t-1)(a(i)-1)}\sigma^d\sigma(s-t)^{2^t} \qquad \textit{if} \quad i = d + 2^s,\ d \neq b_s.$$

§4. The induced homomorphism on the KO-groups of the inclusion $L^{n-1}(2^r)\subset L^n(2^r)$

We use the following notations frequently:

$$(4.1) \qquad L_r^{2n+1} = L^n(2^r), \quad L_r^{2n} = L_0^n(2^r) \qquad (r\geqq 2),$$

where the latter is the $2n$-skeleton in (2.1).

Consider the stable classes

$$(4.2) \qquad \bar{\sigma} = r\sigma = r\eta - 2, \quad \kappa = \rho - 1 \quad \text{in} \quad \widetilde{KO}(L_r^k)$$

of (1.2), where $\sigma\in\tilde{K}(L_r^k)$ is the one of (2.4), $r: K\to KO$ is the real restriction, and ρ is the non-trivial real line bundle over L_r^k, i.e., ρ is the real line bundle over L_r^k whose first Stiefel-Whitney class $w_1(\rho)\in H^1(L_r^k; Z_2)=Z_2$ is non-zero.

LEMMA 4.3. *By the complexification* $c: \widetilde{KO}(L_r^k)\to\tilde{K}(L_r^k)$,

$$c\bar{\sigma} = \sigma^2/(1+\sigma), \quad c\kappa = \sigma(r-1),$$

where $\sigma(r-1)=\eta^{2^{r-1}}-1\in\tilde{K}(L_r^k)$ *is the element in* (2.7).

PROOF. Let $t: K\to K$ be the conjugation. Then $cr=1+t$, $t\eta=\eta^{-1}$ and

$$c\bar{\sigma} = cr\sigma = (1+t)(\eta-1) = \eta + \eta^{-1} - 2 = (\eta-1)^2/\eta = \sigma^2/(1+\sigma).$$

The second equality follows from $c\rho=\eta^{2^{r-1}}$ ([6, Prop. 3.3]). q.e.d.

Now, we can prove the following

PROPOSITION 4.4. (i) *The induced homomorphism*

$$i_k^*: \widetilde{KO}(L_r^k) \longrightarrow \widetilde{KO}(L_r^{k-1}) \quad (i_k: L_r^{k-1} \subset L_r^k)$$

is isomorphic if $k \equiv 7$, 6, 5 *or* 3 mod 8, *epimorphic otherwise and*

$$(4.5) \qquad \operatorname{Ker} i_k^* = \begin{cases} Z_{2^r}\langle 2\bar{\sigma}^{2m+1}\rangle & \text{if } k = 8m+4, \\ Z_2\langle \bar{\sigma}^{2m+1}\rangle & \text{if } k = 8m+2, \\ Z_2\langle \kappa\bar{\sigma}^{2m}\rangle & \text{if } k = 8m+1, \\ Z_{2^r}\langle \bar{\sigma}^{2m}\rangle & \text{if } k = 8m,\ m > 0. \end{cases}$$

(ii) $i_k^*\bar{\sigma} = \bar{\sigma}$, $i_k^*\kappa = \kappa$ *and the ring* $\widetilde{KO}(L_r^k)$ *is generated by* $\bar{\sigma}$ *and* κ.

PROOF. The two equalities in (ii) are clear by definition.
Consider the commutative diagram

$$\begin{array}{ccccccccc}
\widetilde{KO}(S^{2n}) & \xrightarrow{\times 2^r} & \widetilde{KO}(S^{2n}) & \xrightarrow{p^*} & \widetilde{KO}(S^{2n-1} \cup_{2^r} e^{2n}) & \longrightarrow & \widetilde{KO}(S^{2n-1}) & \xrightarrow{\times 2^r} & \widetilde{KO}(S^{2n-1}) \\
 & & \| & & \downarrow p^* & & \downarrow p^* & & \\
 & & \widetilde{KO}(S^{2n}) & \xrightarrow{p^*} & \widetilde{KO}(L_r^{2n}) & \xrightarrow{i_{2n}^*} & \widetilde{KO}(L_r^{2n-1}) & & \\
 & & \downarrow p^* & & \| & & \downarrow i_{2n-1}^* & & \\
 & & \widetilde{KO}(CP^n) & \xrightarrow{\pi^*} & \widetilde{KO}(L_r^{2n}) & & \widetilde{KO}(L_r^{2n-2}) & &
\end{array}$$

of the Puppe exact sequences, where p's are the projections (cf. (2.2)) and π is the second one in (2.3).

(4.6) If $k \equiv 7$, 6, 5 or 3 mod 8, then $\operatorname{Ker} i_k^* = 0$ since $\widetilde{KO}(S^k) = 0$.

Let $n = 4m+2$, $4m+1$ or $4m$. Then $\widetilde{KO}(S^{2n}) = Z$, Z_2 or Z, and

$$p^* s_{2n} = 2y^{2m+1},\ y^{2m+1} \text{ or } y^{2m} \qquad (p^*: \widetilde{KO}(S^{2n}) \longrightarrow \widetilde{KO}(CP^n)),$$

respectively, by the results of B. J. Sanderson [11, Th. (3.9)], where $s_{2n} \in \widetilde{KO}(S^{2n})$ is a generator and $y = r\eta - 2 \in \widetilde{KO}(CP^n)$. On the other hand, $\pi^* y = \bar{\sigma}$ by definition. Thus by the above commutative diagram of the exact sequences, we see that

(4.7) Ker i_{2n}^* ($n - 4m = 2$, 1 or 0) is the cyclic subgroup generated by $2\bar{\sigma}^{2m+1}$, $\bar{\sigma}^{2m+1}$ or $\bar{\sigma}^{2m}$, and its order is a divisor of 2^r, 2 or 2^r, respectively.

For the case $k = 8m+1$, consider the diagram

$$\begin{array}{ccccccc} L_r^k & \xrightarrow{p} & L_r^k/L_r^{8m}(=S^k) & \xleftarrow[\approx]{\phi} & L_r^1\wedge(L_r^{8m}/L_r^{8m-1}) & = & L_r^1\wedge S^{8m} \\ \downarrow{\scriptstyle\Delta} & & & & \downarrow{\scriptstyle i\wedge\pi} & & \downarrow{\scriptstyle i\wedge 1} \\ L_r^k\times L_r^k & \xrightarrow{\tau\times p\pi} & L_r^k\times(CP^{4m}/CP^{4m-1}) & \xrightarrow{q} & L_r^k\wedge(CP^{4m}/CP^{4m-1}) & = & L_r^k\wedge S^{8m}, \end{array}$$

where Δ is the diagonal map, τ and ϕ are the homeomorphisms given by

$$\tau([z_0,\dots, z_{4m-1}, z_{4m}]) = [z_{4m}, z_0,\dots, z_{4m-1}],$$

$$\phi([z], [z_0,\dots, z_{4m}]) = [zz_0,\dots, zz_{4m}] \qquad (z_{4m} \text{ is real} \geqq 0),$$

π's are the ones in (2.3), q is the projection and i is the inclusion. Then we see easily that this diagram is homotopy commutative by the homotopy $h_t\colon L_r^k \to L_r^k\wedge(CP^{4m}/CP^{4m-1})$ given by

$$h_t(\zeta) = ([(t^2+(1-t^2)|z_{4m}|^{-2})^{1/2}z_{4m}, tz_0,\dots, tz_{4m-1}], \pi(\zeta))$$

($\zeta=[z_0,\dots, z_{4m}]\in L_r^k$). Hence by noticing that $\tau^*=1\colon KO(L_r^k)\to KO(L_r^k)$, we have the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}(L_r^k) & \xleftarrow{p^*} & \widetilde{KO}(S^k) & \xrightarrow[\cong]{\phi^*} & \widetilde{KO}(L_r^1\wedge S^{8m}) = \widetilde{KO}(L_r^1)\otimes\widetilde{KO}(S^{8m}) \\ \uparrow{\scriptstyle\cdot} & & & & \uparrow{\scriptstyle i^*\otimes 1} \\ \widetilde{KO}(L_r^k)\otimes\widetilde{KO}(L_r^k) & \xleftarrow{1\otimes\pi^*} & \widetilde{KO}(L_r^k)\otimes\widetilde{KO}(CP^{4m}) & \xleftarrow{1\otimes p^*} & \widetilde{KO}(L_r^k)\otimes\widetilde{KO}(S^{8m}), \end{array}$$

where $\cdot$ is the multiplication in the ring $\widetilde{KO}(L_r^k)$.

In this diagram, $\pi^*p^*s_{8m}=\bar{\sigma}^{2m}$ by the above proof of (4.7), and $i^*\kappa=\kappa$ is a generator of $\widetilde{KO}(L_r^1)=\widetilde{KO}(S^1)=Z_2$ by definition. Thus

(4.8) $\quad$ $\operatorname{Ker} i_{8m+1}^* = \operatorname{Im} p^*$ is generated by $\kappa\bar{\sigma}^{2m}$ whose order is 2 or 1.

Now, we see the latter half of (ii) inductively by (4.6–8), so that the ring homomorphism i_k^* is epimorphic for any k by the equalities in (ii). Thus we have the proposition by (4.6–8) and by showing

(4.9)
$$\begin{aligned} &2^r\bar{\sigma}^{2m+1}\neq 0 \ \text{ in } \ \widetilde{KO}(L_r^{8m+4}), \qquad \bar{\sigma}^{2m+1}\neq 0 \ \text{ in } \ \widetilde{KO}(L_r^{8m+2}),\\ &\kappa\bar{\sigma}^{2m}\neq 0 \ \text{ in } \ \widetilde{KO}(L_r^{8m+1}), \qquad 2^{r-1}\bar{\sigma}^{2m}\neq 0 \ \text{ in } \ \widetilde{KO}(L_r^{8m}). \end{aligned}$$

Since the complexification c is a ring homomorphism, Lemma 4.3 implies

$$c\bar{\sigma}^i = \sigma^{2i}/(1+\sigma)^i, \qquad c(\kappa\bar{\sigma}^i) = \sigma(r-1)\sigma^{2i}/(1+\sigma)^i.$$

Then by using Proposition 2.6, Lemma 2.9 (ii) and (2.7), we see (4.9) as follows:

(4.10) $\quad$ $2^r c\bar{\sigma}^{2m+1} = 2^r\sigma^{4m+2} = -2^{r-1}\sigma^{4m+3}\neq 0 \quad \text{in } \ \widetilde{K}(L_r^{8m+7}),$

so that $2^r\bar{\sigma}^{2m+1}\neq 0$ in $\widetilde{KO}(L_r^{8m+7})$, which implies the first relation in (4.9) by (4.6) for $k\equiv 7, 6, 5 \mod 8$. Hence the second one in (4.9) holds by (4.7) for $n=4m+2$ and (4.6) for $k=8m+3$. Moreover,

$$(4.11) \quad c\bar{\sigma}^{2m+1}=0, \quad c(\kappa\bar{\sigma}^{2m})=\sigma(r-1)\sigma^{4m}=2^{r-1}\sigma^{4m+1}\neq 0 \quad \text{in } \tilde{K}(L_r^{8m+2}),$$

so that $\bar{\sigma}^{2m+1}\neq\kappa\bar{\sigma}^{2m}$ in $\widetilde{KO}(L_r^{8m+2})$, which implies the third one in (4.9) by (4.7) for $n=4m+1$. Further the last one holds since $2^{r-1}c\bar{\sigma}^{2m}=2^{r-1}\sigma^{4m}\neq 0$ in $\tilde{K}(L_r^{8m})$.

Thus we see (4.9), and the proposition is proved completely. q.e.d.

By (i) of the above proposition, we see immediately the following

COROLLARY 4.12. (i) (N. Mahammed [10, Th. 3.4.2], M. Yasuo [13, Th. (0.1)]). *The order of the group* $\widetilde{KO}(L^n(2^r))$ $(r\geqq 2)$ *is equal to* $2^{(r+1)[n/2]+1+\varepsilon}$ *where* ε *is the integer in* (1.3), *i.e.,* $\varepsilon=1$ *if* $n\equiv 1 \mod 4$, $=0$ *otherwise.*

$$\text{(ii)} \qquad \bar{\sigma}^i=0 \quad in \quad \widetilde{KO}(L^n(2^r)) \quad if \quad i>[n/2]+\varepsilon.$$

§5. The complexification

It is known that the complexification c is contained in the Bott exact sequence

$$(5.1) \qquad \widetilde{KO}^1(X)\xrightarrow{\partial}\widetilde{KO}(X)\xrightarrow{c}\tilde{K}(X)$$

(R. Bott [2, (12.2)]), where $\widetilde{KO}^i$ $(\widetilde{KO}^0=\widetilde{KO})$ is the reduced KO-cohomology and ∂ is the natural homomorphism.

By using this sequence, we prove the following proposition on the complexification

$$c\colon \widetilde{KO}(L^n(2^r))\longrightarrow\tilde{K}(L^n(2^r)) \qquad (r\geqq 2),$$

which contains the result of M. Yasuo [12, (A.13)] that

$$(5.2) \qquad c \text{ is monomorphic if } n\equiv 3 \mod 4.$$

PROPOSITION 5.3. *For any* $n=4m+l$, $0\leqq l\leqq 3$, *we set*

$$(5.4) \qquad n'=2[n/2]+1=n \quad \text{if } n \text{ is odd}, \quad =n+1 \quad \text{otherwise}.$$

Then there are monomorphisms of rings

$$(5.5) \qquad \begin{aligned} c_l &: \widetilde{KO}(L^n(2^r))\longrightarrow\tilde{K}(L^{n'}(2^r)) && \text{if } l\neq 1,\\ c_1 &: \widetilde{KO}(L^n(2^r))\longrightarrow\tilde{K}(L^{n'+1}(2^r))/\langle 2\sigma^{n'+1}\rangle && \text{if } l=1,\end{aligned}$$

($r\geqq 2$, *and the last ring is the one in* (3.2)) *such that* $c_3=c$ *and the diagram*

$$(5.6)\qquad \begin{array}{ccccccc} \widetilde{KO}(L_r^{8m+7}) & \xrightarrow{i^*} & \widetilde{KO}(L_r^{8m+5}) & \xrightarrow{i^*} & \widetilde{KO}(L_r^{8m+3}) & \xrightarrow{i^*} & \widetilde{KO}(L_r^{8m+1}) \\ \downarrow c_3=c & \swarrow c_2 & \downarrow c & & \downarrow c_1 & \searrow c & \downarrow c_0 \\ \tilde{K}(L_r^{8m+7}) & \xrightarrow[i^*]{} & \tilde{K}(L_r^{8m+5}) & \xrightarrow[\psi]{} & \tilde{K}(L_r^{8m+5})/\langle 2\sigma^{4m+2}\rangle & \xrightarrow[\psi']{} & \tilde{K}(L_r^{8m+3}) \end{array}$$

is commutative, where i's are the inclusions and ψ, ψ' *are the projections with* $\psi'\psi = i^*$.

PROOF. We prove the proposition inductively by assuming

$$(5.7)\qquad \widetilde{KO}^1(L_r^{8m}) = 0, \quad (\widetilde{KO}^1(L_r^0) = \widetilde{KO}^1(\mathrm{pt}) = 0 \quad \text{is well known}).$$

Consider the Puppe exact sequence

$$\widetilde{KO}(L_r^{2n}) \xrightarrow{i^*} \widetilde{KO}(L_r^{2n-2}) \longrightarrow \widetilde{KO}^1(S^{2n-1}\cup_{2^r} e^{2n}) \xrightarrow{p^*}$$
$$\widetilde{KO}^1(L_r^{2n}) \xrightarrow{i^*} \widetilde{KO}^1(L_r^{2n-2})$$

of the second one in (2.2). Since the left i^* is epimorphic by Proposition 4.4 (i),

$$(5.8)\qquad p^* \quad \textit{is monomorphic for any} \quad n.$$

On the other hand, by the Puppe exact sequence for $\widetilde{KO}^1$ of the first one in the proof of Proposition 4.4, we see easily that

$$(5.9)\qquad \widetilde{KO}^1(S^{2n-1}\cup_{2^r} e^{2n}) = Z_2 \quad \textit{if } n \equiv 1 \textit{ or } 2 \bmod 4, \quad = 0 \quad \textit{otherwise}.$$

Therefore (5.7) implies

$$(5.10)\qquad \widetilde{KO}^1(L_r^{8m+2}) = Z_2.$$

Consider the Bott exact sequences in (5.1) and the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}^1(L_r^{2n-2}) & \xrightarrow{\partial} & \widetilde{KO}(L_r^{2n-2}) & \xrightarrow{c} & \tilde{K}(L_r^{2n-2}) \\ \uparrow i^* & & \uparrow i^* & & \uparrow i^* \\ \widetilde{KO}^1(L_r^{2n}) & \xrightarrow{\partial} & \widetilde{KO}(L_r^{2n}) & \xrightarrow{c} & \tilde{K}(L_r^{2n}) . \end{array}$$

Then by the second equality in (4.5) and the first one in (4.11), (5.10) implies

$$(5.11)\qquad \mathrm{Im}\,\partial = \mathrm{Ker}\,c = Z_2\langle\bar{\sigma}^{2m+1}\rangle = \mathrm{Ker}\, i^*_{8m+2} \quad \text{in} \quad \widetilde{KO}(L_r^{8m+2})$$

(i_k: $L_r^{k-1} \subset L_r^k$). If $\alpha \in \widetilde{KO}^1(L_r^{8m+4})$, then $i^*\partial\alpha = \partial i^*\alpha = a\bar{\sigma}^{2m+1}$ by (5.11), so that

$$\partial\alpha = (a + 2a')\bar{\sigma}^{2m+1} \quad \text{in} \quad \widetilde{KO}(L_r^{8m+4}) \qquad \text{for some } a' \in Z,\ a = 0 \text{ or } 1,$$

by Proposition 4.4 (i) for $k = 8m+4$, $8m+3$. On the other hand, (5.12) $$c\bar{\sigma}^{2m+1} = \sigma^{4m+2} \text{ in } \tilde{K}(L_r^{8m+4}), \quad Z_{2^r}\langle\sigma^{4m+2}\rangle \subset \tilde{K}(L_r^{8m+4}),$$

by Lemma 4.3 and Proposition 2.6. Hence $(a+2a')\sigma^{4m+2}=c\partial\alpha=0$ in $\tilde{K}(L_r^{8m+4})$, so that $a=0$. Thus $\partial i^*\alpha=0$, so that $i^*\alpha=0$ by (5.10–11). Therefore i^*: $\widetilde{KO}^1(L_r^{8m+4})\to\widetilde{KO}^1(L_r^{8m+2})$ is 0, and the first exact sequence and (5.8–9) imply

(5.13) $$\widetilde{KO}^1(L_r^{8m+4}) = Z_2.$$

Similarly, the first equality in (4.5) and (5.12–13) imply that

(5.14) $$\operatorname{Im}\partial = \operatorname{Ker} c = Z_2\langle 2^r\bar{\sigma}^{2m+1}\rangle \quad \text{in} \quad \widetilde{KO}(L_r^{8m+4}).$$

If $\beta\in\widetilde{KO}^1(L_r^{8m+6})$, then $i^*\partial\beta=\partial i^*\beta=2^r b\bar{\sigma}^{2m+1}$ ($b=0$ or 1) by (5.14), and $\partial\beta=2^r b\bar{\sigma}^{2m+1}$ in $\widetilde{KO}(L_r^{8m+6})$ by (4.6). Hence by (4.10) and Proposition 2.6,

$$-2^{r-1}b\sigma^{4m+3} = 2^r b(c\bar{\sigma}^{2m+1}) = c\partial\beta = 0 \quad \text{in} \quad \tilde{K}(L_r^{8m+6}) = \tilde{K}(L_r^{8m+7}),$$

so that $b=0$. Thus $\partial i^*\beta=0$, so that $i^*\beta=0$ by (5. 13–14). Therefore i^*: $\widetilde{KO}^1(L_r^{8m+6})\to\widetilde{KO}^1(L_r^{8m+4})$ is 0, and the first exact sequence and (5. 8–9) imply

(5.15) $$\widetilde{KO}^1(L_r^{8m+6}) = 0, \quad \text{so that } \widetilde{KO}^1(L_r^{8m+7}) = \widetilde{KO}^1(L_r^{8m+8}) = 0$$

by the Puppe exact sequence $\widetilde{KO}^1(S^k)\xrightarrow{p^*}\widetilde{KO}^1(L_r^k)\xrightarrow{i^*}\widetilde{KO}^1(L_r^{k-1})\to\widetilde{KO}^2(S^k)$.

Now, the above proof shows that (5.7), (5. 10–11) *and* (5. 13–15) *are valid for any* m by induction; and these imply the proposition as follows.

$c_3=c$ is monomorphic by (5.15) and the Bott exact sequence, and so is $c_2=c_3 i^{*-1}$ since the upper left i^* in (5.6) is isomorphic by Proposition 4.4 (i).

In the middle of (5.6), we have noticed in (3.2) that

$$\operatorname{Ker}\psi = Z_{2^{r-1}}\langle 2\sigma^{4m+2}\rangle \quad \text{in} \quad \tilde{K}(L_r^{8m+5}).$$

Further i_{8m+5}: $L_r^{8m+4}\subset L_r^{8m+5}$ induces isomorphisms of the $\widetilde{KO}$- and $\tilde{K}$-groups by Propositions 4.4 (i) and 2.6 (ii), and hence (5.12) and (5.14) are valid in $\tilde{K}(L_r^{8m+5})$ and $\widetilde{KO}(L_r^{8m+5})$, respectively. Thus we see that

$$\operatorname{Ker}(\psi c) = Z_{2^r}\langle 2\bar{\sigma}^{2m+1}\rangle \quad \text{in} \quad \widetilde{KO}(L_r^{8m+5}).$$

On the other hand, the upper middle i^* in (5.6) is epimorphic and its kernel is equal to the right hand side of this equality by Proposition 4.4 (i). Therefore $c_1=\psi c i^{*-1}$ is the desired monomorphism.

Similarly, in the right of (5.6), i^* is epimorphic and (5.11) implies $\operatorname{Ker} i^* = \operatorname{Ker} c$ since both i^*_{8m+3}'s on the $\widetilde{KO}$- and $\tilde{K}$-groups are isomorphic. Thus $c_0=ci^{*-1}$ is the desired monomorphism. q. e. d.

As an application of the above proposition, we see the following

COROLLARY 5.16 ([8, Th. 1.4], cf. M. Yasuo [12, Prop. (3.5)]). *In* $\widetilde{KO}(L^n(2^r))$ $(r \geqq 2)$, *the order of the power* $\bar{\sigma}^i$ $(1 \leqq i \leqq [n/2])$ *of* $\bar{\sigma}$ *in* (4.2) *is equal to*

$$2^{r+n-2i} \quad \textit{if } n \textit{ is odd}, \quad 2^{r+n-2i+1} \quad \textit{if } n \textit{ is even};$$

$\bar{\sigma}^{[n/2]+1}$ *is of order* 2 *if* $n \equiv 1 \bmod 4$, *and is* 0 *otherwise*; *and* $\bar{\sigma}^{[n/2]+2}=0$.

PROOF. We see the results for $\bar{\sigma}^i$ $(i>[n/2])$ by Proposition 4.4 (i).

The order of σ^i $(1 \leqq i \leqq n')$ in $\tilde{K}(L^{n'}(2^r))$ is equal to $2^{r+n'-i}$ by [4, Th. 1.1], and we see similarly that the same is true in $\tilde{K}(L^{n'+1}(2^r))/\langle 2\sigma^{n'+1}\rangle$ by using Lemma 3.3 (ii). Furthermore, by Lemma 4.3 and the commutativity of (5.6), the monomorphism c_l in (5.5) satisfies

$$c_l\bar{\sigma}^i = \sigma^{2i}/(1+\sigma)^i.$$

Therefore we see easily the results for $\bar{\sigma}^i$ $(1 \leqq i \leqq [n/2])$. q.e.d.

REMARK 5.17. We can prove that $\widetilde{KO}^1(L_r^k)$ $(r \geqq 2)$ *is equal to*

0 *if* $k \equiv 0, 6$ *or* $7 \bmod 8$, Z_2 *if* $k \equiv 2$ *or* $4 \bmod 8$,

Z *if* $k \equiv 1$ *or* $5 \bmod 8$, $Z_2 \oplus Z_2$ *if* $k \equiv 3 \bmod 8$,

which has been proved by M. Yasuo [13, Th. (0.1–2)]. In fact, the first half is proved in the proof of Proposition 5.3. Then, we see the results for $k \equiv 1, 5 \bmod 8$ by using the exact sequence after (5.15) where p^* is monomorphic, and for $k=8m+3$ by noticing that L_r^k/L_r^{k-2} is homotopy equivalent to $S^{k-1} \vee S^k$ and by studying its Puppe exact sequence.

§6. Some relations in $\widetilde{KO}(L^n(2^r))$

Together with $\bar{\sigma}$ and κ in (4.2), we consider the real restriction

(6.1) $\bar{\sigma}(s) = r\sigma(s) \in \widetilde{KO}(L^n(2^r))$ $(0 \leqq s \leqq r)$, $\bar{\sigma}(0)=\bar{\sigma}$, $\bar{\sigma}(r)=0$,

where $\sigma(s) \in \tilde{K}(L^n(2^r))$ is the element in (2.7).

LEMMA 6.2. (i) *By the complexification* $c\colon \widetilde{KO}(L^n(2^r)) \to \tilde{K}(L^n(2^r))$,

$$c\bar{\sigma}(s) = \sigma(s)^2/(1+\sigma(s)), \quad c(2+\bar{\sigma}(s)) = (2+\sigma(s+1))/(1+\sigma(s)).$$

(ii) (i) *and Lemma* 4.3 *hold also for* c_l *in* (5.5) *instead of* c.

PROOF. (i) The first equality is seen by the same way as the first one in Lemma 4.3, and it implies the second one by (2.8).

(ii) is seen immediately by the commutative diagram (5.6). q.e.d.

Now, we study some relations in $\widetilde{KO}(L^n(2^r))$.

PROPOSITION 6.3. (i) *The following relations hold in* $\widetilde{KO}(L^n(2^r))$ $(r \geqq 2)$:

$$(6.4) \qquad \bar{\sigma}(s) = 4\bar{\sigma}(s-1) + \bar{\sigma}(s-1)^2 \qquad (0 < s \leqq r),$$

i.e., $\bar{\sigma}(s)$ *coincides with the element given by* (1.6).

$$(6.5) \qquad \bar{\sigma}(r-1) = 2\kappa, \quad \kappa^2 = -2\kappa.$$

$$(6.6) \qquad \kappa(2 + \bar{\sigma}(s)) = \sum_{t=s+1}^{r-2}\{(2 + \bar{\sigma}(s))\bar{\sigma}(t)\prod_{u=t+1}^{r-2}(2 + \bar{\sigma}(u))\} \qquad (0 \leqq s \leqq r-2);$$

in particular, $\kappa(2+\bar{\sigma}(r-2))=0$ *and we have* (1.5) *by taking* $s=0$.

(ii) $\widetilde{KO}(L^n(2^r))$ $(r \geqq 2)$ *is generated additively by the elements*

$$\{\kappa, \bar{\sigma}^i: 1 \leqq i \leqq N'\}, \quad N' = \min\{2^{r-1}-1, [n/2] + \varepsilon\},$$

where ε *is the integer in Corollary* 4.12 *and* N' *is the one in Theorem* 1.9 (ii).

PROOF. (i) The second equality in (6.5) follows from $(1+\kappa)^2=\rho^2=1$. By Lemmas 6.2 (i), 4.3 and (2.8), we see easily that

$$c\bar{\sigma}(s) = (2\sigma(s-1) + \sigma(s-1)^2)^2/(1 + \sigma(s-1))^2 = c(4\bar{\sigma}(s-1) + \bar{\sigma}(s-1)^2),$$

$$c\bar{\sigma}(r-1) = 2\sigma(r-1)(1 + \sigma(r-1))/(1 + \sigma(r-1)) = 2c\kappa.$$

Thus we have (6.4) and the first equality in (6.5) for $m \equiv 3 \bmod 4$, since c is monomorphic in this case by (5.2), and so for any n by the equalities in Proposition 4.4 (ii).

Similarly we see (6.6) by showing that the c-images of its both sides are equal as follows: By Lemmas 6.2 (i), 4.3 and (2.8),

$$\begin{aligned} c\{\bar{\sigma}(t)\prod_{u=t+1}^{r-2}(2 + \bar{\sigma}(u))\} &= \sigma(t)^2\prod_{u=t+2}^{r-1}(2 + \sigma(u))/\prod_{u=t}^{r-2}(1 + \sigma(u)) \\ &= \{\sigma(t)(2 + \sigma(t+1)) - \sigma(t+1)\}\prod_{u=t+2}^{r-1}(2 + \sigma(u))/(1 + \sigma(r-1)), \end{aligned}$$

since $\sigma(t)^2(1+\sigma(t))=\sigma(t)(2+\sigma(t+1))-\sigma(t+1)$. Hence the c-image of the right hand side of (6.6) is equal to

$$\begin{aligned} &(2 + \sigma(s+1))\{\sigma(s+1)\prod_{u=s+2}^{r-1}(2 + \sigma(u)) - \sigma(r-1)\}/(1 + \sigma(s))(1 + \sigma(r-1)) \\ &= \{\sigma(r) - (2 + \sigma(s+1))\sigma(r-1)\}/(1 + \sigma(s))(1 + \sigma(r-1)), \end{aligned}$$

which is equal to $c\{\kappa(2+\bar{\sigma}(s))\}$ since $2\sigma(r-1)+\sigma(r-1)^2=\sigma(r)=0$.

(ii) By $\bar{\sigma}(0)=\bar{\sigma}$ and (6.4), we see inductively that

$$\bar{\sigma}(s) = \bar{\sigma}^{2^s} + \sum_{j=1}^{2^s-1} y_{sj}\bar{\sigma}^j \qquad (y_{sj}\text{: even}). \tag{6.7}$$

Hence by the first equality in (6.5), $\bar{\sigma}^{2^{r-1}}$ is a linear combination of

$$\{\kappa, \bar{\sigma}^i\colon 1 \leqq i < 2^{r-1}\}, \tag{6.8}$$

and so is κ^2 by the second one in (6.5). Moreover so is $\kappa\bar{\sigma}$ by (6.6) for $s=0$ whose right hand side is a polynomial on $\bar{\sigma}$ of degree $2^{r-1}-1$ by (6.7).

On the other hand, the ring $\widetilde{KO}(L^n(2^r))$ is generated by $\bar{\sigma}$ and κ by Proposition 4.4 (ii). Thus we see that it is generated additively by (6.8), and we obtain (ii) by Corollary 4.12 (ii). q.e.d.

In the following, we use the monomorphism c_l in (5.5).

LEMMA 6.9 (cf. [8, Lemmas 3.4–5]). *The following relations hold in* $\widetilde{KO}(L^n(2^r))$ $(r\geqq 2)$ *where* $n'=2[n/2]+1$ *is the integer in* (5.4):

(i) *For any integers* $k_0,\dots, k_{s-1}\geqq 0$ *and* $k_s>0$ $(0\leqq s\leqq r)$,

$$2^k\prod_{t=0}^{s}\bar{\sigma}(t)^{k_t} = 0$$

if $k\geqq 0$ *is an integer such that* $2^s(k-r+s)\geqq n'-\sum_{t=0}^{s}2^{t+1}k_t$.

(ii) *For any integers* $k_0,\dots, k_{s-1}\geqq 0$ *and* $k_s>l\geqq 0$ $(0\leqq s<r)$,

$$2^{k'}\alpha\bar{\sigma}(s)^{k_s} = (-1)^l 2^{k'+2l}\alpha\bar{\sigma}(s)^{k_s-l} \qquad \left(\alpha = \prod_{t=0}^{s-1}\bar{\sigma}(t)^{k_t}\right)$$

if $k'\geqq 0$ *is an integer such that* $2^{s+1}(k'-r+s+1)\geqq n'-\sum_{t=0}^{s}2^{t+1}k_t$.

PROOF. (i) By Lemmas 6.2 (ii), 2.9 (i) and 3.3 (ii), we see easily that

$$c_l\left(2^k\prod_{t=0}^{s}\bar{\sigma}(t)^{k_t}\right) = 2^k\prod_{t=0}^{s}\{\sigma(t)^{2k_t}/(1+\sigma(t))^{k_t}\} = 0,$$

where c_l is the monomorphism in (5.5). Thus we see (i).

(ii) We see easily that $2^{k'+2l}\alpha\bar{\sigma}(s)^{k_s-l-2}\bar{\sigma}(s+1)=0$ if $k_s-1>l\geqq 0$ by (i). Thus we have (ii) by (6.4). q.e.d.

§7. Proof of the main theorem

Furthermore, we obtain the following relations in $\widetilde{KO}(L^n(2^r))$ $(r\geqq 2)$.

LEMMA 7.1. (i) *If* $0<s\leqq r-2$, $n<2^s k$ *and* $k\geqq 2$, *then*

$$\sum_{t=0}^{s}2^{r-s-3+2^t k}\bar{\sigma}(s-t) = 0 \qquad in \quad \widetilde{KO}(L^n(2^r)).$$

(ii) *In case* $r \geqq 3$ *or* $n \not\equiv 1 \bmod 4$, *if* $n < 2^{r-1}k$ *and* $k \geqq 2$, *then*

$$2^{k-1}\kappa + \sum_{t=1}^{r-1} 2^{2^t k-2}\bar{\sigma}(r-1-t) = 0 \qquad in \quad \widetilde{KO}(L^n(2^r)).$$

(iii) *If* $n < 2^{r-1}k$, *then* $2^k\kappa = 0$ *in* $\widetilde{KO}(L^n(2^r))$.

PROOF. (i) By Proposition 5.3, it is sufficient to show that

$$(*) \qquad \sum_{t=0}^{s} 2^{r-s-3+2^t k} c_l \bar{\sigma}(s-t) = 0 \qquad \text{in} \begin{cases} \tilde{K}(L^{n'}(2^r)) & \text{if} \quad l \neq 1, \\ \tilde{K}(L^{n'+1}(2^r))/\langle 2\sigma^{n'+1}\rangle & \text{if} \quad l = 1, \end{cases}$$

where $n \equiv l \bmod 4$, c_l is the one in (5.5) and n' is the one in (5.4).

If $t \geqq 2$, then we see easily that

$$2^{r-s-3+2^t k} c_l \bar{\sigma}(s-t) = -2^{r-s-1+2^t(k-1)} c_l \bar{\sigma}(s-t)^{2^{t-1}} \qquad \text{(by Lemma 6.9 (ii))}$$

$$= -2^{r-s-1+2^t(k-1)} \sigma(s-t)^{2^t}(1+\sigma(s-1))/(1+\sigma(s))$$

(by Lemma 6.2 (ii) and (2.7))

$$= -2^{r-s-1+2^t(k-1)} \sigma(s-t)^{2^t}/(1+\sigma(s)) \qquad \text{(by Lemmas 2.9 (i) and 3.3 (ii))}.$$

Similarly, we see easily that $2^{r-s-3+2k} c_l \bar{\sigma}(s-1)$ is equal to

$$2^{r-s-3+2k}(\sigma(s-1)^2 + \sigma(s-1)^3)/(1+\sigma(s)) = -2^{r-s-1+2(k-1)}\sigma(s-1)^2/(1+\sigma(s))$$

by Lemmas 2.9 (ii) and 3.3 (ii); and $2^{r-s-3+k} c_l \bar{\sigma}(s)$ is equal to

$$2^{r-s-3+k}\sigma(s)^2/(1+\sigma(s)) = 2^{r-s-2+k}\sigma(s)/(1+\sigma(s))$$

(by Lemmas 2.9 and 3.3 (ii)).

Thus we see (*) by Lemma 3.5, since $n' < 2^s k$ by $n < 2^s k$, $s > 0$ and (5.4).

(ii) By noticing that $n < 2^{r-1}k$ implies $n'+1 < 2^{r-1}k$ if $n \equiv 1 \bmod 4$ and $r \geqq 3$, we see by the above proof that (i) holds also for $s = r-1$ unless $n \equiv 1 \bmod 4$ and $r = 2$. Thus (ii) holds since $2\kappa = \bar{\sigma}(r-1)$ by (6.5).

(iii) By Lemmas 6.2 (ii), 2.9 (i) and 3.3 (ii), we see that $2^k c_l \kappa = 2^k \sigma(r-1) = 0$. Thus (iii) holds. q.e.d.

Here we attend to the special case $r = 2$.

PROOF OF THEOREM 1.9 (i). In $\widetilde{KO}(L^n(4))$ $(n > 0)$, we see that

$$2^{2[n/2]+1}\bar{\sigma} = 0 \qquad \text{(by Lemma 6.9 (i))},$$

$$2^{[n/2]}(\kappa + 2^{[n/2]}\bar{\sigma}) = 0 \quad \text{if} \quad n \not\equiv 1 \bmod 4 \qquad \text{(by Lemma 7.1 (ii))},$$

and $2^{[n/2]+1}\kappa = 0$ (by Lemma 7.1 (iii)). On the other hand, $3[n/2]+1+\varepsilon$ is the order of $\widetilde{KO}(L^n(4))$ by Corollary 4.12 (i), and we see the desired result by Proposition 6.3 (ii). q.e.d.

To study the case $r \geqq 3$, we consider the first term

$$\bar{\sigma}(s, d) = \bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{s-1}(2 + \bar{\sigma}(t)) \in \widetilde{KO}(L^n(2^r)) \qquad (0 < s \leqq r-2,\ 0 < d < 2^s) \tag{7.2}$$

in the definition of $\bar{\sigma}_i$ for $i = 2^s + d$ in Theorem 1.9 (ii) (cf. (6.4)).

LEMMA 7.3. (i) *If* $0 < s \leqq r-2$, $n < 2d + 2^{s+1}k'$ *and* $r - s - 2 + k' \geqq 0$, *then*

$$2^{r-s-2+k'}\bar{\sigma}(s, d) = 0 \quad \textit{in} \quad \widetilde{KO}(L^n(2^r)).$$

(ii) *If* $k' \geqq 2$ *in addition to* (i), *then*

$$2^{r-s-3+k'}\{\bar{\sigma}(s, d) + \textstyle\sum_{t=1}^{s+1}(-1)^{2^{t-1}}2^{(2^t-1)k'-1}\bar{\sigma}^d\bar{\sigma}(s+1-t)\} = 0$$

in $\widetilde{KO}(L^n(2^r))$, *unless* $n \equiv 1 \bmod 4$, $s = r-2$ *and* $n+1 = 2d + 2^{r-1}k'$.

PROOF. (i) In the same way as the proof of (6.6), we see that

$$c_l\bar{\sigma}(s, d) = \sigma^{2d-2}\sigma(1)\sigma(s+1)/(1+\sigma)^d(1+\sigma(s))$$

by using Lemma 6.2 (ii). Hence we see easily that $2^{r-s-2+k'}c_l\bar{\sigma}(s, d) = 0$ by Lemmas 2.9 (i) and 3.3 (ii). Thus we have (i) by Proposition 5.3.

(ii) Similarly, by using (2.8), we see that

$$2^{r-s-3+k'}c_l\bar{\sigma}(s, d) = 2^{r-s-3+k'}\sigma^{2d}\sigma(s+1)/(1+\sigma)^d(1+\sigma(s)).$$

Also, we see easily that $(-1)^{2^{t-1}}2^{r-s-4+2^tk'}c_l(\bar{\sigma}^d\bar{\sigma}(s+1-t))$ is equal to

$$-2^{r-s-2+2^t(k'-1)}c_l(\bar{\sigma}^d\bar{\sigma}(s+1-t)^{2^{t-1}}) \qquad \text{(by Lemma 6.9 (ii))}$$

$$= -2^{r-s-2+2^t(k'-1)}\sigma^{2d}\sigma(s+1-t)^{2^t}/(1+\sigma)^d(1+\sigma(s))$$

(by Lemma 6.2 (ii) and (2.7)).

Since the assumption $n < 2d + 2^{s+1}k'$ implies $n' < 2d + 2^{s+1}k'$ by (5.4), these and Lemma 3.5 imply that the c_l-image of the left hand side in (ii) is 0. Thus we see (ii) by Proposition 5.3. q.e.d.

Now, we are ready to prove the main theorem.

PROOF OF THEOREM 1.9 (ii). By the definition of $u(i)$ and $\bar{\sigma}_i$ in the theorem, the equality

$$u(i)\bar{\sigma}_i = 0 \quad (0 \leqq i \leqq N') \qquad \text{in} \quad \widetilde{KO}(L^n(2^r))$$

follows from Lemma 6.9 (i) for $n>1$ and $i=1$; from Lemma 7.1 for $i=2^s$ $(s\geqq 1)$ or 0 since

$$n<2^s(a_s+1), \quad \text{and} \quad a_s\geqq 1 \quad \text{if} \quad 2^s\leqq N' \quad \text{or} \quad a_{r-1}\geqq 1 \quad \text{if} \quad n\geqq 2^{r-1};$$

from (4.5) for $n\equiv 1 \bmod 4$ and $i=a_1+1$; and from Lemma 7.3 otherwise since

$$n<2d+2^{s+1}a'(i), \quad a'(i)\geqq 2 \quad \text{if} \quad i=d+2^s\leqq a_1(=[n/2]).$$

On the other hand $\bar{\sigma}(s, d)=\bar{\sigma}^{2^s+d}+\sum_{j=1}^{2^s} y'_{sj}\bar{\sigma}^{j+d-1}$ by (7.2) and (6.7), and hence the definition of $\bar{\sigma}_i$, (6.7) and Corollary 4.12 (ii) imply

$$\bar{\sigma}_0=\kappa+\sum_{j=1}^{N'} z_j\bar{\sigma}^j; \quad \bar{\sigma}_i=\sum_{j=1}^{i} z_{ij}\bar{\sigma}^j, \quad z_{ii}\colon \text{odd} \quad (1\leqq i\leqq N'),$$

since $a'(i)\geqq 2$ as is noticed above. Thus Proposition 6.3 (ii) and Corollary 4.12 (i) imply that $\widetilde{KO}(L^n(2^r))$ is generated additively by

$$\{\bar{\sigma}_i\colon 0\leqq i\leqq N'\}.$$

Furthermore, by the definition of $u(i)$, we see easily that

$$\prod_{i=0}^{N'} u(i)=2^\lambda, \qquad \lambda=(r+1)[n/2]+1+\varepsilon.$$

In fact, this is clear if $n<2$. If $n=2^u+b_u$ $(1\leqq u<r)$, then

$$\begin{aligned}\lambda &= 1+(r-1+2a_1)+\textstyle\sum_{s=0}^{u-2}\{(r-s-3+a_{s+1})2^s+[b_{s+1}/2]\}\\ &\quad +(r-u)[b_u/2]+\varepsilon\\ &= r+(u+1)a_1+(r-u)2^{u-1}-(r-1)+(r-u)(a_1-2^{u-1})+\varepsilon\\ &= (r+1)a_1+1+\varepsilon.\end{aligned}$$

If $n\geqq 2^r$, then we have the above equality by replacing the first term 1 with $a_{r-1}-(a_{r-1}-1)$ and by taking $u=r$.

Moreover the order of $\widetilde{KO}(L^n(2^r))$ is equal to 2^λ by Corollary 4.12 (i).

Thus we have proved the theorem completely. q.e.d.

REMARK 7.4. We notice that the additive base of $\widetilde{KO}(L^n(8))$ given in [7, Prop. (4.3)] is slightly different from that in Theorem 1.9 for $r=3$, but these are related by the relation (6.6) for $s=0$.

Finally, we notice the following

THEOREM 7.5. *For $n>0$, the reduced KO-group of the $2n$-skeleton $L_0^n(2^r)$ $(r\geqq 2)$ of (2.1) is given by*

$$\widetilde{KO}(L_0^n(2^r)) = \begin{cases} \widetilde{KO}(L^n(2^r))/Z_2\langle 2^{r+n-2}\bar{\sigma}\rangle & \text{if} \quad n \equiv 0 \bmod 4, \\ \widetilde{KO}(L^n(2^r)) & \text{otherwise}. \end{cases}$$

Proof. By Proposition 4.4 (i), it is sufficient to show that

$$\kappa\bar{\sigma}^{2m} = 2^{r+4m-2}\bar{\sigma} \quad \text{in} \quad \widetilde{KO}(L^{4m}(2^r)) \qquad (m > 0).$$

This is seen by Proposition 5.3, since $c_0(\kappa\bar{\sigma}^{2m})$ is equal to

$$\sigma^{4m}\sigma(r-1)/(1+\sigma)^{2m} = 2^{r-1}\sigma^{4m+1}/(1+\sigma) = 2^{r+4m-2}\sigma^2/(1+\sigma) = 2^{r+4m-2}c_0\bar{\sigma}$$

in $\widetilde{K}(L^{4m+1}(2^r))$ by Lemmas 6.2 (ii), 2.9, Proposition 2.6 and (2.7). q.e.d.

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