HIROSHIMA MATH. J.
8 (1978), 439–468

# On a Mixed Problem for the Multi-Dimensional Hamilton-Jacobi Equation in a Cylindrical Domain

Yoshihito TOMITA

(Received February 9, 1978)

## 1. Introduction

Let $\Omega$ be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, and $Q$ be the cylinder $(0, \infty)\times\Omega$. We consider the mixed initial and boundary value problem (hereafter called (MP)) for the Hamilton-Jacobi equation in $Q$:

$$(1.1)\qquad u_t + H(t, x, u, u_x) = 0, \qquad (t, x)\in Q,$$

$$(1.2)\qquad u(0, x) = u_0(x), \qquad x\in\bar{\Omega},$$

$$(1.3)\qquad u(t, x) = \phi(x), \qquad (t, x)\in \boldsymbol{R}^+ \times \partial\Omega.$$

Here $\bar{\Omega}$ and $\boldsymbol{R}^+$ denote $\bar{\Omega}=\Omega\cup\partial\Omega$ and $\boldsymbol{R}^+=[0, \infty)$ respectively, $u(t, x)$ is a real-valued function, $H$: $\boldsymbol{R}^+\times\bar{\Omega}\times\boldsymbol{R}^1\times\boldsymbol{R}^n\to\boldsymbol{R}^1$, and $u_x$ denotes the gradient $(u_{x_1},\dots, u_{x_n})$ in the space variables $x$.

The purpose of this paper is to establish the existence and uniqueness of global generalized solutions of (MP). We employ the so-called vanishing viscosity method in proving existence for (MP). The reason for the employment of this method lies in its advantage in estimating the local semi-concavity constant which will be described in the next section. As an intermediate step in the development, we shall solve a mixed problem for a nonlinear second-order parabolic equation by making use of the semigroup approximation theory. The semigroup approach enables us not only to prove the existence of a (generalized) solution of the mixed problem for regularized parabolic equations, but also to employ the vanishing viscosity method.

This investigation is a sequel to our earlier work [20] and is motivated by the works of Aizawa [1, 3] and Kružkov [15]. Aizawa [1] treated the Cauchy problem for the Hamilton-Jacobi equation in one space variable

$$(*)\qquad u_t + f(u_x) = 0, \qquad t > 0, \quad -\infty < x < +\infty,$$

from the viewpoint of the nonlinear semigroup theory, and constructed a global generalized solution, assuming only that $f$ is continuous. He subsequently studied the Cauchy problem for the multi-dimensional equation of this type from the same point of view (cf. [3]). For related works on similar treatments of Cauchy problems, we mention the recent papers of Burch [7] and Tamburro [19]. In these papers existence theorems have been proved under the assumption that $f=f(p)$ is convex in $p=(p_1,\dots, p_n)$. See also the more recent work of Burch and Goldstein [8] in which results concerning the Cauchy problem are refined to study some boundary value problems for $(*)$ in the quadrant $\boldsymbol{R}^+\times\boldsymbol{R}^+$. On the other hand, Kružkov [15] has established the existence and uniqueness of generalized solutions of the Cauchy-Dirichlet problem:

$$\begin{cases} H(x, u, u_x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = \phi. \end{cases}$$

However, his result cannot directly be applied to our problem (MP), since he assumed that $H(x, u, p)$ is nonincreasing in $u$ and strictly convex in $p$.

We also note that some earlier results on mixed problems for Hamilton-Jacobi equations were obtained by Conway and Hopf [9], Aizawa and Kikuchi [4] and Benton [5, 6]. These authors proved the existence by using the variational method assuming that the Hamiltonian is strictly convex in $p$.

The outline of the present paper is as follows. In Section 2 we list the assumptions on $H$, $u_0$ and $\phi$, and define a generalized solution of (MP). Further, in that section, we state two theorems concerning the existence and uniqueness of solutions. In Section 3 we verify the uniqueness and continuous dependence result under the assumption that $H$ is convex in $p$. Sections 4, 5 and 6 are devoted to the study of a mixed problem (denoted by (Pa.MP)) for a nonlinear parabolic equation of the form

$$u_t + H(t, x, u, u_x) = \mu\Delta u \qquad (\mu > 0),$$

where $\Delta$ is the Laplace operator. In Section 4 we state and prove the Generation Theorem which is an appropriately modified form of the Crandall-Pazy theorem [11; Theorem 2.1]. In Section 5, in order to apply this Generation Theorem to (Pa.MP), we investigate boundary value problems for a nonlinear second-order elliptic differential equation. In Section 6 we construct a generalized solution of (Pa.MP). Section 7 contains the proof of our existence theorem for (MP). Here, roughly speaking, our generalized solution of (MP) is obtained as the limit of solutions of (Pa.MP) as $\mu \downarrow 0$.

The author would like to express his hearty gratitude to Professor S. Aizawa for his constant encouragement and many valuable advices during the preparation of this paper.

NOTATIONS. In this paper the letters $x$, $y$, $\Delta x$, $p$ and $q$ are points in $\boldsymbol{R}^n$.

For $p=(p_1,\dots, p_n)$ and $q=(q_1,\dots, q_n)$ in $\boldsymbol{R}^n$, we set $(p, q)=\sum_{i=1}^{n} p_i q_i$ and $|p|^2=(p, p)$. For every $T>0$, let $Q_T$ be the cylinder $(0, T)\times\Omega$. By $U_\delta(y)$ we denote the closed sphere in $\boldsymbol{R}^n$ of center $y$ and radius $\delta$. Similarly, for each compact set $K$ in $\boldsymbol{R}^n$, $U_\delta(K)$ denotes the closed $\delta$-neighborhood of $K$. For every small $\delta>0$, let $\Omega_\delta=\{x\in\Omega;\ \mathrm{dist}(x, \partial\Omega)>\delta\}$ and let $B^\delta$ be the boundary strip, i.e., $B^\delta=\{x\in\Omega;\ \mathrm{dist}(x, \partial\Omega)<\delta\}$. For given $T>0$, $M_0>0$ and $M_1>0$, we define

$$W(T, M_0) = \{(t, x, u, p)\in \boldsymbol{R}^{2n+2};\ (t, x)\in\overline{Q_T},\ |u|\leqq M_0,\ p\in\boldsymbol{R}^n\},$$

$$W(T, M_0, M_1) = \{(t, x, u, p)\in W(T, M_0);\ |p|\leqq M_1\}.$$

We denote by $\mathscr{L}(\bar\Omega)$ the space of Lipschitz continuous functions on $\bar\Omega$. Similarly we define $\mathscr{L}(\overline{Q_T})$. Define by $C^{m+\alpha}(\Omega)$ (resp. $C^{m+\alpha}(\bar\Omega)$) the space of all functions in $C^m(\Omega)$ (resp. $C^m(\bar\Omega)$) whose derivatives of order $m$ are Hölder continuous (with exponent $\alpha$) on compact sets in $\Omega$ (resp. $\bar\Omega$). $g*h$ denotes the convolution of $g$ and $h$.

## 2. Assumptions and results

Throughout this paper we shall assume for simplicity that the Hamiltonian $H(t, x, u, p)$ is real-valued and of class $C^{2+\alpha}$ with respect to all its arguments in $\boldsymbol{R}_t^+\times\bar\Omega\times\boldsymbol{R}_u^1\times\boldsymbol{R}_p^n$ (In fact, with respect to the $t$-derivatives of $H$, it suffices to assume the existence and continuity of $H_t$.) and satisfies the following four assumptions:

(H.I) *For every $T>0$, $M_0>0$ and $M_1>0$, $H$ is strictly convex in $p$ uniformly on $W(T, M_0, M_1)$. That is, there is a positive constant $a_1=a_1(T, M_0, M_1)$ such that*

$$(H_{pp}(t, x, u, p)\xi, \xi)=\sum_{i,j=1}^{n} H_{p_i p_j}(t, x, u, p)\xi_i\xi_j \geqq a_1|\xi|^2$$

*for all $\xi=(\xi_i)\in\boldsymbol{R}^n$ and $(t, x, u, p)\in W(T, M_0, M_1)$;*

(H.II) $\lim_{|p|\to\infty} H(t, x, u, p)/|p|=+\infty$ *holds uniformly on $\overline{Q_T}\times[-M_0, M_0]$ for given $T>0$ and $M_0>0$;*

(H.III) *For every $T>0$ and $M_0>0$, there are two constants $a_2=a_2(T, M_0)$ and $a_3=a_3(T, M_0)$ such that*

$$|H_x(t, x, u, p)|\leqq a_2|p| + a_3 \qquad \text{for} \quad (t, x, u, p)\in W(T, M_0);$$

(H.IV) *For every $T>0$, there is a constant $\omega\geqq 0$ such that*

$$H_u(t, x, u, p)\geqq -\omega \qquad \text{for} \quad (t, x, u, p)\in\overline{Q_T}\times\boldsymbol{R}_u^1\times\boldsymbol{R}_p^n.$$

Now we give the definition of a generalized solution of (MP). It is known that the mixed problem for the Hamilton-Jacobi equation does not have, in general, a global classical solution even if the data are smooth. On the other hand, in the class of weak solutions (Lipschitz continuous functions that satisfy the equation a.e.) uniqueness fails.

DEFINITION 2.1. *A function $u(t, x)$ defined in $Q$ is called a generalized solution of* (MP) *if*

(i) *for every $T>0$, $u \in \mathscr{L}(\overline{Q_T})$ and $u$ satisfies* (1.1) *a.e. in $Q_T$,*

(ii) *$u$ satisfies* (1.2) *and* (1.3),

(iii) *$u$ satisfies a local semi-concavity condition in the following sense. For each compact set $K$ ($\subset\subset\Omega$) and every $\delta>0$ such that $U_{2\delta}(K)\subset\Omega$, there is a nonnegative and continuous function $a_{K,\delta}(t)$ defined in $(0, \infty)$ such that*

$$u(t, x + \Delta x) - 2u(t, x) + u(t, x - \Delta x) \leqq a_{K,\delta}(t)\,|\Delta x|^2$$

*for $t>0$ and $x$, $x+\Delta x$, $x-\Delta x \in U_\delta(K)$ with $|\Delta x|<\delta$.*

It should be noted that the condition (iii) of Definition 2.1 is a modified form of the semi-concavity condition:

$$u(t, x + \Delta x) - 2u(t, x) + u(t, x - \Delta x) \leqq a(t)\,|\Delta x|^2$$

for $x$, $\Delta x \in \boldsymbol{R}^n$, which Douglis [12] and Kružkov [14] imposed on the possible solutions in order to have the uniqueness for the Cauchy problem for the Hamilton-Jacobi equation. In mixed problems, it seems more natural to weaken the semi-concavity condition to our condition (iii). We also note that if we define a generalized solution of (MP) without requiring (iii) then uniqueness may fail.

We now state the assumptions on $u_0$ and $\phi$. Following Kružkov [15], we introduce a concept of local semi-concavity. $E_{loc}(\Omega)$ denotes the set of functions $v$ defined in $\Omega$ such that $v$ satisfies the following condition: *For each compact set $K$ ($\subset\subset\Omega$) and every $\delta>0$ such that $U_{2\delta}(K)\subset\Omega$, there is a constant $C_{K,\delta}$ such that*

$$v(x + \Delta x) - 2v(x) + v(x - \Delta x) \leqq C_{K,\delta}|\Delta x|^2$$

*for $x$, $x+\Delta x$, $x-\Delta x \in U_\delta(K)$ with $|\Delta x|<\delta$.*

We make the following assumptions on the data $\{u_0, \phi\}$:

(B.I) $u_0 \in \mathscr{L}(\bar{\Omega}) \cap E_{loc}(\Omega)$;

(B.II) *There exists a function $\Phi \in \mathscr{L}(\bar{\Omega})$ such that $\Phi(x) \leqq u_0(x)$ for $x \in \bar{\Omega}$, $\Phi(x)=\phi(x)$ for $x \in \partial\Omega$, and*

$$H(t, x, \Phi, \Phi_x) \leqq 0, \qquad a.e. \text{ in } \quad \Omega$$

*for each $t \geqq 0$.*

The theorems described below are the main results of the present paper. For the general existence and uniqueness, we have:

THEOREM 1. *Under the assumptions* (H.I)–(H.IV) *and* (B.I)–(B.II), *there exists a unique generalized solution of* (MP).

Note that the uniqueness for (MP) we shall prove in the next section holds under the assumption that $H$ is merely convex in $p$.

The assumption (B.II) is rather implicit when applied to (MP). In the rest of this section we shall give more explicit sufficient conditions. First we consider the following assumptions.

(H–B) $H_u \geqq 0$, *i.e.*, $\omega = 0$ *in* (H.IV). *Also*, $\phi$ *satisfies*

$$H(t, x, \sup_{x \in \partial\Omega} \phi(x), 0) \leqq 0 \quad \textit{for} \quad (t, x) \in \bar{Q}.$$

Under the assumption (H–B), we can find a constant $L$ such that

$$H(t, x, \sup_{\partial\Omega} \phi, p) \leqq 0 \quad \text{for} \quad (t, x) \in \bar{Q} \quad \text{and} \quad |p| \leqq L. \tag{2.1}$$

THEOREM 2. *Let the assumptions* (H.I)–(H.III), (H–B) *and* (B.I) *be fulfilled. Assume that* $\{u_0, \phi\}$ *satisfies*

$$|\phi(x) - \phi(y)| \leqq L|x - y| \quad \textit{for} \quad x, y \in \partial\Omega, \tag{2.2}$$

$$u_0(x) \geqq \Phi(x) \equiv \max_{y \in \partial\Omega} \{\phi(y) - L|x - y|\} \quad \textit{for} \quad x \in \Omega, \tag{2.3}$$

*where $L$ is the constant satisfying* (2.1). *Then there exists a unique generalized solution of* (MP).

PROOF. It is sufficient to verify that the $\{u_0, \phi\}$ satisfies the assumption (B.II). By the definition of $\Phi$, we have

$$\Phi(x_1) - \Phi(x_2) \leqq L \max_{y \in \partial\Omega} \{|x_2 - y| - |x_1 - y|\} \leqq L|x_1 - x_2|$$

for $x_1, x_2 \in \Omega$. Similarly, $\Phi(x_2) - \Phi(x_1) \leqq L|x_1 - x_2|$. Hence,

$$|\Phi(x_1) - \Phi(x_2)| \leqq L|x_1 - x_2| \quad \text{for} \quad x_1, x_2 \in \Omega.$$

This shows that $\Phi \in \mathscr{L}(\bar{\Omega})$ and $\|\Phi_x\|_\infty \leqq L$. Therefore, from the definition of $L$ and the fact that $\Phi(x) \leqq \sup_{x \in \partial\Omega} \phi(x)$ for $x \in \Omega$, it follows that $H(t, x, \Phi(x), \Phi_x(x)) \leqq 0$ a.e. in $\Omega$ for each $t \geqq 0$.

On the other hand, by (2.2) and (2.3), we see that

$$\Phi(x) \leqq L|x - x_0| + \phi(x_0) \quad \text{for} \quad x \in \Omega, \ x_0 \in \partial\Omega.$$

Then, since $\Phi(x) \geqq \phi(x_0) - L|x - x_0|$, we have $|\Phi(x) - \phi(x_0)| \leqq L|x - x_0|$ for every $x \in \Omega$ and $x_0 \in \partial\Omega$. This implies that $\Phi(x) = \phi(x)$ for $x \in \partial\Omega$. The proof is complete.

Next we assume, in particular, that

(H.IV)* *H is independent of u, i.e.,* $H = H(t, x, p)$; *and satisfies* $H(t, x, 0) \leqq 0$ *for* $(t, x) \in \bar{Q}$.

Note that under the assumption (H.IV)* there is an $L^*$ such that

$$H(t, x, p) \leqq 0 \quad \text{for} \quad (t, x) \in \bar{Q} \quad \text{and} \quad |p| \leqq L^*. \tag{2.4}$$

COROLLARY 1. *In addition to* (H.I)–(H.III), *let* (H.IV)* *be satisfied. Assume that* $\phi$ *satisfies*

$$|\phi(x) - \phi(y)| \leqq L^*|x - y| \quad \textit{for} \quad x, y \in \partial\Omega,$$

*and that* $u_0$ *satisfies* (B.I) *and*

$$u_0(x) \geqq \Phi^*(x) \equiv \max_{y \in \partial\Omega} \{\phi(y) - L^*|x - y|\} \quad \textit{for} \quad x \in \Omega,$$

*where* $L^*$ *is the constant satisfying* (2.4). *Then there exists a unique generalized solution of* (MP).

PROOF. This follows immediately from Theorems 1 and 2.

## 3. Uniqueness

In this section we prove the uniqueness part of Theorem 1 assuming only that $H$ is convex in $p$. Let $T > 0$ be arbitrarily fixed. For each solution $u$, let $M_0$, $M_1$ be constants such that $|u(t, x)| \leqq M_0$ on $\overline{Q_T}$ and $|u_x(t, x)| \leqq M_1$ a.e. in $Q_T$, and let

$$\hat{\omega} = -\min\{H_u(t, x, u, p);\ (t, x, u, p) \in W(T, M_0, M_1)\}.$$

Without loss of generality we can assume $\hat{\omega} \geqq 0$. We now define

$$N_0 = \sup\{[\sum_{i=1}^{n} (H_{p_i}(t, x, u, p))^2]^{1/2};\ (t, x, u, p) \in W(T, M_0, M_1)\}.$$

For $N \geqq N_0$, let $\mathscr{K}$ denote the cone:

$$\mathscr{K} = \{(t, x) \in \boldsymbol{R}^1 \times \boldsymbol{R}^n;\ 0 \leqq t \leqq T,\ |x| \leqq N(T - t)\},$$

and let $S(t)$ be the horizontal plane of $\mathscr{K}$ with altitude $t$.

THEOREM 3 (*Continuous dependence*). *Suppose that H is convex in p,* *i.e., the matrix* $(H_{p_i p_j})$ *is nonnegative. Let* $u$, $v$ *be generalized solutions of* (MP) *with data* $\{u_0, \phi(t, x)\}$ *and* $\{v_0, \psi(t, x)\}$, *respectively. For* $u$ *and* $v$, *let* $M_0$ *be a common absolute bound,* $M_1$ *be a common Lipschitz constant with respect to* $x$, *and let* $\hat{\omega}$ *be the constant mentioned above. Then for* $0 \leqq t \leqq T$,

$$\begin{aligned} &\sup\{|u(t, x) - v(t, x)|;\, x \in S(t) \cap \Omega\} \\ (3.1)\qquad &\leqq e^{\hat{\omega} t}[\sup\{|u_0(x) - v_0(x)|;\, x \in S(0) \cap \Omega\} \\ &\quad + \sup\{|\phi(\tau, y) - \psi(\tau, y)|;\, (\tau, y) \in \bigcup_{0 \leqq \tau \leqq t} \{\tau\} \times (S(\tau) \cap \partial\Omega)\}]. \end{aligned}$$

PROOF. Let $\zeta(t, x)$ be a function in $C_0^\infty(\boldsymbol{R}^{n+1})$ such that $\zeta \geqq 0$, $\zeta(t, x) \doteq 0$ for $t^2 + |x|^2 \geqq 1$ and $\iint_{\boldsymbol{R}^{n+1}} \zeta\, dt dx = 1$, and let $\zeta^\varepsilon(t, x) = \varepsilon^{-(n+1)} \zeta(t/\varepsilon, x/\varepsilon)$ for $\varepsilon > 0$. Let $0 < \rho < \tau < T$ be fixed, and let $\Omega^\delta$ be a subdomain of $\Omega$ such that $\Omega_{2\delta} \subset \Omega^\delta \subset \Omega_\delta$ and the Stokes theorem is valid for $\Omega^\delta$. By a well-known extension theorem, there is a continuous function $\tilde{u}\colon \boldsymbol{R}^{n+1} \to \boldsymbol{R}^1$ such that $\tilde{u} = u$ for $(t, x) \in \overline{Q_T}$ and $|\tilde{u}| \leqq M_0$ for $(t, x) \in \boldsymbol{R}^{n+1}$. Again denoting $\tilde{u}$ by $u$, we set $u^\varepsilon(t, x) = (\zeta^\varepsilon * u)(t, x)$. In a similar fashion we define $v^\varepsilon(t, x) = (\zeta^\varepsilon * v)(t, x)$.

First we note that

$$(3.2)\qquad \begin{aligned} &|u^\varepsilon(t, x)| \leqq M_0 \quad \text{and} \quad |v^\varepsilon(t, x)| \leqq M_0 \\ &|u_x^\varepsilon(t, x)| \leqq M_1 \quad \text{and} \quad |v_x^\varepsilon(t, x)| \leqq M_1 \end{aligned}$$

hold for $(t, x) \in [0, T] \times \Omega^\delta$, provided $\varepsilon < \delta/2$. Secondly, we note that by virtue of (iii) of Definition 2.1 and the result of [15; Lemma 2.4], there is a constant $C$, depending only on $\delta$, $\rho$ and $T$ such that

$$(3.3)\qquad u_{ll}^\varepsilon \leqq C \quad \text{and} \quad v_{ll}^\varepsilon \leqq C$$

for every $(t, x) \in [\rho/2, T] \times \Omega^\delta$ and every $l \in \boldsymbol{R}^n$, where $u_{ll}^\varepsilon$ and $v_{ll}^\varepsilon$ are the second directional derivatives of $u^\varepsilon$ and $v^\varepsilon$ with respect to $l$, respectively.

We put

$$(3.4)\qquad (u^\varepsilon - v^\varepsilon)_t + H(t, x, u^\varepsilon, u_x^\varepsilon) - H(t, x, v^\varepsilon, v_x^\varepsilon) \equiv \beta^\varepsilon(t, x).$$

Now let $\delta(\eta)$ be a function of $C_0^\infty(\boldsymbol{R}^1)$ such that $\delta \geqq 0$, $\delta(\eta) = 0$ for $|\eta| \geqq 1$ and $\int_{\boldsymbol{R}^1} \delta(\eta) d\eta = 1$, and let $\delta_h(\eta) = h^{-1}\delta(\eta/h)$ for any $h > 0$. We define $\Phi^\varepsilon(t, x)$ by

$$\Phi^\varepsilon(t, x) = (\alpha_h(t - \rho) - \alpha_h(t - \tau))\chi_h(t, x)(u^\varepsilon - v^\varepsilon)^{2s-1},$$

where $s$ is a positive integer, $0 < h < \frac{1}{2}\min\{\rho, T - \tau\}$,

$$\chi_h(t, x) \equiv \chi(t, x) = 1 - \alpha_h(|x| + N(t - T) + h)$$

and

$$\alpha_h(\xi) = \int_{-\infty}^{\xi} \delta_h(\eta)d\eta \qquad (\xi \in \boldsymbol{R}^1).$$

It is easy to see that $\chi(t, x)=0$ outside of $\mathscr{K}$, $\chi=\chi_h(t, x)\to 1$ as $h \downarrow 0$ for $(t, x) \in \operatorname{int}(\mathscr{K})$ and

$$\chi_t + N_0|\chi_x| \leqq \chi_t + N|\chi_x| = 0 \qquad \text{for} \quad (t, x) \in \mathscr{K}. \tag{3.5}$$

Multiplying (3.4) by $\Phi^\varepsilon(t, x)$ and integrating over $Q_{\delta,T}=[0, T]\times\Omega^\delta$, we have

$$\begin{aligned}\iint_{Q_{\delta,T}} \Big[ (u^\varepsilon - v^\varepsilon)_t \Phi^\varepsilon + \int_0^1 H_u(\cdots)\, d\lambda (u^\varepsilon - v^\varepsilon)\Phi^\varepsilon \\ + \Big(\sum_{i=1}^n \int_0^1 H_{p_i}(\cdots)\, d\lambda (u^\varepsilon - v^\varepsilon)_{x_i}\Big)\Phi^\varepsilon \Big] dt\,dx = \iint_{Q_{\delta,T}} \beta^\varepsilon \Phi^\varepsilon dt\,dx,\end{aligned} \tag{3.6}$$

where $(\cdots)=(t, x, \lambda u^\varepsilon+(1-\lambda)v^\varepsilon, \lambda u_x^\varepsilon+(1-\lambda)v_x^\varepsilon)$. We first let $\varepsilon \downarrow 0$. By (3.2) and the fact that $u_x^\varepsilon \to u_x$ and $v_x^\varepsilon \to v_x$ a.e. in $Q_{\delta,T}$, we see that the right side of (3.6) converges to zero as $\varepsilon \downarrow 0$. We now estimate the terms on the left side of (3.6) from below. Clearly,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \iint_{Q_{\delta,T}} (u^\varepsilon - v^\varepsilon)_t \Phi^\varepsilon dt\,dx \\ &= -\iint_{Q_{\delta,T}} (\delta_h(t-\rho) - \delta_h(t-\tau))(u - v)^{2s}\chi(t, x)dt\,dx \\ &\quad - \iint_{Q_{\delta,T}} (\alpha_h(t-\rho) - \alpha_h(t-\tau))(u - v)^{2s}\chi_t(t, x)dt\,dx \\ &\quad - (2s-1)\iint_{Q_{\delta,T}} (u - v)_t \Phi(t, x)dt\,dx \\ &= I_{1,h} + I_{2,h} + I_{3,h}\end{aligned}$$

and

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \iint_{Q_{\delta,T}} \Big(\int_0^1 H_u(\cdots)d\lambda\Big)(u^\varepsilon - v^\varepsilon)\Phi^\varepsilon dt\,dx \\ &\geqq -\hat{\omega}\iint_{Q_{\delta,T}} (u - v)\Phi(t, x)dt\,dx,\end{aligned}$$

where $\Phi(t, x)=(\alpha_h(t-\rho)-\alpha_h(t-\tau))\chi_h(t, x)(u-v)^{2s-1}$.

Before estimating the third term on the left side of (3.6), we consider $\Gamma^\varepsilon(t, x)$ defined by

$$\Gamma^\varepsilon(t, x) \equiv \sum_{i,j=1}^n H_{p_ip_j}(\cdots)(\lambda u^\varepsilon + (1-\lambda)v^\varepsilon)_{x_ix_j}$$

for $(t, x) \in [\rho/2, T] \times \Omega^{\delta}$. It is known that there exist $l_1, \dots, l_n$ in $\boldsymbol{R}^n$ such that $(l_i, l_j) = \delta_{ij}$ ($\delta_{ij}$=Kronecker's delta) and

$$\Gamma^{\varepsilon}(t, x) = \sum_{i=1}^{n} \mu_i(t, x)(\lambda u^{\varepsilon} + (1 - \lambda)v^{\varepsilon})_{l_i l_i},$$

where $\mu_i(t, x)$, $i = 1, \dots, n$, are the eigenvalues of the matrix $(H_{p_i p_j})$ at $(t, x)$. Since $H$ is convex in $p$, we have, by (3.2) and (3.3),

$$\Gamma^{\varepsilon}(t, x) \leqq nC_0 C = C_1,$$

where $C_0$ is a positive constant such that $0 \leqq \mu_i(t, x) \leqq C_0$ on $[\rho/2, T] \times \Omega^{\delta}$, $i = 1, \dots, n$. Hence,

$$-\overline{\lim_{\varepsilon \downarrow 0}} \iint_{Q_{\delta,T}} \Big( \sum_{i,j=1}^{n} \Big( \int_0^1 H_{p_i p_j}(\cdots)(\lambda u^{\varepsilon} + (1 - \lambda)v^{\varepsilon})_{x_i x_j} d\lambda \Big) \Big)(u^{\varepsilon} - v^{\varepsilon})\Phi^{\varepsilon} dtdx$$

$$\geqq - C_1 \iint_{Q_{\delta,T}} (u - v)^{2s}(\alpha_h(t - \rho) - \alpha_h(t - \tau))\chi(t, x)dtdx.$$

We now estimate the third term. Integration by parts yields

$$\underline{\lim_{\varepsilon \downarrow 0}} \iint_{Q_{\delta,T}} \Big( \sum_{i=1}^{n} \int_0^1 H_{p_i}(\cdots) d\lambda (u^{\varepsilon} - v^{\varepsilon})_{x_i} \Big) \Phi^{\varepsilon} dtdx$$

$$\geqq - \sqrt{n} N_0 \int_0^T \int_{\partial\Omega^{\delta}} (u - v)^{2s}(\alpha_h(t - \rho) - \alpha_h(t - \tau))\chi(t, \sigma) d\sigma dt$$

$$- \bar{C} \iint_{Q_{\delta,T}} (u - v)^{2s}(\alpha_h(t - \rho) - \alpha_h(t - \tau))\chi dtdx$$

$$- \iint_{Q_{\delta,T}} (\alpha_h(t - \rho) - \alpha_h(t - \tau)) N_0 |\chi_x| (u - v)^{2s} dtdx$$

$$- (2s - 1) \sum_{i=1}^{n} \iint_{Q_{\delta,T}} \Big( \int_0^1 H_{p_i}(-) d\lambda \Big)(u - v)_{x_i} \Phi(t, x) dtdx$$

$$= J_{1,h} + J_{2,h} + J_{3,h} + J_{4,h},$$

where $d\sigma$ is the surface element, $(-) = (t, x, \lambda u + (1 - \lambda)v, \lambda u_x + (1 - \lambda)v_x)$ and $\bar{C} = C_1 + C_2 + 2C_3 M_1$. Here $C_2$, $C_3$ are the constants defined by

$$C_2 = \sup \{ \sum_{i=1}^{n} |H_{p_i x_i}(t, x, u, p)|; (t, x, u, p) \in W(T, M_0, M_1) \},$$

$$C_3 = \sup \{ \sum_{i=1}^{n} |H_{p_i u}(t, x, u, p)|; (t, x, u, p) \in W(T, M_0, M_1) \}.$$

It follows from (3.5) that $I_{2,h} + J_{3,h} \geqq 0$. In view of (i) of Definition 2.1, we have

$$I_{3,h}+J_{4,h}\geqq-(2s-1)\hat{\omega}\iint_{Q_{\delta,T}}(u-v)\Phi(t,x)dtdx.$$

Thus letting $\varepsilon\downarrow 0$ and then $h\downarrow 0$ in (3.6), we have

$$\begin{aligned}&\int_{S(\tau)\cap\Omega^{\delta}}(u(\tau,x)-v(\tau,x))^{2s}dx-\int_{S(\rho)\cap\Omega^{\delta}}(u(\rho,x)-v(\rho,x))^{2s}dx\\ &\quad-(\bar{C}+2s\hat{\omega})\int_{\rho}^{\tau}\int_{S(t)\cap\Omega^{\delta}}(u(t,x)-v(t,x))^{2s}dxdt\\ &\quad-\sqrt{n}N_0\int_{\rho}^{\tau}\int_{S(t)\cap\partial\Omega^{\delta}}(u(t,\sigma)-v(t,\sigma))^{2s}d\sigma dt\leqq 0.\end{aligned}\tag{3.7}$$

We now put

$$F(t;s,\delta)\equiv\int_{S(t)\cap\Omega^{\delta}}(u(t,x)-v(t,x))^{2s}dx$$

and

$$G(t;s,\delta)\equiv\int_{S(t)\cap\partial\Omega^{\delta}}(u(t,\sigma)-v(t,\sigma))^{2s}d\sigma.$$

Using the Gronwall's inequality and raising both sides to the power $1/2s$, we have

$$\begin{aligned}F(\tau;s,\delta)^{1/2s}&\leqq e^{\frac{(\bar{C}+2s\hat{\omega})}{2s}(\tau-\rho)}\Big\{F(\rho;s,\delta)^{1/2s}\\ &\quad+(\sqrt{n}N_0)^{1/2s}\Big(\int_{\rho}^{\tau}G(\eta;s,\delta)e^{-(\bar{C}+2s\hat{\omega})(\eta-\rho)}d\eta\Big)^{1/2s}\Big\}\end{aligned}\tag{3.8}$$

for every positive integer $s$. We next let $s\to\infty$ in (3.8). Using the well-known fact that if $\Omega$ is bounded then $\lim_{p\to\infty}\|u\|_p=\|u\|_\infty$ for $u\in L^\infty(\Omega)$, we have

$$\begin{aligned}&\sup\{|u(\tau,x)-v(\tau,x)|;\,x\in S(\tau)\cap\Omega^{\delta}\}\\ &\leqq e^{\hat{\omega}(\tau-\rho)}[\sup\{|u(\rho,x)-v(\rho,x)|;\,x\in S(\rho)\cap\Omega^{\delta}\}\\ &\quad+\sup\{|u(t,x)-v(t,x)|;\,(t,x)\in\bigcup_{\rho\leqq t\leqq\tau}\{t\}\times(S(t)\cap\partial\Omega^{\delta})\}]\end{aligned}\tag{3.9}$$

for every $0<\rho<\tau<T$. Letting $\rho\downarrow 0$ in (3.9) and then $\delta\downarrow 0$, we obtain the desired inequality (3.1). The proof is complete.

As a consequence of Theorem 3, we have:

COROLLARY (*Uniqueness*). *Let $H$ be convex in $p$. Then there is at most one generalized solution of* (MP).

## 4. Generation Theorem

We now turn our consideration to the existence part for (MP). Our first task is to construct a (generalized) solution of (Pa.MP). As indicated in the introduction, we shall treat (Pa.MP) from a semigroup point of view in Sections 5 and 6. The main tool we use is the following Generation Theorem which is an extension of the Crandall-Pazy theorem (cf. [11; Theorem 2.1]). We note that the proof given below is essentially due to Crandall and Pazy.

Let $X$ be a Banach space with the norm $\| \ \|$. A subset $A$ of $X \times X$ is in the class of $\mathscr{A}(\omega)$ if for each $\lambda>0$ such that $\lambda\omega<1$ and each pair $[x_i, y_i] \in A$, $i=1, 2$, we have

$$(1-\lambda\omega)\|x_1 - x_2\| \leqq \|(x_1+\lambda y_1)-(x_2+\lambda y_2)\|.$$

For $\lambda>0$ and $t \geqq 0$, let $J_\lambda(t)=(I+\lambda A(t))^{-1}$ and $D(J_\lambda(t))=R(I+\lambda A(t))$.

Generation Theorem. *Let $A(t)$ satisfy*

(I) $A(t) \in \mathscr{A}(\omega)$ *for* $0 \leqq t \leqq T$;

(II) $D(A(t)) = \mathscr{D}$ *is independent of* $t$;

(III) *For each* $x \in \mathscr{D}$, *there is a* $\lambda_x > 0$ *satisfying the following* (a) *and* (b):

(a) $x \in R(I+\lambda A(t))$ *for every* $0<\lambda<\lambda_x$ *and* $t \in [0, T]$,

(b) $\prod_{i=1}^{k} J_\lambda(t_i)x$ *is uniquely determined for every* $\lambda \in (0, \lambda_x)$ *and every finite family of real numbers* $\{t_i\}_{i=1}^{k}$ *such that* $0 \leqq t_i \leqq T$, $i=1, 2, \ldots, k$;

(IV) *There exists an operator* $b(\cdot)\colon \mathscr{D} \to \mathbf{R}^+$ *such that*

(a) *for* $x \in \mathscr{D}$, $0<\lambda<\lambda_x$ *and* $k \geqq 1$ *with* $s+k\lambda \leqq T$,

$$b\Big(\prod_{i=1}^{k} J_\lambda(s+i\lambda)x\Big) \leqq (1+\lambda C_0)^k C_1,$$

*where* $C_0$ *and* $C_1$ *are constants independent of* $\lambda$; *and*

(b) *for* $x \in \mathscr{D}$,

$$\|J_\lambda(t)x - J_\lambda(s)x\| \leqq \lambda L(\|J_\lambda(t)x\|, b(J_\lambda(t)x))\,|t-s|,$$

*where* $L(r_1, r_2)\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ *is nondecreasing in* $(r_1, r_2)$, *that is,* $L(r_1, r_2) \leqq L(r_1', r_2')$ *if* $r_1 \leqq r_1'$ *and* $r_2 \leqq r_2'$.

*Then* $\{A(t)\}$ *determines an evolution operator* $U(t, s)$ *on* $\bar{\mathscr{D}}$ *such that*

$$\|U(t, s)x - U(t, s)y\| \leqq e^{\omega(t-s)}\|x-y\| \tag{4.1}$$

*for* $x, y \in \bar{\mathscr{D}}$ *and* $0 \leqq s \leqq t \leqq T$, *that is,* (i) $U(s, s)=I$ *(the identity operator),* $U(t, s)U(s, r)=U(t, r)$ *for* $0 \leqq r \leqq s \leqq t \leqq T$; *and* (ii) *for* $x \in \bar{\mathscr{D}}$, $U(t, s)x$ *is con-*

*tinuous in* $(t, s)$ *on the triangle* $0 \leqq s \leqq t \leqq T$.

*Moreover, for* $x \in \mathscr{D}$ *we have*

(i) $U(t, s)x$ *is given by*

$$U(t, s)x = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t-s}{n}}\left(s + i\frac{t-s}{n}\right)x \qquad (0 \leqq s \leqq t \leqq T), \tag{4.2}$$

(ii) $U(t, 0)x$ *is Lipschitz continuous in* $t$ *on* $[0, T]$.

PROOF. The reader is referred to [11; the proof of Theorem 2.1]. Let $x \in \mathscr{D}$ and $0 < \mu < \lambda < \lambda_x < \lambda_0$, where $\lambda_0$ is a constant such that $\lambda_0 \omega < 1$. Set

$$P_{\lambda,k} = P_{\lambda,k}x = \prod_{i=1}^{k} J_\lambda(s + i\lambda)x,$$

$$a_{k,l} = \|P_{\lambda,k} - P_{\mu,l}\|,$$

$$b_{k,l} = \|J_\mu(s + l\mu)P_{\mu,l-1} - J_\mu(s + k\lambda)P_{\mu,l-1}\|.$$

Proceeding in the same way as in [11] yields

$$a_{k,l} \leqq \alpha a_{k-1,l-1} + \beta a_{k,l-1} + b_{k,l},$$

where $\alpha = \mu\lambda^{-1}(1-\mu\omega)^{-1}$ and $\beta = (\lambda-\mu)\lambda^{-1}(1-\mu\omega)^{-1}$. By the condition (IV)-(b), we have $b_{k,l} \leqq \mu L(\|P_{\mu,l}\|, b(P_{\mu,l}))|l\mu - k\lambda|$. By [11; Lemma 2.2] and the condition (IV)-(a), we have $\|P_{\mu,l}\| \leqq K_0$ and $b(P_{\mu,l}) \leqq K_1$ for some constants $K_0$ and $K_1$ independent of $l$ and $\mu$. Hence

$$b_{k,l} \leqq \mu L(K_0, K_1)|l\mu - k\lambda| \equiv \mu\rho(|l\mu - k\lambda|), \tag{4.3}$$

where $\rho(r) = L(K_0, K_1)r$ for $r \geqq 0$. Remarking (4.3) and following the idea of Crandall and Pazy (cf. [11; p. 68]), we have for any $\delta > 0$

$$\begin{aligned} a_{m,n} \leqq K\{&[(n\mu - m\lambda)^2 + n\mu(\lambda - \mu)]^{1/2} + [(n\mu - m\lambda)^2 + m\lambda(\lambda - \mu)]^{1/2} \\ &+ n\mu\rho(|n\mu - m\lambda|) + n\mu\rho(\delta) + n^2\mu^2(\lambda - \mu)\delta^{-2}\}, \end{aligned} \tag{4.4}$$

where $K$ can be taken to depend only on $\|x\|$, $b(x)$, $C_0$, $C_1$, $\lambda_0$, $\omega$ and $T$. Notice that (4.4) corresponds to the estimate (2.25) of [11]. Therefore we find that

$$U(t, s)x = \lim_{m\to\infty} \prod_{i=1}^{m} J_{\lambda_m}(s + i\lambda_m)x, \qquad x \in \mathscr{D}$$

exists if $\{\lambda_m\}$ is a sequence such that $0 \leqq m\lambda_m \leqq t - s$ and $m\lambda_m \to t - s$ as $m \to \infty$. Moreover, according to the condition (I), we have (4.1) for $x, y \in \mathscr{D}$. Thus we can extend $U(t, s)$ to the operator (denoted by $U(t, s)$ again) defined on $\bar{\mathscr{D}}$ satisfying (4.1). An argument similar to the proofs of [11; Propositions 2.1 and 2.2] implies that $U(t, s)x$ is continuous in $(t, s)$ for $x \in \mathscr{D}$. Noting this fact, we can verify by a simple calculation that $U(t, s)x$ is continuous in $(t, s)$ for $x \in \bar{\mathscr{D}}$.

Finally we observe the Lipschitz continuity of $U(t, 0)x$ in $t$ on $[0, T]$. Putting $\lambda = \tau/m$ and $\mu = t/n$ in (4.4) where $s=0$, letting $n, m \to \infty$ and then letting $\delta \downarrow 0$, we have

$$\|U(t, 0)x - U(\tau, 0)x\| \leqq L_0|t - \tau|, \quad L_0 = K(2 + TL(K_0, K_1)).$$

The proof is complete.

## 5. The evolution operator approach to parabolic problems

We consider a mixed problem (hereafter called (Pa.MP)) for quasi-linear second order parabolic equations:

$$(5.1) \qquad u_t + H(t, x, u, u_x) = \mu \Delta u \quad \text{in } Q,$$

$$(5.2) \qquad u(0, x) = u_0(x) \quad \text{on } \bar{\Omega},$$

$$(5.3) \qquad u(t, x) = \phi(x) \quad \text{on } [0, \infty) \times \partial\Omega.$$

Here, as before, we assume that $u_0(x) = \phi(x)$ for $x \in \partial\Omega$, and that $\Omega$ is a bounded domain of $\boldsymbol{R}^n$ whose boundary $\partial\Omega$ is of class $C^3$. It is known that if the normal curvatures of $\partial\Omega \in C^3$ are bounded in absolute value by $\kappa$ then the distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$ is of class $C^2$ and satisfies $|d_x(x)| \geqq d_0 > 0$ at all points whose distance from $\partial\Omega$ is less than $\delta_0$, where $d_0$ and $\delta_0$ are appropriate positive constants such that $\delta_0 < 1/\kappa$ (cf. Serrin [18; Lemma 3.1]).

We now state the definition of a generalized solution of (Pa.MP).

Definition 5.1. *A function u defined in Q is called a generalized solution of* (Pa.MP) *if*: (i) *for every* $T > 0$, $u \in \mathscr{L}(\overline{Q_T})$ *satisfies* (5.2) *and* (5.3), *and* (ii) *u satisfies* (5.1) *in the distribution sense, that is, for every* $T > 0$ *and every* $\psi \in C_0^\infty(Q_T)$, *we have*

$$\int_0^T \int_\Omega \{-u\psi_t + H(t, x, u, u_x)\psi + \mu(u_x, \psi_x)\} dxdt = 0.$$

In this and next sections we shall apply the Generation Theorem stated in the previous section in order to construct a generalized solution of (Pa.MP). Let $[0, T]$ be arbitrarily fixed. In what follows we assume that $H$ satisfies the assumptions (H.I)–(H.IV). Also we make the following assumptions on $\{u_0, \phi\}$:

(B.I)* $u_0 \in C^{2+\alpha}(\Omega) \cap C^2(\bar{\Omega})$;

(B.II)* *There exists a function* $\Phi \in C^{2+\alpha}(\bar{\Omega})$ *such that* $\Phi(x) \leqq u_0(x)$ *for*

*$x \in \bar{\Omega}$, $\Phi(x)=\phi(x)$ for $x \in \partial\Omega$ and*

$$H(t, x, \Phi, \Phi_x) - \mu\Delta\Phi(x) \leqq 0 \quad \text{for} \quad t \geqq 0 \quad \text{and} \quad x \in \Omega. \tag{5.4}$$

Notice that there exists a constant $\mu_0$ such that

$$\mu \sup\{|\Delta\Phi(x)| + |\Delta d(x)|;\ x \in B^{\delta_0}\} \leqq 1 \tag{5.5}$$

for all $0<\mu<\mu_0$. Because we shall employ the vanishing viscosity method for proving the existence of a generalized solution of (MP), we may assume without loss of generality that $\mu$ in (5.1) is small. Henceforth it is assumed that (5.5) holds (see Remark 6.1).

Let us work in the Banach space $C(\bar{\Omega})$ of all real-valued continuous functions $v$ on $\bar{\Omega}$ with norm: $\|v\|_0=\max\{|v(x)|;\ x \in \bar{\Omega}\}$.

Define

$$\hat{\mathscr{D}} = \{v \in C^{2+\alpha}(\Omega) \cap C^1(\bar{\Omega});\ v \geqq \Phi \text{ on } \bar{\Omega},\ v = \phi \text{ on } \partial\Omega\}.$$

We start by defining the operators $A(t)$ and $b(\cdot)$ associated with (Pa.MP) in $C(\bar{\Omega})$.

DEFINITION 5.2 (*Definition of $A(t)$*). *We define $A(t)$ by $v \in D(A(t))$, $A(t)v=w$ if and only if*: (i) $v \in \hat{\mathscr{D}}$, (ii) $w \in C(\bar{\Omega})$ *and* (iii) $H(t, x, v, v_x)-\mu\Delta v=w$ *in* $\Omega$.

REMARK 5.1. If $v \in \hat{\mathscr{D}}$ and $H(t, x, v, v_x)-\mu\Delta v=w$ in $\Omega$, then the following conditions are equivalent.

(ii) $\quad w \in C(\bar{\Omega})$.

(ii)′ $\quad \Delta v \in C(\bar{\Omega})$.

(ii)″ $\quad v + \lambda_0 w \in C(\bar{\Omega})$ for some $\lambda_0 > 0$.

(ii)‴ $\quad v + \lambda w \in C(\bar{\Omega})$ for every $\lambda > 0$.

To see this, it suffices to note that $v \in \hat{\mathscr{D}} \subset C^1(\bar{\Omega})$ implies $H(t, x, v, v_x) \in C(\bar{\Omega})$.

DEFINITION 5.3 (*Definition of $b(\cdot)$*). *Define the operator $b(\cdot)$: $\hat{\mathscr{D}} \to \boldsymbol{R}^+$ by*

$$b(v) = \|v_x\|_0 = \sup\{[\sum_{i=1}^{n} v_{x_i}(x)^2]^{1/2};\ x \in \bar{\Omega}\} \qquad \text{for} \quad v \in \hat{\mathscr{D}}.$$

From Definition 5.2 and Remark 5.1 it follows immediately that $\{A(t)\}$ satisfies the condition (II) in the Generation Theorem. Thus we may denote $\mathscr{D}=D(A(t))$. Note that $\mathscr{D} \subset \hat{\mathscr{D}}$ and $\mathscr{D}$ is a convex set. From now on we are going to prove that $\{A(t)\}$ satisfies the conditions (I), (III) and (IV). To this end, we state without proof the following lemma which is a version of the maximum principle.

LEMMA 5.1. *Let* $a \in C(\Omega)$ *be positive in* $\Omega$, $a_i \in C(\Omega)$, $i=1,\dots, n$, *and* $\varepsilon>0$. *If* $v \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfies*

$$a(x)v(x) + \sum_{i=1}^{n} a_i(x)v_{x_i}(x) - \varepsilon \Delta v(x) \geqq 0 \qquad \text{for} \quad x \in \Omega$$

*and* $v \geqq 0$ *on* $\partial\Omega$, *then* $v(x) \geqq 0$ *for* $x \in \bar{\Omega}$.

Throughout this section we choose a positive number $\lambda_0$ such that $\lambda_0 \omega < 1$ and fix it, where $\omega$ is the constant in the assumption (H.IV). To verify the condition (I) in the Generation Theorem, we shall prove:

PROPOSITION 5.1. *Let* $0<\lambda<\lambda_0$. *If* $u, v \in \mathscr{D}$ *satisfy* $u+\lambda A(t)u=h$ *and* $v+\lambda A(t)v=g$, *respectively, then*

$$\|u - v\|_0 \leqq (1 - \lambda\omega)^{-1} \|h - g\|_0.$$

PROOF. Since

$$u + \lambda[H(t, x, u, u_x) - \mu \Delta u] = h$$

and

$$v + \lambda[H(t, x, v, v_x) - \mu \Delta v] = g$$

in $\Omega$, the difference $w=u-v$ satisfies

$$\begin{aligned} w &+ \lambda H_u(t, x, a_\theta(x), p_\theta(x))w \\ &+ \lambda(H_p(t, x, a_\theta(x), p_\theta(x)), w_x) - \lambda\mu\Delta w = h - g, \end{aligned}$$

where $a_\theta(x)=v+\theta(u-v)$, $p_\theta(x)=v_x+\theta(u_x-v_x)$ and $0<\theta=\theta(x)<1$.

Suppose that $w$ has a positive maximum at $x_0 \in \Omega$ (note that $\Omega$ is open). Then, by the assumption (H.IV),

$$\|h - g\|_0 \geqq w(x_0) + \lambda H_u(t, x_0, a_\theta(x_0), p_\theta(x_0))w(x_0) \geqq (1 - \lambda\omega)w(x_0).$$

This implies $w(x_0) \leqq (1-\lambda\omega)^{-1}\|h-g\|_0$, since $0<\lambda<\lambda_0$. Similarly we can show that if $w$ has a negative minimum at $x_1 \in \Omega$ then $w(x_1) \geqq -(1-\lambda\omega)^{-1}\|h-g\|_0$. Remarking that $w$ vanishes on $\partial\Omega$, we have the desired inequality. Thus the proof is complete.

As an immediate consequence of Proposition 5.1, we have:

COROLLARY. *For* $h \in \hat{\mathscr{D}}$, *there is at most one solution* $u \in D(A(t))$ *of*

$u+\lambda A(t)u=h.$

We next prove that $\{A(t)\}$ satisfies the condition (III). Of course, the condition (III)–(a) means that for $h\in\mathscr{D}$ there is a positive constant $\lambda_h$ such that for every $0<\lambda<\lambda_h$ and every $t\in[0, T]$ we can prove the existence of a classical solution $u\in\mathscr{D}$ of the boundary value problem (hereafter called (BVP)):

$$u+\lambda[H(t, x, u, u_x)-\mu\Delta u]=h, \qquad x\in\Omega, \tag{5.6}$$

$$u(x)=\phi(x), \qquad x\in\partial\Omega. \tag{5.7}$$

Before proceeding further, we want to note that the estimates appearing in this section are independent of $\lambda$ and $\mu$.

LEMMA 5.2. *For* $h\in\hat{\mathscr{D}}$ *and* $0<\lambda<\lambda_0$, *let* $u\in C^2(\Omega)\cap C(\bar{\Omega})$ *satisfy* (5.6) *and* (5.7). *Then*

$$u(x)\geqq\Phi(x) \qquad for \quad x\in\bar{\Omega}. \tag{5.8}$$

PROOF. Since $h\in\hat{\mathscr{D}}$, we have

$$\Phi+\lambda[H(t, x, \Phi, \Phi_x)-\mu\Delta\Phi]-h\leqq 0, \qquad x\in\Omega, \tag{5.9}$$

by the assumption (B.II)*. Hence we find that the difference $\tilde{w}=u-\Phi$ satisfies $\tilde{w}(x)=0$ for $x\in\partial\Omega$ and

$$\tilde{w}+\lambda[H_u(\cdots)\tilde{w}+\sum_{i=1}^{n}H_{p_i}(\cdots)\tilde{w}_{x_i}-\mu\Delta\tilde{w}]\geqq 0, \qquad x\in\Omega,$$

where $(\cdots)=(t, x, \Phi+\theta\tilde{w}, \Phi_x+\theta\tilde{w}_x)$, $0<\theta=\theta(x)<1$. Since $1+\lambda H_u(\cdots)\geqq 1-\lambda\omega>0$, we have $\tilde{w}(x)=u(x)-\Phi(x)\geqq 0$ for $x\in\bar{\Omega}$ by Lemma 5.1. The proof is complete.

LEMMA 5.3. *For* $h\in\hat{\mathscr{D}}$ *and* $0<\lambda<\lambda_0$, *let* $u\in C^2(\Omega)\cap C(\bar{\Omega})$ *be a classical solution of* (BVP). *Then there exists a function* $\Psi(x)\in C^\infty(\bar{\Omega})$ *such that* $u(x)\leqq\Psi(x)$ *for* $x\in\bar{\Omega}$ *and*

$$\Psi+\lambda[H(t, x, \Psi, \Psi_x)-\mu\Delta\Psi]\geqq h, \qquad (t, x)\in\overline{Q_T}. \tag{5.10}$$

PROOF. By the assumption (H.II), we can choose a vector $l^0=(l_1^0,\ldots, l_n^0)\in\boldsymbol{R}^n$ such that for all $(t, x)\in\overline{Q_T}$

$$H(t, x, 0, l^0)\geqq((\operatorname{diam}\Omega)|l^0|+\|h\|_0)\omega.$$

Here $\operatorname{diam}\Omega$ denotes the diameter of the domain $\Omega$. We define

$$\Psi(x)\equiv(l^0, x)+a=\sum_{i=1}^{n}l_i^0x_i+a, \qquad x\in\Omega, \tag{5.11}$$

where $a=\|h\|_0-\min\{(l^0, x); x\in\bar{\Omega}\}$. It is evident that

$$(5.12) \qquad 0 \leqq \|h\|_0 \leqq \Psi(x) \leqq (\operatorname{diam}\Omega)|l^0| + \|h\|_0, \qquad x\in\bar{\Omega}.$$

It follows from (5.11), (5.12) and the assumption (H.IV) that

$$(5.13) \qquad H(t, x, \Psi, \Psi_x) - \mu\Delta\Psi \geqq -\omega\Psi(x) + H(t, x, 0, l^0) \geqq 0$$

for all $t\in[0, T]$ and $x\in\bar{\Omega}$, and hence, by using (5.12) again, we obtain (5.10). Therefore, the argument similar to the one at the end of the proof of Lemma 5.2 implies that $\Psi(x)-u(x)\geqq 0$ for $x\in\bar{\Omega}$. Consequently,

$$(5.14) \qquad u(x) \leqq \Psi(x) \leqq (\operatorname{diam}\Omega)|l^0| + \|h\|_0.$$

The proof is complete.

Next we shall establish an a priori estimate for the first derivatives of solutions of (BVP). We are now in a position to give a comment concerning the restriction of $\lambda$ occurring in (BVP). According to Lemmas 5.2 and 5.3, for each given $h\in\hat{\mathscr{D}}$ there is a positive constant $M_0$ such that

$$(5.15) \qquad \|u\|_0 \leqq M_0 \equiv \max\{\|\Phi\|_0, \|\Psi\|_0\}$$

for all classical solutions $u$ of (BVP). Hence, by the assumption (H.III), there are positive constants $a_2$ and $a_3$, depending on $h$, such that $|H_x(t, x, u, p)|\leqq a_2|p| + a_3$ for $(t, x, u, p)\in W(T, M_0)$. For such $a_2$ and $a_3$, we can choose a positive constant $\lambda_h(<\lambda_0)$ such that for all $\lambda\in(0, \lambda_h)$

$$(5.16) \quad (1-(a_2+\omega)\lambda)^{-1} \leqq 1+(a_2+\omega+1)\lambda, \quad a_3(a_2+\omega+1)\lambda \leqq 1.$$

For later applications, we consider (BVP) with $0<\lambda<\lambda_h$ and $0<\mu<\mu_0$ (cf. (5.5)).

LEMMA 5.4. *Let $h\in\hat{\mathscr{D}}$, and let $\lambda_h$ be as mentioned above. Suppose that for $\lambda\in(0, \lambda_h)$, $u\in C^3(\Omega)\cap C^1(\bar{\Omega})$ satisfies (5.6) and (5.7). Then we have*

$$\|u_x\|_0 \leqq (1+\lambda C_0)C_1,$$

*where $C_0=a_2+a_3+\omega+2$ and $C_1$ is a constant depending only on $H$, $\Omega$ and $h$.*

PROOF. First we choose a $\sigma_0$ such that $\sigma_0\geqq\|\Phi_x\|_0+\|h_x\|_0$. Note that the inequality

$$(5.17) \qquad \Phi(x)+\sigma d(x) \geqq h(x), \qquad x\in B^{\delta_0}$$

holds, provided $\sigma\geqq\sigma_0$. In fact, since $\Phi(x)=h(x)=\phi(x)$ for $x\in\partial\Omega$, we have $|\Phi(x)-h(x)|\leqq(\|\Phi_x\|_0+\|h_x\|_0)d(x)\leqq\sigma d(x)$, if $\sigma\geqq\sigma_0$.

We also take two sequences $\{\sigma_m\}$ and $\{\delta_m\}$ such that

(i) $\sigma_m \geqq \sigma_0$ and $\delta_m \leqq \delta_0$, $m = 1, 2, \ldots$,

(ii) $\sigma_m \uparrow \infty$ and $\delta_m \downarrow 0$ as $m \to \infty$,

(iii) for each $m$, the inequality

$$\overline{M} - \underline{M} \leqq \sigma_m \delta_m \leqq \overline{M} - \underline{M} + 1 \tag{5.18}$$

holds, where $\underline{M} = \min_{x \in \bar{\Omega}} \Phi(x)$ and $\overline{M} = \max_{x \in \bar{\Omega}} \Psi(x)$.

It is not difficult to see that we have $|\Phi(x) + \sigma_m d(x)| \leqq \|\Phi\|_0 + \sigma_m \delta_m \leqq \|\Phi\|_0 + \overline{M} - \underline{M} + 1 = \tilde{M}$ for all $x \in B^{\delta_m}$ and $m = 1, 2, \ldots$; and

$$\lim_{m \to \infty} \frac{H(t, x, u, \Phi_x + \sigma_m d_x)}{\sigma_m} = \lim_{m \to \infty} \frac{H(t, x, u, \Phi_x + \sigma_m d_x)}{\sigma_m |d_x|} |d_x| = +\infty$$

for $(t, x, u) \in [0, T] \times B^{\delta_m} \times [-\tilde{M}, \tilde{M}]$.

Hence there are positive constants $\sigma_1$ $(= \sigma_{m_1}) > 1$ and $\delta_1$ $(= \delta_{m_1}) < \delta_0$, independent of $\lambda$ and $\mu$, such that

$$H(t, x, \Phi + \sigma_1 d, \Phi_x + \sigma_1 d_x) - \mu(\Delta\Phi + \sigma_1 \Delta d) \geqq 0 \tag{5.19}$$

for $(t, x) \in [0, T] \times B^{\delta_1}$ $(\subset [0, T] \times B^{\delta_0})$. In fact, we have only to choose $\sigma_1$ so large that

$$\frac{H(t, x, \Phi + \sigma_1 d, \Phi_x + \sigma_1 d_x)}{\sigma_1 |d_x|} |d_x| \geqq 1 \qquad \text{for } (t, x) \in [0, T] \times B^{\delta_1}.$$

From (5.17) and (5.19) it follows that $\hat{\Phi} \equiv \Phi + \sigma_1 d$ satisfies

$$\hat{\Phi} + \lambda[H(t, x, \hat{\Phi}, \hat{\Phi}_x) - \mu\Delta\hat{\Phi}] - h \geqq 0, \qquad x \in \overline{B^{\delta_1}}. \tag{5.20}$$

Furthermore, using (5.18) and Lemma 5.3, we can verify that $\hat{\Phi}(x) \geqq \overline{M} \geqq u(x)$ if $d(x) = \delta_1$ and $\hat{\Phi}(x) = \phi(x)$ if $d(x) = 0$, and hence

$$\hat{\Phi}(x) \geqq u(x) \qquad \text{for } x \in \partial B^{\delta_1}. \tag{5.21}$$

Proceeding in the same way as in the proof of Lemma 5.3, and noting (5.20) and (5.21), we have

$$u(x) \leqq \hat{\Phi}(x) = \Phi(x) + \sigma_1 d(x) \qquad \text{for } x \in \overline{B^{\delta_1}}. \tag{5.22}$$

Combining this with Lemma 5.2 yields

$$\Phi(x) - \sigma_1 d(x) \leqq \Phi(x) \leqq u(x) \leqq \Phi(x) + \sigma_1 d(x), \qquad x \in \overline{B^{\delta_1}}.$$

Then, since

$$\left|\frac{1}{h_i}(u(x_1,\dots, x_i + h_i,\dots, x_n) - u(x_1,\dots, x_n))\right| \leqq \sigma_1 + \|\Phi_x\|_0 \leqq 2\sigma_1$$

for $x \in \partial\Omega$ and every small $h_i$ such that $(x_1,\dots, x_i + h_i,\dots, x_n) \in B^{\delta_1}$, $i=1,\dots, n$, we obtain $|u_{x_i}(x)| \leqq 2\sigma_1$ for $x \in \partial\Omega$, $i=1,\dots, n$. Consequently we have the boundary estimate:

$$\|u_x\|_{C(\partial\Omega)} = \sup_{x\in\partial\Omega}[\sum_{i=1}^{n} u_{x_i}(x)^2]^{1/2} \leqq 2n\sigma_1 \equiv C_1. \tag{5.23}$$

We can now establish an interior estimate for $u_x$ with the aid of (5.23). Differentiating both sides of (5.6) with respect to $x_i$, multiplying by $u_{x_i}$ and summing from $i=1$ to $n$, we have

$$(u_x, u_x) + \lambda(H_x, u_x) + \lambda H_u(u_x, u_x)$$
$$+ \lambda \sum_{i,j=1}^{n} H_{p_j} u_{x_j x_i} u_{x_i} - \lambda\mu \sum_{i=1}^{n} u_{x_i} \Delta u_{x_i} - (h_x, u_x) = 0.$$

Setting $z(x) \equiv (u_x(x), u_x(x))$, we have

$$(1 + \lambda H_u)z + \lambda(H_x, u_x) + \frac{\lambda}{2}(H_p, z_x)$$
$$- \frac{\lambda\mu}{2}\Delta z + \lambda\mu \sum_{i,j=1}^{n} u_{x_i x_j}^2 - (h_x, u_x) = 0.$$

Suppose that $z$ has a positive relative maximum $z(x_0)$ at $x_0 \in \Omega$. Then

$$(1 + \lambda H_u)z(x_0) + \lambda(H_x, u_x(x_0)) - (h_x(x_0), u_x(x_0)) \leqq 0.$$

By the assumption (H.III) and the fact that $1 - \lambda\omega > 0$ for $0 < \lambda < \lambda_0$, we get

$$|u_x(x_0)| \leqq \|h_x\|_0 + \lambda[(a_3 + 1) + (a_2 + \omega + 1)\|h_x\|_0],$$

since $0 < \lambda < \lambda_h$. Here we have used the fact that both inequalities in (5.16) hold for $\lambda \in (0, \lambda_h)$. Now we must treat two cases separately.

Case 1: $\|h_x\|_0 \leqq 1$. In this case, we have, by noting $C_1 \geqq 1$,

$$|u_x(x_0)| \leqq 1 + \lambda(a_2 + a_3 + \omega + 2) = 1 + \lambda C_0 \leqq (1 + \lambda C_0)C_1.$$

Case 2: $1 \leqq \|h_x\|_0 (\leqq C_1)$. In this case, we have

$$|u_x(x_0)| \leqq (1 + \lambda(a_2 + a_3 + \omega + 2))\|h_x\|_0 \leqq (1 + \lambda C_0)C_1.$$

Consequently,

$$|u_x(x_0)| \leqq (1 + \lambda C_0)C_1,$$

and hence, by (5.23), we have $\|u_x\|_0 \leqq (1+\lambda C_0)C_1$. Thus the proof is complete.

We are now able to prove the following result, which implies that $\{A(t)\}$ satisfies the condition (III).

PROPOSITION 5.2. *For $h \in \hat{\mathscr{D}}$, there exists a $\lambda_h$ $(0<\lambda_h<\lambda_0)$ such that* (a) *$h \in R(I+\lambda A(t))$ for all $0<\lambda<\lambda_h$ and $0 \leqq t \leqq T$; and* (b) *for every $\{\lambda_k\}_{k=1}^N$ with $0<\lambda_k<\lambda_h$ and every $\{t_k\}_{k=1}^N$ with $0 \leqq t_k \leqq T$, there exists a sequence $\{u^k\}$ of solutions of*

$$\begin{cases} u + \lambda_k[H(t_k, x, u, u_x) - \mu \Delta u] = u^{k-1}, & x \in \Omega, \\ u(x) = \phi(x), & x \in \partial\Omega, \end{cases}$$

*where $u^0 = h$.*

PROOF. Let us prove this by showing that (a) and (b) hold with the $\lambda_h$ obtained in deriving (5.16). We can prove (a) by using the a priori estimates obtained in Lemmas 5.2–5.4, and by using the Tychonoff fixed point theorem (cf. [17] or [18]). Here we note that in order to be able to seek a solution in $C^4(\Omega)$ we assume that $\hat{\mathscr{D}} \subset C^{2+\alpha}(\Omega)$ and $H \in C^{2+\alpha}$.

It remains to prove (b). To this end, we first verify a simple (but basic) result that under the assumption of its existence, each $u^k$ satisfies

$$\Phi(x) \leqq u^k(x) \leqq \Psi(x) \qquad \text{for} \quad x \in \bar{\Omega}, \tag{5.24}$$

where $\Phi(x)$, $\Psi(x)$ are the functions appearing in the assumption (B.II)*, and given by (5.11), respectively. We prove this by induction on $k$. Lemmas 5.2 and 5.3 imply that (5.24) holds for $k=1$. Assume that (5.24) is already proved for the integers less than or equal to $k-1$. Remarking (5.4) and (5.13), we have by the hypothesis of induction

$$\Phi + \lambda_k[H(t_k, x, \Phi, \Phi_x) - \mu \Delta \Phi] - u^{k-1} \leqq 0$$

and

$$\Psi + \lambda_k[H(t_k, x, \Psi, \Psi_x) - \mu \Delta \Psi] - u^{k-1} \geqq 0$$

for $x \in \bar{\Omega}$. Therefore the arguments used in the proofs of Lemmas 5.2 and 5.3 can be employed to obtain (5.24) for $u^k$.

By virtue of (5.24), we have $\|u^k\|_0 \leqq M_0$ where $M_0$ is the same constant as in (5.15). This implies that the $\lambda_h$ may be taken as a $\lambda_{u^k}$, $k=1,\dots, N$, since we can take the same $a_2$ and $a_3$ as before (cf. (5.16)). Now the proof of the existence of $u^k$ can be carried out in a similar way as in the proof of (a). (For the a priori estimates of $\|u_x^k\|_0$, see the next proposition.) The proof is complete.

The following propositions make the observation that $\{A(t)\}$ satisfies the condition (IV) in the Generation Theorem. Let

$$u^k = \prod_{i=1}^{k} J_\lambda(t + i\lambda)h.$$

PROPOSITION 5.3. *Let $h \in \hat{\mathscr{D}}$ and $0<\lambda<\lambda_h$. Then,*

$$b(\prod_{i=1}^{k} J_\lambda(t + i\lambda)h) = \|u_x^k\|_0 \leqq (1 + \lambda C_0)^k C_1 \tag{5.25}$$

*for every integer $k$ such that $t+k\lambda \leqq T$, where $C_0$, $C_1$ are the same constants as in Lemma 5.4.*

PROOF. Let $\hat{\Phi}(x)=\Phi(x)+\sigma_1 d(x)$ be as in the proof of Lemma 5.4. Since, by the choice of $\sigma_1$ and $\delta_1$ in the proof of Lemma 5.4,

$$H(t + 2\lambda, x, \hat{\Phi}, \hat{\Phi}_x) - \mu\Delta\hat{\Phi} \geqq 0 \qquad \text{on} \quad \overline{B^{\delta_1}}$$

and $\hat{\Phi}(x) \geqq u^1(x)$ for $x \in \overline{B^{\delta_1}}$ (cf. (5.22)), $\hat{\Phi}$ satisfies (5.20) with $t$ and $h$ replaced by $t+2\lambda$ and $u^1$, respectively. Hence, by Lemma 5.1 and (5.24),

$$\Phi(x) \leqq u^2(x) \leqq \hat{\Phi}(x) = \Phi(x) + \sigma_1 d(x) \qquad \text{for} \quad x \in \overline{B^{\delta_1}}.$$

This yields with the same constant $C_1$ as in (5.23)

$$\|u_x^2\|_{C(\partial\Omega)} \leqq C_1. \tag{5.26}$$

Calculating in the same way as before, we have

$$\|u_x^2\|_0 \leqq (1 + \lambda C_0)^2 C_1,$$

by using the estimates $\|u_x^1\|_0 \leqq (1+\lambda C_0)C_1$ and (5.26).

Proceeding similarly step by step, we complete the proof of Proposition 5.3.

PROPOSITION 5.4. *Let $J_\lambda(t)h=u$ and $J_\lambda(s)h=v$ for $h \in \hat{\mathscr{D}}$, $0<\lambda<\lambda_h$ and $0 \leqq t, s \leqq T$. Then*

$$\|u - v\|_0 \leqq \lambda L(\|u\|_0, b(u))\,|t - s|, \tag{5.27}$$

*where $L(r_1, r_2)=C\sup\{|H_t(t, x, u, p)|; (t, x, u, p) \in W(T, r_1, r_2)\}$ with a positive constant $C$ independent of $t$, $s$ and $h$.*

PROOF. Clearly, the difference $w=u-v$ satisfies

$$0 = w + \lambda[H(t, x, u, u_x) - H(s, x, v, v_x)] - \lambda\mu\Delta w$$

$$= w + \lambda[H_t(\tilde{t}, x, u, u_x)(t - s) + H_u(s, x, \tilde{a}(x), u_x)w$$

$$+ (H_p(s, x, v, \tilde{p}(x)), w_x)] - \lambda\mu\Delta w,$$

where $\tilde{t}$, $\tilde{a}(x)$ and $\tilde{p}(x)$ are determined by the mean value theorem.

Suppose $w$ has a positive maximum at $x_0 \in \Omega$. Then

$$w(x_0) + \lambda H_u(s, x_0, \tilde{a}(x_0), u_x(x_0))w(x_0) \leqq \lambda(\sup|H_t|)|t - s|.$$

Here the supremum is taken over all $(t, x, z, p) \in W(T, \|u\|_0, b(u))$. Consequently,

$$w(x_0) \leqq \lambda C(\sup|H_t|)|t - s| \equiv \lambda L(\|u\|_0, b(u))|t - s|,$$

where $C$ is an appropriate constant such that $(1-\lambda\omega)^{-1} \leqq C$ for $0<\lambda<\lambda_0$. (Since we may assume without loss of generality that $\lambda_0\omega<1/2$, we can take $C=2$.) Similarly, we see that if $w$ has a negative minimum at $x_1 \in \Omega$ then $w(x_1) \geqq -\lambda L(\|u\|_0, b(u))|t-s|$. Remarking that $w$ vanishes on $\partial\Omega$, we have (5.27). The proof is complete.

Combining the results obtained above, we conclude:

THEOREM 4. *Suppose that $H$ satisfies the assumptions* (H.I)–(H.IV). *Let $\{A(t)\}$ be a family of operators of Definition* 5.2. *Then $\{A(t)\}$ determines an evolution operator $U(t, s)$ on $\bar{\mathscr{D}}$.*

*Moreover, we have*

(i) *For each given $u_0 \in \mathscr{D}$ and each $0<\varepsilon<\lambda_{u_0}$, the problem*

$$\text{(5.28)} \qquad \begin{cases} \varepsilon^{-1}(u(t) - u(t-\varepsilon)) + A([t/\varepsilon]\varepsilon)u(t) = 0, & t \geqq 0, \\ u(t) = u_0, & t < 0, \end{cases}$$

*has a unique solution $u^\varepsilon(t)$ on $[0, \infty)$ and $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t) = U(t, 0)u_0$ uniformly in $t$ on compact sets, where $[t/\varepsilon]$ is the greatest integer in $t/\varepsilon$.*

(ii) *If $v \in \mathscr{D}$, then $U(t, 0)v$ is locally Lipschitz continuous in $t$.*

(iii) *$U(t) = U(t, 0)$ satisfies*

$$\|U(t)u - U(t)v\|_0 \leqq e^{\omega t}\|u - v\|_0 \quad \textit{for} \quad u, v \in \bar{\mathscr{D}}.$$

## 6. Relationship between the evolution operator and (Pa.MP)

The main aim of this section is to show the existence of a generalized solution of (Pa.MP). Our approach to this problem depends much on the theory of nonlinear evolution equations in a Banach space. We associate (Pa.MP) with the initial value problem for an abstract quasi-linear parabolic equation of the form

$$\text{(ACP)} \qquad \begin{cases} du(t)/dt + A(t)u(t) = 0, & 0 \leqq t \leqq T, \\ u(0) = u_0 \end{cases}$$

in the Banach space $C(\bar{\Omega})$, where $T$ is a given positive number.

For each given $u_0 \in \mathscr{D}$ and each $\varepsilon$ such that $0<\varepsilon<\lambda_{u_0}$, let $u^\varepsilon(t)$ be the solution of (5.28), i.e., $u^\varepsilon(t)=\prod_{i=0}^{[t/\varepsilon]} J_\varepsilon(i\varepsilon)u_0$. Put $(u^\varepsilon(t))(x)\equiv u^\varepsilon(t, x)$. It should be noted that (5.24) and (5.25) imply

$$\|u^\varepsilon(t, \cdot)\|_0 \leqq M_0 \quad \text{and} \quad \|u_x^\varepsilon(t, \cdot)\|_0 \leqq M_1 \tag{6.1}$$

for all $t \in [0, T]$, where $M_0$ and $M_1$ are independent of $\varepsilon$, $\mu$.

In order to prove that $(U(t)u_0)(x)\equiv u(t, x)$ is a generalized solution of (Pa.MP), we intend to verify that there exists a subsequence $\{\varepsilon(i)\}$ such that $u_x^{\varepsilon(i)} \to u_x$ a.e. in $Q_T$. To do so, we shall make use of the concept of local semi-concavity. Before stating a lemma, we list some notations.

Define

$$\eta(r) = \begin{cases} 1 & 0 \leqq r \leqq 1, \\ \exp[(r-1)^3/(r-2)] & 1 \leqq r \leqq 2, \\ 0 & 2 \leqq r. \end{cases} \tag{6.2}$$

Clearly, $\eta \in C^2(\boldsymbol{R}^+)$. For $v \in C^2(\Omega)$, $y \in \Omega$ and $\delta>0$ with $U_{2\delta}(y) \subset \Omega$, we set

$$|v|_{E(U_\delta(y))} = \sup\{v_{ll}(x);\ x \in U_\delta(y),\ l \in \boldsymbol{R}^n\},$$

$$|\tilde{v}|_{E(U_{2\delta}(y))} = \sup\{\eta(|x-y|/\delta)v_{ll}(x);\ x \in U_{2\delta}(y),\ l \in \boldsymbol{R}^n\},$$

$$|\tilde{v}|^\#_{E(U_{2\delta}(y))} = \max\{|\tilde{v}|_{E(U_{2\delta}(y))}, 1\}.$$

The following lemma plays an essential role in our later discussions.

LEMMA 6.1. *Let $h \in \hat{\mathscr{D}}$ and $0<\mu<\mu_0$ $(<1)$. Then for each $y \in \Omega$ and every $\delta>0$ such that $U_{2\delta}(y) \subset \Omega$, there exist positive constants $\hat{C}=\hat{C}(\delta)$ and $\hat{\lambda}_h=\hat{\lambda}_h(\delta)$, independent of $\mu$, such that*

$$|\tilde{u}^k|^\#_{E(U_{2\delta}(y))} \leqq (1+\lambda\hat{C})^k |\tilde{h}|^\#_{E(U_{2\delta}(y))} \tag{6.3}$$

*for $0<\lambda<\hat{\lambda}_h$ and $k=1, 2, \ldots, [T/\lambda]$, where $u^k=\prod_{i=1}^{k} J_\lambda(i\lambda)h$.*

PROOF. We shall prove this by induction on $k$. Let us first prove (6.3) for $u^1$. For simplicity, we denote $u^1$ by $u$. Let $l=(l_1, \ldots, l_n) \in \boldsymbol{R}^n$ with $|l|=1$ be arbitrarily fixed. By definition, $u$ satisfies

$$u + \lambda[H(t, x, u, u_x) - \mu\Delta u] = h, \qquad x \in \Omega. \tag{6.4}$$

Carrying out the second directional differentiation with respect to $l$ in (6.4), we have easily

$$
(6.5)\quad
\begin{aligned}
&u_{ll} + \lambda\Big[\sum_{i,j=1}^{n} H_{x_i x_j} l_i l_j + 2\sum_{i=1}^{n} H_{x_i u} l_i u_l + H_{uu}(u_l)^2 \\
&+ 2\sum_{i,j=1}^{n} H_{x_i p_j} l_i u_{x_j l} + 2\sum_{i=1}^{n} H_{u p_i} u_l u_{x_i l} + H_u u_{ll} \\
&+ \sum_{i,j=1}^{n} H_{p_i p_j} u_{x_i l} u_{x_j l} + \sum_{i=1}^{n} H_{p_i} u_{x_i ll} - \mu \Delta u_{ll}\Big] = h_{ll}.
\end{aligned}
$$

By virtue of (6.1), we have

$$
\sup\{|\sum_{i,j=1}^{n} H_{x_i x_j} l_i l_j| + 2|\sum_{i=1}^{n} H_{x_i u} l_i u_l| + |H_{uu}|(u_l)^2\} \leqq C_2,
$$

where the constant $C_2$ is independent of $\lambda$ and $\mu$. Here we take the supremum over all $(t, x, u, p) \in W(T, M_0, M_1)$ in order to be able to proceed on with our argument.

Set $w = u_{ll}$. Since $H$ is strictly convex in $p$, the inequality

$$
\sum_{i,j=1}^{n} H_{p_i p_j} u_{x_i l} u_{x_j l} = (H_{pp}(u_l)_x, (u_l)_x) \geqq a_1 |(u_l)_x|^2
$$

holds with a constant $a_1 = a_1(M_0, M_1) > 0$. By the Schwarz inequality, we have

$$
2|\sum_{i,j=1}^{n} H_{x_i p_j} l_i u_{x_j l}| \leqq \frac{a_1}{4} |(u_l)_x|^2 + \frac{4}{a_1} |H_{px} l|^2,
$$

$$
2|\sum_{i=1}^{n} H_{u p_i} u_l u_{x_i l}| \leqq \frac{a_1}{4} |(u_l)_x|^2 + \frac{4}{a_1} |H_{pu} u_l|^2.
$$

Therefore, from (6.5) it follows that

$$
(6.6)\quad w + \lambda H_u w + \frac{1}{2}\lambda a_1 |(u_l)_x|^2 + \lambda(H_p, w_x) - \lambda\mu\Delta w - h_{ll} \leqq \lambda C_4,
$$

where $C_4 = C_2 + C_3$ and

$$
C_3 = \frac{4}{a_1} \sup\{|H_{px} l|^2 + |H_{up} u_l|^2; (t, x, u, p) \in W(T, M_0, M_1)\}.
$$

Multiplying both sides of (6.6) by $(\eta(|x-y|/\delta))^2$ and setting $z = \eta w$, we have

$$
(6.7)\quad
\begin{aligned}
&(1 + \lambda H_u) z\eta + \frac{1}{2}\lambda a_1 z^2 + \lambda\eta(H_p, z_x) - \lambda(H_p, \eta_x) z - \lambda\mu\eta\Delta z \\
&+ 2\lambda\mu(\eta_x, z_x) + \lambda\mu(\Delta\eta - (2|\eta_x|^2/\eta)) z - \eta^2 h_{ll} \leqq \lambda C_4 \eta^2,
\end{aligned}
$$

since $w(x)^2 \leqq |(u_l)_x|^2$.

We now suppose that $z$ has a maximum $z(x_0)$ $(>1)$ on $U_{2\delta}(y)$. Since $z$

vanishes on $\partial U_{2\delta}(y)$, $x_0$ is an interior point of $U_{2\delta}(y)$. Using the Schwarz inequality again, we get

$$(1+\lambda H_u)z\eta+\frac{1}{2}\lambda a_1 z^2-\lambda\left\{\frac{a_1}{4}z^2+\frac{1}{a_1}(H_p,\eta_x)^2\right\}$$
$$-\lambda\left\{\frac{a_1}{4}z^2+\frac{1}{a_1}(\varDelta\eta-(2|\eta_x|^2/\eta))^2\right\}-\eta^2 h_{ll}\leqq\lambda C_4\eta^2,$$

whence

$$(1-\lambda\omega)z(x_0)\eta(|x_0-y|/\delta)\leqq\eta^2 h_{ll}(x_0)+\lambda C_4\eta^2+\lambda C_5\eta,$$

where $C_5=C_5(\delta)$ is a constant independent of $\lambda$ and $\mu$. Here we have used the fact that there is a constant $C(\delta)$, depending only on $\delta$, such that $|\eta_x|^2\leqq C(\delta)\eta$ and $(\varDelta\eta-(2|\eta_x|^2/\eta))^2\leqq C(\delta)\eta$. Thus we have

$$z\leqq(1-\lambda\omega)^{-1}(1+\lambda C_6)|\tilde{h}|^{\#}_{E(U_{2\delta}(y))} \tag{6.8}$$

for $0<\lambda<\lambda_0$, where $C_6=C_6(\delta)=C_4+C_5$. But a simple calculation allows us to choose $\hat{\lambda}_h=\hat{\lambda}_h(\delta)$ small enough so that $(1-\lambda\omega)^{-1}\leqq 1+(\omega+1)\lambda$ and $\lambda(\omega+1)C_6\leqq 1$ hold for all $0<\lambda<\hat{\lambda}_h$. Hence, it follows from (6.8) that for every $\lambda\in(0,\hat{\lambda}_h)$ we have

$$z\leqq(1+\lambda\hat{C})|\tilde{h}|^{\#}_{E(U_{2\delta}(y))},$$

where $\hat{C}=\hat{C}(\delta)=\omega+C_6+2$. Consequently we have

$$|\tilde{u}|^{\#}_{E(U_{2\delta}(y))}\leqq(1+\lambda\hat{C})|\tilde{h}|^{\#}_{E(U_{2\delta}(y))}$$

for every $0<\lambda<\hat{\lambda}_h$.

Next we prove (6.3) for $u^k$ under the assumption that (6.3) holds for $u^{k-1}$. Let $w^k=u^k_{ll}$ and $z^k=\eta(|x-y|/\delta)w^k$. Then, by virtue of (6.1), we see that $z^k$ satisfies (6.8) with $h_{ll}$ replaced by $u^{k-1}_{ll}$. Hence the argument similar to the proof for $u^1$ $(=u)$ implies that

$$|\tilde{u}^k|^{\#}_{E(U_{2\delta}(y))}\leqq(1+\lambda\hat{C})|\tilde{u}^{k-1}|^{\#}_{E(U_{2\delta}(y))}\leqq(1+\lambda\hat{C})^k|\tilde{h}|^{\#}_{E(U_{2\delta}(y))}$$

for $0<\lambda<\hat{\lambda}_h$. This completes the proof of Lemma 6.1.

From now on we will verify that $u(t,x)$ is a generalized solution for (Pa.MP). Let $K$ be an arbitrary compact subset of $\Omega$ and $\delta>0$ be so small that $U_{2\delta}(K)\subset\Omega$. Denote

$$|u_0|_{E(U_{2\delta}(K))}=\sup\{(u_0)_{ll}(x);\, x\in U_{2\delta}(K),\, l\in\boldsymbol{R}^n\}.$$

Since we may assume without loss of generality that $|u_0|_{E(U_{2\delta}(K))}\geqq 1$, Lemma 6.1

shows that for every $y \in K$ and $0<\lambda<\hat{\lambda}_h$

$$\begin{aligned} |u^k|_{E(U_\delta(y))} &\leqq |\tilde{u}^k|^{\#}_{E(U_{2\delta}(y))} \\ &\leqq (1+\lambda\hat{C})^k |\tilde{u}_0|^{\#}_{E(U_{2\delta}(y))} \\ &\leqq (1+\lambda\hat{C})^k |u_0|_{E(U_{2\delta}(K))}, \end{aligned}$$

whence for every $0 \leqq t \leqq T$

$$|u^\varepsilon(t, \cdot)|_{E(U_\delta(K))} \leqq e^{\hat{c}t}|u_0|_{E(U_{2\delta}(K))} \equiv a_{K,\delta}(t).$$

From this it follows that

$$u^\varepsilon(t, x+\Delta x) - 2u^\varepsilon(t, x) + u^\varepsilon(t, x-\Delta x) \leqq a_{K,\delta}(t)|\Delta x|^2 \tag{6.9}$$

for $t \in [0, T]$ and $x, x+\Delta x, x-\Delta x \in U_\delta(K)$ with $|\Delta x|<\delta$.

Now, as in [3], we use the next lemma concerning the convergence of a sequence of locally semi-concave functions.

LEMMA 6.2 (Kružkov). *Let $\{u^m\}_{m=1}^\infty$ be a sequence of Lipschitz continuous functions on $\bar{\Omega}$ such that*
(i) $\|u^m\|_0 \leqq M_0$ *and* $\|u_x^m\|_\infty \leqq M_1$, $m = 1, 2,\dots,$
(ii) *for each compact $K \subset\subset \Omega$ and $\delta>0$ such that $U_{2\delta}(K) \subset \Omega$,*

$$u^m(x+\Delta x) - 2u^m(x) + u^m(x-\Delta x) \leqq a_{K,\delta}|\Delta x|^2, \qquad m = 1, 2,\dots$$

*with a constant $a_{K,\delta}$ for $x, x+\Delta x, x-\Delta x \in U_\delta(K)$: $|\Delta x|<\delta$.*

*Then there exist $u \in \mathscr{L}(\bar{\Omega})$ and a subsequence $\{u^{m(i)}\}$ such that $u^{m(i)} \to u$ uniformly on $\bar{\Omega}$, $u_x^{m(i)} \to u_x$ in $L^1(\Omega)$ and $u_x^{m(i)} \to u_x$ a.e. in $\Omega$. Moreover, the limit $u$ satisfies* (i) *and* (ii) *with the same constants.*

PROOF. See [15; Lemma 3.1].

Since $U(t)u_0$ is Lipschitz continuous in $t$ on $[0, T]$ and $(U(t)u_0)(x)=u(t, x)$ is Lipschitz continuous in $x$ with the Lipschitz constant $M_1$ for each $t \geqq 0$, $u(t, x)$ is Lipschitz continuous in $(t, x)$, and hence $u$ is differentiable at almost all points of $Q_T$. Furthermore, by (6.9) and Lemma 6.2, we find a subsequence $\{u^{\varepsilon(i)}\}$ such that $\{u_x^{\varepsilon(i)}\}$ converges to $u_x$ a.e. in $Q_T$ as $\varepsilon(i) \downarrow 0$. Multiply (5.28) by arbitrary $\psi \in C_0^\infty(Q_T)$ and integrate over $Q_T$. Integrating by parts and letting $\varepsilon \downarrow 0$ through the subsequence $\{\varepsilon(i)\}$ yield

$$\iint_{Q_T} \{-u\psi_t + H(t, x, u, u_x)\psi + \mu(u_x, \psi_x)\} dtdx = 0,$$

since $[t/\varepsilon] \to t$ as $\varepsilon \downarrow 0$. It is easy to see that $u$ satisfies (5.2) and (5.3).

Thus we conclude:

THEOREM 5. *Let $H$ satisfy the assumptions* (H.I)–(H.IV), *and let $U(t)$ be the evolution operator on $\bar{\mathscr{D}}$ obtained in Theorem* 4. *Suppose that $\{u_0, \phi\}$ satisfies* (B.I)* *and* (B.II)*. *Then $u(t, x)=(U(t)u_0)(x)$ is a generalized solution of* (Pa.MP).

REMARK 6.1. Under the same assumptions as in Theorem 5, we can prove the existence for (Pa.MP) without requiring that $\mu>0$ is small. In fact, our restriction on $\mu$ (cf. (5.5)) was used in Lemma 5.4 to derive the a priori estimate, independent of $\mu$, for the first derivatives of a solution of (BVP). For this purpose, however, we have only to take a positive constant $\sigma_1>1$ such that

$$H(t, x, \Phi+\sigma_1 d, \Phi_x+\sigma_1 d_x) \geqq \mu\sigma_1 \sup\{|\Delta\Phi(x)|+|\Delta d(x)|;\ x\in\overline{B^{\delta_0}}\}$$

for all $(t, x)\in[0, T]\times\overline{B^{\delta_0}}$. Notice that, in general, $\sigma_1$ depends on $\mu$, $\Phi$ and $\Omega$.

## 7. Proof of Theorem 1

This section is devoted to the verification of the existence part of Theorem 1. First recall that $\Omega$ is assumed to be a bounded domain whose boundary $\partial\Omega$ is of class $C^3$. Let the normal curvatures of $\partial\Omega$ be bounded in absolute value by $\kappa$. As was carried out by Kružkov [15], we approximate $\Omega$ by a sequence $\{\Omega^m\}$ of domains with the following properties:

(i) $\Omega_{1/m}\subset\Omega^m\subset\Omega_{1/2m}$ and $\partial\Omega^m\in C^\infty$, $\quad m=m_0, m_0+1,\dots.$

(ii) For each $m\geqq m_0$, the distance function $d^m(x)$ corresponding to $\Omega^m$ is of class $C^2$ and satisfies $|d_x^m(x)|\geqq\tilde{d}_0>0$ in the boundary strip $B^m=\{x\in\Omega^m;\ d^m(x)<\tilde{\delta}_0\}$, where $\tilde{d}_0$ and $\tilde{\delta}_0$ are constants such that $\tilde{\delta}_0<1/\kappa$. (In (i) and (ii), it is assumed that $m_0$ is sufficiently large.)

In what follows, let $m\geqq m_0$. Put

$$\hat{u}_0(x)\equiv u_0(x)-\Phi(x).$$

Note that $\hat{u}_0(x)\geqq 0$ for $x\in\bar{\Omega}$ and $\hat{u}_0(x)=0$ for $x\in\partial\Omega$ from the assumption (B.II). Let $\zeta^m(x)$ be a function in $C_0^\infty(\boldsymbol{R}^n)$ such that $\zeta^m(x)=1$ if $x\in\Omega_{5/m}$, $\zeta^m(x)=0$ if $x\in\boldsymbol{R}^n-\Omega_{3/m}$, $\zeta^m\geqq 0$ and $\|\zeta_x^m\|_0\leqq k_1 m$ with a constant $k_1$ independent of $m$. Furthermore, we set

$$\hat{u}_0^m(x)\equiv\hat{u}_0(x)\zeta^m(x)$$

and let $\hat{u}_0^{m,\varepsilon}$ and $\Phi^\varepsilon$ be mollified functions of $\hat{u}_0^m$ and $\Phi$, respectively, where $\varepsilon<1/2m$. (Take $\rho\in C_0^\infty(\boldsymbol{R}^n)$ such that $\rho\geqq 0$, $\rho=0$ for $|x|\geqq 1$ and $\int\rho(x)dx=1$; and set $\rho_\varepsilon(x)=\varepsilon^{-n}\rho(x/\varepsilon)$ for $\varepsilon>0$. Define $\hat{u}_0^{m,\varepsilon}=\rho_\varepsilon*\hat{u}_0^m$ and $\Phi^\varepsilon=\rho_\varepsilon*\Phi$.)

We consider the following mixed problem:

$$(\text{Pa.MP})^m \quad \begin{cases} u_t + H(t, x, u, u_x) - \dfrac{1}{m} = \mu_m \Delta u & \text{in} \quad Q_T^m = (0, T) \times \Omega^m, \\ u(0, x) = \hat{u}_0^{m,\varepsilon}(x) + \Phi^\varepsilon(x) \equiv \tilde{u}_0^m(x) & \text{on} \quad \overline{\Omega^m}, \\ u(t, x) = \phi^\varepsilon(x) & \text{on} \quad [0, T] \times \partial\Omega^m, \end{cases}$$

where $\phi^\varepsilon$ is the restriction of $\Phi^\varepsilon$ to $\partial\Omega^m$ ($\varepsilon$ and $\mu_m$ will be determined below).

The following lemma allows us to show that $H_m \equiv H - 1/m$, $\tilde{u}_0^m$ and $\phi^\varepsilon$ satisfy the hypotheses of Theorem 5.

LEMMA 7.1. (1) *If* $\varepsilon < 1/2m$ *then* $\tilde{u}_0^m(x) \geqq \Phi^\varepsilon(x)$ *for* $x \in \overline{\Omega^m}$, $\tilde{u}_0^m(x) = \phi^\varepsilon(x)$ *for* $x \in \partial\Omega^m$, *and*

$$\|\tilde{u}_0^m\|_{C^1(\overline{\Omega^m})} = \max\{|\tilde{u}_0^m(x)| + |(\tilde{u}_0^m)_x(x)|;\ x \in \overline{\Omega^m}\} \leqq \tilde{C}$$

*with a constant* $\tilde{C}$ *independent of* $m$.

(2) *For each* $m \geqq m_0$, *there are constants* $\varepsilon = \varepsilon(m)$ *and* $\mu_m > 0$ *such that*

$$H(t, x, \Phi^\varepsilon, \Phi_x^\varepsilon) - \frac{1}{m} - \mu_m \Delta\Phi^\varepsilon \leqq 0, \qquad x \in \overline{\Omega^m}, \tag{7.1}$$

$$\mu_m \sup\{|\Delta\Phi^\varepsilon(x)| + |\Delta d^m(x)|;\ x \in \overline{B^m}\} \leqq 1. \tag{7.2}$$

(3) *Let* $K$ *be a compact subset of* $\Omega$ *and* $\delta$ *be a positive number such that* $U_{2\delta}(K) \subset \Omega$. *Then there exists a constant* $a_{K,\delta}$, *independent of* $m$, *such that*

$$\tilde{u}_0^m(x + \Delta x) - 2\tilde{u}_0^m(x) + \tilde{u}_0^m(x - \Delta x) \leqq a_{K,\delta}|\Delta x|^2$$

*for* $x, x+\Delta x, x-\Delta x \in U_\delta(K)$ *with* $|\Delta x| < \delta$, *provided* $U_{2\delta}(K) \subset \Omega_{6/m}$.

(4) $H_m(t, x, u, p) = H(t, x, u, p) - 1/m$ *satisfies the assumptions* (H.I)–(H.IV) *with all the constants corresponding to* $a_1$, $a_2$, $a_3$ *and* $\omega$ *being independent of* $m$.

PROOF. (1) and (4) are clear. Also, $u_0 \in E_{loc}(\Omega)$ implies immediately (3). We now give only the proof of (2). For each given $m$ we first take $\varepsilon = \varepsilon(m)$ so small that

$$|H(t, x, \Phi^\varepsilon, \Phi_x^\varepsilon) - H(t, x, \Phi, \Phi_x^\varepsilon)| < 1/4m.$$

Since $H$ is convex in $p$ and continuous, we see that

$$H(t, x, \Phi, \Phi_x^\varepsilon) \leqq \varepsilon^{-n}\int \rho\left(\frac{x-y}{\varepsilon}\right) H(t, y, \Phi(y), \Phi_y(y))dy + \frac{1}{4m} \leqq \frac{1}{4m}$$

by making $\varepsilon = \varepsilon(m)$ smaller if necessary. Here we have used the assumption (B.II).

Hence,

$$H(t, x, \Phi^{\varepsilon}, \Phi^{\varepsilon}_x) - \frac{1}{m} \leqq -\frac{1}{2m}.$$

Fix such an $\varepsilon=\varepsilon(m)>0$. We next choose $\mu_m>0$ small enough to insure that (7.2) and $-1/2m+\mu_m \sup\{|\Delta\Phi^{\varepsilon}(x)|;\ x\in\Omega^m\}<0$ hold.

The proof of Lemma 7.1 is complete.

Notice that we may suppose $\mu_m \downarrow 0$ as $m\to\infty$. Lemma 7.1 and Theorem 5 imply that it is possible to construct a generalized solution $u^m(t, x)$ of $(\mathrm{Pa.MP})^m$ via the Generation Theorem, and that there are constants $M_0$ and $M_1$ satisfying $|u^m(t, x)|\leqq M_0$ for $(t, x)\in\overline{Q^m_T}$ and $|u^m_x(t, x)|\leqq M_1$ a.e. in $Q^m_T$, respectively. Moreover, it is easily shown that if $K$ is a compact set in $\Omega$ and $\delta>0$ is such that $U_{2\delta}(K)\subset\Omega_{6/m}$ then

$$u^m(t, x+\Delta x) - 2u^m(t, x) + u^m(t, x-\Delta x) \leqq a_{K,\delta}(t)\,|\Delta x|^2$$

for $x$, $x+\Delta x$, $x-\Delta x\in U_\delta(K)$ with $|\Delta x|<\delta$, where $a_{K,\delta}(t)$ is a positive and non-decreasing function of $t$ (cf. (6.9)).

Since $\{\Omega^m\}$ converges to $\Omega$ as $m\to\infty$, by using Lemma 6.2 and a diagonal argument, we can find a subsequence $\{u^{m(i)}\}$ and $u\in\mathscr{L}(\overline{Q_T})\cap E_{loc}(Q_T)$ such that $u^{m(i)}\to u$ uniformly on any compact set of $Q_T$, $u^{m(i)}_x\to u_x$ a.e. in $Q_T$ and $u(t, x)=\phi(x)$ on $[0, T]\times\partial\Omega$. ($E_{loc}(Q_T)$ denotes the space of all $v$ such that $v$ satisfies the condition (iii) of Definition 2.1.)

We next prove that $u$ satisfies (1.1). For arbitrary $\psi\in C^\infty_0(Q_T)$ there is an $m_1$ such that

$$\iint_{Q_T}\left\{-u^m(t, x)\psi_t + \left(H(t, x, u^m, u^m_x) - \frac{1}{m}\right)\psi + \mu_m(u^m_x, \psi_x)\right\}dtdx = 0$$

for all $m\geqq m_1$. Letting $m\to\infty$ in the above yields

$$\begin{aligned} 0 &= \iint_{Q_T}\{-u\psi_t + H(t, x, u, u_x)\psi\}dtdx \\ &= \iint_{Q_T}\{u_t + H(t, x, u, u_x)\}\psi dtdx, \end{aligned}$$

since $u\in\mathscr{L}(\overline{Q_T})$. Hence $u$ satisfies (1.1) a.e. in $Q_T$. It is clear that $u$ satisfies (1.2) and (1.3). Therefore, the limit function $u(t, x)$ is a generalized solution of (MP). Finally we note that $\{u^m\}$ itself converges to $u$ because of the uniqueness for (MP). The proof of Theorem 1 has been completed.

## References

[1] S. Aizawa, A semigroup treatment of the Hamilton-Jacobi equation in one space variable, Hiroshima Math. J., **3** (1973), 367–386.

[2] S. Aizawa, On the uniqueness of global generalized solutions for the equation $F(x, u, \text{grad } u)=0$, Proc. Japan Acad., **51** (1975), 147–150.

[3] S. Aizawa, A semigroup treatment of the Hamilton-Jacobi equation in several space variables, Hiroshima Math. J., **6** (1976), 15–30.

[4] S. Aizawa and N. Kikuchi, A mixed initial and boundary-value problem for the Hamilton-Jacobi equation in several space variables, Funkcial. Ekvac., **9** (1966), 139–150.

[5] S. H. Benton, A general space-time boundary value problem for the Hamilton-Jacobi equation, J. Diff. Equations, **11** (1972), 425–435.

[6] S. H. Benton. Global variational solutions of Hamilton-Jacobi boundary value problems, J. Diff. Equations, **13** (1973), 468–480.

[7] B. C. Burch, A semigroup treatment of the Hamilton-Jacobi equation in several space variables, J. Diff. Equations, **23** (1977), 107–124.

[8] B. C. Burch and J. A. Goldstein, Two boundary value problems for the Hamilton-Jacobi equation, to appear.

[9] E. D. Conway and E. Hopf, Hamilton's theory and generalized solutions of the Hamilton-Jacobi equation, J. Math. Mech., **13** (1964), 939–986.

[10] M. G. Crandall, The semigroup approach to first order quasilinear equations in several space variables, Israel J. Math., **12** (1972), 108–132.

[11] M. G. Crandall and A. Pazy, Nonlinear evolution equations in Banach spaces, Israel J. Math., **11** (1972), 57–94.

[12] A. Douglis, Solutions in the large for multi-dimensional, non-linear partial differential equations of first order, Ann. Inst. Fourier, Grenoble, **15–2** (1965), 1–35.

[13] A. Douglis, Layering methods for nonlinear partial differential equations of first order, Ann. Inst. Fourier, Grenoble, **22–3** (1972), 141–227.

[14] S. N. Kružkov, Generalized solutions of nonlinear equations of first order with several variables I, Mat. Sb., **70** (112) (1966), 394–415 (in Russian).

[15] S. N. Kružkov, Generalized solutions of Hamilton-Jacobi equations of eikonal type, Mat. Sb., **98** (140) (1975), 450–493 (in Russian).

[16] K. Masuda, Evolution equations, Kinokuniya, Tokyo (1975), (in Japanese).

[17] M. Nagumo, On principally linear elliptic differential equations of the second order, Osaka Math. J., **6** (1954), 207–229.

[18] J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, Phil. Trans. Royal Soc. London, Ser. A, **264** No 1153 (1969), 413–496.

[19] M. B. Tamburro, The evolution operator solution of the Cauchy problem for the Hamilton-Jacobi equation, Israel J. Math., **26** (1977), 232–264.

[20] Y. Tomita, A semigroup treatment of the mixed problem for the Hamilton-Jacobi equation in one space variable, Hiroshima Math. J., **7** (1977), 183–199.

*Department of Mathematics,*
*Kobe University of Mercantile Marine*