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On Group Theoretic Properties of Cocommutative Hopf Algebras

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In a previous paper [7] we gave an answer to some questions on "Isomorphism Theorems" of formal groups over a field raised by J. Dieudonné in his book [1]. We treated there the case of infinitesimal formal groups which corresponds to the colocal case in terms of Hopf algebras. Precisely we showed that "Isomorphism Theorems" of colocal Hopf algebras over a field, analogous to those in classical group theory, can be given.

The first aim of this paper is to generalize the results in [7] to any case. In other words we shall show that the above mentioned questions by J. Dieudonné can be solved completely for any formal groups over a field. Our method for proofs is very similar to that in [7], and in particular the main theorem in [3] by K. Newman, which means essentially "First Isomorphism Theorem" of cocommutative Hopf algebras, plays an important role. As applications of "Isomorphism Theorems" we shall give some basic properties of cocommutative Hopf algebras which are analogous to those in classical group theory. In § 2 we shall obtain results on commutative diagrams and direct products of cocommutative Hopf algebras. In particular "Five Lemma" will be given. In § 3 we show that the smash product of cocommutative Hopf algebras defined in [2] or in [4] coincides with the semi-direct product of them in the sense of [1]. Moreover certain properties of split exact sequences of Hopf algebras are shown. In the last section nilpotent and solvable Hopf algebras are studied. Although almost all results in § 4 can be obtained from those in § 1 by imitating classical group theory, we shall give detailed proofs because of unfamiliarity of our theory.

Our terminology and notations follow those in the book [6].

§ 1. Isomorphism Theorems

First we show the following

LEMMA 1. *Let (C, Δ, ε) be a coalgebra over a field k and let e be a group-like element of C. If x is an element of $C^+ = \ker \varepsilon$, then we have*

$$\Delta(x) - x \otimes e - e \otimes x \in C^+ \otimes C^+.$$

PROOF. Since $\Delta(e) = e \otimes e$, we have $\varepsilon(e) = 1$ and hence

$$(\varepsilon \otimes 1_C)(\Delta(x) - x \otimes e - e \otimes x) = 1 \otimes x - 1 \otimes x = 0.$$

This means that

$$\Delta(x) - x \otimes e - e \otimes x \in \operatorname{Ker}(\varepsilon \otimes 1_C) = C^+ \otimes C.$$

Similarly we see

$$\Delta(x) - x \otimes e - e \otimes x \in C \otimes C^+.$$

Since $C = C^+ \oplus ke$, we have $(C^+ \otimes C) \cap (C \otimes C^+) = C^+ \otimes C^+$. Therefore we have our assertion. q.e.d.

PROPOSITION 1. *Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be cocommutative coalgebras over k, and let ρ be a coalgebra homomorphism of C to C'. Assume that C has a grouplike element e and let D' be a subcoalgebra of C' containing $\rho(e)$. If we put $D = \{x \in C \mid (\rho \otimes 1_C)\Delta(x) - \rho(e) \otimes x \in D'^+ \otimes C\}$ where $D'^+ = D' \cap \ker \varepsilon$, D is a subcoalgebra of C. Moreover a subcoalgebra E of C is contained in D if and only if $\rho(E)$ is contained in D'.*

PROOF. The first assertion and the fact that $\rho(D) \subset D'$ are shown in a similar way to the proof of Prop. 4.8 in [6], and so we omit the detail. Now let E be a subcoalgebra of C such that $\rho(E) \subset D'$. Since $\rho(e) \in D'$, we may assume, replacing E with $E + ke$ if necessary, that E contains e. Then by Lemma 1 we see

$$\Delta(x) - x \otimes e - e \otimes x \in E^+ \otimes E^+ \qquad \text{for} \quad x \in E \cap \ker \varepsilon = E^+.$$

Since $\rho(E^+) \subset D' \cap \ker \varepsilon' = D'^+$ from $\varepsilon' \rho(E^+) = \varepsilon(E^+) = 0$, we see

$$(\rho \otimes 1_C)\Delta(x) - \rho(e) \otimes x \in D'^+ \otimes E \subset D'^+ \otimes C.$$

This means $E^+ \subset D$ and hence $E = ke \oplus E^+ \subset D$. q.e.d.

We call D in Proposition 1 the *h-inverse* of D' by ρ and denote it by $h\text{-}\rho^{-1}(D')$. It is clear that D is independent of the choice of a grouplike element e satisfying $\rho(e) \in D'$. Now if B is a bigebra over k, B has a natural left B-module structure defined by the multiplication of B. Then we have the following

COROLLARY 1. *Let B be a cocommutative Hopf algebra over a field k and let C' be a cocommutative coalgebra over k with a left B-module structure. If ρ is a coalgebra homomorphism of B to C' such that ρ is also a left B-module homomorphism, then the h-inverse of $k\rho(1)$ by ρ is a subbigebra of B.*

This is a generalization of Corollary to Proposition of § 1 in [7], and the same proof is available. Therefore we omit it.

COROLLARY 2. *Let B and B' be cocommutative bigebras (resp. Hopf* *algebras) over a field k, and let ρ be a bigebra homomorphism of B to B'. If D' is a subbigebra (resp. Hopf subalgebra) of B', then the h-inverse D of D' by ρ is a subbigebra (resp. Hopf subalgebra) of B.*

PROOF. Since 1 is a grouplike element of B and $\rho(1)=1$ is contained in D', there exists the h-inverse D of D' by ρ. To see that D is a subbigebra of B, it is sufficient by Prop. 1 to show that D is a subalgebra of B. If x and y are in D, then we have

$$(\rho \otimes 1_B)\Delta(x) = 1 \otimes x + x_1 \quad \text{with} \quad x_1 \in D'^+ \otimes B \quad \text{and}$$

$$(\rho \otimes 1_B)\Delta(y) = 1 \otimes y + y_1 \quad \text{with} \quad y_1 \in D'^+ \otimes B,$$

where Δ is the comultiplication of B. Since Δ and ρ are algebra homomorphisms, we see

$$\begin{aligned}(\rho \otimes 1_B)\Delta(xy) &= (\rho \otimes 1_B)\Delta(x)(\rho \otimes 1_B)\Delta(y)\\ &= (1 \otimes x + x_1)(1 \otimes y + y_1)\\ &= 1 \otimes xy + (1 \otimes x)y_1 + x_1(1 \otimes y) + x_1 y_1,\end{aligned}$$

and hence

$$(\rho \otimes 1_B)\Delta(xy) - 1 \otimes xy \in D'^+ \otimes B.$$

This means that xy is contained in D, so D is a subalgebra of B. Lastly we see $c(D) \subset D$ for Hopf algebra cases in the same way as the proof of Prop. 4.8 if c is the antipode of B. Therefore D is a Hopf subalgebra of B. q.e.d.

The following two lemmas are generalizations of Lemmas 1 and 2 in [7] for non-colocal cases and their proofs are exactly the same ones.

LEMMA 2. *Let B and B' be cocommutative Hopf algebras over a field k, and let ρ be a Hopf algebra homomorphism of B to B'. Then the h-inverse D of a normal Hopf subalgebra D' of B' by ρ is normal in B.*

LEMMA 3. *Let B, B' and ρ be as above, and assume that ρ is surjective. Then if D is a normal Hopf subalgebra of B, so is $\rho(D)$ in B'.*

Now we have obtained preliminary results in the above to prove "Isomorphism Theorems" of not necessarily colocal cocommutative Hopf algebras over a field k. The following results are generalizations for non-colocal cocommutative Hopf algebras of the ones for colocal cases which were shown in §2 of the paper [7]. The previous proofs given in [7] of "Isomorphism Theorems" for colocal cases are also available for general cocommutative cases without any change, if we use the above obtained results instead of the ones of §1 in [7].

Therefore we shall state only the results without proofs.

Let B be a cocommutative Hopf algebra over a field k and let D be a Hopf subalgebra of B. If B^+ is the kernel of the counit ε of B, we put $D^+=D\cap B^+$. Then it is easy to see that the left ideal BD^+ of B generated by D^+ is a coideal of B. Therefore the quotient space B/BD^+ is a coalgebra over k with a natural left B-module structure and the canonical map π of B to B/BD^+ is a coalgebra and left B-module homomorphism. Then the following theorem was given by K. Newman in his paper [3].

THEOREM 1. *Let B, D and π be as above. Then D is the h-kernel of π. Conversely let ρ be a surjective coalgebra and left B-module homomorphism of B to a coalgebra C with a left B-module structure, and let E be the h-kernel of ρ. Then the kernel of ρ is BE^+ and C is isomorphic to B/BE^+ as coalgebras and left B-modules.*

In particular if D is a normal Hopf subalgebra of B, we know that BD^+ is equal to D^+B and hence that BD^+ is a Hopf ideal (cf. Lemma 14.8 in [6]). Therefore the quotient space B/BD^+ is a Hopf algebra over k which is called *the Hopf quotient algebra of B by D* and will be denoted by B/D. If ρ_D is the canonical map of B to $B/D=B/BD^+$, ρ_D is a Hopf algebra homomorphism. Then Theorem 1 gives the following corollary which would correspond to "First isomorphism theorem" in group theory.

COROLLARY. *Let B, D, B/D and ρ_D be as above. Then the h-kernel of the Hopf algebra homomorphism ρ_D is D. If ρ is a surjective Hopf algebra homomorphism of B to a Hopf algebra B' over k, then the h-kernel E of ρ is a normal Hopf subalgebra of B and B' is isomorphic to $B/E=B/BE^+$ as Hopf algebras.*

THEOREM 2. *Let B and B' be cocommutative Hopf algebras over a field k, and let ρ be a surjective Hopf algebra homomorphism of B to B'. If C is the h-inverse of a Hopf subalgebra C' of B' by ρ, B/BC^+ is isomorphic to $B'/B'C'^+$ as coalgebras and left B-modules, where the left B-module structure of $B'/B'C'^+$ is the one obtained naturally from its left B'-module structure through ρ.*

As a direct consequence of this theorem we have the following Hopf algebra version of "Second isomorphism theorem" in group theory.

COROLLARY 1. *Let B, B' and ρ be as above. If C' is a normal Hopf subalgebra of B', so is the h-inverse C of C' by ρ in B. Moreover the quotient Hopf algebras B/C and B'/C' are isomorphic to each other.*

COROLLARY 2. *Let B, B' and ρ be as above, and let D be the h-kernel of ρ. Then there is a bijective correspondence between Hopf subalgebras C' of*

B' and Hopf subalgebras C of B containing D such that C' (resp. C) corresponds to h-$\rho^{-1}(C')$ (resp. $\rho(C)$). Moreover the Hopf quotient algebra C/D is isomorphic to $C'=\rho(C)$.

REMARK. If a Hopf subalgebra C of B corresponds to C' of B' in the above corollary, *C is normal in B if and only if C' is normal in B'* by Lemmas 2 and 3.

The following theorem could be called "Third isomorphism theorem".

THEOREM 3. *Let B be a cocommutative Hopf algebra over a field k, and let $J(C, D)$ and $I(C, D)$ be the join and the intersection of Hopf subalgebras C and D of B, respectively. Then if D is normal in B, $I(C, D)$ and D are normal in C and $J(C, D)$, respectively, and the Hopf quotient algebras $J(C, D)/D$ and $C/I(C, D)$ are isomorphic to each other.*

§2. Commutative diagrams and direct products

In this section we show certain results on commutative diagrams of cocommutative Hopf algebras whose corresponding ones in group theory are well known and useful. Furthermore we give some elementary properties of tensor product Hopf algebras of cocommutative Hopf algebras.

In the following we assume that a map of a Hopf algebra over a field k to another one is always a Hopf algebra homomorphism if otherwise specified.

A sequence

$$\cdots \longrightarrow B_{i-1} \xrightarrow{f_{i-1}} B_i \xrightarrow{f_i} B_{i+1} \longrightarrow \cdots$$

of cocommutative Hopf algebras over a field k is called *exact*, if the h-kernel of f_i is equal to the image of f_{i-1} for each i. From the definition of h-kernels it is easy to see that *a Hopf algebra homomorphism f of B to B' is injective (resp. surjective) if and only if*

$$k \xrightarrow{i} B \xrightarrow{f} B' \quad (\text{resp. } B \xrightarrow{f} B' \xrightarrow{\varepsilon'} k)$$

is exact, where i and ε' are the identity of B and the coidentity of B', respectively.

PROPOSITION 2. *Assume that the commutative diagram of cocommutative Hopf algebras over a field k*

$$\begin{array}{ccccc} B & \xrightarrow{u} & C & \xrightarrow{v} & D \\ {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow \\ B' & \xrightarrow{u'} & C' & \xrightarrow{v'} & D' \end{array}$$

has both rows exact. Then we have the followings:

(i) *If u', b and d are injective, so is c.*
(ii) *If v, b and d are surjective, so is c.*
(iii) *If c is injective b and v surjective, d is injective.*
(iv) *If c is surjective d and u' injective, b is surjective.*

PROOF. (i) If we put $E=h\text{-ker}\, c$, then $c(E)=k$. So we see $k=v'c(e)=dv(E)$. This means $v(E)=k$ by injectivity of d. Since $u(B)=h\text{-ker}\ v$ contains E, there is a Hopf subalgebra E' of B such that $u(E')=E$ by Cor. 2 to Th. 2. Then we have $u'b(E')=cu(E')=c(E)=k$ and hence $E'=k$ by injectivity of u' and b. Therefore we see $E=u(E')=k$ and hence c is injective.
(ii) If we put $E'=c(C)$, then we have $v'(E')=v'c(C)=dv(C)=D'$ by surjectivity of v and d, and hence $C'=h\text{-}v'^{-1}(D')=J(E', u'(B'))$ by Cor. 2 to Th. 2 and exactness of the lower row of the diagram. On the other hand we see $u'(B')=u'b(B)=cu(B)\subset c(C)=E'$. This means that c is surjective.
(iii) Let E be the h-kernel of d. Then there is a Hopf subalgebra E' of C such that $v(E')=E$ by Cor. 2 to Th. 2 and surjectivity of v. Since we have $k=d(E)=dv(E')=v'c(E')$, $c(E')$ is contained in $h\text{-ker}\, v'$. Therefore there is a Hopf subalgebra E'' of B such that $u'b(E'')=c(E')$ by Cor. 2 to Th. 2, exactness of the lower row of the diagram and surjectivity of b. Then since $c(E')=u'b(E'')=cu(E'')$, we have $E'=u(E'')$ by Cor. 2 to Th. 2 and injectivity of c. Therefore we see $E=v(E')=vu(E'')=k$ by exactness of the upper row of the diagram. This means that d is injective.
(iv) Since c is surjective, there is a Hopf subalgebra E of C such that $u'(B')=c(E)$ by Cor. 2 to Th. 2. Then we have $dv(E)=v'c(E)=v'u'(B')=k$ by exactness of the lower row of the diagram, and hence $v(E)=k$ by injectivity of d. This means that E is contained in $h\text{-ker}\, v=u(B)$. Therefore there is a Hopf subalgebra E' of B such that $u(E')=E$ by Cor. 2 to Th. 2. Then we have $u'b(E')=cu(E')=c(E)=u'(B')$ and hence $b(E')=B'$ by injectivity of u'. This means that b is surjective. q. e. d.

PROPOSITION 3. *Assume that the commutative diagram of cocommutative Hopf algebras over a field k*

$$\begin{array}{ccccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & D & \xrightarrow{x} & E \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & D' & \xrightarrow{x'} & E' \end{array}$$

has both rows exact. Then we have the followings:
(i) *If a is surjective b and d injective, then c is injective.*
(ii) *If e is injective b and d surjective, then c is surjective.*

PROOF. (i) If we put $X=h\text{-ker}\, c$, then we see $dw(X)=w'e(X)=k$ and hence $w(X)=k$ by injectivity of d. Therefore there is a Hopf subalgebra Y of B such that $v(Y)=X$ by Cor. 2 to Th. 2 and exactness of the upper row of the diagram. Since we have $v'b(Y)=cv(Y)=c(X)=k$, there is a Hopf subalgebra Z of A such that $u'a(Z)=b(Y)$ by Cor. 2 to Th. 2, exactness of the lower row and surjectivity of a. Then we see $bu(Z)=u'a(Z)=b(Y)$ and hence $u(Z)=Y$ by injectivity of b. Therefore we have $X=v(Y)=vu(Z)=k$ by exactness of the upper row. This means that c is injective.

(ii) By surjectivity of d and Cor. 2 to Th. 2 there is a Hopf subalgebra X of D such that $d(X)=w'(C')$. Since $w'(C')=h\text{-ker}\, x'$ by our assumption, we see $ex(X)=x'd(X)=x'w'(C')=k$ and hence $x(X)=k$ by injectivity of e. Therefore there is a Hopf subalgebra Y of C such that $w(Y)=X$ by Cor. 2 to Th. 2 and exactness of the upper row. Then we have $w'c(Y)=dw(Y)=d(X)=w'(C')$ and hence $J(c(Y), v'(B'))=C'$ by Cor. 2 to Th. 2 and exactness of the lower row. This means obviously $J(c(C), v'(B'))=C'$. On the other hand we see $v'(B')=v'b(B)=cv(B)\subset c(C)$ by surjectivity of b. Therefore we have $C'=J(c(C), v'(B'))=c(C)$. q.e.d.

COROLLARY (*Five Lemma*). *In the commutative diagram of Prop.* 3 *assume further that b and d are isomorphic, that a is surjective and that e is injective. Then c is isomorphic.*

Though the following results on tensor product Hopf algebras are already known (cf. Chap. I, §3, no. 12, 14, 19 in [1]), we give proofs for convenience' sake.

PROPOSITION 4. *Let B and C be cocommutative Hopf algebras over a field k. Then the tensor product Hopf algebra $B\otimes C$ of B and C is the direct product of them in the category of cocommutative Hopf algebras over k.*

PROOF. Let E be $B\otimes C$, and let ρ_B and ρ_C be the canonical Hopf algebra projections of E to B and C respectively. Then it is sufficient to show that if F is a cocommutative Hopf algebra over k and if f and g are Hopf algebra homomorphisms of F to B and C respectively, then there is a unique Hopf algebra homomorphism h of F to E satisfying $f=\rho_B h$ and $g=\rho_C h$. If Δ is the comultiplication of F, then Δ is a Hopf algebra homomorphism by cocommutativity of F and so $h=(f\otimes g)\Delta$ is a Hopf algebra homomorphism of F to E. Now put $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ for x in F. Then we have

$$(\rho_B h)(x) = \rho_B(\sum_{(x)} f(x_{(1)})\otimes g(x_{(2)})) = \sum_{(x)} \varepsilon_B(g(x_{(2)}))f(x_{(1)})$$

$$= f(\sum_{(x)} \varepsilon_F(x_{(2)})x_{(1)}) = f(x),$$

where ε_B and ε_F are the coidentities of B and F, respectively. This means $\rho_B h=f$. Similarly we see $\rho_C h=g$. Next we show the uniqueness of h. Let j_B and j_C be the Hopf algebra homomorphisms of B and C to E respectively such that $j_B(x)=x\otimes 1$ for x in B and $j_C(y)=1\otimes y$ for y in C. Then we can see easily that

$$m_E(j_B\rho_B\otimes j_C\rho_C)\Delta_E=1_E,$$

where m_E and Δ_E are the multiplication and the comultiplication of E respectively. Let h' be a Hopf algebra homomorphism of F to E satisfying $\rho_B h'=f$ and $\rho_C h'=g$. Then we have

$$\begin{aligned} m_E(j_B\otimes j_C)(f\otimes g)\Delta &= m_E(j_B\otimes j_C)(\rho_B\otimes\rho_C)(h'\otimes h')\Delta \\ &= m_E(j_B\rho_B\otimes j_C\rho_C)\Delta_E h' = h'. \end{aligned}$$

This means obviously $h=h'$. q.e.d.

COROLLARY. *Let B and C be as in Prop. 4, and let D be a Hopf subalgebra of $B\otimes C$. If B' and C' are the images of D by the canonical Hopf algebra projections ρ_B and ρ_C of $B\otimes C$ to B and C respectively, then D is contained in $B'\otimes C'$, where $B'\otimes C'$ is considered naturally as a Hopf subalgebra of $B\otimes C$.*

PROOF. Let q_B and q_C be the restrictions of ρ_B and ρ_C to E respectively, and j_B and j_C be the canonical injections of B' and C' to B and C respectively. If $\rho_{B'}$ and $\rho_{C'}$ are the canonical Hopf algebra projections of $B'\otimes C'$ to B' and C' respectively, then there is a unique Hopf algebra homomorphism h of D to $B'\otimes C'$ satisfying $q_B=\rho_{B'}h$ and $q_C=\rho_{C'}h$ by Prop. 4. If j_D is the canonical injection of D to $B\otimes C$, then we have $\rho_B j_D=j_B q_B=j_B\rho_{B'}h$ and $\rho_C j_D=j_C q_C=j_C\rho_{C'}h$. On the other hand if $j_{B'\otimes C'}$ is the canonical injection of $B'\otimes C'$ to $B\otimes C$, then we see $\rho_B j_{B'\otimes C'}=j_B\rho_{B'}$ and $\rho_C j_{B'\otimes C'}=j_C\rho_{C'}$. Therefore we have $\rho_B j_D=j_B\rho_{B'}h=\rho_B j_{B'\otimes C'}h$ and $\rho_C j_D=j_C\rho_{C'}h=\rho_C j_{B'\otimes C'}h$, and hence $j_D=j_{B'\otimes C'}h$ by Prop. 4. This means clearly that D is contained in $B'\otimes C'$. q.e.d.

THEOREM 4. *Let B be a cocommutative Hopf algebra over a field k, and let C and D be Hopf subalgebras of B. Then the followings are equivalent:*

(i) B is isomorphic to $C\otimes D$.

(ii) The join $J(C, D)$ of C and D is B, the intersection $I(C, D)$ of C and D is k, and C commutes with D.

(iii) The join $J(C, D)$ of C and D is B, the intersection $I(C, D)$ of C and D is k, and C and D are normal in B.

PROOF. The implications (i)⇒(ii) and (i)⇒(iii) are trivial. (iii)⇒(ii). Let E be any cocommutative coalgebra over k and consider $\mathrm{Hom}_{coal}(E, C)$ and $\mathrm{Hom}_{coal}(E, D)$ naturally as subgroups of the group $\mathrm{Hom}_{coal}(E, B)$. If f and

g are elements of $\mathrm{Hom}_{coal}(E, C)$ and $\mathrm{Hom}_{coal}(E, D)$ respectively, then we see $f*g*f^{-1} \in \mathrm{Hom}_{coal}(E, D)$ and $g*f^{-1}*g^{-1} \in \mathrm{Hom}_{coal}(E, C)$ by Prop. 12.1 in [6] and our assumption. Therefore $f*g*f^{-1}*g^{-1}$ is contained in $\mathrm{Hom}_{coal}(E, C) \cap \mathrm{Hom}_{coal}(E, D) = \mathrm{Hom}_{coal}(E, I(C, D)) = \mathrm{Hom}_{coal}(E, k)$. This means $f*g = g*f$, and C commutes with D by Prop. 13.1 in [6]. (ii)$\Rightarrow$(i). By Th. 3 and our assumption we see that $B/D = J(D, C)/D$ is isomorphic to $C/I(C, D) = C$. This means that the sequence

$$k \longrightarrow D \xrightarrow{j} B \xrightarrow{\rho} C \cong B/D \longrightarrow k$$

is exact, where j and ρ are the canonical injection and surjection respectively. Now let α be the linear map of $D \otimes C$ to B satisfying $\alpha(x \otimes y) = xy$ for $x \in C$ and $y \in D$. Since C commutes with D, we see α is an algebra homomorphism and hence a Hopf algebra one. Moreover we have the following commutative diagram of cocommutative Hopf algebras

$$\begin{array}{ccccccccc} k & \longrightarrow & D & \xrightarrow{j_D} & C \otimes D & \longrightarrow & C & \xrightarrow{\rho_C} & k \\ & & \downarrow{\scriptstyle 1_D} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1_C} & & \\ k & \longrightarrow & D & \xrightarrow{j} & B & \xrightarrow{\rho} & C & \longrightarrow & k \end{array}$$

where j_D and ρ_C are the canonical injection and projection respectively. Since the upper row is also exact by Prop. 1 in [5], α must be an isomorphism by Cor. to Prop. 3. q.e.d.

§ 3. Semi-direct products and split exact sequences

First we recall the definitions of B-module bigebras and smash products of cocommutative Hopf algebras over a field k (cf. 2.1 and 2.13 in [2]).

Let $(B, m_B, i_B, \Delta_B, \varepsilon_B)$ and $(D, m_D, i_D, \Delta_D, \varepsilon_D)$ be bigebras over a field k, and assume that a k-linear map f of $B \otimes D$ to D gives a left B-module structure of D. Then we call *D a left B-module bigebra with respect to f*, if the followings are satisfied for any elements x in B and a, b in D:

(1) $$f(x \otimes ab) = \sum_{(x)} f(x_{(1)} \otimes a) f(x_{(2)} \otimes b),$$

(2) $$f(x \otimes 1) = \varepsilon_B(x) 1,$$

(3) $$\Delta_D(f(x \otimes a)) = \sum_{(x),(a)} f(x_{(1)} \otimes a_{(1)}) \otimes f(x_{(2)} \otimes a_{(2)}),$$

and

(4) $$\varepsilon_D(f(x \otimes a)) = \varepsilon_B(x) \varepsilon_D(a),$$

where $\Delta_B(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and $\Delta_D(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$.

Now let B be a cocommutative bigebra over k and let D be a left B-module bigebra with respect to f. The *smash product* or the *semi-direct product* of D and B with respect to f, written $D\#_f B$ or simply $D\# B$, is a bigebra over k defined as follows:

(i) As a vector space $D\#_f B$ is $D\otimes B$. Elements $a\otimes x$ will be written $a\# x$.

(ii) The multiplication is defined by

$$(a\# x)(b\# y) = \sum_{(x)} a(f(x_{(1)}\otimes b))\#(x_{(2)}y), \tag{5}$$

where $\Delta_B(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. The unit is $1\# 1$.

(iii) The comultiplication Δ and the coidentity ε are defined by

$$\Delta(a\# x) = \sum_{(a),(x)} (a_{(1)}\# x_{(1)})\otimes(a_{(2)}\# x_{(2)}) \quad \text{and} \tag{6}$$

$$\varepsilon(a\# x) = \varepsilon_D(a)\varepsilon_B(x), \tag{7}$$

where $\Delta_B(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ and $\Delta_D(a)=\sum_{(a)} a_{(1)}\otimes a_{(2)}$.

Then it is known and not difficult to verify that $D\#_f B$ is a bigebra over k. Moreover if B and D are Hopf algebras with the antipodes c_B and c_D respectively, then it can be shown by a routine calculation that $D\#_f B$ is also a Hopf algebra over k with antipode c defined by

$$c(a\# x) = f(c_B(x_{(1)})\otimes c_D(a))\# c_B(x_{(2)}), \tag{8}$$

where $\Delta_B(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. If j_D and j_B are linear maps of D and B to $D\#_f B$ defined by $j_D(a)=a\# 1$ and $j_B(x)=1\# x$ respectively, then it is easy to see that j_D and j_B are injective Hopf algebra homomorphisms. Now we have the following

LEMMA 4. *Let B and D be cocommutative Hopf algebras over a field k, and assume that D is a B-module bigebra with respect to f. If we put $D'=j_D(D)$ and $B'=j_B(B)$, then we have the followings:*

(i) $J(D', B') = D\#_f B$ and $I(D', B') = k$.

(ii) $\phi_{D\#_f B}((1\# x)\otimes(a\# 1))=f(x\otimes a)\# 1$ for x in B and a in D, where $\phi_{D\#_f B}$ is the adjoint map of $D\#_f B$ given in the beginning of §12 *in* [6]. *In particular D' is normal in $D\#_f B$.*

PROOF. It is clear that $I(D', B')=D'\cap B'=k$. Since we have $(a\# 1)(1\# x)$ $=a(f(1\otimes 1))\# x=a\# x$, $J(D', B')$ must be equal to $D\#_f B$. Next we see by (2), (5) and (8)

$$\begin{aligned}\phi_{D\#_f B}((1\# x)\otimes(a\# 1)) &= \sum_{(x)}(f(x_{(1)}\otimes a)\# x_{(2)})(1\# c_B(x_{(3)}))\\ &= \sum_{(x)} f(x_{(1)}\otimes a)f(x_{(2)}\otimes 1)\# x_{(3)}c_B(x_{(4)})\end{aligned}$$

$$= \sum_{(x)} f(x_{(1)} \otimes a)\varepsilon_B(x_{(2)})\#\varepsilon_B(x_{(3)})$$

$$= \sum_{(x)} f(x_{(1)} \otimes a)\#\varepsilon_B(x_{(2)}) = f(x \otimes a)\#1,$$

where $\Delta_B(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $(\Delta_B \otimes 1_B)\Delta_B(x) = (1_B \otimes \Delta_B)\Delta_B(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, and $(\Delta_B \otimes \Delta_B)\Delta_B(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)}$. Therefore we have $\phi_{D\#_f B}(B' \otimes D') \subset D'$, and hence we see easily $\phi_{D\#_f B}((D\#_f B) \otimes D') \subset D'$ by Prop. 12.4 in [6] and $J(D', B') = D\#_f B$. In other words D' is normal in $D\#_f B$. q.e.d.

COROLLARY. *Let B, D and f be as above. If ρ_B is a k-linear map of $D\#_f B$ to B defined by $\rho_B(a\#x) = \varepsilon_D(a)x$, then the sequence*

$$k \xrightarrow{i_D} D \xrightarrow{j_D} D\#_f B \xrightarrow{\rho_B} B \xrightarrow{\varepsilon_B} k$$

is an exact one of Hopf algebras.

PROOF. Since ρ_B is the canonical projection of the tensor product coalgebra $D \otimes B$ to B, ρ_B is a surjective coalgebra homomorphism. On the other hand we see by (4) and (5)

$$\rho_B((a\#x)(b\#y)) = \sum_{(x)} \rho_B(af(x_{(1)} \otimes b)\#x_{(2)}y)$$

$$= \sum_{(x)} \varepsilon_D(a)\varepsilon_B(x_{(1)})\varepsilon_D(b)x_{(2)}y = \rho_B(a\#x)\rho_B(b\#y).$$

This means that ρ_B is an algebra homomorphism and hence a Hopf algebra one. Since $D\#_f B/D'$ is isomorphic to $B' \simeq B$ by Lemma 4 and Th. 3, we see easily that our sequence is exact. q.e.d.

LEMMA 5. *Let $(E, m, i, \Delta, \varepsilon, c)$ be a cocommutative Hopf algebra over k, and let D and B be Hopf subalgebras of E. Assume that D is normal in E and let f be the restriction to $B \otimes D$ of the adjoint map ϕ_E of E. Then D is a left B-module bigebra with respect to f.*

PROOF. It is clear by normality of D in E that $f(B \otimes D)$ is contained in D. Then we see for x, y in B and a in D

$$f(1_B \otimes f)(x \otimes y \otimes a) = f(x \otimes \sum_{(y)} y_{(1)}ac(y_{(2)}))$$

$$= \sum_{(x),(y)} x_{(1)}y_{(1)}ac(y_{(2)})c(x_{(2)})$$

$$= \sum_{(xy)} (xy)_{(1)}ac((xy)_{(2)}) = f(m|_{B \otimes B} \otimes 1_D)(x \otimes y \otimes a),$$

where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ etc.. This means that D is a left B-module. Next we have for x in B and a, b in D

$$\sum_{(x)} f(x_{(1)} \otimes a) f(x_{(2)} \otimes b) = \sum_{(x)} x_{(1)} a c(x_{(2)}) x_{(3)} b c(x_{(4)})$$
$$= \sum_{(x)} x_{(1)} a \varepsilon(x_{(2)}) b c(x_{(3)}) = \sum_{(x)} x_{(1)} a b c(x_{(2)}) = f(x \otimes ab),$$

where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $(\Delta \otimes 1_E)\Delta(x) = (1_E \otimes \Delta)\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ and $(\Delta \otimes \Delta)\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)}$. Therefore f satisfies (1). Similarly we can easily see that f satisfies (2). Moreover we have by cocommutativity of E

$$\Delta(f(x \otimes a)) = \Delta(\sum_{(x)} x_{(1)} a c(x_{(2)}))$$
$$= \sum_{(x),(a)} x_{(1)} a_{(1)} c(x_{(2)}) \otimes x_{(3)} a_{(2)} c(x_{(4)})$$
$$= \sum_{(x),(a)} f(x_{(1)} \otimes a_{(1)}) \otimes f(x_{(2)} \otimes a_{(2)}),$$

and hence f satisfies (3). Similarly we see that f satisfies (4). Therefore D is a left B-module bigebra. q.e.d.

COROLLARY. *Let E, D, B and f be as in Lemma 5. If g is a k-linear map of $D\#_f B$ to $J(D, B)$ defined by $g(a\#x) = m(a \otimes x) = ax$, then g is a surjective Hopf algebra homomorphism of $D\#_f B$ to $J(D, B)$. Moreover g is an isomorphism if and if $I(D, B) = k$.*

PROOF. First we see from (5)

$$g((a\#x)(b\#y)) = g(\sum_{(x)} a f(x_{(1)} \otimes b)\#x_{(2)} y)$$
$$= \sum_{(x)} g(a x_{(1)} b c(x_{(2)})\#x_{(3)} y)$$
$$= \sum_{(x)} a x_{(1)} b \varepsilon(x_{(2)}) y = g(a\#x) g(b\#y),$$

where x, y in B with $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and $(1_B \otimes \Delta)\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, and where a, b in D. Since $g(1\#1) = 1$, g is a k-algebra homomorphism. Similarly we can see easily that g is a coalgebra homomorphism using (6) and (7). Therefore g is a Hopf algebra homomorphism of $D\#_f B$ onto $J(D, B)$. Now assume $I(D, B) = k$. Then we have the following commutative diagram of cocommutative Hopf algebras:

$$\begin{array}{ccccccccc} k & \longrightarrow & D & \xrightarrow{j_D} & D\#_f B & \xrightarrow{\rho_B} & B & \longrightarrow & k \\ {\scriptstyle 1_k}\downarrow & & {\scriptstyle 1_D}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle 1_B}\downarrow & & {\scriptstyle 1_k}\downarrow \\ k & \longrightarrow & D & \longrightarrow & J(D, B) & \longrightarrow & B & \longrightarrow & k, \end{array}$$

where the upper row is the exact sequence given in Cor. to Lemma 4 and where the lower one is also an exact sequence obtained naturally from the assumption $I(D, B)=k$ by Th. 3. Then we see by Cor. to Prop. 3 that g is an isomorphism. Conversely if g is an isomorphism, Hopf subalgebras D' and B' of $D\sharp_f B$ given in Lemma 4 correspond to D and B in $J(E, B)$ by g, respectively. Therefore we see $I(D, B)=k$, because we have $I(D', B')=k$ by Lemma 4. q.e.d.

Now we recall that an exact sequence

$$(*) \qquad k \longrightarrow D \xrightarrow{j} E \xrightarrow{\rho} B \longrightarrow k$$

of cocommutative Hopf algebras over k is said to be *split*, if there is a Hopf algebra homomorphism λ of B to E such that $\rho\lambda=1_B$. Then we have the following

PROPOSITION 5. *The exact sequence* (*) *of cocommutative Hopf algebras over k is split if and only if there is a Hopf subalgebra C of E satisfying $J(j(D), C)=E$ and $I(j(D), C)=k$.*

PROOF. Assume that the sequence (*) is split, and let λ be a Hopf algebra homomorphism of B to E such that $\rho\lambda=1_B$. Then $C=\lambda(B)$ is a Hopf subalgebra of E. Since $J(C, j(D))\supset j(D)$ and $\rho(J(C, j(D)))=\rho(C)=B$, we see $J(C, j(D))=E$ by Cor. 2 to Th. 2. On the other hand we see by Th. 3

$$C/I(C, j(D)) \simeq J(C, j(D))/j(D) = E/j(D) \simeq B,$$

and hence the h-kernel of the restriction $\rho|_C$ of ρ to C is $I(C, j(D))$. But it is clear that $\rho|_C$ is an isomorphism. Therefore $I(C, j(D))$ must be k.

Conversely assume that there is a Hopf subalgebra C of E satisfying $J(C, j(D))=E$ and $I(C, j(D))=k$. Then we see by Th. 1 and Th. 3

$$B \simeq E/j(D) = J(C, j(D))/j(D) \simeq C/I(C, j(D)) \simeq C$$

and this isomorphism is given by the restriction of ρ to C. Therefore we may define λ to be the inverse map of this isomorphism. q.e.d.

Summarizing the above results, we have the following

THEOREM 5. *Let B, D and E be cocommutative Hopf algebras over a field k. Then the followings are equivalent:*
(i) *There are Hopf subalgebras B' and D' of E isomorphic to B and D respectively such that D' is normal in E and that the join and the intersection of B' and D' are E and k respectively.*
(ii) *D is a left B-module bigebra with respect to f and E is isomorphic to $D\sharp_f B$.*
(iii) *There is a split exact sequence of cocommutative Hopf algebras:*

$$k \longrightarrow D \longrightarrow E \longrightarrow B \longrightarrow k.$$

PROOF. This is a direct consequence of Prop. 5, Lemma 4 and Cor. to Lemma 5.

REMARK. (i) The equivalence between (ii) and (iii) is given by R. K. Molnar (cf. 3.6, (c) and 4.1 in [2]). But his proof is somewhat different from ours, because he did not use isomorphism theorems.
(ii) J. Dieudonné gave the notion of semi-direct products of formal groups over k in his book [1] (Chap. 1, §3, no. 15) and his definition of semi-direct products in terms of bigebras is essentially our assertion (i) in Th. 5. Therefore the equivalence between (i) and (ii) in Th. 5 means that the definition of semi-direct products of cocommutative Hopf algebras given by R. K. Molnar in [2] coincides with one by J. Dieudonné.

We shall terminate this section by giving a result on relations between semi-direct products and h-kernels of homomorphisms of cocommutative Hopf algebras. For this purpose we need the next lemma and its corollary.

LEMMA 6. *The sequence*

$$k \longrightarrow D \xrightarrow{j} E \xrightarrow{\rho} B$$

of cocommutative Hopf algebras over k is exact if and only if the induced sequence

$$\{e\} \longrightarrow \mathrm{Hom}_{coal}(C, D) \xrightarrow{j_*} \mathrm{Hom}_{coal}(C, E) \xrightarrow{\rho_*} \mathrm{Hom}_{coal}(C, B)$$

of groups is exact for any cocommutative coalgebra C over k.

PROOF. The necessity can be shown in the exactly same way as in the proof of Prop. 14.12 in [6] and so we omit the proof. Conversely we assume that the second sequence is exact for any cocommutative coalgebra C over k. In particular if $C=h\text{-ker}\, j$ and f is the canonical injection of C into D, then the image of $j_*(f)=jf$ is k. Therefore jf is the neutral element of the group $\mathrm{Hom}_{coal}(C, E)$ and hence f is the neutral element of $\mathrm{Hom}_{coal}(C, D)$ by injectivity of j_*. This means $C=k$, i.e., j is injective. Next if $C=D$ and $f=1_D$, then $\rho_* j_*(F)$ is the neutral element of $\mathrm{Hom}_{coal}(C, B)$. Therefore $j(D)$ is contained in $h\text{-ker}\,\rho$. On the other hand if $C=h\text{-ker}\,\rho$ and f is the canonical injection of C into E, then $\rho_*(f)$ is the neutral element of $\mathrm{Hom}_{coal}(C, B)$, and hence there is a coalgebra homomorphism g of C to D such that $f=j_*(g)=jg$. This means that C is contained in $j(D)$. Therefore we see $j(D)=h\text{-ker}\,\rho$. q.e.d.

COROLLARY. *If the sequence*

$$(*) \qquad k \longrightarrow D \xrightarrow{j} E \xrightarrow{\rho} B \longrightarrow k$$

of cocommutative Hopf algebras over k is exact and split, then the induced sequence

$$(**)\quad \{e\} \longrightarrow \mathrm{Hom}_{coal}(C, D) \xrightarrow{j_*} \mathrm{Hom}_{coal}(C, E) \xrightarrow{\rho_*} \mathrm{Hom}_{coal}(C, B) \longrightarrow \{e\}$$

of groups is exact and split for any cocommutative coalgebra C over k. Conversely if (**) is exact and split for any cocommutative coalgebra C over k, then (*) is exact.

PROOF. Assume that the first sequence is exact and split. Then there is a Hopf algebra homomorphism λ of B to E such that $\rho\lambda = 1_B$. If C is any cocommutative coalgebra over k and if f is any coalgebra homomorphism of C to B, then we have $f = \rho\lambda f = \rho_*(\lambda f)$. Therefore ρ_* is surjective. On the other hand we see that $\rho_*\lambda_* = 1_{\mathrm{Hom}_{coal}(C,B)}$. This means by Lemma 6 that the second sequence is exact and split. Conversely assume that the second sequence is exact and split for any cocommutative coalgebra C over k. If $C = B$ and $f = 1_B$, then there is a coalgebra homomorphism g of C to E such that $f = \rho_*(g)$, i.e., $1_B = \rho_g$. Therefore ρ is surjective and so the first sequence is exact. q.e.d.

LEMMA 7. *Let G_1 and G_2 be groups, and let N and H be subgroups of G_1 such that G_1 is the semi-direct product of N with H. If f is a group homomorphism of G_1 to G_2 such that $f(N) \cap f(H) = \{e_2\}$, then* $\ker f$ *is the semi-direct product of* $\ker f|_N$ *with* $\ker f|_H$.

PROOF. Since we have $G \rhd N$ and $\ker f|_N = \ker f \cap N$, $\ker f|_N$ is normal in $\ker f$. On the other hand we see $\ker f|_N \cap \ker f|_H = \{e_1\}$ from $N \cap H = \{e_1\}$. Since any element g in $\ker f$ is written uniquely $g = nh$ with n in N and h in H, we have $e_2 = f(g) = f(n)f(h)$. By the assumption that $f(N) \cap f(H) = \{e_2\}$, we have $f(n) = f(h) = e_2$. This means that n and h are contained in $\ker f|_N$ and $\ker f|_H$ respectively, and hence that $\ker f = (\ker f|_N)(\ker f|_H)$. Therefore $\ker f$ is the semi-direct product of $\ker f|_N$ with $\ker f|_H$. q.e.d.

Now let D and B be Hopf subalgebras of a cocommutative Hopf algebra E over a field k such that $J(D, B) = E$ and $I(D, E) = k$. When D is normal in E, we may say from Th. 5 that *E is the semi-direct product of D with B.* Then we have the following Hopf algebra version of Lemma 7.

PROPOSITION 6. *Let E_1 and E_2 be cocommutative Hopf algebras over a field k. Let D and B be Hopf subalgebras of E_1 such that E_1 is the semi-direct product of D with B, and let f be a Hopf algebra homomorphism of E_1 and E_2 such that $I(f(D), f(B)) = k$. Then* h-$\ker f$ *is the semi-direct product of* h-$\ker f|_D$ *with* h-$\ker f|_B$.

PROOF. Since E_1 is the semi-direct product of D with B, we have a split exact sequence of cocommutative Hopf algebras:

$$k \longrightarrow D \xrightarrow{j_D} E_1 \underset{j_B}{\overset{\rho}{\rightleftarrows}} B \longrightarrow k,$$

where j_D and j_B are the canonical injections of D and B into E_1 respectively and where $\rho j_B = 1_B$. Moreover if we put $K = h\text{-}\ker f$, $K_D = h\text{-}\ker f|_D$ and $K_B = h\text{-}\ker f|_B$, then we have the following commutative diagram of cocommutative Hopf algebras:

$$\begin{array}{ccccccccc}
 & & k & \xrightarrow{1_k} & k & \xrightarrow{1_k} & k & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_D & \xrightarrow{j_D|K_D} & K & \xrightarrow{\rho|K} & K_B & & \\
 & & {\scriptstyle j_K|K_D}\downarrow & & {\scriptstyle j_K}\downarrow & & {\scriptstyle j_K|K_B}\downarrow & & \\
k & \longrightarrow & D & \xrightarrow{j_D} & E_1 & \underset{j_B}{\overset{\rho}{\rightleftarrows}} & B & \longrightarrow & k \\
 & & {\scriptstyle f|D}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \\
 & & E_2 & \xrightarrow{1_{E_2}} & E_2 & \xrightarrow{1_{E_2}} & E_2 & &
\end{array}$$

where j_K is the canonical injection of K into E_1 and where three columns are exact. If C is any cocommutative coalgebra over k, we have the following commutative diagram of groups from the above one:

$$\begin{array}{ccccccccc}
 & & \{e\} & \longrightarrow & \{e\} & \longrightarrow & \{e\} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Hom}_{coal}(C, K_D) & \longrightarrow & \mathrm{Hom}_{coal}(C, K) & \longrightarrow & \mathrm{Hom}_{coal}(C, K_B) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{e\} & \longrightarrow & \mathrm{Hom}_{coal}(C, D) & \longrightarrow & \mathrm{Hom}_{coal}(C, E_1) & \rightleftarrows & \mathrm{Hom}_{coal}(C, K_B) & \longrightarrow & \{e\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Hom}_{coal}(C, E_2) & \longrightarrow & \mathrm{Hom}_{coal}(C, E_2) & \longrightarrow & \mathrm{Hom}_{coal}(C, E_2) & &
\end{array}$$

In this diagram three columns are exact by Lemma 6 and the third row is exact and split by Cor. to Lemma 6. Therefore the sequence

$$\{e\} \longrightarrow \mathrm{Hom}_{coal}(C, K_D) \xrightarrow{(j_D|K)_*} \mathrm{Hom}_{coal}(C, K)$$
$$\xrightarrow{(\rho|K)_*} \mathrm{Hom}_{coal}(C, K_B) \longrightarrow \{e\}$$

of groups is exact and split for any cocommutative coalgebra C by Lemma 7, and so the sequence

$$k \longrightarrow K_D \xrightarrow{j_D|K} K \xrightarrow{\rho_K} K_B \longrightarrow k$$

of cocommutative Hopf algebras is exact by Cor. to Lemma 6. This means that K_D is normal in K. Since $\rho(K_B)=\rho j_B(K_B)=1_B(K_B)=K_B$, we see $J(K_D, K_B)=K$ by Cor. 2 to Th. 2. On the other hand we see $I(K_B, K_D)=k$ from $I(D, B)=k$. Therefore K is the semi-direct product of K_D with K_B. q.e.d.

§4. Nilpotent and solvable Hopf algebras

The purpose of this section is to define nilpotent and solvable cocommutative Hopf algebras over a field k and to get analogous results to those in classical group theory.

Let $(B, m, i, \Delta, \varepsilon, c)$ be a cocommutative Hopf algebra over a field k, and let E and F be Hopf subalgebras of B. Then we defined in §14 of [6] the *commutator* $[E, F]$ of E and F which is the smallest Hopf subalgebra of B containing all the elements $\sum_{(x),(y)} x_{(1)}y_{(1)}c(x_{(2)})c(y_{(2)})$ where x in E with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ and y in F with $\Delta(y)=\sum_{(y)} y_{(1)}\otimes y_{(2)}$. The next basic lemma is necessary in the following.

LEMMA 8. *Let B and B' be cocommutative Hopf algebras over a field k, and let D be a normal Hopf subalgebra of B. Then we have the followings:*
(i) *$f([E, F])=[f(E), f(F)]$ for any Hopf subalgebras E and F of B and for any Hopf algebra homomorphism f of B to B'.*
(ii) *$[B, D]\subset D$.*
(iii) *The Hopf quotient algebra B/D of B by D is commutative if and only if $[B, B]\subset D$.*

PROOF. (i) Let Δ and c be the comultiplication and the antipode of B respectively. Since $[E, F]$ is the subalgebra of B generated by the elements $\sum_{(x),(y)} x_{(1)}y_{(1)}c(x_{(2)})c(y_{(2)})$ where x and y run over E and F respectively (cf. Proof of Prop. 14.2 in [6]), it is easy to see $f([E, F])=[f(E), f(F)]$.
(ii) Since D is normal in B, we see that $\sum_{(x)} x_{(1)}yc(x_{(2)})$ is contained in D for any elements x in B and y in D. Therefore $\sum_{(x),(y)} x_{(1)}y_{(1)}c(x_{(2)})c(y_{(2)})$ is contained in D for any x in B and y in D. This means that $[B, D]$ is contained in D.
(iii) Let π be the canonical Hopf algebra homomorphism of B to B/D. If B/D is commutative, then we see from (i) that $\pi([B, B])=[\pi(B), \pi(B)]=k$. Since D is the h-kernel of π by Th. 1, we see that $[B, B]$ is contained in D. Conversely if $[B, B]\subset D$, then we see from (i) $[\pi(B), \pi(B)]=\pi([B, B])\subset\pi(D)=k$. This means by Prop. 13.1 and Prop. 14.1 in [6] that $B/D=\pi(B)$ is commutative.

q.e.d.

Now we shall consider a chain of Hopf subalgebras of a cocommutative Hopf algebra B over a field k:

$$B = D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_n,$$

where each D_i is normal in D_{i-1}. We call this chain *subinvariant*. In particular if each D_i is normal in B, the sequence is called *a normal chain*. A cocommutative Hopf algebra B over k is said to be *nilpotent* if it possesses a finite normal chain $B=D_0\supset D_1\supset\cdots\supset D_n=k$ such that $[B, D_i]\subset D_{i+1}$ for $0\leq i<n$.

Next we shall define the lower and upper central series of a Hopf algebra. If B is as above, then we put $\mathscr{C}^0B=B$ and $\mathscr{C}^{i+1}B=[B, \mathscr{C}^iB]$ inductively. We call the sequence

$$B = \mathscr{C}^0B \supset \mathscr{C}^1B \supset \mathscr{C}^2B \supset\cdots\supset \mathscr{C}^iB \supset\cdots$$

the lower central series of B. By Prop. 14.16 in [6] we see that each $\mathscr{C}^iB$ is normal in B. On the other hand put $\mathscr{C}_0B=k$ and let $\mathscr{C}_1B$ be the center of $B=B/\mathscr{C}_0B$. It is clear that $\mathscr{C}_0B$ and $\mathscr{C}_1B$ are normal in B. Let E be the center of the Hopf quotient algebra $B/\mathscr{C}_1B$ and π_1 the canonical Hopf algebra homomorphism of B to $B/\mathscr{C}_1B$. If $\mathscr{C}_2B$ is the h-inverse of E by π_1, then $\mathscr{C}_2B$ is a normal Hopf subalgebra of B by Lemma 2. Next let π_2 be the canonical Hopf algebra homomorphism of B to $B/\mathscr{C}_2B$ and $\mathscr{C}_3B$ the h-inverse of the center of $B/\mathscr{C}_2B$ by π_2. Similarly we define π_i and $\mathscr{C}_{i+1}B$ for $i\geq 3$ inductively. In other words $\mathscr{C}_{i+1}B$ is the unique Hopf subalgebra of B containing $\mathscr{C}_iB$ such that $\mathscr{C}_{i+1}B/\mathscr{C}_iB$ is the center of $B/\mathscr{C}_iB$. Then we have an ascending chain of normal Hopf subalgebras of B:

$$k = \mathscr{C}_0B \subset \mathscr{C}_1B \subset \mathscr{C}_2B \subset\cdots\subset \mathscr{C}_iB \subset\cdots,$$

which is called *the upper central series of B.*

PROPOSITION 7. *Let B a cocommutative Hopf algebra over a field k. Then the followings are equivalent:*

(i) *B is nilpotent.*

(ii) *There is a positive integer n such that* $\mathscr{C}^nB=k$.

(iii) *There is a positive integer n such that* $\mathscr{C}_nB=B$.

PROOF. (i)$\Rightarrow$(ii) and (iii). Let

$$B = D_0 \supset D_1 \supset\cdots\supset D_n = k$$

be a normal chain of Hopf subalgebras of B such that $[B, D_i]\subset D_{i+1}$ for $0\leq i<n$. We show $D_i\supset\mathscr{C}^iB$ by induction on i. For $i=0$ we see $D_0=B=\mathscr{C}^0B$. If we assume that $D_i\supset\mathscr{C}^iB$, then we see $D_{i+1}\supset[B, D_i]\supset[B, \mathscr{C}^iB]=\mathscr{C}^{i+1}B$. This means that $k=D_n=\mathscr{C}^iB$. Next we show $\mathscr{C}_iB\subset D_{n-i}$ by induction on i. For $i=0$ we have $\mathscr{C}_0B=k=D_n$. Assume $\mathscr{C}_iB\supset D_{n-i}$. Since $\mathscr{C}_{i+1}B/\mathscr{C}_iB$ is the center of $B/\mathscr{C}_iB$, we see $[B, \mathscr{C}_{i+1}B]\subset\mathscr{C}_iB$ from Prop. 13.1 and Prop. 14.1

in [6] and Lemma 8, (i). Similarly we see easily that if $[D, B]\subset\mathscr{C}_iB$ for a Hopf subalgebra D of B, $\mathscr{C}_{i+1}B$ contains D by the definition of centers. Since we have $[B, D_{n-i-1}]\subset D_{n-i}\subset\mathscr{C}_iB$, we see $D_{n-i-1}\subset\mathscr{C}_{i+1}B$. Therefore we have $\mathscr{C}_nB=B$. The implication (ii)⇒(i) is trivial. (iii)⇒(i). Assume that $\mathscr{C}_nB=B$. Since $\mathscr{C}_1B$ is the center of B, we see $[B, \mathscr{C}_1B]=k=\mathscr{C}_0B$. If E_i is the center of $B/\mathscr{C}_iB$, $\mathscr{C}_{i+1}B$ is the h-inverse of E_i by π_i: $B\to B/\mathscr{C}_iB$ and hence $B/\mathscr{C}_{i+1}B$ is isomorphic to $(B/\mathscr{C}_iB)/E_i$ by Cor. 1 to Th. 2. Therefore we have $\pi_i([B, \mathscr{C}_{i+1}B])=[\pi_i(B), \pi_i(\mathscr{C}_{i+1}B)]=[B/\mathscr{C}_iB, E_i]=k$ by Lemma 8, (i). Since the h-kernel of π_i is $\mathscr{C}_iB$ by Cor. to Th. 1, we see $[B, \mathscr{C}_{i+1}B]\subset\mathscr{C}_iB$. Therefore B is nilpotent. q.e.d.

COROLLARY. *If B is a non-trivial nilpotent cocommutative Hopf algebra over a field k, then the center of B is also non-trivial.*

PROPOSITION 8. *Hopf subalgebras and Hopf quotient algebras of a nilpotent cocommutative Hopf algebra over a field k are also nilpotent. Conversely let D be a normal Hopf subalgebra of a cocommutative Hopf algebra B over k such that D is contained in the center of B. If B/D is nilpotent, then so is B.*

PROOF. Let B be a nilpotent cocommutative Hopf algebra over k. If D is a Hopf subalgebra of B, then we see $\mathscr{C}^iD\subset\mathscr{C}^iB$. Therefore we have $\mathscr{C}^nD=k$ for some $n>0$ by Prop. 7, i.e., D is nilpotent. On the other hand assume that D is normal in B, and let π be the canonical Hopf algebra homomorphism of B to B/D. By Lemma 8, (i) we see easily $\mathscr{C}^i(B/D)=\pi(\mathscr{C}^iB)$. Therefore we see similarly to the above that $\mathscr{C}^n(B/D)=k$ for some $n>0$. Therefore B/D is nilpotent. Conversely assume that D is contained in the center of B and that B/D is nilpotent. By Prop. 7 there is a positive integer n such that $\mathscr{C}^n(B/D)=k$. By Lemma 8, (i) and Cor. to Th. 1 we see $\mathscr{C}^nB\subset D$. This means $\mathscr{C}^{n+1}B\subset [B, D]=k$ by Prop. 13.1 and Prop. 14.1 in [6]. Therefore B is nilpotent by Prop. 7. q.e.d.

PROPOSITION 9. *Tensor product Hopf algebras of nilpotent cocommutative Hopf algebras over a field k are also nilpotent.*

PROOF. Let B and D be nilpotent cocommutative Hopf algebras over k and put $E=B\otimes D$. Let p_B and p_D be the canonical projections of E to B and D as Hopf algebras respectively. Then we see $p_B(\mathscr{C}^nE)=\mathscr{C}^nB=p_D(\mathscr{C}^nE)=\mathscr{C}^nD=k$ for some $n>0$ by Lemma 8, (i) and Prop. 7. Since the h-kernel of p_B is $k\otimes D$ as seen easily, we have $\mathscr{C}^nE\subset k\otimes D$. Similarly we see $\mathscr{C}^nE\subset B\otimes k$ and hence $\mathscr{C}^nE\subset(B\otimes k)\cap(k\otimes D)=k$. This means that E is nilpotent by Prop. 7. q.e.d.

PROPOSITION 10. *Let B be a nilpotent cocommutative Hopf algebra over a field k and D a Hopf subalgebra of B different from B. Then the normalizer*

$N_B(D)$ *of* D *in* B *is not* D.

PROOF. Let C be any cocommutative coalgebra over k. Now we define Hopf subalgebra D_i of B as follows: Put $D_0=D$ and let D_i be the normalizer $N_B(D_{i-1})$ of D_{i-1} in B for $i\geq 1$. Then we shall show $D_i\supset \mathscr{C}_iB$ for each $i\geq 0$. For $i=0$ we have $D_0\supset \mathscr{C}_0B=k$. Assume that $D_i\supset \mathscr{C}_iB$, and let f and g be elements in $\mathrm{Hom}_{coal}(C, \mathscr{C}_{i+1}B)$ and $\mathrm{Hom}_{coal}(C, D_i)$ respectively. Then we see $[f, g]=f*g*f^{-1}*g^{-1}\in \mathrm{Hom}_{coal}(C, \mathscr{C}_iB)\subset \mathrm{Hom}_{coal}(C, D_i)$, and hence $f*g*f^{-1}\in g*\mathrm{Hom}_{coal}(C, D_i)=\mathrm{Hom}_{coal}(C, D_i)$. This means that $\mathrm{Hom}_{coal}(C, \mathscr{C}_{i+1}B)$ is contained in the normalizer of $\mathrm{Hom}_{coal}(C, D_i)$ in $\mathrm{Hom}_{coal}(C, B)$. Therefore $\mathscr{C}_{i+1}B$ is a Hopf subalgebra of $N_B(D_i)=D_{i+1}$ by Prop. 12.1 and Prop. 12.4 in [6]. In particular we have $D_n\supset \mathscr{C}_nB=B$ for some n by Prop. 7. If $N_B(D)=D_1=D$, then we see $D_n=D_{n-1}=\cdots=D_1=D$. But this contradicts the assumption $B\neq D$. q.e.d.

Next we define a solvable Hopf algebra. For this purpose we need first to define the i-th derived Hopf subalgebra of a cocommutative Hopf algebra B over k. For $i=0$ we put $\mathscr{D}^0B=B$ and for $i>0$ we define $\mathscr{D}^iB=[\mathscr{D}^{i-1}B, \mathscr{D}^{i-1}B]$ inductively. Then we call $\mathscr{D}^iB$ *the i-th derived Hopf subalgebra of* B. If $\mathscr{D}^nB=k$ for some $n>0$, B is called *solvable*. It is clear that *a nilpotent Hopf algebra is solvable.*

PROPOSITION 11. *Hopf subalgebras and Hopf quotient algebras of a solvable cocommutative Hopf algebra over a field k are solvable. Conversely let D be a normal Hopf subalgebra of a cocommutative Hopf algebra B over k. If D and B/D are solvable, then so is B. In particular the tensor product Hopf algebra of solvable Hopf algebras is solvable.*

Proof is similar to that of Prop. 8 and we omit the detail.

PROPOSITION 12. *Let B be a cocommutative Hopf algebra over a field k. Then the followings are equivalent:*
(i) *B is solvable.*
(ii) *There is a normal chain of B:*

$$B = D_0 \supset D_1 \supset \cdots \supset D_n = k$$

where D_i/D_{i+1} *is commutative for* $1\leq i<n$.
(iii) *There is a subinvariant chain of B:*

$$B = D'_0 \supset D'_1 \supset \cdots \supset D'_m = k$$

where D'_i/D'_{i+1} *is commutative for* $1\leq i<m$.

PROOF. (i)⇒(ii). We may put $D_i=\mathscr{D}^iB$ for each i. The implication (ii)

$\Rightarrow$(iii) is trivial. (iii)$\Rightarrow$(i). Since $B/D_1' = D_0'/D_1'$ is commutative, we see $D_1' \supset [B, B] = \mathscr{D}^1 B$ by Lemma 8, (iii). Similarly if $D_{i-1}' \supset \mathscr{D}^{i-1}B$, then we have $D_i' \supset [D_{i-1}^i, D_{i-1}'] \supset [\mathscr{D}^{i-1}B, \mathscr{D}^{i-1}B] = \mathscr{D}^i B$. Hence, ultimately we have $k = D_m' \supset \mathscr{D}^m B$. Therefore B is solvable. q.e.d.

The following lemma is necessary to define the radical of a Hopf algebra.

LEMMA 9. *Let D and E be normal and solvable Hopf subalgebras of a cocommutative Hopf algebras B over a field k. Then $J(D, E)$ is also normal in B and solvable.*

PROOF. Since $J(D, E)/E \simeq D/I(D, E)$ by Th. 3, $J(D, E)/E$ is solvable by Prop. 11. Then by the same proposition, we see that $J(D, E)$ is also solvable. q.e.d.

Let B be as above. If there is a maximal Hopf subalgebra D of B which is normal in B and solvable, then D is the largest one satisfying the same conditions by Lemma 9. We call such D *the radical of B*.

PROPOSITION 13. *If a cocommutative Hopf algebra B over k has the radical D, then D is the smallest normal Hopf subalgebra E of B such that the radical of B/E is k.*

PROOF. Let E be a normal Hopf subalgebra of B such that the radical of B/E is k, and let π_E be the canonical Hopf algebra homomorphism of B to B/E. If D_1 is any normal and solvable Hopf subalgebra of B, then $\pi_E(D_1)$ is also normal and solvable Hopf subalgebra of B/E by Lemma 3 and Prop. 11. Therefore we see $\pi_E(D_1) = k$ and hence $D_1 \subset E$ by Cor. to Th. 1. This means by Lemma 9 that D is contained in E. On the other hand let π_D be the canonical Hopf algebra homomorphism of B to B/D and let E' be a normal and solvable Hopf subalgebra of B/D. If E is the h-inverse of E' by π_D, then we see $E' \simeq E/D$ by Cor. 1 to Th. 2. Therefore E is solvable and normal in B by Prop. 11 and Lemma 2. This means $E = D$ and hence $E' = k$ by the maximality of D. Therefore the radical of B/D is k. This completes the proof. q.e.d.

PROPOSITION 14. *Let B_i be a cocommutative Hopf algebra over a field k for $1 \leq i \leq n$. Then $B = B_1 \otimes \cdots \otimes B_n$ has the radical D if and only if each B_i has the radical D_i for $1 \leq i \leq n$. Furthermore, then, we have $D = D_1 \otimes \cdots \otimes D_n$.*

PROOF. Assume that B_i has the radical D_i for $1 \leq i \leq n$. Then we see easily that $D_1 \otimes \cdots \otimes D_n$ is normal in B, and that $D_1 \otimes \cdots \otimes D_n$ is solvable by Prop. 11. Let E be a normal and solvable Hopf subalgebra of B. If E_i is the image of E by the canonical projection π_i of B to B_i as Hopf algebras, then E_i is normal in B_i and solvable by Lemma 3 and Prop. 11. Therefore we see $E_i \subset D_i$ for $1 \leq i \leq n$

and hence $E \subset D_1 \otimes \cdots \otimes D_n$ by Cor. to Prop. 4. This means that $D_1 \otimes \cdots \otimes D_n$ is the radical of B. Conversely assume that B has the radical D. Then $D_i = \pi_i(D)$ is normal in B_i and solvable for $1 \leq i \leq n$ by Lemma 3 and Prop. 11. Let E_i be a normal and solvable Hopf subalgebra of B_i. If E_i is not contained in D_i, then the join of D and $E'_i = k \otimes \cdots \otimes k \otimes E_i \otimes k \otimes \cdots \otimes k$ is larger than D. However this is a contradiction to the maximality of D from Lemma 9, because E'_i is normal in B and solvable. Therefore we see $D_i \subset E_i$, and hence D_i is the radical of B_i. The last assertion in our proposition is already shown in the above. q.e.d.

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