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Extension Theorems on Some Generalized Nilpotent Lie Algebras

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1. Introduction

Let L be a Lie algebra over a field. It is well known that if I is a nilpotent ideal of L and L/I^2 is nilpotent, then L is nilpotent. In group theory a theorem asserting that for a normal nilpotent subgroup N of a group G a property of G/N' passes to G is termed one of Hall's type, and it has been shown for several properties including the nilpotency ([5], [6, p. 57]). In connection with these, it seems interesting for us to investigate the Lie-theoretic analogue of a theorem of Hall's type. The aim of this paper is to show the following extension theorem in Lie algebras: Let $\mathfrak{X}$ be one of the classes $\mathfrak{Z}$, $\mathfrak{E}$, $\mathrm{L}\mathfrak{N}$, $\mathfrak{F}\mathrm{t}$, $\mathfrak{B}$, $\mathfrak{G}\mathrm{r}$ of Lie algebras. If I is a nilpotent ideal of L and L/I^2 lies in $\mathfrak{X}$, then L lies in $\mathfrak{X}$.

2. Notations

Throughout this paper we consider Lie algebras over an arbitrary field Φ which are not necessarily finite-dimensional.

Let L be a Lie algebra and H be a subalgebra of L. We use the following notations as usual.

H si L: H is a subideal of L.

H asc L: H is an ascendant subalgebra of L, i.e., there exists an ascending series $\{H_\beta: 0 \leqq \beta \leqq \alpha\}$ of subalgebras of L, indexed by ordinals $\beta \leqq \alpha$, such that $H_0 = H$, $H_\alpha = L$, $H_\beta \lhd H_{\beta+1}$ for all $\beta < \alpha$, and $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ for all limit ordinals $\lambda \leqq \alpha$.

$\zeta_\alpha(L)$: the α-th term of the upper central series of L (α: an ordinal). In particular $\zeta_1(L)$ is the center of L.

$\zeta_*(L)$: the hypercenter of L.

We say that $x \in L$ is a right Engel element if for each $y \in L$ there exists a non-negative integer $n = n(x, y)$ such that $[x, {}_n y] = 0$.

$\mathrm{r}(L)$: the set of right Engel elements of L.

Let us recall several classes of Lie algebras.

$\mathfrak{N}$: the class of nilpotent Lie algebras.

$\mathfrak{Z}$: the class of hypercentral Lie algebras.

$\mathrm{L}\mathfrak{N}$: the class of locally nilpotent Lie algebras.

$\mathfrak{Fit}$: the class of Lie algebras which are the sum of nilpotent ideals.

$\mathfrak{B}$: the class of Lie algebras L such that $x \in L$ implies $<x>$ si L.

$\mathfrak{Gr}$: the class of Lie algebras L such that $x \in L$ implies $<x>$ asc L.

$\mathfrak{E}$: the class of Lie algebras L satisfying the condition that for every x, $y \in L$ there exists a positive integer $n = n(x, y)$ such that $[x, {}_n y] = 0$.

When $L \in \mathfrak{Fit}$ (resp. $\mathfrak{B}$, $\mathfrak{Gr}$, $\mathfrak{E}$), L is called a Fitting (resp. a Baer, a Gruenberg, an Engel) algebra.

It is to be noted that we defined $\mathfrak{Gr}$ for an arbitrary base field Φ, though it is defined only for a field of characteristic zero in [1, Chap. 6].

Any notation not explained here may be found in [1].

3. The case of $\mathfrak{Z}$

This case is the Lie-theoretic analogue of a result of Betten [2].

LEMMA 3.1. *If $L \in \mathfrak{Z}$ and I is a non-zero ideal of L, then $I \cap \zeta_1(L) \neq 0$.*

PROOF. Let $\{\zeta_\beta(L) : 0 \leqq \beta \leqq \alpha\}$ be the upper central series of L with $L = \zeta_\alpha(L)$. Denote by S the set of all ordinals $\beta \leqq \alpha$ for which $I \cap \zeta_\beta(L) \neq 0$. Clearly $S \neq \phi$. Let $\gamma = \min S$. It is easily seen that γ is neither 0 nor a limit ordinal. Hence $I \cap \zeta_{\gamma-1}(L) = 0$ and $I \cap \zeta_\gamma(L) \neq 0$. So we have

$$[I \cap \zeta_\gamma(L), L] \subseteqq I \cap \zeta_{\gamma-1}(L) = 0,$$

which means that $0 \neq I \cap \zeta_\gamma(L) \leqq \zeta_1(L)$. Therefore $I \cap \zeta_1(L) \neq 0$.

If H is a subalgebra of a Lie algebra L, then the centralizer of H in L is $C_L(H) = \{y \in L : [H, y] = 0\}$. Evidently, if $H \lhd L$ then $C_L(H) \lhd L$.

LEMMA 3.2. *Let L be a Lie algebra and I be an ideal of L. If $I/\zeta_1(I) \neq 0$ and $L/\zeta_1(I) \in \mathfrak{Z}$, then $I^2 \cap \zeta_1(L) \neq 0$ and in particular $\zeta_1(L) \neq 0$.*

PROOF. By Lemma 3.1 we have $I/\zeta_1(I) \cap \zeta_1(L/\zeta_1(I)) \neq 0$. Hence we can find $x \in I \backslash \zeta_1(I)$ such that $[x, L] \subseteqq \zeta_1(I)$. It is easy to see that $\zeta_1(I) + <x> \lhd L$. From the remark above $C_L(\zeta_1(I) + <x>) \lhd L$ and therefore $I \cap C_L(\zeta_1(I) + <x>) \lhd L$. We also have $I \cap C_L(<x>) = I \cap C_L(\zeta_1(I) + <x>)$, whence $I \cap C_L(<x>) \lhd L$. Since $x \in I$, we have $\zeta_1(I) \leqq I \cap C_L(<x>)$. By the assumption that $L/\zeta_1(I) \in \mathfrak{Z}$, we have $L/(I \cap C_L(<x>)) \in \mathrm{Q}\mathfrak{Z} = \mathfrak{Z}$. From the fact that $x \notin \zeta_1(I)$ it follows that $I \cap C_L(<x>) \lneqq I$, i.e., that $I/(I \cap C_L(<x>))$ is a non-zero ideal of $L/(I \cap C_L(<x>))$. By Lemma 3.1 we now obtain $I/(I \cap C_L(<x>)) \cap \zeta_1(L/(I \cap C_L(<x>))) \neq 0$. Hence there exists $y \in I \backslash C_L(<x>)$ such that $[y, L] \subseteqq I \cap C_L(<x>)$. Evidently $[x, y] \neq 0$. Owing to the Jacobi identity,

$$\begin{aligned}[[x, y], L] &\subseteqq [[x, L], y] + [x, [y, L]] \\ &\subseteqq [\zeta_1(I), y] + [x, C_L(<x>)] \\ &= 0.\end{aligned}$$

Namely $[x, y] \in \zeta_1(L)$. Therefore $I^2 \cap \zeta_1(L) \neq 0$. This proves the lemma.

By making use of Lemma 3.2, we shall prove the following

THEOREM 3.1. *Let L be a Lie algebra and I be an ideal of L. If I is nilpotent and L/I^2 is hypercentral, then L is hypercentral.*

PROOF. We use induction on the nilpotency class k of I. If $k=1$, then $I^2=0$ and the assertion is trivial. Let $k>1$ and assume that the assertion is true for $k-1$. Let $I \in \mathfrak{N}_k$ and $I \notin \mathfrak{N}_1$. Then $I/\zeta_1(I) \in \mathfrak{N}_{k-1}$ and

$$(L/\zeta_1(I))/(I/\zeta_1(I))^2 \simeq L/(I^2 + \zeta_1(I)) \simeq (L/I^2)/((I^2 + \zeta_1(I))/I^2) \in \mathrm{Q}\mathfrak{Z} = \mathfrak{Z}.$$

By induction hypothesis it follows that $L/\zeta_1(I) \in \mathfrak{Z}$. Since $I \notin \mathfrak{N}_1$, $I/\zeta_1(I)$ is a non-zero ideal of $L/\zeta_1(I)$. Hence by Lemma 3.2 we have $\zeta_1(L) \neq 0$. Now, suppose that $L \neq \zeta_*(L)$. Since I is an $\mathfrak{N}_k$-ideal of L, $(I+\zeta_*(L))/\zeta_*(L)$ is also an $\mathfrak{N}_k$-ideal of $L/\zeta_*(L)$ and

$$(L/\zeta_*(L))/((I + \zeta_*(L))/\zeta_*(L))^2 \simeq (L/I^2)/((I^2 + \zeta_*(L))/I^2) \in \mathrm{Q}\mathfrak{Z} = \mathfrak{Z}.$$

Since $L/\zeta_*(L) \notin \mathfrak{Z}$, $(I+\zeta_*(L))/\zeta_*(L) \notin \mathfrak{N}_1$. By the fact shown above, we have $\zeta_1(L/\zeta_*(L)) \neq 0$, which contradicts the definition of $\zeta_*(L)$. Therefore we conclude that $L=\zeta_*(L)$. Thus the proof is complete.

4. The cases of $\mathfrak{E}$ and $\mathrm{L}\mathfrak{N}$

THEOREM 4.1. *Let L be a Lie algebra and I be an ideal of L. If I is nilpotent and L/I^2 is Engel, then L is Engel.*

PROOF. We use induction on the nilpotency class k of I. If $k=1$, then the assertion is trivial. Let $k>1$ and suppose that the assertion is true for $k-1$. Let $I \in \mathfrak{N}_k$. Then $I/\zeta_1(I) \in \mathfrak{N}_{k-1}$ and

$$(L/\zeta_1(I))/(I/\zeta_1(I))^2 \simeq (L/I^2)/((I^2 + \zeta_1(I))/I^2) \in \mathrm{Q}\mathfrak{E} = \mathfrak{E}.$$

By induction hypothesis we have $L/\zeta_1(I) \in \mathfrak{E}$.

Now we claim that $I^2 \subseteqq \mathfrak{r}(L)$. In fact, let $x, y \in I$ and $z \in L$. Since $L/\zeta_1(I) \in \mathfrak{E}$, we can find positive integers m and n such that $[x, {}_m z] \in \zeta_1(I)$ and $[y, {}_n z] \in \zeta_1(I)$. By the Jacobi identity,

$$[[x, y], {}_{m+n}z] = \sum_{i+j=m+n} \binom{m+n}{i} [[x, {}_iz], [y, {}_jz]] = 0.$$

Hence $[x, y] \in \mathfrak{r}(L)$. Therefore $I^2 \subseteqq \mathfrak{r}(L)$, as claimed.

Let $v, w \in L$. Since $L/I^2 \in \mathfrak{E}$, there exists a positive integer p such that $[v, {}_pw] \in I^2$. Since $I^2 \subseteqq \mathfrak{r}(L)$, we can find a positive integer q such that $[[v, {}_pw], {}_qw]=0$. It follows that $[v, {}_{p+q}w]=0$. Hence $L \in \mathfrak{E}$. This completes the proof.

Although $\mathrm{L}\mathfrak{N}$ is not E-closed in general, it is known [1, p. 336] that with respect to $\mathfrak{E}$ $\mathrm{L}\mathfrak{N}$ is E-closed. Namely, we have

LEMMA 4.1. *Let $L \in \mathfrak{E}$ and I be an ideal of L. If I and L/I are locally nilpotent, then L is locally nilpotent.*

Now we have the following theorem as a consequence of Theorem 4.1:

THEOREM 4.2. *Let L be a Lie algebra and I be an ideal of L. If I is nilpotent and L/I^2 is locally nilpotent, then L is locally nilpotent.*

PROOF. Since $\mathrm{L}\mathfrak{N} \leqq \mathfrak{E}$, it follows from Theorem 4.1 that $L \in \mathfrak{E}$. Hence by Lemma 4.1 we have $L \in \mathrm{L}\mathfrak{N}$.

5. The cases of $\mathfrak{F}\mathfrak{t}$, $\mathfrak{B}$ and $\mathfrak{G}\mathfrak{r}$

LEMMA 5.1. *Let L be a Lie algebra and I be an ideal of L. If I and L/I^2 are nilpotent, then L is nilpotent.*

PROOF. See [3, Theorem 2] (or [1, Proposition 7.1.1 (c)]).

LEMMA 5.2. *Let L be a Lie algebra.*

(1) *If I is a nilpotent ideal of L and H is a nilpotent subideal of L, then $I+H$ is a nilpotent subideal of L.*

(2) *If I is a hypercentral ideal of L and H is an ascendant hypercentral subalgebra of L, then $I+H$ is an ascendant hypercentral subalgebra of L.*

PROOF. (1) Obviously $I+H$ si L. Let $I \in \mathfrak{N}_c$, $H \in \mathfrak{N}_d$ and $H \lhd^n L$. Put $m=d+c(n+d)+1$. Then $(I+H)^m$ is the sum of all $[W_1, W_2,\dots, W_m]$ with $W_i=I$ or H. Since $I \lhd L$ and $I \in \mathfrak{N}_c$, we may suppose that I appears in $[W_1, W_2,\dots, W_m]$ at most c times. Noting that

$$[L, {}_{n+d}H] = [[L, {}_nH], {}_dH] \subseteqq [H, {}_dH] = 0,$$

we see that $[W_1,\dots, W_m]=0$. Hence $(I+H)^m=0$.

(2) See [4, Proposition 3].

THEOREM 5.1. *Let L be a Lie algebra and I be a nilpotent ideal of L.*

(1) *If L/I^2 is Fitting, then L is Fitting.*
(2) *If L/I^2 is Baer, then L is Baer.*
(3) *If L/I^2 is Gruenberg, then L is Gruenberg.*

PROOF. (1) By the definition of $\mathfrak{Ft}$,

$$L/I^2 = \sum\{J/I^2 : J \lhd L, J/I^2 \in \mathfrak{N}\}.$$

By Fitting's Theorem (see [1, Theorem 1.2.5])

$$(J + I)/I^2 = J/I^2 + I/I^2 \in \mathfrak{N}.$$

By Lemma 5.1 we have $J+I \in \mathfrak{N}$ and therefore $J \in \mathfrak{N}$. Consequently

$$L = \sum\{H : H \lhd L, H \in \mathfrak{N}\}.$$

Therefore $L \in \mathfrak{Ft}$.

(2) Let $x \in L$. Then

$$(<x> + I^2)/I^2 \text{ si } L/I^2 \quad \text{and} \quad (<x> + I)/I^2 = (<x> + I^2)/I^2 + I/I^2.$$

By Lemma 5.2 (1) we have $(<x>+I)/I^2 \in \mathfrak{N}$ and $<x>+I$ si L. Using Lemma 5.1 we obtain $<x>+I \in \mathfrak{N}$, and hence $<x>$ si $<x>+I$. Therefore we have $<x>$ si L. Thus $L \in \mathfrak{B}$.

(3) Let $x \in L$. Then

$$(<x> + I^2)/I^2 \text{ asc } L/I^2 \quad \text{and} \quad (<x> + I)/I^2 = (<x> + I^2)/I^2 + I/I^2.$$

By Lemma 5.2 (2) we have $(<x>+I)/I^2 \in \mathfrak{Z}$ and $<x>+I$ asc L. Using Theorem 3.1 we obtain $<x>+I \in \mathfrak{Z}$. It follows that $<x>$ asc $<x>+I$, and therefore $<x>$ asc L. Thus $L \in \mathfrak{Gr}$.

6. Remarks

All classes observed in the above theorems are subclasses of $\mathfrak{E}$. Let P be a vector space over Φ with basis $e_0, e_1, e_2, \ldots$ and regard P as an abelian Lie algebra. Let z be the identity transformation of P and let L be a split extension of P by $<z>$. Let $\mathfrak{X}$ be any class in the theorems. Then clearly $P \in \mathfrak{X}$, $P \lhd L$ and $L/P \in \mathfrak{X}$. But since $[e_i, {}_n z] = e_i$ for any positive integer n, $L \notin \mathfrak{E}$, and therefore $L \notin \mathfrak{X}$. This tells us that any class in the above theorems is not E-closed.

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