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A Two Point Connection Problem

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§1. Introduction

A two point connection problem for linear ordinary differential equations in the whole complex plane is to seek the explicit connection formulas between two fundamental sets of solutions locally defined and to analyze global behaviors of solutions. In this paper we shall be concerned with the system of linear differential equations

$$t\frac{dX}{dt} = (A_0 + A_1 t + \cdots + A_q t^q)X, \tag{1.1}$$

where the coefficients A_i $(i=0, 1,\dots, q)$ are n by n constant matrices, and derive the connection formulas between two fundamental sets of solutions in neighborhoods of $t=0$ and $t=\infty$.

The origin $t=0$ is a regular singularity of (1.1). According to the local theory of systems of linear differential equations (see W. Wasow [23, Chapters II and V]), an application of a finite number of constant transformations and the so-called shearing transformations reduces (1.1) to a system of linear differential equations in which the leading coefficient matrix is of the following Jordan canonical form:

$$\hat{A}_0 = \begin{pmatrix} \mathscr{A}_1 & & & 0 \\ & \mathscr{A}_2 & & \\ & & \ddots & \\ 0 & & & \mathscr{A}_\nu \end{pmatrix}, \quad \mathscr{A}_i = \rho_i + J_i{}^{*)} \qquad (i = 1, 2,\dots, \nu), \tag{1.2}$$

*) Throughout this paper, as in this expression, use will be made of the notation that a scalar in the matrix representation denotes a diagonal matrix whose diagonal elements equal that scalar.

where $\rho_i-\rho_j\neq a$ nonzero integer $(i\neq j)$ and

$$(1.3)\qquad J_i=\begin{pmatrix} 0 & & & 0\\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \\ 0 & & & 1\ 0\end{pmatrix}\qquad (i=1,2,\dots,\nu).$$

Then the reduced system of linear differential equations possesses, corresponding to each n_i by n_i Jordan block $\mathscr{A}_i$, a set of convergent power series solutions near $t=0$, which are expressed in terms of the column vectorial form

$$(1.4)\qquad X_l^i(t)=\sum_{j=1}^{l}\frac{1}{(l-j)!}(\log t)^{l-j}\hat{X}_j^i(t)\qquad (l=1,2,\dots,n_i)$$

$$(1.5)\qquad \hat{X}_j^i(t)=t^{\rho_{ij}}\sum_{m=0}^{\infty}G_j^i(m)t^m\qquad (\rho_{ij}=\rho_i;\ j=1,2,\dots,n_i),$$

and it turns out that a fundamental set of solutions near $t=0$ of (1.1) consists of ν sets of convergent power series solutions with the expressions similar to (1.4) and (1.5), where for each fixed i every characteristic constant ρ_{ij} differs from the others by integers.

On the other hand, $t=\infty$ is a singularity of Poincaré's rank q for (1.1). In order to see whether $t=\infty$ is an irregular singularity of (1.1) or not, we have to apply W. B. Jurkat and D. A. Lutz-J. Moser's theorem [7, 13] to (1.1). However, if the integer $q\geqq 1$ and the matrix A_q is similar to

$$(1.6)\qquad \hat{A}_q=\begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix},\ \lambda_k\neq 0,\ \lambda_k\neq\lambda_j\qquad (k\neq j),$$

then $t=\infty$ is certainly the irregular singularity of (1.1) at which we can immediately find formal power series solutions of the column vectorial form

$$(1.7)\qquad X^k(t)=\exp\left(\frac{\lambda_k}{q}t^q+\frac{\alpha_{q-1}^k}{q-1}t^{q-1}+\cdots+\alpha_1^k t\right)t^{\mu_k}\sum_{s=0}^{\infty}H^k(s)t^{-s}$$

$$(k=1,2,\dots,n).$$

In more general cases when $t=\infty$ is the irregular singularity and the matrix A_q has multiple eigenvalues, applying the ingenious theorem established first by M. Hukuhara [5, 6] and independently by H. L. Turrittin [22], we can reduce (1.1) to the canonical system of linear differential equations

$$(1.8)\qquad t\frac{dX}{dt}=\{\delta_{kj}(p_k(t)+\mu_k+J_k)+B_1t^{-1}+B_2t^{-2}+\cdots+B_Nt^{-N}\}X$$

$$(k,\ j=1,\ 2,\dots,\ \nu),$$

where δ_{kj} denotes the Kronecker delta, i.e., the first term in braces is the block-diagonal matrix, $p_k(t)\,(k=1, 2,\dots, \nu)$ are polynomials of the form

$$p_k(t)=\lambda_k t^h+\alpha^k_{h-1}t^{h-1}+\cdots+\alpha^k_1 t\qquad(k=1, 2,\dots, \nu),$$

h being a positive integer, and J_k $(k=1, 2,\dots, \nu)$ are n_k by n_k shifting matrices stated in (1.3). From (1.8) we can also easily find the explicit expressions of formal power sereis solutions at infinity.

Considering the above fact, we deal with two systems of linear differential equations in this paper. In Section 5 we first consider the two point connection problem for (1.1) which possesses convergent power series solutions of the form (1.4–5) and formal power series solutions of the form (1.7). For simplicity, we from the outset assume that $A_0=\rho+J$ and $A_q=\hat{A}_q$. Next in Section 6 we treat the canonical system of linear differential equations (1.8) which is assumed to possess a fundamental set of convergent power series solutions involving no logarithmic terms near the origin. The two point connection problem for systems of linear differential equations of more general types will then be solved by a slight modification of these considerations.

Now we shall shortly explain our method of attack in the former case. Our method is based on the expansion of convergent power series solutions in series of functions whose global behaviors can be somewhat easily analyzed.

Putting $n_i=n$, $\rho_{ij}=\rho$ and dropping the superscript i from $X^i(t)$, $\hat{X}^i_j(t)$ and $G^i_j(m)$ in the expressions (1. 4–5), we can write a fundamental matrix of solutions of (1.1) in the form

$$(1.9)\qquad (X_1(t),\ X_2(t),\dots,\ X_n(t))=(\hat{X}_1(t),\ \hat{X}_2(t),\dots,\ \hat{X}_n(t))t^{J_*},$$

where J_* is the transposed matrix of the n by n shifting matrix J. Hereafter we use the notation that a matrix A_* denotes the transposed matrix of A. The functions $\hat{X}_j(t)\,(j=1, 2,\dots, n)$ defined by the convergent power series (1.5) satisfy the systems of homogeneous and nonhomogeneous linear differential equations

$$(1.10)\qquad t\frac{d\hat{X}_1}{dt}=(A_0+A_1t+\cdots+A_qt^q)\hat{X}_1,$$

$$(1.11)\qquad t\frac{d\hat{X}_j}{dt}=(A_0+A_1t+\cdots+A_qt^q)\hat{X}_j-\hat{X}_{j-1}(t)\qquad(j=2,\ 3,\dots,\ n).$$

Substituting (1.5) into (1.10–11) and identifying coefficients of like powers of t, we see that the coefficients $G_j(m)\,(j=1, 2,\dots, n)$ are determined by the following systems of linear difference equations of the q-th order

$$(1.12)\qquad \begin{cases} (m+\rho-A_0)G_1(m) \\ \qquad = A_1G_1(m-1)+A_2G_1(m-2)+\cdots+A_qG_1(m-q), \\ (\rho-A_0)G_1(0)=0,\ G_1(r)=0\quad (r<0), \end{cases}$$

$$(1.13)\qquad \begin{cases} (m+\rho-A_0)G_j(m) \\ \qquad = A_1G_j(m-1)+A_2G_j(m-2)+\cdots+A_qG_j(m-q)-G_{j-1}(m), \\ (\rho-A_0)G_j(0)=-G_{j-1}(0),\ G_j(r)=0\quad (r<0)\qquad (j=2,3,\dots,n). \end{cases}$$

Similarly, we see by the same procedure as above that the coefficients $H^k(s)$ ($k=1, 2,\dots, n$) of the formal solutions (1.7) are determined by the systems of linear difference equations of the q-th order

$$(1.14)\qquad \begin{cases} (A_q-\lambda_k)H^k(s)+(A_{q-1}-\alpha_{q-1}^k)H^k(s-1)+\cdots \\ \qquad +(A_1-\alpha_1^k)H^k(s-q+1)+(A_0+s-q-\mu_k)H^k(s-q)=0, \\ (A_q-\lambda_k)H^k(0)=0,\quad H^k(r)=0\quad (r<0)\qquad (k=1,2,\dots,n), \end{cases}$$

where we can put

$$(1.15)\qquad H^1(0)=\begin{pmatrix}1\\0\\ \vdots\\ \vdots\\ 0\end{pmatrix},\ H^2(0)=\begin{pmatrix}0\\1\\0\\ \vdots\\ 0\end{pmatrix},\dots,\ H^n(0)=\begin{pmatrix}0\\ \vdots\\ \vdots\\ 0\\ 1\end{pmatrix}.$$

We then establish the following expansion theorem, together with showing a method of determining the constant coefficients $T_l^{ki}(i, k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ called the Stokes multipliers:

$$(1.16)\qquad (G_1(m), G_2(m),\dots, G_n(m)) = \sum_{s=0}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{q} H^k(s)(T_l^{k1}, T_l^{k2},\dots, T_l^{kn})\mathscr{G}_l^k(m+s)_*,$$

$$(1.17)\qquad (X_1(t), X_2(t),\dots, X_n(t)) = \sum_{m=0}^{\infty}(G_1(m), G_2(m),\dots, G_n(m))t^{m+\rho+J_*} = \sum_{s=0}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{q} H^k(s)(T_l^{k1}, T_l^{k2},\dots, T_l^{kn})Y_l^k(t, s)_*t^{J_*},$$

where the n by n matrix functions $Y_l^k(t, s)$ are defined by

(1.18) $$Y_l^k(t, s) = t^\rho \sum_{m=0}^{\infty} \mathscr{G}_l^k(m+s)t^m \qquad (k = 1, 2, \dots, n;\ l = 1, 2, \dots, q).$$

We call the matrix functions $Y_l^k(t, s)$ the fundamental matrix functions associated with the two point connection problem to be considered from the fact that they are defined only by the characteristic constant λ_k, $\alpha_{q-1}^k, \dots, \mu_k$ and ρ, and have just the same behaviors in the respective neighborhoods of two singular points as the desired global solutions of (1.1). From (1.17) and the global behaviors of the associated fundamental matrix functions $Y_l^k(t, s)$ we can prove that in every sectorial neighborhood of infinity $S(l_1, l_2, \dots, l_n)$ there exists a fundamental set of actual solutions $\hat{X}_s^k(t)(k=1, 2, \dots, n)$ such that $\hat{X}_s^k(t) \sim X^k(t)$ as $t \to \infty$ and the connection formula

(1.19) $$(X_1(t), X_2(t), \dots, X_n(t)) = (X_S^1(t), X_S^2(t), \dots, X_S^n(t)) \begin{pmatrix} T_{l_1}^{11} T_{l_1}^{12} \cdots T_{l_1}^{1n} \\ T_{l_2}^{21} T_{l_2}^{22} \cdots T_{l_2}^{2n} \\ \vdots \quad \vdots \quad \vdots \\ T_{l_n}^{n1} T_{l_n}^{n2} \cdots T_{l_n}^{nn} \end{pmatrix}$$

holds. Since $S(l_1, l_2, \dots, l_n)(1 \leqq l_1, l_2, \dots, l_n \leqq q)$ cover the whole complex t-plane, from (1.19) we can immediately clear up the Stokes phenomenon.

The method stated above was established first by K. Okubo [14] and has been extended in a series of papers [15, 9, 10]. However, in those papers we did not explain the fact that the error terms in the asymptotic expansion formulas of their final results can be removed, and also did not give the concrete method of evaluation of the Stokes multipliers. As one of our objectives of this paper, we shall show that the error terms can be dropped, and moreover that the exact values of the Stokes multipliers can be evaluated if the asymptotic behaviors of the coefficients $G_j(m)(j=1, 2, \dots, n)$ as $m \to \infty$ are known.

The asymptotic bahaviors of the coefficients $\mathscr{G}_l^k(m)$ of the associated fundamental matrix functions (1.18) for large m play a basic role in the analysis to follow, for instance, in proving (1.16) and in evaluating the Stokes multipliers. The functions $\mathscr{G}_l^k(m)(l=1, 2, \dots, q)$ form a fundamental set of solutions of the system of linear difference equations

(1.20) $$(m+q+\rho-\mu_k+J)\mathscr{G}^k(m+q) - \alpha_1^k \mathscr{G}^k(m+q-1) - \cdots - \alpha_{q-1}^k \mathscr{G}^k(m+1) - \lambda_k \mathscr{G}^k(m) = 0.$$

In the next section, in order to obtain the asymptotic behaviors of $\mathscr{G}_l^k(m)$, we shall investigate the relations between $\mathscr{G}_l^k(m)(l=1, 2, \dots, q)$ and a fundamental set of solutions $\Phi_l^k(m)(l=1, 2, \dots, q)$ of the system of linear difference equations

$$(1.21)\qquad \lambda_k\Phi^k(m+q)+\alpha^k_{q-1}\Phi^k(m+q-1)+\cdots + \alpha^k_1\Phi^k(m+1)-(m+\rho-\mu_k+J)\Phi^k(m)=0.$$

Since the solution $\Phi^k_l(m)$ can be expressed in terms of a modified gamma function and its derivatives, we consequently obtain the desired results for $\mathscr{G}^k_l(m)$ with the help of the study of the modified gamma function by N. G. de Bruijn [2]. Moreover, taking account of the relations between $\mathscr{G}^k_l(m)$ and $\Phi^k_l(m)$ just derived, we can obtain the well-formed results as to the global behaviors of the associated fundamental matrix functions $Y^k_l(t, s)$ in Section 3.

In guaranteeing the validity of the expansion (1.16), we have to estimate the coefficients $H^k(s)(k=1, 2,\dots, n)$ for sufficiently large positive values of s. The derivation of their estimates is a difficult and complicated work since there is no general way to obtain the growth order of solutions for such systems of linear difference equations (1.14) with a singular matrix as their coefficient of the highest order. In Section 4 we shall explain a method to reduce (1.14) to normal systems of linear difference equations and then, applying O. Perron- H. Poincaré's theorem, we shall obtain the estimates of $H^k(s)$, together with showing how to determine the characteristic constants $\alpha^k_{q-1},\dots, \alpha^k_1$ and μ_k.

Section 7 deals with the evaluation of the Stokes multipliers and in the last section we shall apply our theory established to the solution of the two point connection problem for the extended Airy equation.

§2. Relations between two systems of linear difference equations

As will be seen in the later sections, the matrix function $\Phi(m)$, which is defined as a solution of the matrix form of the system of linear difference equations (1.21), appears as the Stokes multiplier in the study of the asymptotic behavior of the associated fundamental function $Y(t, s)$ and on the other hand, the asymptotic behavior of the matrix function $\mathscr{G}(m)$ as $m\to\infty$ is needed in several stages of our analysis. So in this section we shall make clear the explicit expressions of the matrix functions $\Phi(m)$ and $\mathscr{G}(m)$, obtaining the important relations between them. In fact, we can express the matrix function $\Phi(m)$ in terms of a modified gamma function and its derivatives.

From now on we assume that m is the complex variable since an integral m is regarded as an integral value on the real axis of the complex m-plane.

First we summarize some results derived in the previous paper [10].

Assuming for the time being that $\operatorname{Re}\nu>0$, a modified gamma function is defined by the integral

$$(2.1)\qquad \Gamma(\nu:\gamma_k)=\Gamma(\nu:\gamma_{q-1},\dots,\gamma_k,\dots,\gamma_1)$$

$$= \int_0^\infty \exp\left(-\frac{\eta^q}{q} - \sum_{k=1}^{q-1} \frac{\gamma_k}{k}\eta^k\right)\eta^{\nu-1}d\eta,$$

where the abbreviation that the dependence on $q-1$ complex parameters $\gamma_{q-1},\dots, \gamma_k,\dots, \gamma_1$ is represented only by the k-th parameter γ_k has been used and hereafter will be used throughout this paper.

By partial integration we easily obtain the q-th order linear difference equation

$$(2.2) \qquad \nu\Gamma(\nu: \gamma_k) = \Gamma(\nu + q: \gamma_k) + \gamma_{q-1}\Gamma(\nu + q - 1: \gamma_k) + \cdots + \gamma_1\Gamma(\nu + 1: \gamma_k)$$

and from this, we immediately see that the modified gamma function $\Gamma(\nu: \gamma_k)$ can be analytically continued over the whole complex plane except for $\nu=0, -1, -2,\dots$. Therefore the above condition imposed on ν can be replaced by the condition that $\nu \neq$ a non-positive integer.

If we put

$$(2.3) \qquad \phi_l(m) = \Gamma(m + \rho - \mu: \alpha_k\lambda^{-k/q}\omega^{k(l-1)})(\lambda^{-1/q}\omega^{l-1})^{m+\rho-\mu} \qquad (l = 1, 2,\dots, q),$$

where $\lambda^{-1/q}=|\lambda|^{-1/q}\exp(-i\arg\lambda/q)(-\pi \leqq \arg\lambda<\pi)$ and $\omega=\exp(2\pi i/q)$, then we find from (2.2) that $\Phi_l(m)(l=1, 2,\dots, q)$ are particular solutions of the linear difference equation

$$(2.4) \qquad \lambda\phi(m + q) + \alpha_{q-1}\phi(m + q - 1) + \cdots + \alpha_1\phi(m + 1) = (m + \rho - \mu)\phi(m).$$

With the aid of the detailed study of the modified gamma function by N. G. de Bruijn [2; Chapter 6], we can prove the following

Proposition 2.1. *Under the assumption that $\rho-\mu \neq$ an integer, $\phi_l(m)$ $(l=1, 2,\dots, q)$ form a fundamental set of solutions of the q-th order linear difference equation* (2.4). *Each solution $\phi_l(m)$ has the asymptotic behavior*

$$(2.5) \quad \phi_l(m) \sim (\lambda^{-1/q}\omega^{l-1})^{m+\rho-\mu}\left(\frac{2\pi}{qm}\right)^{1/2}\exp\left(\frac{m}{q}\log m - \frac{m}{q} + m\sum_{k=1}^{\infty} c_{lk}m^{-k/q}\right)$$

$$\times \left(\sum_{k=0}^{\infty} d_{lk}m^{-k/q}\right) \text{ as } m \longrightarrow \infty \quad \text{in} \quad |\arg(m + \rho - \mu)| < \pi - \delta,$$

where $d_{l0}=1$ and δ is an arbitrarily small positive number.

The latter part of the above proposition is due to N. G. de Bruijn and then the former part can be proved by using these asymptotic expansions (2.5).

The relation between a fundamental set of solutions of (2.4) and that of the linear difference equation

$$(2.6) \qquad -(m + q + \rho - \mu)g(m + q) + \alpha_1 g(m + q - 1) + \cdots + \alpha_{q-1}g(m + 1)$$

$$+ \lambda g(m) = 0$$

is very important. Let us denote a fundamental set of solutions of (2.6) by

$$g_1(m),\ g_2(m),\dots,\ g_q(m), \tag{2.7}$$

and its Casorati determinant by $\mathscr{C}_g(m)$ as follows:

$$\mathscr{C}_g(m) = \begin{vmatrix} g_1(m) & g_2(m) & \cdots g_q(m) \\ g_1(m+1) & g_2(m+1) & \cdots g_q(m+1) \\ \vdots & \vdots & \vdots \\ g_1(m+q-1) & g_2(m+q-1) & \cdots g_q(m+q-1) \end{vmatrix}. \tag{2.8}$$

We denote the cofactor of the (j, k) element $g_j(m+k-1)$ in the above representation of the Casorati determinant by $\Delta_g^{(j,k)}(m)$. Similarly, $\mathscr{C}_\phi(m)$ and $\Delta_\phi^{(j,k)}(m)$ denote the Casorati determinant constructed by a fundamental set of solutions $\phi_l(m)(l=1, 2,\dots, q)$ of (2.4) and the cofactor corresponding to the (j, k) element $\phi_j(m+k-1)$, respectively. We then have derived the following results.

PROPOSITION 2.2. *Let us denote*

$$E_j(m) = \frac{\Delta_\phi^{(j,1)}(m)}{-(m+\rho-\mu)\mathscr{C}_\phi(m)} \qquad (j = 1, 2,\dots, q) \tag{2.9}$$

and

$$\mathscr{E}_j(m) = \frac{\Delta_g^{(j,1)}(m-q)}{\lambda\mathscr{C}_g(m-q)} \qquad (j = 1, 2,\dots, q). \tag{2.10}$$

Then the functions $E_j(m)$ form a fundamental set of solutions of (2.6), *and conversely, the functions $\mathscr{E}_j(m)$ form a fundamental set of solutions of* (2.4). *Moreover, for each j $(j=1, 2,\dots, q)$, we have the following relations*:

$$-(m+k+\rho-\mu)E_j(m+k) + \alpha_1 E_j(m+k-1) + \cdots + \alpha_k E_j(m) = \frac{\Delta_\phi^{(j,k+1)}(m)}{\mathscr{C}_\phi(m)} \qquad (k = 0, 1,\dots, q-1), \tag{2.11}$$

$$\lambda\mathscr{E}_j(m+k) + \alpha_{q-1}\mathscr{E}_j(m+k-1) + \cdots + \alpha_{q-k}\mathscr{E}_j(m) = \frac{\Delta_g^{(j,k+1)}(m-q)}{\mathscr{C}_g(m-q)} \qquad (k = 0, 1,\dots, q-1). \tag{2.12}$$

Taking account of Propositions 2.1 and 2.2, we shall from now on take

$$g_j(m) = -E_j(m) \qquad (j = 1, 2,\dots, q) \tag{2.13}$$

as a fundamental set of solutions of (2.6), and then we can easily obtain the explicit asymptotic expansion formulas for $g_j(m)$, together with the relations needed later. We summarize them in the form of

PROPOSITION 2.3. *We have*

$$(2.14)\qquad g_j(m) \sim \frac{\lambda^{-1/q}}{q}\omega^{j-1}\frac{m^{-(q-1)/q}}{\phi_j(m+1)}\left\{1+O(m^{-1/q})\right\}\qquad (j=1,2,\dots,q)$$

and

$$(2.15)\qquad \frac{g_j(m+r)}{g_j(m)} \sim (\lambda^{-1/q}\omega^{j-1})^{-r}m^{-r/q}\{1+O(m^{-1/q})\}\qquad (j=1,2,\dots,q)$$

for sufficiently large values of m in the sector $|\arg(m+\rho-\mu)|<\pi-\delta$, δ *being an arbitrarily small positive number. Moreover we have the following important relations*:

$$(2.16)\qquad \begin{aligned}&[\lambda g_j(m-1)]\phi_l(m+q-1)\\ &+[\lambda g_j(m-2)+\alpha_{q-1}g_j(m-1)]\phi_l(m+q-2)\\ &+\\ &\vdots\\ &+[\lambda g_j(m-q)+\alpha_{q-1}g_j(m-q+1)+\cdots+\alpha_1 g_j(m-1)]\phi_l(m)\\ &=\begin{cases}1 & for\ \ j=l,\\ 0 & for\ \ j\neq l\qquad (j,l=1,2,\dots,q).\end{cases}\end{aligned}$$

PROOF. The asymptotic expansion for $g_j(m)$ can easily be derived from (2.5) and (2.9), and the asymptotic formula (2.15) is an immediate consequence of (2.14). As for (2.16), it follows from the relation (2.11) that

$$\begin{aligned}&\lambda g_j(m-q+k-1)+\alpha_{q-1}g_j(m-q+k)+\cdots+\alpha_k g_j(m-1)\\ &\quad=-\lambda E_j(m-q+k-1)-\alpha_{q-1}E_j(m-q+k)-\cdots-\alpha_k E_j(m-1)\\ &\quad=\alpha_{k-1}E_j(m)+\alpha_{k-2}E_j(m+1)+\cdots+\alpha_1 E_j(m+k-2)\\ &\qquad-(m+k-1+\rho-\mu)E_j(m+k-1)\\ &\quad=\frac{\Delta_\phi^{(j,k)}(m)}{\mathscr{C}_\phi(m)}.\end{aligned}$$

Hence the left hand side of (2.16) is equal to

$$\frac{1}{\mathscr{C}_\phi(m)}\{\Delta_\phi^{(j,q)}(m)\phi_l(m+q-1)+\Delta_\phi^{(j,q-1)}(m)\phi_l(m+q-2)+\cdots$$

$$+ \Delta_{\phi}^{(j,1)}(m)\phi_l(m)\},$$

thereby obtaining the desired result.

Now we shall investigate the properties of solutions of two systems of linear difference equations (1.20) and (1.21), omitting the index k.

Setting

$$\Phi(m) = \begin{pmatrix} \phi^1(m) \\ \phi^2(m) \\ \vdots \\ \phi^n(m) \end{pmatrix}, \quad \mathscr{G}(m) = \begin{pmatrix} g^1(m) \\ g^2(m) \\ \vdots \\ g^n(m) \end{pmatrix}$$

and substituting them into (1.21) and (1.20), respectively, we have the following linear difference equations of a scalar type:

$$(2.17) \quad \begin{cases} \lambda\phi^1(m+q) + \alpha_{q-1}\phi^1(m+q-1) + \cdots + \alpha_1\phi^1(m+1) \\ \qquad = (m+\rho-\mu)\phi^1(m), \\ \lambda\phi^j(m+q) + \alpha_{q-1}\phi^j(m+q-1) + \cdots + \alpha_1\phi^j(m+1) \\ \qquad = (m+\rho-\mu)\phi^j(m) + \phi^{j-1}(m) \qquad (j = 2, 3,\dots, n), \end{cases}$$

$$(2.18) \quad \begin{cases} (m+\rho-\mu)g^1(m) = \alpha_1 g^1(m-1) + \cdots + \alpha_{q-1}g^1(m-q+1) \\ \qquad + \lambda g^1(m-q), \\ (m+\rho-\mu)g^j(m) + g^{j-1}(m) = \alpha_1 g^j(m-1) + \cdots \\ \qquad + \alpha_{q-1}g^j(m-q+1) + \lambda g^j(m-q) \qquad (j = 2, 3,\dots, n). \end{cases}$$

We first collect some easily verified results.

Lemma 2.1. *Particular solutions of the nonhomogeneous linear difference equations in* (2.17) *and* (2.18) *for* $j=2, 3,\dots, n$ *are given by*

$$(2.19) \qquad \phi^j(m) = \frac{1}{(j-1)!}\frac{d^{j-1}}{dm^{j-1}}(\phi^1(m))$$

and

$$(2.20) \qquad g^j(m) = \frac{1}{(j-1)!}\frac{d^{j-1}}{dm^{j-1}}(g^1(m)) \qquad (j = 2, 3,\dots, n),$$

respectively, where $\phi^1(m)$ *and* $g^1(m)$ *are solutions of the respective homogeneous linear difference equations.*

Proof. The proof will be immediately done by induction, considering the relations

$$(\phi^j(m))' = j\phi^{j+1}(m),$$

$$(m+\rho-\mu)(\phi^j(m))' + \phi^j(m) + (\phi^{j-1}(m))' = \lambda(\phi^j(m+q))' + \cdots + \alpha_1(\phi^j(m+1))'$$

and the like, where the prime denotes differentiation with respect to m.

LEMMA 2.2. *Let $\phi^1(m)$ and $g^1(m)$ be solutions of the homogeneous linear difference equations in (2.17) and (2.18), respectively. Then we have particular matrix solutions of (1.21) and (1.20) expressed in the form*

$$\Phi(m) = \exp\left(J\frac{d}{dm}\right)\phi^1(m) = \phi^1(m) + \phi^2(m)J + \cdots + \phi^n(m)J^{n-1} \tag{2.21}$$

and

$$\mathscr{G}(m) = \exp\left(J\frac{d}{dm}\right)g^1(m) = g^1(m) + g^2(m)J + \cdots + g^n(m)J^{n-1}, \tag{2.22}$$

respectively.

Moreover, if we put

$$\Phi(m+r)\mathscr{G}(m) = \Delta(m: r), \tag{2.23}$$

r being an arbitrary number, then $\Delta(m: r)$ can be written in the form

$$\Delta(m: r) = \delta^1(m: r) + \delta^2(m: r)J + \cdots + \delta^n(m: r)J^{n-1}, \tag{2.24}$$

where

$$\delta^j(m: r) = \sum_{k=1}^{j} \phi^{j+1-k}(m+r)g^k(m), \tag{2.25}$$

$$\delta^j(m: r) = \frac{1}{(j-1)!}\frac{d^{j-1}}{dm^{j-1}}\delta^1(m: r) \qquad (j = 1, 2, \ldots, n). \tag{2.26}$$

PROOF. We observe from Lemma 2.1 that $\Phi(m)$ satisfies the system of linear difference equations (1.21) as follows:

$$\begin{aligned}
&(m+\rho-\mu+J)\Phi(m)\\
&= (m+\rho-\mu+J)(\phi^1(m) + \phi^2(m)J + \cdots + \phi^n(m)J^{n-1})\\
&= (m+\rho-\mu)\phi^1(m) + \{(m+\rho-\mu)\phi^2(m) + \phi^1(m)\}J + \cdots\\
&\quad + \{(m+\rho-\mu)\phi^n(m) + \phi^{n-1}(m)\}J^{n-1} + \phi^n(m)J^n\\
&= \{\alpha_1\phi^1(m+1) + \cdots + \lambda\phi^1(m+q)\} + \{\alpha_1\phi^2(m+1) + \cdots + \lambda\phi^2(m+q)\}J
\end{aligned}$$

$$+\cdots+\{\alpha_1\phi^n(m+1)+\cdots+\lambda\phi^n(m+q)\}J^{n-1}$$

$$=\alpha_1\Phi(m+1)+\cdots+\lambda\Phi(m+q).$$

Similarly, we see that $\mathscr{G}(m)$ defined as (2.22) satisfies (1.20). Since the formulas (2.25) can be derived directly by multiplication, we shall prove the formulas (2.26). From the relation

$$(\phi^j(m+r)g^k(m))'=j\phi^{j+1}(m+r)g^k(m)+k\phi^j(m+r)g^{k+1}(m),$$

it follows that

$$\begin{aligned}(\delta^j(m\colon r))' &= \sum_{k=1}^{j}\{(j+1-k)\phi^{j+2-k}(m+r)g^k(m)+k\phi^{j+1-k}(m+r)g^{k+1}(m)\}\\ &= j\phi^{j+1}(m+r)g^1(m)+\sum_{k=2}^{j}(j+1-k)\phi^{j+2-k}(m+r)g^k(m)\\ &+\sum_{k=2}^{j}(k-1)\phi^{j+2-k}(m+r)g^k(m)+j\phi^1(m+r)g^{j+1}(m)\\ &= j\sum_{k=1}^{j+1}\phi^{j+2-k}(m+r)g^k(m)\\ &= j\delta^{j+1}(m\colon r)\end{aligned}$$

and hence the required formulas (2.26) are obtained.

We now take the functions $\phi_l^1(m)\,(l=1,2,\ldots,q)$ defined by (2.3), and then take the functions $g_j^1(m)\,(j=1,2,\ldots,q)$ defined by (2.13) as the respective fundamental sets of solutions of the homogeneous linear difference equations in (2.17) and (2.18). Then we have the following important results.

THEOREM 2.1. *The qn column vectors of the matrices*

$$\Phi_l(m)=\exp\left(J\frac{d}{dm}\right)\phi_l^1(m)\qquad(l=1,2,\ldots,q)\tag{2.27}$$

and

$$\mathscr{G}_j(m)=\exp\left(J\frac{d}{dm}\right)g_j^1(m)\qquad(j=1,2,\ldots,q)\tag{2.28}$$

form the respective fundamental sets of solutions of the systems of linear difference equations (1.21) *and* (1.20). *Moreover we have the relations*

$$\begin{aligned}&\Phi_l(m+q-1)[\lambda\mathscr{G}_j(m-1)]\\ &+\Phi_l(m+q-2)[\lambda\mathscr{G}_j(m-2)+\alpha_{q-1}\mathscr{G}_j(m-1)]\\ &+\end{aligned}\tag{2.29}$$

$$\vdots$$
$$+ \Phi_l(m)[\lambda\mathscr{G}_j(m-q) + \alpha_{q-1}\mathscr{G}_j(m-q+1) + \cdots + \alpha_1\mathscr{G}_j(m-1)]$$
$$= \begin{cases} 1 & \textit{for} \quad j = l, \\ 0 & \textit{for} \quad j \neq l \qquad (j,\, l = 1, 2, \ldots, q). \end{cases}$$

PROOF. In order to prove that the qn column vectors of $\Phi_l(m)$ and $\mathscr{G}_l(m)$ ($l=1, 2,\ldots, q$) form a fundamental set of solutions of (1.21) and (1.20), respectively, we need only show the nonvanishing of the respective Casorati determinants $\mathscr{C}_\Phi(m)$ and $\mathscr{C}_\mathscr{G}(m)$. After simple calculations, we obtain

$$\mathscr{C}_\Phi(m) = \mathscr{C}_\phi(m)\mathscr{C}_\phi(m+1)\cdots\mathscr{C}_\phi(m+q-1),$$
$$\mathscr{C}_\mathscr{G}(m) = \mathscr{C}_g(m)\mathscr{C}_g(m+1)\cdots\mathscr{C}_g(m+q-1),$$

the right hand sides of which never vanish from the assumptions, and we have thus proved the first part of the theorem. As for the relations (2.29), taking Lemma 2.2 into consideration, we can write the left hand side of (2.29) in the form

$$\lambda\Delta_{lj}(m-1\colon q)$$
$$+ \lambda\Delta_{lj}(m-2\colon q) + \alpha_{q-1}\Delta_{lj}(m-1\colon q-1)$$
$$+$$
$$\vdots$$
$$+ \lambda\Delta_{lj}(m-q\colon q) + \alpha_{q-1}\Delta_{lj}(m-q+1\colon q-1) + \cdots + \alpha_1\Delta_{lj}(m-1\colon 1)$$
$$= \hat{\delta}^1_{lj}(m) + \hat{\delta}^2_{lj}(m)J + \cdots + \hat{\delta}^n_{lj}(m)J^{n-1},$$

where

$$\hat{\delta}^1_{lj}(m) = \phi^1_l(m+q-1)[\lambda g^1_j(m-1)]$$
$$+ \phi^1_l(m+q-2)[\lambda g^1_j(m-2) + \alpha_{q-1}g^1_j(m-1)]$$
$$+$$
$$\vdots$$
$$+ \phi^1_l(m)[\lambda g^1_j(m-q) + \alpha_{q-1}g^1_j(m-q+1) + \cdots + \alpha_1 g^1_j(m-1)]$$

and

$$\hat{\delta}^k_{lj}(m) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dm^{k-1}}\hat{\delta}^1_{lj}(m) \qquad (k = 2, 3,\ldots, n).$$

Since, according to Proposition 2.3,

$$\hat{\delta}^1_{lj}(m) = \begin{cases} 1 & \text{for} \quad l = j, \\ 0 & \text{for} \quad l \neq j \qquad (l,\ j = 1, 2,\dots, q) \end{cases}$$

and hence $\hat{\delta}^k_{lj}(m)=0$ $(k=2, 3,\dots, n)$, we obtain the required result.

REMARK 1. Let l take only q consecutive integers. For such l we define $\phi_l(m)$ by (2.3). Then the q functions again form a fundamental set of solutions of the linear difference equation (2.4). If we define $g_j(m)$ associated with such q functions by (2.13), then the relations of the form (2.16) also hold.

REMARK 2. Let $l'=l \bmod q$, i.e., $l'=pq+l$, p being an integer. Then we have $\phi_{l'}(m)=\phi_l(m)\omega^{pq(m+\rho-\mu)}$ and hence the left hand side of the foumula (2.16) in which $\phi_l(m)$ is replaced by $\phi_{l'}(m)$ is equal to $\omega^{pq(m+\rho-\mu)}$ for $l=j$, and to zero for $l\neq j$. From this fact, if $\Phi_l(m)$ is replaced by $\Phi_{l'}(m)=\exp\left(J\frac{d}{dm}\right)\phi_{l'}(m)$ in (2.29), then the left hand side of the formula (2.29) is equal to $\omega^{pq(J+m+\rho-\mu)}$ for $l=j$, and to zero for $l\neq j$.

Finally, we note that, taking account of the expression (2.28), the asymptotic behaviors of $\mathscr{G}_l(m)(l=1, 2,\dots, q)$ as $m\to\infty$ can be derived from the asymptotic expansions (2.14) and the differentiation of them with respect to m.

§3. Associated fundamental matrix functions

We introduce the function $Y(t, s)$ associated with the two point connection problem to be considered, and anew write down its definition as follows:

$$(3.1) \qquad Y(t, s) = t^\rho \sum_{m=0}^{\infty} \mathscr{G}(m+s)t^m,$$

where the matrix coefficients $\mathscr{G}(m)$ satisfy the linear difference equation

$$(3.2) \qquad (m+\rho-\mu+J)\mathscr{G}(m) = \alpha_1\mathscr{G}(m-1)+\cdots + \alpha_{q-1}\mathscr{G}(m-q+1)+\lambda\mathscr{G}(m-q).$$

In Sections 5 and 6 we shall attempt to expand convergent power series solutions of systems of linear ordinary differential equations in terms of a sequence $\{Y(t, s);\ s=0, 1,\dots\}$ and then to obtain their global behaviors in the whole complex t-plane from those of $Y(t, s)$.

For that purpose, we shall investigate the global behavior of the associated fundamental matrix function $Y(t, s)$ in detail in this section.

We begin with the study of scalar functions defined by the definite integrals

$$(3.3) \qquad z(t\colon v\colon \gamma_k) = z(t\colon v\colon \gamma_1,\dots, \gamma_k,\dots, \gamma_{q-1})$$

$$= \int_0^1 \exp\left(\frac{1}{q}t^q(1-\tau^q) + \sum_{k=1}^{q-1} \frac{\gamma_k}{k} t^k(1-\tau^k)\right)\tau^{\nu-1}d\tau,$$

(3.4) $$z_F(t:\nu:\gamma_k) = \int_0^1 \exp\left(\frac{1}{q}t^q(1-\tau^q) + \sum_{k=1}^{q-1} \frac{\gamma_k}{k} t^k(1-\tau^k)\right)F(\tau)\tau^{\nu-1}d\tau,$$

where the function $F(\tau)$ is holomorphic at least in the closed disk $|\tau| \leqq 1$. It is easy to see that the definite integral $z(t:\nu:\gamma_k)$, together with $z_F(t:\nu:\gamma_k)$, is well-defined under the assumption that $\mathrm{Re}\,\nu > 0$, but by partial integration and then using the principle of analytic continuation we can replace the condition $\mathrm{Re}\,\nu > 0$ by a weaker condition that $\nu \neq 0, -1, -2, \ldots$.

We now consider the behavior of (3.3) for sufficiently large values of t. Suppose that $|t| > t_0$, t_0 being a sufficiently large positive number and t lies in the sector

(3.5) $$S_1:\ -\frac{\pi}{q} \leqq \arg t < \frac{\pi}{q}.$$

Putting $\eta = t\tau$ and changing the path of integration which is the line segment from the origin to t into the following two paths of integration:

(i) the positive real axis from the origin to infinity;
(ii) the circular arc $|\eta| = |t|$ from t to $|t|$ and then the positive real axis from $|t|$ to infinity,

we have

(3.6) $$\begin{aligned} z(t:\nu:\gamma_k) &= \int_0^t \exp(p(t) - p(\eta))\left(\frac{\eta}{t}\right)^\nu \frac{d\eta}{\eta} \\ &= \exp(p(t))t^{-\nu}\int_{0\,(\mathrm{i})}^{\infty} \exp(-p(\eta))\eta^{\nu-1}d\eta \\ &\quad - \int_{t\,(\mathrm{ii})}^{\infty} \exp(p(t) - p(\eta))\left(\frac{\eta}{t}\right)^\nu \frac{d\eta}{\eta} \\ &= \Gamma(\nu:\gamma_k)\exp(p(t))t^{-\nu} - \hat{z}(t:\nu:\gamma_k), \end{aligned}$$

where we set

$$p(t) = \frac{t^q}{q} + \sum_{k=1}^{q-1} \frac{\gamma_k}{k} t^k$$

and $\Gamma(\nu:\gamma_k)$ is the modified gamma function defined in (2.1). From this formula and the observation of the analytic continuation of the modified gamma function described in the preceeding section, we can see again the well-definedness of the definite integral (3.3) under the condition that $\nu \neq 0, -1, -2, \ldots$.

We define the function $\hat{z}_a(t:\nu:\gamma_k)$ by the integral

$$\hat{z}_a(t: \nu: \gamma_k) = \int_t^\infty \exp(p(t) - p(\eta)) \left(\frac{\eta}{t}\right)^\nu \frac{d\eta}{\eta} \tag{3.7}$$

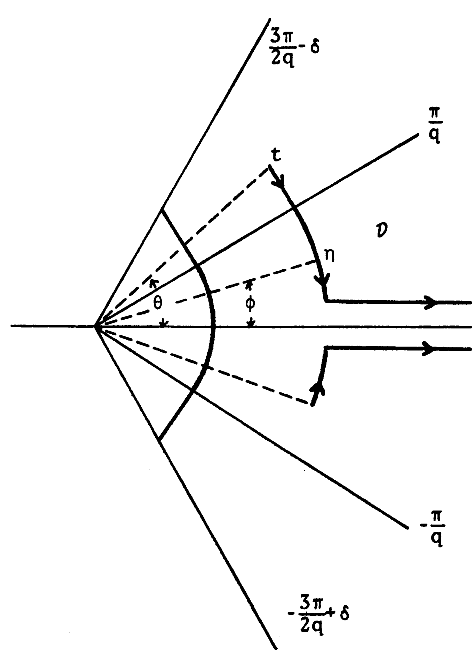
in the domain $\mathscr{D} = \{t;\ |t| \geqq t_0 > 0,\ |\arg t| \leqq 3\pi/2q - \delta,\ \delta$ being an arbitrarily small positive number$\} - \{t;\ |t|\cos(\arg t - \pi/q) < t_0,\ |\arg t| > \pi/q\}$. In the above integral, the path of integration is taken the so-called Friedrichs' path [3] as follows: when t lies in the sector $|\arg t| \leqq \pi/q$, η moves along the path of integration (ii) stated above, and when t lies in the sector $\pi/q < |\arg t| \leqq 3\pi/2q - \delta$, η moves along the straight line orthogonal to the line $|\arg \eta| = \pi/q$ from t to $|t|\cos(\arg t - \pi/q)e^{\pm\pi i/q}$ and then along the same path (ii) from that point to infinity. We immediately see that the function $\hat{z}_a(t: \nu: \gamma_k)$ is an analytic continuation of the function $\hat{z}(t: \nu: \gamma_k)$, and then we can prove the following

LEMMA 3.1. *The function $\hat{z}_a(t: \nu: \gamma_k)$ is bounded in the domain $\mathscr{D}$.*

PROOF. We here only show the boundedness of the integral when η lies on the line orthogonal to $\arg \eta = \pm\pi/q$ or on the circular arc. In this case, if we put $t = |t|e^{i\theta}$, the variable of integration η can be written in the form

$$\eta = c(|\phi|)|t|e^{i\phi} \qquad \left(0 \leqq |\phi| \leqq |\theta| \leqq \frac{3\pi}{2q} - \delta\right),$$

where



$$c(|\phi|) = \begin{cases} \dfrac{\cos\left(|\theta| - \dfrac{\pi}{q}\right)}{\cos\left(|\phi| - \dfrac{\pi}{q}\right)} & \text{if} \quad \dfrac{\pi}{q} \leqq |\phi| \leqq |\theta| \leqq \dfrac{3\pi}{2q} - \delta, \\ 1 & \text{if} \quad 0 \leqq |\phi| \leqq |\theta| \leqq \dfrac{\pi}{q}, \end{cases}$$

and hence

$$0 < \cos\left(\frac{\pi}{2q} - \delta\right) \leqq c(|\phi|) \leqq 1.$$

Then, setting

$$t^q Q(\phi\colon \theta\colon t) = \mathrm{Re}\,(p(t) - p(\eta)),$$

we easily see that if t_0 is taken a sufficiently large positive number, we have for an arbitrarily small positive number ε

$$Q(\theta\colon \theta\colon t) = 0,$$

$$Q(\phi\colon \theta\colon t) \leqq \frac{1}{2q}\{\cos q\theta - c(|\theta| - \varepsilon)^q \cos(q(|\theta| - \varepsilon))\} < 0$$

$$(0 \leqq |\phi| \leqq |\theta| - \varepsilon),$$

which implies that

$$|\exp(p(t) - p(\eta))| \leqq 1$$

on that path of integration.

Hence we have

$$\left|\int_t^{|t|} \exp(p(t) - p(\eta))\left(\frac{\eta}{t}\right)^{\nu} \frac{d\eta}{\eta}\right|$$

$$\leqq \int_0^{|\theta|} \exp(|\mathrm{Re}\,\nu||\log c(|\phi|)| + (\theta - \phi)|\mathrm{Im}\,\nu|) \frac{d\phi}{\left(\cos\left(\dfrac{\pi}{2q} - \delta\right)\right)^2}$$

$$\leqq M,$$

where M is a constant depending on the parameter ν. Similarly, we have the desired boundedness of the integral when η moves on the positive real axis, and thus the proof of the lemma is completed.

From the above lemma we obtain

$$z(t\colon \nu\colon \gamma_k) = \Gamma(\nu\colon \gamma_k)\exp(p(t))t^{-\nu} + O(1) \tag{3.7}$$

for sufficiently large values of t in the sector S_1.

If t lies in the sector

$$S_l:\ -\frac{3\pi}{q}+\frac{2\pi}{q}l \leqq \arg t < -\frac{\pi}{q}+\frac{2\pi}{q}l,$$

l being an arbitrary integer, we change the path of integration along the line segment from the origin to t in the integral (3.6) into the path $\eta=\xi\omega^{l-1}(\omega=\exp(2\pi i/q))$, where ξ moves along the paths of integration (i) and (ii). Then by the same considerations as stated above we have

$$z(t:\nu:\gamma_k)=\Gamma(\nu:\gamma_k\omega^{k(l-1)})\omega^{\nu(l-1)}\exp(p(t))t^{-\nu}+O(1) \tag{3.8}$$

$$\text{as}\quad t\longrightarrow\infty\quad \text{in}\quad S_l.$$

We summarize the above results, together with the behavior of $z_F(t:\nu:\gamma_k)$, in the following

LEMMA 3.2. *Let l be any integer. Then for sufficiently large values of t in the sector*

$$S_l:\ -\frac{3\pi}{q}+\frac{2\pi}{q}l \leqq \arg t < -\frac{\pi}{q}+\frac{2\pi}{q}l, \tag{3.9}$$

the function $z(t:\nu:\gamma_k)$ has the behavior

$$z(t:\nu:\gamma_k)=\Gamma(\nu:\gamma_k\omega^{k(l-1)})\omega^{\nu(l-1)}\exp(p(t))t^{-\nu}+O(1) \tag{3.10}$$

and the function $z_F(t:\nu:\gamma_k)$ has the behavior

$$z_F(t:\nu:\gamma_k)=\left(\sum_{n=0}^{m-1}\frac{F^{(n)}(0)}{n!}\Gamma(\nu+n:\gamma_k\omega^{k(l-1)})\omega^{(\nu+n)(l-1)}t^{-n}\right) \times\exp(p(t))t^{-\nu}+O(\exp(p(t))t^{-\nu-m})+O(1). \tag{3.11}$$

We here note that the q sectors $S_l\ (l=1, 2,\dots, q)$ cover the whole complex t-plane and according to Lemma 3.1, the last O-term in the statements (3.10) and (3.11) holds in a more wide sector

$$-\frac{7\pi}{2q}+\delta+\frac{2\pi}{q}l \leqq \arg t < -\frac{\pi}{2q}-\delta+\frac{2\pi}{q}l.$$

Now we shall turn to the investigation of the associated fundamental matrix function $Y(t, s)$. From its definition we immediately obtain the following two relations: For an arbitrary positive integer r,

$$Y(t, s-r)=\mathscr{G}(s-r)t^{\rho}+\mathscr{G}(s-r+1)t^{\rho+1}+\cdots \tag{3.12}$$

$$+ \mathscr{G}(s-1)t^{\rho+r-1} + t^r Y(t, s).$$

(3.13) $$t\frac{dY(t, s)}{dt} = \alpha_1 Y(t, s-1) + \cdots + \alpha_{q-1}Y(t, s-q+1) + \lambda Y(t, s-q) + (\mu - s - J)Y(t, s).$$

Substituting the formulas (3.12) for $r=1, 2,\ldots, q$ into the right hand side of (3.13), we then have the system of nonhomogeneous linear differential equations of the first order

(3.14) $$\begin{aligned} t\frac{dY(t, s)}{dt} = &\{\lambda t^q + \alpha_{q-1}t^{q-1} + \cdots + \alpha_1 t + (\mu - s - J)\}Y(t, s) \\ &+ [\lambda\mathscr{G}(s-1)]t^{\rho+q-1} \\ &+ [\lambda\mathscr{G}(s-2) + \alpha_{q-1}\mathscr{G}(s-1)]t^{\rho+q-2} \\ &+ \\ &\vdots \\ &+ [\lambda\mathscr{G}(s-q) + \alpha_{q-1}\mathscr{G}(s-q+1) + \cdots + \alpha_1\mathscr{G}(s-1)]t^{\rho}. \end{aligned}$$

By quadrature, we therefore obtain the integral representation of $Y(t, s)$ as a particular matrix solution of (3.14) with the form (3.1) as follows:

(3.15) $$\begin{aligned} Y(t, s) = &Z(\lambda^{1/q}t: s - \mu + \rho + q - 1: \gamma_k)t^{\rho+q-1}[\lambda\mathscr{G}(s-1)] \\ &+ Z(\lambda^{1/q}t: s - \mu + \rho + q - 2: \gamma_k)t^{\rho+q-2}[\lambda\mathscr{G}(s-2) + \alpha_{q-1}\mathscr{G}(s-1)] \\ &+ \\ &\vdots \\ &+ Z(\lambda^{1/q}t: s - \mu + \rho: \gamma_k)t^{\rho}[\lambda\mathscr{G}(s-q) + \alpha_{q-1}\mathscr{G}(s-q+1) + \cdots + \alpha_1\mathscr{G}(s-1)], \end{aligned}$$

where we put $\gamma_k = \alpha_k\lambda^{-k/q}$ $(k=1, 2,\ldots, q-1)$ and

(3.16) $$Z(t: \nu: \gamma_k) = \int_0^1 \exp\left(\frac{1}{q}t^q(1-\tau^q) + \sum_{k=1}^{q-1}\frac{\gamma_k}{k}t^k(1-\tau^k)\right)\tau^{J+\nu-1}d\tau.$$

This integral representation plays an important role in the derivation of the global behavior of $Y(t, s)$ on its Riemann surface. To see this, we first investigate the global behavior of the matrix function $Z(t: \nu: \gamma_k)$. As in the case of a scalar function, assuming that t lies in the sector S_1 and taking (i) and (ii) as the path of integration after the change of variables $\eta = t\tau$, we can rewrite (3.16) in the form

(3.17) $$Z(t: \nu: \gamma_k) = \exp(p(t))t^{-J-\nu}\int_{0\,(\mathrm{i})}^{\infty} \exp(-p(\eta))\eta^{J+\nu-1}d\eta$$

$$-\int_{t \atop (ii)}^{\infty} \exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{J+\nu}\frac{d\eta}{\eta}$$

$$= \Psi_1(\nu:\gamma_k)\exp(p(t))t^{-J-\nu} - \hat{Z}_1(t:\nu:\gamma_k),$$

where we put

$$\Psi_1(\nu:\gamma_k) = \int_0^{\infty}\exp(-p(\eta))\eta^{J+\nu-1}d\eta \tag{3.18}$$

and we have used the fact that $\Psi_1(\nu:\gamma_k)$ commutes with J. By the same considerations as in the proof of Lemma 3.1, we have

$$\hat{Z}_1(t:\nu:\gamma_k) = O(1) \quad \text{as } t \longrightarrow \infty \text{ in } \mathscr{D} \tag{3.19}$$

since the matrix function $\hat{Z}_1(t:\nu:\gamma_k)$ has the form

$$\hat{Z}_1(t:\nu:\gamma_k) = \int_{t \atop (ii)}^{\infty}\exp(p(t)-p(\eta))\left\{1+\frac{1}{1!}\log\left(\frac{\eta}{t}\right)J+\cdots + \frac{1}{(n-1)!}\left(\log\left(\frac{\eta}{t}\right)\right)^{n-1}J^{n-1}\right\}\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}$$

and the logarithmic terms do not essentially influence on the estimates of the integrals. Hence we have

$$Z(t:\nu:\gamma_k) = \Psi_1(\nu:\gamma_k)\exp(p(t))t^{-J-\nu} + O(1) \quad \text{as } t \longrightarrow \infty \text{ in } S_1 \tag{3.20}$$

and similarly, for any integer l we have

$$Z(t:\nu:\gamma_k) = \Psi_l(\nu:\gamma_k)\exp(p(t))t^{-J-\nu} + O(1) \quad \text{as } t \longrightarrow \infty \text{ in } S_l, \tag{3.21}$$

where we put

$$\begin{aligned}\Psi_l(\nu:\gamma_k) &= \int_0^{\infty}\exp(-p(\eta\omega^{l-1}))(\eta\omega^{l-1})^{J+\nu}\frac{d\eta}{\eta} \\ &= \Psi_1(\nu:\gamma_k\omega^{k(l-1)})\omega^{(l-1)(J+\nu)}.\end{aligned} \tag{3.22}$$

Concerning the function $\Psi_l(\nu:\gamma_k)$, it is easily seen by partial integration that $\Psi_l(\nu:\gamma_k)$ satisfies the linear difference equation

$$\begin{aligned}&\Psi(\nu+q:\gamma_k)+\gamma_{q-1}\Psi(\nu+q-1:\gamma_k)+\cdots+\gamma_1\Psi(\nu+1:\gamma_k) \\ &= (J+\nu)\Psi(\nu:\gamma_k).\end{aligned} \tag{3.23}$$

Moreover if we write

$$\Psi_l(\nu:\gamma_k) = \int_0^{\infty}\exp(-p(\eta\omega^{l-1}))\left\{1+\frac{\log(\eta\omega^{l-1})}{1!}J+\cdots\right.$$

$$+\frac{(\log(\eta\omega^{l-1}))^{n-1}}{(n-1)!}J^{n-1}\Big\}(\eta\omega^{l-1})^{\nu}\frac{d\eta}{\eta}$$

$$=\psi_l^1(\nu:\gamma_k)+\psi_l^2(\nu:\gamma_k)J+\cdots+\psi_l^n(\nu:\gamma_k)J^{n-1},$$

then we have

$$\psi_l^1(\nu:\gamma_k)=\Gamma(\nu:\gamma_k\omega^{k(l-1)})\omega^{\nu(l-1)}, \tag{3.24}$$

$$\psi_l^j(\nu:\gamma_k)=\frac{1}{(j-1)!}\frac{d^{j-1}}{d\nu^{j-1}}\psi_l^1(\nu:\gamma_k)\qquad(j=2,3,\dots,n), \tag{3.25}$$

where $\Gamma(\nu:\gamma_k)$ is the modified gamma function defined in (2.1). Then, from the results derived in Section 2 and the considerations given above, we can prove

LEMMA 3.3. *Assume that $\rho-\mu\neq$ an integer. Let l be an arbitrary integer. Then we have*

$$\begin{aligned}&Z(\lambda^{1/q}t:s+\rho-\mu:\alpha_k\lambda^{-k/q})\\&=\exp\left(\frac{\lambda}{q}t^q+\frac{\alpha_{q-1}}{q-1}t^{q-1}+\cdots+\alpha_1 t\right)t^{-J+\mu-\rho-s}\Phi_l(s)+O(1)\end{aligned} \tag{3.26}$$

for sufficiently large values of t in the sector

$$S_l(\lambda):\ -\frac{3\pi}{q}+\frac{2\pi}{q}l\leqq\arg\lambda^{1/q}t<-\frac{\pi}{q}+\frac{2\pi}{q}l, \tag{3.27}$$

where the matrix function $\Phi_l(s)=\Psi_1(s+\rho-\mu:\alpha_k\lambda^{-k/q}\omega^{k(l-1)})(\lambda^{-1/q}\omega^{l-1})^{J+s+\rho-\mu}$ is exactly the same function defined in Section 2, i.e.,

$$\Phi_l(s)=\exp\left(J\frac{d}{ds}\right)\Gamma(s+\rho-\mu:\alpha_k\lambda^{-k/q}\omega^{k(l-1)})(\lambda^{-1/q}\omega^{l-1})^{s+\rho-\mu} \tag{3.28}$$

satisfying the system of linear difference equations

$$\lambda\Phi(s+q)+\alpha_{q-1}\Phi(s+q-1)+\cdots+\alpha_1\Phi(s+1)=(s+\rho-\mu+J)\Phi(s).$$

In the statement (3.26) *the last O-term holds in a more wide sector*

$$\hat{S}_l(\lambda):\ -\frac{7\pi}{2q}+\delta+\frac{2\pi}{q}l\leqq\arg\lambda^{1/q}t\leqq-\frac{\pi}{2q}-\delta+\frac{2\pi}{q}l, \tag{3.29}$$

δ being an arbitrarily small positive number.

Moreover we can prove just the same result as that of a scalar function $z_F(t:\nu:\gamma_k)$, which is needed later.

LEMMA 3.4. *Let $Z_F(t: \nu: \gamma_k)$ be a matrix function defined by the definite integral*

$$(3.30)\quad Z_F(t: \nu: \gamma_k) = \int_0^1 \exp\Big(\frac{1}{q}t^q(1-\tau^q) + \sum_{k=0}^{q-1}\frac{\gamma_k}{k}t^k(1-\tau^k)\Big)\tau^{J+\nu-1}\mathscr{F}(\tau)d\tau,$$

where the matrix function $\mathscr{F}(\tau)$ is holomorphic at least in the closed disk $|\tau|\leqq 1$. Then we have

$$(3.31)\quad Z_F(\lambda^{1/q}t: s+\rho-\mu: \alpha_k\lambda^{-k/q})$$

$$= \exp\Big(\frac{\lambda}{q}t^q + \frac{\alpha_{q-1}}{q-1}t^{q-1} + \cdots + \alpha_1 t\Big)t^{-J+\mu-\rho-s}\Big\{\sum_{n=0}^{m-1}\frac{t^{-n}}{n!}\Phi_l(s+n)\mathscr{F}^{(n)}(0)\Big\}$$

$$+ O\Big(\exp\Big(\frac{\lambda}{q}t^q + \frac{\alpha_{q-1}}{q-1} + \cdots + \alpha_1 t\Big)t^{-J+\mu-\rho-s-m}\Big) + O(1)$$

$$\text{as } t \longrightarrow \infty \text{ in } S_l(\lambda).$$

We now define q associated fundamental matrix functions

$$(3.32)\qquad Y_j(t, s) = t^\rho \sum_{m=0}^{\infty}\mathscr{G}_j(m+s)t^m \qquad (j=1, 2,\dots, q),$$

where

$$\mathscr{G}_j(m) = \exp\Big(J\frac{d}{dm}\Big)g_j^1(m) \qquad (j=1, 2,\dots, q)$$

which are defined in Theorem 2.1.

Taking the relations (2.29) in Theorem 2.1 into consideration, if t lies in the sector $S_l(\lambda)\,(l=1, 2,\dots, q)$, we have

$$(3.33)\quad Y_j(t, s) = \delta_{jl}\exp\Big(\frac{\lambda}{q}t^q + \frac{\alpha_{q-1}}{q-1}t^{q-1} + \cdots + \alpha_1 t\Big)t^{-J+\mu-s}$$

$$- \hat{Z}_l(\lambda^{1/q}t: s-\mu+\rho+q-1: \alpha_k\lambda^{-k/q})t^{\rho+q-1}[\lambda\mathscr{G}_j(s-1)]$$

$$- \hat{Z}_l(\lambda^{1/q}t: s-\mu+\rho+q-2: \alpha_k\lambda^{-k/q})t^{\rho+q-2}[\lambda\mathscr{G}_j(s-2) + \alpha_{q-1}\mathscr{G}_j(s-1)]$$

$$-$$

$$\vdots$$

$$- \hat{Z}_l(\lambda^{1/q}t: s-\mu+\rho: \alpha_k\lambda^{-k/q})t^{\rho}[\lambda\mathscr{G}_j(s-q) + \alpha_{q-1}\mathscr{G}_j(s-q+1) + \cdots$$

$$+ \alpha_1\mathscr{G}_j(s-1)]$$

$$= \delta_{jl}\exp\Big(\frac{\lambda}{q}t^q + \frac{\alpha_{q-1}}{q-1} + \cdots + \alpha_1 t\Big)t^{-J+\mu-s} + \hat{Y}_{jl}(t, s),$$

where δ_{jl} is the Kronecker symbol and $\hat{Z}_l(t: v: \gamma_k)$ denotes the integral defined in S_l of the same type as $\hat{Z}_1(t: v: \gamma_k)$ in S_1.

From Lemma 3.3 we then have

$$\hat{Y}_{jl}(t, s) = O(t^{\rho+q-1}) \quad \text{as } t \longrightarrow \infty \text{ in } \hat{S}_l(\lambda). \tag{3.34}$$

But we can get a more precise information about the asymptotic behavior of $\hat{Y}_{jl}(t, s)$. In fact, substituting (3.33) into the relation (3.12), we have

$$\begin{aligned} Y_j(t, s) &= \delta_{jl} \exp\left(\frac{\lambda}{q} t^q + \frac{\alpha_{q-1}}{q-1} t^{q-1} + \cdots + \alpha_1 t\right) t^{-J+\mu-s} + \hat{Y}_{jl}(t, s) \\ &= \mathscr{G}_j(s-1)t^{\rho-1} + \mathscr{G}_j(s-2)t^{\rho-2} + \cdots + \mathscr{G}_j(s-r)t^{\rho-r} + t^{-r} Y_j(t, s-r) \\ &= \mathscr{G}_j(s-1)t^{\rho-1} + \mathscr{G}_j(s-2)t^{\rho-2} + \cdots + \mathscr{G}_j(s-r)^{\rho-r} \\ &\quad + t^{-r}\delta_{jl} \exp\left(\frac{\lambda}{q} t^q + \frac{\alpha_{q-1}}{q-1} t^{q-1} + \cdots + \alpha_1 t\right) t^{-J+\mu-s+r} + t^{-r}\hat{Y}_{jl}(t, s-r), \end{aligned}$$

which implies that

$$\begin{aligned} \hat{Y}_{jl}(t, s) &= \mathscr{G}_j(s-1)t^{\rho-1} + \mathscr{G}_j(s-2)t^{\rho-2} + \cdots + \mathscr{G}_j(s-r)t^{\rho-r} \\ &\quad + t^{-r}\hat{Y}_{jl}(t, s-r). \end{aligned} \tag{3.35}$$

Let σ be an arbitrary positive integer and put $r = q + \sigma$. From the estimate (3.34) for $Y_{jl}(t, s-r)$, we obtain

$$\begin{aligned} \hat{Y}_{jl}(t, s) &= \mathscr{G}_j(s-1)t^{\rho-1} + \cdots + \mathscr{G}_j(s-\sigma)t^{\rho-\sigma} \\ &\quad + \mathscr{G}_j(s-\sigma-1)t^{\rho-\sigma-1} + \cdots + \mathscr{G}_j(s-r)t^{\rho-r} + t^{-q-\sigma}O(t^{\rho+q-1}) \\ &= \mathscr{G}_j(s-1)t^{\rho-1} + \cdots + \mathscr{G}_j(s-\sigma)t^{\rho-\sigma} + O(t^{\rho-\sigma-1}) \\ &\sim t^\rho\{\mathscr{G}_j(s-1)t^{-1} + \mathscr{G}_j(s-2)t^{-2} + \cdots\} \qquad \text{as } t \longrightarrow \infty \text{ in } \hat{S}_l(\lambda). \end{aligned} \tag{3.36}$$

Thus we have established the following important theorem for the global behaviors of the associated fundamental matrix functions $Y_j(t, s)$.

THEOREM 3.1. *Assume that $\rho - \mu \neq$ an integer. Each associated fundamental matrix function $Y_j(t, s)$ $(j = 1, 2, \ldots, q)$ has the asymptotic behavior as* follows:

$$\begin{aligned} Y_j(t, s) &\sim \delta_{jl} \exp\left(\frac{\lambda}{q} t^q + \frac{\alpha_{q-1}}{q-1} t^{q-1} + \cdots + \alpha_1 t\right) t^{-J+\mu-s} \\ &\quad + t^\rho\{\mathscr{G}_j(s-1)t^{-1} + \mathscr{G}_j(s-2)t^{-2} + \cdots\} \\ &\qquad \text{as } t \longrightarrow \infty \text{ in } S_l(\lambda) \qquad (l = 1, 2, \ldots, q), \end{aligned} \tag{3.37}$$

where the q *sectors* $S_l(\lambda)(l=1, 2,\dots, q)$ *cover the whole complex* t*-plane. Strictly speaking, the asymptotic power series in the statement* (3.37) *holds in* $\hat{S}_l(\lambda)$.

Finally, we note that in order to obtain the global behavior of $Y_j(t, s)$ on its Riemann surface, we have only to apply Theorem 3.1 after changing suitably the variable of integration. For instance, if t lies in the sector $S_{pq+l}(\lambda)$ for any integer p, then we have $Y_j(t, s)=Y_j(t\omega^{-pq}, s)\omega^{pq\rho}$ and apply (3.37) to its right hand side, obtaining

$$(3.38)\qquad Y_j(t, s) = \delta_{jl}\exp\left(\frac{\lambda}{q}t^q + \frac{\alpha_{q-1}}{q-1}t^{q-1} + \cdots + \alpha_1 t\right)t^{-J+\mu-s}\omega^{pq(J+s+\rho-\mu)}$$
$$+ O(t^{\rho-1}) \quad \text{as } t \longrightarrow \infty \text{ in } S_{pq+l}(\lambda).$$

But this fact is also an immediate consequence of Remark 2 in Section 2.

§4. Growth order of coefficients of formal solutions

Now we shall be concerned with the growth order of the coefficients $H^k(s)$ $(k=1, 2,\dots, n)$ of formal power series solutions (1.7) for sufficiently large positive integral values of s. The column vectorial coefficients $H^k(s)(k=1, 2,\dots, n)$ satisfy the systems of linear difference equations (1.14) whose coefficients of the highest order are singular matrices. From those linear difference equations just as they are, we cannot obtain the estimates of $H^k(s)(k=1, 2,\dots, n)$. But we have to go through the first process of determining the characteristic constants α^k_{q-1}, $\dots$, α^k_1 and μ_k $(k=1, 2,\dots, n)$, and this determination leads to the reduction of the systems of linear difference equations (1.14) to those of a regular type, that is, systems of linear difference equations with nonsingular matrices as their coefficients of the highest order. More precisely speaking, we can reduce (1.14) to systems of linear difference equations of the Perron-Poincaré type and then obtain the estimates of $H^k(s)(k=1, 2,\dots, n)$ by applying O. Perron-H. Poincaré's theorem [16, 17]. To show this, we consider the system of linear difference equations for $H^1(s)$

$$(4.1)\qquad (A_q - \lambda_1)H^1(s+q) + (A_{q-1} - \alpha^1_{q-1})H^1(s+q-1) + \cdots$$
$$+ (A_1 - \alpha^1_1)H^1(s+1) + (A_0 - \mu_1 + s)H^1(s) = 0,$$

subject to the intial conditions

$$(4.2)\qquad (A_q - \lambda_1)H^1(0) = 0, \quad H^1(r) = 0 \qquad \text{for } \; -(q-1) \leqq r < 0.$$

It is easy to see that the other systems of linear difference equations for $H^k(s)$

($k=2, 3,\dots, n$) can be transformed to those of just the same form as (4.1) by the so-called elementary transformations.

We here put

$$H^1(s)=\begin{pmatrix} h_1^1(s) \\ \hat{H}^1(s) \end{pmatrix}, \qquad \hat{H}^1(s)=\begin{pmatrix} h_2^1(s) \\ h_3^1(s) \\ \vdots \\ h_n^1(s) \end{pmatrix},$$

$$\Lambda^1=\begin{pmatrix} (\lambda_1-\lambda_2)^{-1} & & & 0 \\ & (\lambda_1-\lambda_3)^{-1} & & \\ & & \ddots & \\ 0 & & & (\lambda_1-\lambda_n)^{-1} \end{pmatrix},$$

$$A_i=\begin{pmatrix} a_i^{11} & \beta_i^1 \\ \gamma_i^1 & \hat{A}_i^1 \end{pmatrix} \qquad (i=0, 1,\dots, q-1),$$

where β_i^1 and γ_i^1 ($i=0, 1,\dots, q-1$) are ($n-1$)-dimensional row and column vectors, respectively, i.e.,

$$\beta_i^1=(a_i^{12}, a_i^{13},\dots, a_i^{1n}),$$

$$\gamma_i^1=\begin{pmatrix} a_i^{21} \\ a_i^{31} \\ \vdots \\ a_i^{n1} \end{pmatrix} \qquad (i=0, 1,\dots, q-1),$$

and $\hat{A}_i^1$ ($i=0, 1,\dots, q-1$) denote ($n-1$) by ($n-1$) matrices constructed by the remaining elements of the matrices A_i ($i=0, 1,\dots, q-1$). Moreover we put

$$\mathscr{A}_i^1=\Lambda^1(\hat{A}_i^1-\alpha_i^1) \qquad (i=1, 2,\dots, q-1),$$

$$\mathscr{A}_0^1=\Lambda^1(\hat{A}_0^1-\mu_1),$$

and

$$\eta_i^1=\Lambda^1\gamma_i^1 \qquad (i=0, 1,\dots, q-1).$$

Then we can rewrite the linear difference equation (4.1) in the form

$$\begin{aligned} (4.3) \qquad & (a_{q-1}^{11}-\alpha_{q-1}^1)h_1^1(s+q-1)+(a_{q-2}^{11}-\alpha_{q-2}^1)h_1^1(s+q-2)+\cdots \\ & +(a_0^{11}-\mu_1+s)h_1^1(s)+\beta_{q-1}^1\hat{H}^1(s+q-1)+\beta_{q-2}^1\hat{H}^1(s+q-2)+ \\ & \cdots+\beta_0^1\hat{H}^1(s)=0, \end{aligned}$$

$$\begin{aligned}(4.4)\qquad \hat{H}^1(s+q) &= \mathscr{A}^1_{q-1}\hat{H}^1(s+q-1)+\eta^1_{q-1}h^1_1(s+q-1)\\ &+\mathscr{A}^1_{q-2}\hat{H}^1(s+q-2)+\eta^1_{q-2}h^1_1(s+q-2)\\ &+\\ &\vdots\\ &+\mathscr{A}^1_1 H^1(s+1)+\eta^1_1 h^1_1(s+1)\\ &+(\mathscr{A}^1_0+\Lambda^1 s)\hat{H}^1(s)+\eta^1_0 h^1_1(s).\end{aligned}$$

Substituting the second formula (4.4) into the first formula (4.3) one after another, we obtain a relation between $h^1_1(s-r)(0\leqq r\leqq q)$ and $\hat{H}^1(s-r)(1\leqq r\leqq q)$, together with the formulas to determine the characteristic constants $\alpha^1_{q-1},\dots,\alpha^1_1$ and μ_1. In fact, dropping the index 1 from now on, we have

$$\begin{aligned}&\beta_{q-1}\hat{H}(s+q-1)+\beta_{q-2}\hat{H}(s+q-2)+\cdots+\beta_0\hat{H}(s)\\ &=\beta_{q-1}\{\sum_{r=1}^{q-1}\mathscr{A}_{q-r}\hat{H}(s+q-1-r)+(\mathscr{A}_0+\Lambda(s-1))\hat{H}(s-1)\\ &\quad+\sum_{r=1}^{q}\eta_{q-r}h(s+q-1-r)\}+\beta_{q-2}\hat{H}(s+q-2)+\cdots+\beta_0\hat{H}(s)\\ &=\sum_{r=1}^{q-1}(\beta_{q-1}\mathscr{A}_{q-r}+\beta_{q-1-r})\hat{H}(s+q-1-r)\\ &\quad+\beta_{q-1}(\mathscr{A}_0+\Lambda(s-1))\hat{H}(s-1)\\ &\quad+\sum_{r=1}^{q-1}\beta_{q-1}\eta_{q-r}h(s+q-1-r)+\beta_{q-1}\eta_0 h(s-1)\end{aligned}$$

and then replace the term of the highest order $\hat{H}(s+q-2)$ by the right hand member of (4.4) again. If ν times above procedures lead to

$$\begin{aligned}(4.5)\qquad &\beta_{q-1}\hat{H}(s+q-1)+\beta_{q-2}\hat{H}(s+q-2)+\cdots+\beta_0\hat{H}(s)\\ &=\sum_{r=1}^{q-\nu}P(\nu: r)\hat{H}(s+q-\nu-r)+\sum_{r=1}^{\nu}Q(\nu: r: s)\hat{H}(s-r)\\ &\quad+\sum_{r=1}^{q-1}R_1(\nu: r)h(s-1+r)+\sum_{r=1}^{\nu}R_2(\nu: r)h(s-r),\end{aligned}$$

where $P(\nu: r)$ are $(n-1)$-dimensional row vectors not depending on s, $Q(\nu: r: s)$ are $(n-1)$-dimensional row vectors depending on s and $R_i(\nu: r)(i=1, 2)$ are scalar constants, then we have the following recurrence relations: For $0\leqq r\leqq q-1$,

$$(4.6)\qquad P(\nu+1: r)=P(\nu: 1)\mathscr{A}_{q-r}+P(\nu: r+1)\qquad (1\leqq r\leqq q-\nu-1),$$

$$(4.7)\quad \begin{cases} Q(\nu+1: r: s) = P(\nu: 1)\mathscr{A}_{\nu+1-r} + Q(\nu: r: s) & (1 \leqq r \leqq \nu), \\ Q(\nu+1: \nu+1: s) = P(\nu: 1)(\mathscr{A}_0 + \Lambda(s-\nu-1)), \end{cases}$$

$$(4.8)\quad \begin{cases} R_1(\nu+1: r) = P(\nu: 1)\eta_{\nu+r} + R_1(\nu: r) & (1 \leqq r \leqq q-\nu-1), \\ R_1(\nu+1: r) = R_1(\nu: r) & (q-\nu \leqq r \leqq q-1), \end{cases}$$

$$(4.9)\quad \begin{cases} R_2(\nu+1: r) = P(\nu: 1)\eta_{\nu+1-r} + R_2(\nu: r) & (1 \leqq r \leqq \nu), \\ R_2(\nu+1: \nu+1) = P(\nu: 1)\eta_0, \end{cases}$$

where the formulas have no meaning when the relations between ν and r in the round parentheses do not hold.

From the above recurrence formulas, putting

$$P(0: 1) = \beta_{q-1}, \quad P(0: 2) = \beta_{q-2}, \ldots, P(0: q) = \beta_0$$

and

$$R_1(0: 1) = R_1(0: 2) = \cdots = R_1(0: q-1) = 0,$$

we can successively evaluate the values of $P(\nu: r)$, $Q(\nu: r: s)$, $R_1(\nu: r)$ and $R_2(\nu: r)$ for $1 \leqq \nu \leqq q$, thereby finally obtaining the formula

$$(4.10)\quad \beta_{q-1}\hat{H}(s+q-1) + \beta_{q-2}\hat{H}(s+q-2) + \cdots + \beta_0\hat{H}(s)$$
$$= \sum_{r=1}^{q} Q(q: r: s)\hat{H}(s-r) + \sum_{r=1}^{q-1} R_1(q: r)h(s-1+r) + \sum_{r=1}^{q} R_2(q: r)h(s-r).$$

Consequently, substituting (4.10) into (4.3), we have

$$(4.11)\quad \begin{aligned} &(a_{q-1} - \alpha_{q-1})h(s+q-1) \\ &+ (a_{q-2} - \alpha_{q-2} + R_1(q: q-1))h(s+q-2) \\ &+ \\ &\vdots \\ &+ (a_0 - \mu + R_1(q: 1) + s)h(s) \\ &+ \sum_{r=1}^{q} Q(q: r: s)\hat{H}(s-r) + \sum_{r=1}^{q} R_2(q: r)h(s-r) = 0. \end{aligned}$$

Considering the initial conditions that $h(0)=1$, $h(r)=0$ $(r<0)$ and $\hat{H}(r)=0$ $(r \leqq 0)$, we immediately obtain the formulas to determine the characteristic constants $\alpha_{q-1}, \ldots, \alpha_1$ and μ as follows:

$$(4.12)\qquad \begin{cases} \alpha_{q-1} = a_{q-1} \\ \alpha_{q-2} = a_{q-2} + R_1(q: q-1), \\ \vdots \\ \alpha_1 = a_1 + R_1(q: 2), \\ \mu = a_0 + R_1(q: 1). \end{cases}$$

These relations then yield the formula

$$(4.13)\qquad sh(s) + \sum_{r=1}^{q} Q(q: r: s)\hat{H}(s-r) + \sum_{r=1}^{q} R_2(q: r)h(s-r) = 0.$$

Combining (4.13) with (4.4), we have thus obtained the required system of linear difference equations of a regular type:

$$(4.14)\qquad H(s) = B_{q-1}(s)H(s-1) + B_{q-2}(s)H(s-2) + \cdots + B_0(s)H(s-q),$$

where

$$(4.15)\qquad B_{q-r}(s) = \begin{pmatrix} -\frac{1}{s}R_2(q: r) & -\frac{1}{s}Q(q: r: s) \\ \eta_{q-r} & \mathscr{A}_{q-r} \end{pmatrix} \qquad (1 \leqq r \leqq q-1),$$

$$(4.16)\qquad B_0(s) = \begin{pmatrix} -\frac{1}{s}R_2(q: q) & -\frac{1}{s}Q(q: q: s) \\ \eta_0 & \mathscr{A}_0 + \Lambda(s-q) \end{pmatrix}.$$

Now, in order to derive the estimate of $H(s)$, we shall investigate the behaviors of $Q(q: r: s)(1 \leqq r \leqq q)$ for sufficiently large positive integral values of s. We can prove the following

LEMMA 4.1.

$$(4.17)\qquad \lim_{s\to\infty} \frac{1}{s}Q(q: r: s) = P(r-1: 1)\Lambda \qquad (1 \leqq r \leqq q)$$

hold.

PROOF. Since $P(v: 1)(0 \leqq v \leqq q-1)$ are constant vectors, it immediately follows from (4.7) that

$$(4.18)\qquad \lim_{s\to\infty} \frac{1}{s}Q(v+1: r: s) = \lim_{s\to\infty} \frac{1}{s}Q(v: r: s) \qquad (1 \leqq r \leqq v)$$

and

$$(4.19)\qquad \lim_{s\to\infty} \frac{1}{s}Q(v+1: v+1: s) = P(v: 1)\Lambda \qquad (0 \leqq v \leqq q-1).$$

Then, applying (4.18) in the first place and (4.19) in the next place, we have the required formula (4.17) as follows:

$$\lim_{s\to\infty}\frac{1}{s}Q(q\colon r\colon s) = \lim_{s\to\infty}\frac{1}{s}Q(q-1\colon r\colon s)$$
$$= \lim_{s\to\infty}\frac{1}{s}Q(q-2\colon r\colon s)$$
$$= \vdots$$
$$= \lim_{s\to\infty}\frac{1}{s}Q(r\colon r\colon s) = P(r-1\colon 1)\Lambda \qquad (1 \leqq r \leqq q).$$

We here define the norm of an n by m matrix $A=(a^{ij})$ by

$$\|A\| = \max_{1\leqq i\leqq n}\{\sum_{j=1}^{m}|a^{ij}|\}.$$

Then, according to Lemma 4.1, we have for $s\geqq N>q$, N being a sufficiently large positive integer,

$$\|B_{q-r}(s)\| \leqq \|\mathscr{B}_{q-r}\| + 1 = b_{q-r} \qquad (1 \leqq r \leqq q-1), \tag{4.20}$$

where we have put

$$\lim_{s\to\infty} B_{q-r}(s) = \mathscr{B}_{q-r} = \begin{pmatrix} 0 & -P(r-1\colon 1)\Lambda \\ \eta_{q-r} & \mathscr{A}_{q-r} \end{pmatrix} \qquad (1 \leqq r \leqq q-1).$$

In particular, since

$$B_0(s) = \begin{pmatrix} O\left(\frac{1}{s}\right) & -P(q-1\colon 1)\Lambda + O\left(\frac{1}{s}\right) \\ \eta_0 & (s-q)\Lambda + \mathscr{A}_0 \end{pmatrix} \qquad \text{as} \quad s \longrightarrow \infty,$$

we have for $s\geqq N>q$

$$\|B_0(s)\| \leqq (s-q)|\hat{\lambda}_1 - \lambda_1|^{-1} + \|A_0\|, \tag{4.21}$$

where

$$|\hat{\lambda}_1 - \lambda_1| = \min_{j\neq 1}|\lambda_j - \lambda_1| > 0. \tag{4.22}$$

Taking account of the estimates (4.20) and (4.21), we obtain from (4.14)

$$\|H(s)\| \leqq b_{q-1}\|H(s-1)\| + b_{q-2}\|H(s-2)\| + \cdots + b_1\|H(s-q+1)\| \tag{4.23}$$

$$+ ((s-q)|\hat{\lambda}_1 - \lambda_1|^{-1} + \|A_0\|)\|H(s-q)\| \qquad (s \geqq N)$$

and hence we have only to consider the growth order of a solution of the single linear difference equation

$$\mathscr{H}(s) = b_{q-1}\mathscr{H}(s-1) + b_{q-2}\mathscr{H}(s-2) + \cdots + b_1\mathscr{H}(s-q+1) + ((s-q)|\hat{\lambda}_1 - \lambda_1|^{-1} + \|A_0\|)\mathscr{H}(s-q), \tag{4.24}$$

subject to the initial conditions

$$\mathscr{H}(N-1) = \|H(N-1)\|,\ \mathscr{H}(N-2) = \|H(N-2)\|, \dots, \mathscr{H}(N-q) = \|H(N-q)\| \tag{4.25}$$

since we easily see from (4.23) and (4.25) that

$$\|H(s)\| \leqq \mathscr{H}(s) \qquad (s \geqq N). \tag{4.26}$$

Obviously, the linear difference equation (4.24) is of the Perron-Poincaré type (see [16, 17]) and the corresponding Newton-Puiseux polygon constructed by the coordinates (0, 0), (1, 0),..., $(q-1, 0)$ and $(q, 1)$ is a straight line with the directional coefficient $1/q$. Applying O. Perron-H. Poincaré's theorem to (4.24), we have

$$\overline{\lim_{s\to\infty}} \left| \frac{\mathscr{H}(s)}{\Gamma(s+1)^{1/q}} \right|^{1/s} = |\gamma|, \tag{4.27}$$

where γ is a root of the algebraic equation

$$\gamma^q - |\hat{\lambda}_1 - \lambda_1|^{-1} = 0, \quad \text{i.e.,}$$

$$|\gamma| = |\hat{\lambda}_1 - \lambda_1|^{-1/q}. \tag{4.28}$$

Combining (4.26) and (4.27–8), we therefore obtain the required growth order of $H(s)$.

We state the result derived above in the form of

THEOREM 4.1. *The coefficients* $H^k(s)$ $(k=1, 2,\dots, n)$ *of the formal solutions* (1.7) *have the following growth order*:

$$\overline{\lim_{s\to\infty}} \left(\frac{\|H^k(s)\|}{|\Gamma(s+1)^{1/q}|} \right)^{1/s} \leqq \frac{1}{|\hat{\lambda}_k - \lambda_k|^{1/q}} \qquad (k = 1, 2, \dots, n), \tag{4.29}$$

where

$$|\hat{\lambda}_k - \lambda_k| = \min_{j \neq k} |\lambda_j - \lambda_k| > 0. \tag{4.30}$$

§ 5. Connection problem for a system of differential equations with multiple characteristic constants at a regular singularity

In this section we shall solve the two point connection problem for the system of linear ordinary differential equations (1.1). For each k $(k=1, 2,\dots, n)$, let $\Phi_l^k(m)\,(l=1, 2,\dots, q)$ be a fundamental set of solutions of the system of linear difference equations (1.21) and then take $\mathscr{G}_l^k(m)\,(l=1, 2,\dots, q)$ defined in association with $\Phi_l^k(m)$ in Section 2 as a fundamental set of solutions of the system of linear difference equations (1.20). Let us denote

$$(5.1)\qquad \mathscr{G}_l^k(m) = \exp\left(J\frac{d}{dm}\right)g_l^{k1}(m)$$

$$= g_l^{k1}(m) + g_l^{k2}(m)J + \cdots + g_l^{kn}(m)J^{n-1}$$

$$(k = 1, 2,\dots, n:\ l = 1, 2,\dots, q).$$

Now we shall begin with the definition of column vectorial functions $F_l^{ki}(m)$ expressed in terms of the series

$$(5.2)\qquad F_l^{ki}(m) = \sum_{s=0}^{\infty} H^k(s)g_l^{ki}(m+s) \qquad (i,\ k = 1, 2,\dots, n;\ l = 1, 2,\dots, q),$$

where $H^k(s)\,(k=1, 2,\dots, n)$ are the coefficients of formal solutions (1.7). In order to prove the well-definedness of the functions $F_l^{ki}(m)$, that is, the convergence of the series in the right hand side of (5.2), we need exact informations on the asymptotic behaviors of the functions $g_l^{ki}(m)$ for sufficiently large values of m in a sector including the positive real axis. Taking account of (2.5), (2.14) and (2.20) and applying the theorem of termwise differentiation of an asymptotic expansion, we have

$$(5.3)\quad g_l^{ki}(m) \sim (\lambda_k^{-1/q}\omega^{l-1})^{-m}\left(\log \lambda_k^{1/q}\omega^{1-l} - \frac{1}{q}\log m\right)^{i-1}$$

$$\times \exp\left\{-\frac{m}{q}\log m + \frac{m}{q} - \frac{1}{2}\log m + mO(m^{-1/q})\right\}\{d_l^{ki} + O(m^{-1/q})\}$$

$$(i,\ k = 1, 2,\dots, n;\ l = 1, 2,\dots, q)$$

for sufficiently large values of m in the sector $|\arg(m+\rho-\mu_k)|<\pi-\delta'$, δ' being a small positive number larger than δ in (2.5), where d_l^{ki} $(i, k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ are constants. Moreover it follows from (5.3) that for an arbitrary number r, there hold in the above sector

$$(5.4)\qquad \frac{g_l^{ki}(m+r)}{g_l^{ki}(m)} \sim (\lambda_k^{-1/q}\omega^{l-1})^{-r}m^{-r/q}\{1 + O(m^{-1/q})\} \quad \text{as} \quad m \longrightarrow \infty$$

$$(i,\ k=1,2,\dots,n;\ l=1,2,\dots,q).$$

By means of Abel's transformation, using the asymptotic relations (5.4), we can prove the following lemma in exactly the same manner as in the paper [10] and we here omit its proof.

LEMMA 5.1. *Suppose that the series which define the functions* $F_l^{ki}(m_0)$ *for a certain number* m_0 *are absolutely convergent. Then the series defining* $F_l^{ki}(m)$ *are also absolutely convergent in the right half-plane* $\mathrm{Re}\, m \geqq \mathrm{Re}\, m_0+\varepsilon$, ε *being an arbitrarily small positive number. Moreover we have the asymptotic relations*

$$(5.5)\qquad F_l^{ki}(m) \sim H^k(0) g_l^{ki}(m)\{1+O(m^{-1/q})\}$$

$$(i,\ k=1,2,\dots,n;\ l=1,2,\dots,q)$$

for sufficiently large values of m in that right half-plane.

From this lemma, using Theorem 4.1 and (5.3), we can show the well-definedness of $F_l^{ki}(m)$ as follows:

THEOREM 5.1. *Let* m_0 *be an arbitrary number. Then, under the assumption that*

$$(5.6)\qquad 0<|\lambda_k|<|\lambda_j-\lambda_k| \qquad (i \neq k;\ j=1,2,\dots,n),$$

the functions $F_l^{ki}(m)(i,\ k=1,2,\dots,n;\ l=1,2,\dots,q)$ *are well-defined in the right half-plane* $\mathrm{Re}\, m > \mathrm{Re}\, m_0$.

PROOF. Considering Lemma 5.1, we here have only to prove the absolute convergence of the series defining $F_l^{ki}(m_0)$. In order to apply Cauchy's test to these series, we evalutate the values of

$$(5.7)\qquad \overline{\lim_{s\to\infty}}\,(\|H^k(s)\|\,|g_l^{ki}(s+m_0)|)^{1/s}$$

$$\leqq \overline{\lim_{s\to\infty}}\left(\frac{\|H^k(s)\|}{|\Gamma(s+1)^{1/q}|}\right)^{1/s} \overline{\lim_{s\to\infty}}\,(|g_l^{ki}(s+m_0)|\,|\Gamma(s+1)^{1/q}|)^{1/s}.$$

From (5.3) and the asymptotic behavior of the gamma function, we have for sufficiently large s

$$g_l^{ki}(s+m_0)\Gamma(s+1)^{1/q} \sim (\lambda_k^{-1/q}\omega^{l-1})^{-s}\left(\log \lambda_k^{1/q}\omega^{1-l} - \frac{1}{q}\log(s+m_0)\right)^{i-1}$$

$$\times \exp\left\{-\frac{s}{q}\log s + \frac{s}{q} - \left(\frac{m_0}{q}+\frac{1}{2}\right)\log s + sO(s^{-1/q})\right\}$$

$$\times \exp\left\{\frac{1}{q}\left(s+\frac{1}{2}\right)\log s - \frac{s}{q}\right\}\{\hat{d}_l^{ki} + O(s^{-1/q})\}$$

$$\sim (\lambda_k^{-1/q}\omega^{l-1})^{-s}\left(\log \lambda_k^{1/q}\omega^{1-l} - \frac{1}{q}\log(s+m_0)\right)^{i-1}$$

$$\times \exp\left\{\left(\frac{1}{2q} - \frac{m_0}{q} - \frac{1}{2}\right)\log s + sO(s^{-1/q})\right\}\{\hat{d}_l^{ki} + O(s^{-1/q})\},$$

$\hat{d}_l^{ki}$ being a constant, which implies that

$$\text{(5.8)} \qquad \overline{\lim_{s\to\infty}}\,(|g_l^{ki}(s+m_0)||\Gamma(s+1)^{1/q}|)^{1/s} \leqq |\lambda_k^{1/q}\omega^{1-l}| = |\lambda_k|^{1/q}.$$

Combining (5.8) and (4.29), we therefore obtain

$$\text{(5.9)} \qquad \overline{\lim_{s\to\infty}}\,(\|H^k(s)\||g_l^{ki}(s+m_0)|)^{1/s} \leqq \frac{|\lambda_k|^{1/q}}{|\hat{\lambda}_k - \lambda_k|^{1/q}} < 1$$

from the assumption (5.6) and the definition of $|\hat{\lambda}_k - \lambda_k|$ as stated in (4.30) and thus we have completed the proof of Theorem 5.1.

From now on, we let the variable m take only integral values. We first obtain the following important theorem.

THEOREM 5.2. *Assume that $\rho-\mu_k \neq$ an integer $(k=1, 2,\dots, n)$. Then the functions $F_l^{k1}(m)(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ form a fundamental set of solutions of the linear difference equation* (1.12) *for $m \geqq -q+1$. From this fact, the coefficients $G_1(m)$ can be expressed in terms of the linear combination*

$$\text{(5.10)} \qquad G_1(m) = \sum_{k=1}^{n}\sum_{l=1}^{q} T_l^{k1}F_l^{k1}(m),$$

where the constant coefficients $T_l^{k1}(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ are determined by the system of linear equations

$$\text{(5.11)} \quad G_1(r) = \sum_{k=1}^{n}\sum_{l=1}^{q} T_l^{k1}F_l^{k1}(r) \qquad (r = -q+1,\ -q+2,\dots,\ -1,\ 0).$$

PROOF. From (1.14) and (2.18) we easily see that

$$\text{(5.12)}\ (m+\rho-A_0)F_l^{k1}(m)$$

$$= \sum_{s=0}^{\infty}\{(m+s+\rho-\mu_k) + (\mu_k - s - A_0)\}H^k(s)g_l^{k1}(m+s)$$

$$= \sum_{s=0}^{\infty} H^k(s)\{\alpha_1^k g_l^{k1}(m+s-1) + \alpha_2^k g_l^{k1}(m+s-2) + \cdots + \lambda_k g_l^{k1}(m+s-q)\}$$

$$+\sum_{s=-q}^{\infty}\{(A_q-\lambda_k)H^k(s+q)+(A_{q-1}-\alpha_{q-1}^k)H^k(s+q-1)+\cdots$$
$$+(A_1-\alpha_1^k)H^k(s+1)\}g_l^{k1}(m+s)$$
$$=A_q\sum_{s=0}^{\infty}H^k(s)g_l^{k1}(m-q+s)+A_{q-1}\sum_{s=0}^{\infty}H^k(s)g_l^{k1}(m-q+1+s)$$
$$+\cdots+A_1\sum_{s=0}^{\infty}H^k(s)g_l^{k1}(m-1+s)$$
$$=A_qF_l^{k1}(m-q)+A_{q-1}F_l^{k1}(m-q+1)+\cdots+A_1F_l^{k1}(m-1),$$

where we have used the initial conditions that $H^k(r)=0$ for $r<0$. This implies that $F_l^{k1}(m)\,(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ are particular solutions of the system of linear difference equations (1.12). In order to prove that those particular solutions form a fundamental set of solutions of (1.12), we have only to show the nonvanishing of the Casorati determinant constructed from them:

(5.13) $$\mathscr{C}_F(m)=$$
$$\begin{vmatrix} F_1^{11}(m) & \cdots F_q^{11}(m) & F_1^{21}(m) & \cdots & F_q^{21}(m) & \cdots F_1^{n1}(m) & \cdots F_q^{n1}(m) \\ F_1^{11}(m+1) & \cdots F_q^{11}(m+1) & F_1^{21}(m+1) & \cdots & F_q^{21}(m+1) & \cdots F_1^{n1}(m+1) & \cdots F_q^{n1}(m+1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ F_1^{11}(m+q-1) & \cdots F_q^{11}(m+q-1) & F_q^{21}(m+q-1) & \cdots & F_q^{21}(m+q-1) & \cdots F_1^{n1}(m+q-1) & \cdots F_q^{n1}(m+q-1) \end{vmatrix}.$$

It is easily verified from (5.12) that the Casorati determinant satisfies the first order linear difference equation

(5.14) $$(m+q-1)^n\mathscr{C}_F(m)=(-1)^{n^2(q-1)}\prod_{k=1}^{n}\lambda_k\mathscr{C}_F(m-1),$$

whence we have

(5.15) $$\mathscr{C}_F(m)=\frac{(-1)^{n^2(q-1)(m+q-1)}(\prod_{k=1}^{n}\lambda_k)^{m+q-1}}{(\Gamma(m+q))^n}\mathscr{C}_F(-q+1).$$

From (5.14), if we could prove that $\mathscr{C}_F(m)\neq 0$ for a certain value of m, then we have $\mathscr{C}_F(m-1)\neq 0$ and similarly, by successive applications of (5.14), we finally obtain $\mathscr{C}_F(m)\neq 0$ for $m\geqq -q+1$.

Let m be a sufficiently large number. Then it follows from (5.5) that

$$(5.16)\qquad \mathscr{C}_F(m) \sim$$

$$\begin{vmatrix} H^1(0)g_1^{11}(m) & \cdots H^1(0)g_q^{11}(m) & H^2(0)g_1^{21}(m) & \cdots H^2(0)g_q^{21}(m) & \cdots H^n(0)g_1^{n1}(m) & \cdots H^n(0)g_q^{n1}(m) \\ H^1(0)g_1^{11}(m+1) & \cdots H^1(0)g_q^{11}(m+1) & H^2(0)g_1^{21}(m+1) & \cdots H^2(0)g_q^{21}(m+1) & \cdots H^n(0)g_1^{n1}(m+1) & \cdots H^n(0)g_q^{n1}(m+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H^1(0)g_1^{11}(m+q-1) & \cdots H^1(0)g_q^{11}(m+q-1) & H^2(0)g_1^{21}(m+q-1) & \cdots H^2(0)g_q^{21}(m+q-1) & \cdots H^n(0)g_1^{n1}(m+q-1) & \cdots H^n(0)g_q^{n1}(m+q-1) \end{vmatrix}$$

$$\times \{1 + O(m^{-1/q})\}$$

$$\sim (-1)^{\frac{n(n-1)}{2}\,\frac{q(q-1)}{2}}$$

$$\begin{vmatrix} g_1^{11}(m) & \cdots g_q^{11}(m) & & & & & \\ g_1^{11}(m+1) & \cdots g_q^{11}(m+1) & & & & \text{\Large 0} & \\ \vdots & \vdots & & & & & \\ g_1^{11}(m+q-1) & \cdots g_q^{11}(m+q-1) & & & & & \\ & & g_1^{21}(m) & \cdots g_q^{21}(m) & & & \\ & & g_1^{21}(m+1) & \cdots g_q^{21}(m+1) & & & \\ & & \vdots & \vdots & & & \\ & & g_1^{21}(m+q-1) & \cdots g_q^{12}(m+q-1) & & & \\ & & & & \ddots & & \\ & & & & & g_1^{n1}(m) & \cdots g_q^{n1}(m) \\ & \text{\Large 0} & & & & g_1^{n1}(m+1) & \cdots g_q^{n1}(m+1) \\ & & & & & \vdots & \vdots \\ & & & & & g_1^{n1}(m+q-1) & \cdots g_q^{n1}(m+q-1) \end{vmatrix}$$

$$\times \{1 + O(m^{-1/q})\}$$

$$\sim (-1)^{\frac{n(n-1)}{2}\,\frac{q(q-1)}{2}} \mathscr{C}_{g^1}(m)\mathscr{C}_{g^2}(m)\cdots\mathscr{C}_{g^n}(m)\{1 + O(m^{-1/q})\}.$$

Since for each k, $g_l^{k1}(m)\,(l=1, 2,\ldots, q)$ form a fundamental set of solutions of the linear difference equation

$$(5.17)\qquad (m + \rho - \mu_k)g^{k1}(m)$$
$$= \alpha_1^k g^{k1}(m-1) + \cdots + \alpha_{q-1}^k g^{k1}(m-q+1) + \lambda_k g^{k1}(m-q),$$

the Casorati determinant $\mathscr{C}_{g^k}(m)$ does not vanish. We therefore conclude that $\mathscr{C}_F(m)\neq 0$ for a sufficiently large value of m.

Moreover we can calculate the exact value of $\mathscr{C}_F(-q+1)$, together with obtaining an invariant relation between the characteristic constants μ_k and ρ. In fact, we see again from (5.17) that the Casorati determinant $\mathscr{C}_{g^k}(m)$ satisfies the first order linear difference equation

$$(5.18)\qquad (m+q-1+\rho-\mu_k)\mathscr{C}_{g^k}(m)=(-1)^{q-1}\lambda_k\mathscr{C}_{g^k}(m-1),$$

which implies that

$$(5.19)\qquad \mathscr{C}_{g^k}(m)=\frac{((-1)^{q-1}\lambda_k)^m\Gamma(q+\rho-\mu_k)}{\Gamma(m+q+\rho-\mu_k)}\mathscr{C}_{g^k}(0).$$

On the other hand, from the asymptotic behaviors of $g_l^{k1}(m)(l=1,2,\dots,q)$ we have for sufficiently large values of m

$$(5.20)\qquad \mathscr{C}_{g^k}(m)\sim\prod_{l=1}^{q}g_l^{k1}(m)$$

$$\times\begin{vmatrix} 1 & 1 & \cdots & 1\\ \lambda_k^{1/q}m^{-1/q} & \lambda_k^{1/q}\omega^{-1}m^{-1/q} & \cdots & \lambda_k^{1/q}\omega^{-(q-1)}m^{-1/q}\\ \vdots & \vdots & & \vdots\\ \lambda_k^{(q-1)/q}m^{-(q-1)/q} & \lambda_k^{(q-1)/q}\omega^{-(q-1)}m^{-(q-1)/q} & \cdots & \lambda_k^{(q-1)/q}\omega^{-(q-1)(q-1)}m^{-(q-1)/q}\end{vmatrix}$$

$$\times\{1+O(m^{-1/q})\}$$

$$\sim\Big(\prod_{l=1}^{q}g_l^{k1}(m)\Big)\lambda_k^{(q-1)/2}m^{-(q-1)/2}V_q(1,\omega^{-1},\dots,\omega^{-(q-1)})\{1+O(m^{-1/q})\},$$

where $V_q(x_1,x_2,\dots,x_q)$ denotes the so-called Vandermonde determinant, i.e.,

$$V_q(x_1,x_2,\dots,x_q)=\begin{vmatrix}1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_q\\ \vdots & \vdots & & \vdots\\ x_1^{q-1} & x_2^{q-1} & \cdots & x_q^{q-1}\end{vmatrix}.$$

Since we moreover obtain

$$(5.21)\qquad \prod_{l=1}^{q}g_l^{k1}(m)\sim\Big(\frac{\lambda_k^{-1/q}}{q}\Big)^q\omega^{q(q-1)/2}m^{-(q-1)}\Big(\prod_{l=1}^{q}\frac{1}{\phi_l^{k1}(m+1)}\Big)\{1+O(m^{-1/q})\}$$

$$\sim\Big(\frac{\lambda_k^{-1/q}}{q}\Big)^q\omega^{q(q-1)/2}m^{-(q-1)}\Big(\frac{2\pi}{q}\Big)^{-q/2}(\lambda_k^{-1}\omega^{q(q-1)/2})^{-(m+1-\mu_k+\rho)}$$

$$\times\exp\Big\{-\Big(m+1-\mu_k+\rho-\frac{q}{2}\Big)\log m+m+R(\lambda_k,\alpha_{q-1}^k,\dots,\alpha_1^k)\Big\}$$

$$\times \{1 + O(m^{-1/q})\},$$

where $R(\lambda_k, \alpha_{q-1}^k, \dots, \alpha_1^k)$ is a constant (see [10]), we can evaluate the exact value of $\mathscr{C}_{g^k}(0)$, letting $m \to \infty$ in (5.19), as follows:

$$(5.22) \quad \mathscr{C}_{g^k}(0) = \frac{1}{\Gamma(q+\rho-\mu_k)} \frac{\Gamma(m+q+\rho-\mu_k)}{((-1)^{q-1}\lambda_k)^m} \mathscr{C}_{g^k}(m)$$

$$\sim \frac{(2\pi q)^{-q/2}}{\Gamma(q+\rho-\mu_k)} (\lambda_k^{-1}\omega^{q(q-1)/2})^{\mu_k-\rho} \lambda_k^{(q-1)/2} V_q(1, \omega^{-1}, \dots, \omega^{-(q-1)})$$

$$\times \exp(R(\lambda_k, \alpha_{q-1}^k, \dots, \alpha_1^k)) m^{-3(q-1)/2} \Gamma(m+q+\rho-\mu_k)$$

$$\times \exp\left\{-(m+1-\mu_k+\rho-\frac{q}{2})\log m + m\right\}\{1 + O(m^{-1/q})\}$$

$$\sim \frac{(2\pi)^{-(q-1)/2} q^{-q/2}}{\Gamma(q+\rho-\mu_k)} \lambda_k^{(q-1)/2+\rho-\mu_k} \omega^{q(q-1)(\mu_k-\rho)/2}$$

$$\times V_q(1, \omega^{-1}, \dots, \omega^{-(q-1)}) \exp(R(\lambda_k, \alpha_{q-1}^k, \dots, \alpha_1^k))\{1 + O(m^{-1/p})\},$$

which implies that $\mathscr{C}_{g^k}(0)$ is equal to the constant in the last expression. Then, taking account of (5.16), (5.19) and letting $m \to \infty$ in (5.15), we have

$$(5.23) \quad \mathscr{C}_F(-q+1)$$

$$\sim \frac{(\Gamma(m+q))^n (-1)^{\frac{n(n-1)}{2}\frac{q(q-1)}{2}} \mathscr{C}_{g^1}(m)\mathscr{C}_{g^2}(m)\cdots\mathscr{C}_{g^n}(m)}{(-1)^{n^2(q-1)(m+q-1)}(\prod_{k=1}^n \lambda_k)^{m+q-1}}\{1 + O(m^{-1/q})\}$$

$$\sim (-1)^{\frac{n(n-1)}{2}\frac{q(q-1)}{2} - n^2(q-1)^2}\left\{\prod_{k=1}^n \frac{\Gamma(q+\rho-\mu_k)}{\lambda_k^{q-1}} \mathscr{C}_{g^k}(0)\right\}(-1)^{-n(n-1)(q-1)m}$$

$$\times \left\{\prod_{k=1}^n \frac{\Gamma(m+q)}{\Gamma(m+q+\rho-\mu_k)}\right\}\{1 + O(m^{-1/q})\}$$

$$\sim (-1)^{\frac{n(n-1)}{2}\frac{q(q-1)}{2} - n^2(q-1)^2}\left\{\prod_{k=1}^n \frac{\Gamma(q+\rho-\mu_k)}{\lambda_k^{q-1}} \mathscr{C}_{g^k}(0)\right\} m^{-\sum_{k=1}^n(\rho-\mu_k)}$$

$$\times \{1 + O(m^{-1/q})\}.$$

This implies that

$$(5.24) \qquad \sum_{k=1}^n \mu_k = n\rho$$

and

(5.25)
$$\mathscr{C}_F(-q+1) = (-1)^{\frac{n(n-1)}{2}\frac{q(q-1)}{2}-n^2(q-1)^2}\left\{\prod_{k=1}^{n}\frac{\Gamma(q+\rho-\mu_k)}{\lambda_k^{q-1}}\mathscr{C}_{g^k}(0)\right\}$$

since the Casorati determinant $\mathscr{C}_F(-q+1)$ is a nonzero constant not depending on m. The invariant identity (5.24) will also be obtained from (4.12), considering the formulas (4.6–9), although the algebraic calculation is not easy.

Now, from the theory of linear difference equations, we have

(5.26)
$$G_1(m) = \sum_{k=1}^{n}\sum_{l=1}^{q} T_l^{k1}(m)F_l^{k1}(m),$$

where $T_l^{k1}(m)(k=1, 2,..., n; l=1, 2,..., q)$ are periodic functions of period 1. However, since the variable m takes only integral values, we may regard $T_l^{k1}(m)$ as constant coefficients and from the fact that $\mathscr{C}_F(-q+1)\neq 0$, we can determine the constant coefficients $T_l^{k1}(k=1, 2,..., n; l=1, 2,..., q)$ by the linear equation (5.11) subject to the initial conditions that $G_1(0)=1$ and $G_1(r)=0$ for $r=-q+1, -q+2,..., -1$. Thus the proof of Theorem 5.2 is completed.

Next we consider the functions $F_l^{ki}(m)(k=1, 2,..., n; l=1, 2,..., q)$ for $i=2, 3,..., n$. Simple calculations similar to (5.12) lead to

(5.27)
$$(m+\rho-A_0)F_l^{ki}(m) = A_1F_l^{ki}(m-1)+\cdots+A_qF_l^{ki}(m-q)-F_l^{ki-1}(m) \quad (i=2, 3,..., n).$$

Multiplying both sides of

$$(m+\rho-A_0)F_l^{k2}(m) = A_1F_l^{k2}(m-1)+\cdots+A_qF_l^{k2}(m-q)-F_l^{k1}(m)$$

by T_l^{k1} and summing over k and l from 1 to n and from 1 to q, respectively, we have from (5.10)

$$(m+\rho-A_0)\hat{G}_2(m) = A_1\hat{G}_2(m-1)+\cdots+A_q\hat{G}_2(m-q)-G_1(m),$$

where we have put

$$\hat{G}_2(m) = \sum_{k=1}^{n}\sum_{l=1}^{q} T_l^{k1}F_l^{k2}(m),$$

which implies that $\hat{G}_2(m)$ is a particular solution of the nonhomogeneous linear difference equation (1.13) for $j=2$. Hence, from the general theory of linear difference equations, the coefficients $G_2(m)$ can be expressed in the form

$$G_2(m) = \hat{G}_2(m) + \sum_{k=1}^{n}\sum_{l=1}^{q} T_l^{k2}F_l^{k1}(m),$$

where the constant T_l^{k2} are determined by the linear equation

$$G_2(r) - \hat{G}_2(r) = \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{k2} F_l^{k1}(r) \qquad (r = -q+1, -q+2, \dots, -1, 0).$$

Follwing the same procedure as above, we can finally obtain

THEOREM 5.3. *The coefficients* $G_j(m)(j=1, 2, \dots, n)$ *are expressed in the form*

$$G_j(m) = \sum_{i=1}^{j} \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj+1-i} F_l^{ki}(m) \qquad (j = 1, 2, \dots, n), \tag{5.28}$$

where the constant coefficients $T_l^{kj}(k=1, 2, \dots, n;\ l=1, 2, \dots, q)$ *are determined by the linear equation*

$$G_j(r) - \sum_{i=2}^{j} \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj+1-i} F_l^{ki}(r) = \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj} F_l^{k1}(r) \tag{5.29}$$

$$(r = -q+1, -q+2, \dots, -1, 0),$$

successively.

PROOF. The proof will be done by induction. Suppose that the formulas for $G_1(m), G_2(m), \dots, G_j(m)$ in the form of (5.28) are obtained, together with determining the constant coefficients $T_l^{ki}(k=1, 2, \dots, n;\ l=1, 2, \dots, q)$ for $i \leqq j$. Multiplying both sides of (5.27) by T_l^{kj+2-i} and summing over k, l and i from 1 to n, from 1 to q and from 2 to $j+1$, respectively, we have a particular solution of the nonhomogeneous linear difference equation (1.13) for $G_{j+1}(m)$ as follows:

$$\begin{aligned}(m + \rho - A_0)\hat{G}_{j+1}(m) &= A_1 \hat{G}_{j+1}(m-1) + \cdots + A_q \hat{G}_{j+1}(m-q) \\ &\quad - \sum_{i=2}^{j+1} \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj+2-i} F_l^{ki-1}(m) \\ &= A_1 \hat{G}_{j+1}(m-1) + \cdots + A_q \hat{G}_{j+1}(m-q) - G_j(m),\end{aligned}$$

where we have put

$$\hat{G}_{j+1}(m) = \sum_{i=2}^{j+1} \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj+2-i} F_l^{ki}(m).$$

Hence we can write $G_{j+1}(m)$ in the form

$$G_{j+1}(m) = \hat{G}_{j+1}(m) + \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj+1} F_l^{k1}(m),$$

determining the constant coefficients $T_l^{kj+1}(k=1, 2, \dots, n;\ l=1, 2, \dots, q)$ by the linear equation

$$G_{j+1}(r) - \hat{G}_{j+1}(r) = \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kj+1} F_l^{k1}(r)$$

$$(r = -q+1, -q+2, \dots, -1, 0).$$

This completes the proof of Theorem 5.3.

We can now rewrite (5.28) in the form

$$\text{(5.30)} \qquad G_j(m) = \sum_{n=1}^{k} \sum_{l=1}^{q} \sum_{i=1}^{j} T_l^{kj+1-i} F_l^{ki}(m)$$

$$= \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (\sum_{i=1}^{j} T_l^{kj+1-i} g_l^{ki}(m+s))$$

$$= \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (T_l^{k1}, T_l^{k2}, \dots, T_l^{kn}) (g_l^{kj}(m+s),$$

$$g_l^{kj-1}(m+s), \dots, g_l^{k1}(m+s), 0, \dots, 0)_* \qquad (j = 1, 2, \dots, n),$$

and hence we have

$$\text{(5.31)} \qquad (G_1(m), G_2(m), \dots, G_n(m))$$

$$= \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (T_l^{k1}, T_l^{k2}, \dots, T_l^{kn}) \mathscr{G}_l^k(m+s)_*.$$

From this relation we can immediately prove the following expansion theorem.

THEOREM 5.4. *The convergent power series solutions* $X_j(t)$ $(j=1, 2, \dots, n)$ *of the linear differential equation* (1.1) *can be expanded in terms of the associated fundamental matrix functions* $Y_l^k(t, s)$ $(k=1, 2, \dots, n;\ l=1, 2, \dots, q)$ *as follows*:

$$\text{(5.32)} \qquad (X_1(t), X_2(t), \dots, X_n(t))$$

$$= \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (T_l^{k1}, T_l^{k2}, \dots, T_l^{kn}) Y_l^k(t, s)_* t^{J_*},$$

where

$$\text{(5.33)} \qquad Y_l^k(t, s) = \sum_{m=0}^{\infty} \mathscr{G}_l^k(m+s) t^{m+\rho} \qquad (k = 1, 2, \dots, n;\ l = 1, 2, \dots, q).$$

PROOF. From the relation (5.5) we easily find that functions of the form

$$\text{(5.34)} \qquad X_l^{ki}(t) = \sum_{m=0}^{\infty} F_l^{ki}(m) t^{m+\rho} \qquad (i, k = 1, 2, \dots, n;\ l = 1, 2, \dots, q)$$

are well-defined for $|t| < \infty$ and moreover we have

$$(5.35) \qquad X_l^{ki}(t) = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} (H^k(s) g_l^{ki}(m+s)) t^{m+\rho}$$

$$= \sum_{s=0}^{\infty} H^k(s) (\sum_{m=0}^{\infty} g_l^{ki}(m+s) t^{m+\rho})$$

since the interchangeability of the order of sums over s and m is guaranteed by the absolute convergence of the double series. In fact, if we put for a sufficiently large positive integer σ

$$R_l^{ki}(m: \sigma) = \frac{1}{|g_l^{ki}(m+\sigma)|} \sum_{s=\sigma+1}^{\infty} \|H^k(s)\| |g_l^{ki}(m+s)|,$$

then we can prove the uniform boundedness of $R_l^{ki}(m: \sigma)$ by means of Abel's transformation and the behavior of $g_l^{ki}(m)$, the fact of which also has been used in the proof of Lemma 5.1 (see [10]). Hence the absolute convergence of the double series in (5.35) immediately follows from the inequality

$$\sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \|H^k(s)\| \, |g_l^{ki}(m+s) t^{m+\rho}|$$

$$\leqq \sum_{s=0}^{\sigma} \|H^k(s)\| \sum_{m=0}^{\infty} |g_l^{ki}(m+s) t^{m+\rho}| + \sum_{m=0}^{\infty} R_l^{ki}(m: \sigma) |g(m+\sigma) t^{m+\rho}|.$$

From the above consideration we have

$$(X_1(t), X_2(t), \ldots, X_n(t)) = (\hat{X}_1(t), \hat{X}_2(t), \ldots, \hat{X}_n(t)) t^{J_*}$$

$$= \sum_{m=0}^{\infty} (G_1(m), G_2(m), \ldots, G_n(m)) t^{m+\rho+J_*}$$

$$= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (T_l^{k1}, T_l^{k2}, \ldots, T_l^{kn}) \mathscr{G}_l^k(m+s)_* t^{m+\rho+J_*}$$

$$= \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (T_l^{k1}, T_l^{k2}, \ldots, T_l^{kn}) \sum_{m=0}^{\infty} (\mathscr{G}_l^k(m+s) t^{m+\rho})_* t^{J_*}$$

$$= \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s) (T_l^{k1}, T_l^{k2}, \ldots, T_l^{kn}) Y_l^k(t, s)_* t^{J_*}.$$

Thus we have established the expansion theorem.

We are now in a position to state our main theorem.

THEOREM 5.5. *Suppose that*

(i) $\rho - \mu_k \neq$ *an integer* $(k = 1, 2, \ldots, n)$, *and*

(ii) $0 < \dfrac{|\lambda_k|}{|\lambda_j - \lambda_k|} < 1 \quad (j \neq k) \qquad (j, k = 1, 2, \ldots, n).$

If t is sufficiently large in the sector

$$(5.36)\qquad S(l_1, l_2,\dots, l_n) = S_{l_1}(\lambda_1) \cap S_{l_2}(\lambda_2) \cap \cdots \cap S_{l_n}(\lambda_n) \qquad (l_1, l_2,\dots, l_n = 1, 2,\dots, q),$$

where

$$(5.37)\qquad S_l(\lambda_k):\ -\frac{3\pi}{q}+\frac{2\pi}{q}l \leqq \arg \lambda_k^{1/q}t < -\frac{\pi}{q}+\frac{2\pi}{q}l \qquad (k = 1, 2,\dots, n;\ l = 1, 2,\dots, q),$$

then we have

$$(5.38)\qquad (X_1(t), X_2(t),\dots, X_n(t)) \sim \sum_{k=1}^{n} X^k(t)(T_{l_k}^{k1}, T_{l_k}^{k2},\dots, T_{l_k}^{kn}).$$

PROOF. Let p and σ be arbitrarily large positive integers. Putting $p'=p+q$ and using the relation (3.12) and the integral representation (3.15), we have

$$(5.39)\qquad (X_1(t), X_2(t),\dots, X_n(t))$$

$$= \sum_{s=0}^{\infty}\sum_{l=1}^{n}\sum_{l=1}^{q} H^k(s)(T_l^{k1}, T_l^{k2},\dots, T_l^{kn})\{t^{\rho}(\mathscr{G}_l^k(s-1)t^{-1} + \mathscr{G}_l^k(s-2)t^{-2}$$

$$+\cdots+ \mathscr{G}_l^k(s-p')t^{-p'}) + t^{-p'}Y_l^k(t, s-p')\}_* t^{J_*}$$

$$= t^{\rho}\sum_{r=1}^{p'}(G_1(-r), G_2(-r),\dots, G_n(-r))t^{-r+J_*}$$

$$+ \sum_{s=0}^{\sigma}\sum_{k=1}^{n}\sum_{l=1}^{q} H^k(s)(T_l^{k1}, T_l^{k2},\dots, T_l^{kn})Y_l^k(t, s-p')_* t^{-p'+J_*}$$

$$+ \sum_{s=\sigma+1}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{q}\{Y_l^k(t, s-p')(T_l^{k1}, T_l^{k2},\dots, T_l^{kn})_* H^k(s)_*\}_* t^{-p'+J_*}$$

$$= t^{\rho}\sum_{r=1}^{p'}(G_1(-r), G_2(-r),\dots, G_n(-r))t^{-r+J_*}$$

$$+ \sum_{s=0}^{\sigma}\sum_{k=1}^{n}\sum_{l=1}^{q} H^k(s)(T_l^{k1}, T_l^{k2},\dots, T_l^{kn})Y_l^k(t, s-p')_* t^{-p'+J_*}$$

$$+ \{\sum_{k=1}^{n}(t^{\rho+q-1}[\lambda_k Z_{\mathscr{F}_{-1}^k}(\lambda_k^{1/q}t:\ \sigma - p' + q + \rho - \mu_k:\ \alpha_j^k\lambda_k^{-j/q})]$$

$$+ t^{\rho+q-2}[\lambda_k Z_{\mathscr{F}_{-2}^k}(\lambda_k^{1/q}t:\ \sigma - p' + q - 1 + \rho - \mu_k:\ \alpha_j^k\lambda_k^{-j/q})$$

$$+ \alpha_{q-1}^k Z_{\mathscr{F}_{-1}^k}(\lambda_k^{1/q}t:\ \sigma - p' + q - 1 + \rho - \mu_k:\ \alpha_j^k\lambda_k^{-j/q})]$$

$$+$$

$$\vdots$$

$$+ t^{\rho}[\lambda_k Z_{\mathscr{F}^k_{\pm q}}(\lambda_k^{1/q}t\colon \sigma - p' + 1 + \rho - \mu_k\colon \alpha_j^k\lambda_k^{-j/q})$$

$$+ \alpha_{q-1}^k Z_{\mathscr{F}^k_{\pm q+1}}(\lambda_k^{1/q}t\colon \sigma - p' + 1 + \rho - \mu_k\colon \alpha_j^k\lambda_k^{-j/q}) + \cdots$$

$$+ \alpha_1^k Z_{\mathscr{F}^k_{\pm 1}}(\lambda_k^{1/q}t\colon \sigma - p' + 1 + \rho - \mu_k\colon \alpha_j^k\lambda_k^{-j/q})])\}_* t^{-p'+J_*},$$

where we have put

$$(5.40) \qquad \mathscr{F}_i^k(\tau) = \sum_{l=1}^{q}\sum_{s=1}^{\infty}\mathscr{G}_l^k(s+\sigma+1+i)(T_l^{k1}, T_l^{k2},\ldots, T_l^{kn})_* H^k(s+\sigma+1)_* t^s$$

and used the notation (3.30). In the above calculation we also have used the termwise integration the validity of which is easily seen, taking account of the proof of Theorem 5.1, from the fact that the power seies (5.40) is absolutely and uniformly convergent in any compact set of

$$|\tau| < \frac{|\hat{\lambda}_k - \lambda_k|^{1/q}}{|\lambda_k|^{1/q}}, \quad |\hat{\lambda}_k - \lambda_k| = \min_{j\neq k}|\lambda_j - \lambda_k|$$

and hence, in the closed unit disk $|\tau| \leqq 1$.
We here apply Theorem 3.1 and Lemma 3.4 to (5.39) and obtain

$$(X_1(t), X_2(t),\ldots, X_n(t))$$

$$\sim t^{\rho}\sum_{r=1}^{p'}(G_1(-r), G_2(-r),\ldots, G_n(-r))t^{-r+J_*}$$

$$+ \sum_{s=0}^{\sigma}\sum_{k=1}^{n}\sum_{l=1}^{q} H^k(s)(T_l^{k1}, T_l^{k2},\ldots, T_l^{kn})$$

$$\times \left\{\delta_{ll_k}\exp\left(\frac{\lambda_k}{q}t^q + \frac{\alpha_{q-1}^k}{q-1}t^{q-1} + \cdots + \alpha_1^k t\right)t^{-J+\mu_k-s+p'} + O(t^{\rho-1})\right\}_* t^{-p'+J_*}$$

$$+ \sum_{k=1}^{n}\left\{O\left(\exp\left(\frac{\lambda_k}{q}t^q + \frac{\alpha_{q-1}^k}{q-1}t^{q-1} + \cdots + \alpha_1^k t\right)t^{-J+\mu_k-\sigma-1+p'}\right) + O(t^{\rho+q-1})\right\}_* t^{-p'+J_*}$$

$$\sim t^{\rho}\{\sum_{r=1}^{p}(G_1(-r), G_2(-r),\ldots, G_n(-r))t^{-r} + O(t^{-p-1})\}t^{J_*}$$

$$+ \sum_{k=1}^{n}\exp\left(\frac{\lambda_k}{q}t^q + \frac{\alpha_{q-1}^k}{q-1}t^{q-1} + \cdots + \alpha_1^k t\right)t^{\mu_k}$$

$$\times \{\sum_{s=0}^{\sigma} H^k(s)(T_{l_k}^{k1}, T_{l_k}^{k2},\ldots, T_{l_k}^{kn})t^{-s} + O(t^{-\sigma-1})\},$$

the first expression of which means the asymptotically zero expansion since from the initial conditions, $G_j(r)=0$ $(j=1, 2,\ldots, n)$ for all $r<0$. We have thus completed the proof of Theorem 5.5.

Lastly we make some remarks.

REMARK 3. Obviously, the sectors $S(l_1, l_2,\dots, l_n)(l_1, l_2,\dots, l_n=1, 2,\dots, q)$ cover the whole complex t-plane and hence, we have completely analyzed the global behaviors of $X_j(t)(j=1, 2,\dots, n)$ in the whole complex t-plane. On the other hand, although the global behaviors of them on their Riemann surfaces, that is the same thing, the determination of the Stokes multipliers in other planes, will be obtained directly from Theorem 5.5, we are led to the same results, only following Remark 2 in Section 2.

REMARK 4. We can rewrite Theorem 5.5 in the following form; There exists a fundamental set of solutions $\hat{X}_S^k(t)(k=1, 2,\dots, n)$ in the sectorial neighborhood $S(l_1, l_2,\dots, l_n)$ of infinity such that

$$\hat{X}_S^k(t) \sim X^k(t) \quad \text{as} \quad t \longrightarrow \infty \quad \text{in} \quad S(l_1, l_2,\dots, l_n) \tag{5.41}$$

and the connection formula

$$(X_1(t), X_2(t),\dots, X_n(t)) = (\hat{X}_S^1(t), \hat{X}_S^2(t),\dots, \hat{X}_S^n(t)) \begin{pmatrix} T_{l_1}^{11} & T_{l_1}^{12} \cdots T_{l_1}^{1n} \\ T_{l_2}^{21} & T_{l_2}^{22} \cdots T_{l_2}^{2n} \\ \vdots & \vdots \quad\ \vdots \\ T_{l_n}^{n1} & T_{l_n}^{n2} \cdots T_{l_n}^{nn} \end{pmatrix} \tag{5.42}$$

holds. In fact, if we put

$$\begin{cases} \hat{X}_S^1(t) = C_{11}X_1(t) + C_{21}X_2(t) + \cdots + C_{n1}X_n(t), \\ \hat{X}_S^2(t) = C_{12}X_1(t) + C_{22}X_2(t) + \cdots + C_{n2}X_n(t), \\ \cdots\cdots \\ \hat{X}_S^n(t) = C_{1n}X_1(t) + C_{2n}X_2(t) + \cdots + C_{nn}X_n(t), \end{cases}$$

where the matrix $\{C_{ij}\}$ is the inverse matrix of $\{T_{l_i}^{ij}\}$, i.e.,

$$\begin{pmatrix} T_{l_1}^{11} & T_{l_1}^{12} \cdots T_{l_1}^{1n} \\ T_{l_2}^{21} & T_{l_2}^{22} \cdots T_{l_2}^{2n} \\ \vdots & \vdots \quad\ \vdots \\ T_{l_n}^{n1} & T_{l_n}^{n2} \cdots T_{l_n}^{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \cdots C_{1n} \\ C_{21} & C_{22} \cdots C_{2n} \\ \vdots & \vdots \quad\ \vdots \\ C_{n1} & C_{n2} \cdots C_{nn} \end{pmatrix} = 1,$$

then we easily see that $\hat{X}_S^k(t)(k=1, 2,\dots, n)$ form a fundamental set of solutions and have the properties (5.41) and (5.42).

§6. Connection problem for a system of differential equations with multiple characteristic constants at an irregular singularity

In the case when the constant matrix A_q in (1.1) has multiple eigenvalues, according to H. L. Turrittin–M. Hukuhara's theorem, an appropriate non-singular transformation

$$X = C(t)Y = \left(\sum_{k=0}^{N} C_k t^{-k/p}\right)Y, \tag{6.1}$$

where N and p are suitably chosen positive integers and the change of variables $\tau = t^{1/p}$ reduce the system of linear differential equations (1.1) to the canonica system of linear differential equations similar to (1.8) for $Y(\tau)$(see [5, 6, 22]). In this section we therefore consider the two point connection problem for the system of linear differential equations

$$\begin{aligned} t\frac{dX}{dt} &= \{\delta_{kj}(p_k(t) + \mu_k + J_k) + \sum_{r=1}^{N} B_r t^{-r}\}X \\ &= \{\sum_{r=1}^{q} A_r t^r + \sum_{r=0}^{N} B_r t^{-r}\} \qquad (j,\ k = 1, 2,\dots, \nu), \end{aligned} \tag{6.2}$$

where $p_k(t)\,(k=1, 2,\dots, \nu)$ are polynomials of degree $q\ (q \geqq 1)$ with the form

$$p_k(t) = \lambda_k t^q + \alpha_{q-1}^k t^{q-1} + \cdots + \alpha_1^k t \qquad (k = 1, 2,\dots, \nu)$$

and $J_k\ (k=1, 2,\dots, \nu)$ are n_k by n_k shifting matrices stated in (1.3), i.e.,

$$A_q = \begin{pmatrix} \overbrace{\lambda_1 \ddots \lambda_1}^{n_1} & & & 0 \\ & \overbrace{\lambda_2 \ddots \lambda_2}^{n_2} & & \\ & & \ddots & \\ 0 & & & \overbrace{\lambda_\nu \ddots \lambda_\nu}^{n_\nu} \end{pmatrix},$$

$$A_r = \begin{pmatrix} \overbrace{\alpha_r^1 \ddots \alpha_r^1}^{n_1} & & & 0 \\ & \overbrace{\alpha_r^2 \ddots \alpha_r^2}^{n_2} & & \\ & & \ddots & \\ 0 & & & \overbrace{\alpha_r^\nu \ddots \alpha_r^\nu}^{n_\nu} \end{pmatrix} \qquad (r = 1, 2,\dots, q-1),$$

$$A_0 = \begin{pmatrix} \overbrace{\begin{matrix} \mu_1 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad \mu_1 \end{matrix}}^{n_1} & & & 0 \\ & \overbrace{\begin{matrix} \mu_2 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad \mu_2 \end{matrix}}^{n_2} & & \\ & & \ddots & \\ 0 & & & \overbrace{\begin{matrix} \mu_\nu & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad \mu_\nu \end{matrix}}^{n_\nu} \end{pmatrix}, \qquad (n_k \geqq 1\,;\, n_1+n_2+\cdots+n_\nu = n)\,.$$

We hereafter assume that $\lambda_k \neq \lambda_i$ $(k \neq i)$, $\lambda_k \neq 0$ $(k, i=1, 2,\dots, \nu)$. In appearance, the origin $t=0$ is not a regular singularity, but from the fact that the regular singularity is invariant under such a linear transformation of a fractional power polynomial as (6.1)**) we may assume that the system of linear differential equations (6.2) has a fundamental set of solutions of the form

$$(6.3) \qquad X_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \qquad (j = 1, 2,\dots, n),$$

where for simplicity $\rho_i - \rho_j \neq$ an integer $(i \neq j;\ i, j=1, 2,\dots, n)$ have been assumed. We then see that the coefficients $G_j(m)$ $(j=1, 2,\dots, n)$ satisfy the systems of linear difference equations

$$(6.4) \qquad \begin{cases} (\rho_j + m) G_j(m) = \sum_{r=0}^{q} A_r G_j(m-r) + \sum_{r=1}^{N} B_r G_j(m+r) \\ G_j(0) \neq 0,\ G_j(r) = 0\ (r<0) \qquad (j = 1, 2,\dots, n). \end{cases}$$

On the other hand, we can obtain ν sets of formal power series solutions with the following form:

$$(6.5) \qquad \begin{aligned} &(X^{k1}(t), X^{k2}(t),\dots, X^{kn_k}(t)) \\ &\quad = (\hat{X}^{k1}(t), \hat{X}^{k2}(t),\dots, \hat{X}^{kn_k}(t)) t^{-J_k^*} \exp(p_k(t)), \end{aligned}$$

where

$$(6.6) \qquad \hat{X}^{k\eta}(t) = \sum_{s=0}^{\infty} H^{k\eta}(s) t^{-s+\mu_k} \qquad (\eta = 1, 2,\dots, n_k),$$

$$(6.7) \qquad p_k(t) = \frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t \quad (k=1, 2,\dots, \nu).$$

**) As a matter of course, the coefficient matrices in (6.2) must satisfy the Jurkat-Lutz conditions for the regular singularity.

Direct calculations show that the power series $\hat{X}^{k\eta}(t)(\eta=1, 2,\dots, n_k)$ formally satisfy the systems of homogeneous and nonhomogeneous linear differential equations

$$(6.8)\qquad t\frac{d\hat{X}^{k1}}{dt} = \{\sum_{r=0}^{q} A_r t^r + \sum_{r=1}^{N} B_r t^{-r} - \not{p}_k(t)\}\hat{X}^{k1},$$

$$(6.9)\qquad t\frac{d\hat{X}^{k\eta}}{dt} = \{\sum_{r=0}^{q} A_r t^r + \sum_{r=1}^{N} B_r t^{-r} - \not{p}_k(t)\}\hat{X}^{k\eta} + \hat{X}^{k\eta-1}(t) \qquad (\eta = 2, 3,\dots, n_k).$$

From these it is easily seen that the coefficients $H^{k\eta}(s)(\eta=1, 2,\dots, n_k)$ satisfy the following systems of homogeneous and nonhomogeneous linear difference equations

$$(6.10)\quad \begin{cases} (A_q - \lambda_k)H^{k1}(s+q) + (A_{q-1} - \alpha_{q-1}^k)H^{k1}(s+q-1) + \cdots \\ + (A_0 - \mu_k + s)H^{k1}(s) + B_1 H^{k1}(s-1) + \cdots + B_N H^{k1}(s-N) = 0, \\ H^{k1}(0) \neq 0,\ H^{k1}(r) = 0 \qquad (r < 0), \end{cases}$$

$$(6.11)\quad \begin{cases} (A_q - \lambda_k)H^{k\eta}(s+q) + (A_{q-1} - \alpha_{q-1}^k)H^{k\eta}(s+q-1) + \cdots \\ + (A_0 - \mu_k + s)H^{k\eta}(s) + B_1 H^{k\eta}(s-1) + \cdots + B_N^{k\eta}(s-N) + H^{k\eta-1}(s) = 0, \\ H^{k\eta}(0) \neq 0,\ H^{k\eta}(r) = 0 \quad (r < 0) \qquad (\eta = 2, 3,\dots, n_k). \end{cases}$$

We can then give the explicit forms of particular solutions of (6.11) by means of $H^{k1}(s)$ as follows:

$$(6.12)\qquad H^{k\eta}(s) = \frac{1}{(\eta-1)!}\frac{d^{\eta-1}}{ds^{\eta-1}}(H^{k1}(s)) \qquad (\eta = 2, 3,\dots, n_k).$$

Now we shall introduce fundamental functions associated with this two point connection problem. Let $g_{jl}^{k1}(m)(l=1, 2,\dots, q)$ be a fundamental set of solutions of the linear difference equation

$$(6.13)\qquad (m + \rho_j - \mu_k)g_j^{k1}(m) = \alpha_1^k g_j^{k1}(m-1) + \cdots + \alpha_{q-1}^k g_j^{k1}(m-q+1) + \lambda_k g_j^{k1}(m-q).$$

We then put

$$(6.14)\qquad \mathscr{g}_{jl}^k(m) = \exp\left(J_k \frac{d}{dm}\right) g_{jl}^{k1}(m) = g_{jl}^{k1}(m) + g_{jl}^{k2}(m)J_k + \cdots + g_{jl}^{kn_k}(m)J_k^{n_k-1} \qquad (l = 1, 2,\dots, q),$$

whence $g_{jl}^{k\eta}(m)(\eta=2, 3,\dots, n_k)$ satisfy the nonhomogeneous linear difference equations

$$(6.15)\qquad (m+\rho_j-\mu_k)g_{jl}^{k\eta}(m)=\alpha_1^k g_{jl}^{k\eta}(m-1)+\cdots+\alpha_{q-1}^k g_{jl}^{k\eta}(m-q+1)$$
$$+\lambda_k g_{jl}^{k\eta}(m-q)-g_{jl}^{k\eta-1}(m)\qquad (\eta=2,3,\dots,n_k).$$

As we have seen in Section 2, the qn_k column vectors of the n_k by n_k matrices $\mathscr{G}_{jl}^k(m)\,(l=1,2,\dots,q)$ form a fundamental set of solutions of the linear difference equation

$$(6.16)\qquad (m+\rho_j-\mu_k+J_k)\mathscr{G}_j^k(m)$$
$$=\alpha_1^k\mathscr{G}_j^k(m-1)+\cdots+\alpha_{q-1}^k\mathscr{G}_j^k(m-q+1)+\lambda_k\mathscr{G}_j^k(m-q).$$

We now define associated fundamental functions by the power series

$$(6.17)\qquad Y_{jl}^k(t,s)=\sum_{m=0}^{\infty}\mathscr{G}_{jl}^k(m+s)t^{m+\rho_j}$$
$$(j=1,2,\dots,n;\ k=1,2,\dots,\nu;\ l=1,2,\dots,q).$$

If we choose $g_{jl}^{k1}(m)\,(l=1,2,\dots,q)$ as (2.13) which are expressed in terms of the modified gamma functions, then all the remarkable results in regard to the global behaviors of $Y_{jl}^k(t,s)$ are immediately obtained in exactly the same manner as in Section 3.

We here define

$$(6.18)\qquad F_{jl}^{k\eta}(m)=\sum_{\kappa=1}^{\eta}\sum_{s=0}^{\infty}H^{k\kappa}(s)g_{jl}^{k\eta+1-\kappa}(m+s)$$
$$(j=1,2,\dots,n;\ k=1,2,\dots,\nu;\ \eta=1,2,\dots,n_k;\ l=1,2,\dots,q).$$

Postponing the proof of the well-definedness of those functions, we first show that for each j the functions $F_{jl}^{k\eta}(m)$ satisfy the system of linear difference equations (6.4). In fact, it follows from (6.10–11), (6.13) and (6.15) that

$$(6.19)\qquad (m+\rho_j)F_{jl}^{k\eta}(m)$$
$$=\sum_{\kappa=1}^{\eta}\sum_{s=0}^{\infty}\{(m+s+\rho_j-\mu_k)+(\mu_k-s)\}H^{k\kappa}(s)g_{jl}^{k\eta+1-\kappa}(m+s)$$
$$=\sum_{\kappa=1}^{\eta}\sum_{s=0}^{\infty}H^{k\kappa}(s)\{\sum_{r=1}^{q-1}\alpha_r^k g_{jl}^{k\eta+1-\kappa}(m+s-r)$$
$$+\lambda_k g_{jl}^{k\eta+1-\kappa}(m+s-q)-g_{jl}^{k\eta-\kappa}(m+s)\}$$
$$+\sum_{\kappa=1}^{\eta}\sum_{s=0}^{\infty}\{(A_q-\lambda_k)H^{k\kappa}(s+q)+\sum_{r=1}^{q-1}(A_r-\alpha_r^k)H^{k\kappa}(s+r)+A_0H^{k\kappa}(s)$$

$$+ \sum_{r=1}^{N} B_r H^{k\kappa}(s-r) + H^{k\kappa-1}(s)\} g_{jl}^{k\eta+1-\kappa}(m+s)$$

$$= \sum_{r=1}^{q-1} \alpha_r^k F_{jl}^{k\eta}(m-r) + \lambda_k F_{jl}^{k\eta}(m-q) - \sum_{\kappa=1}^{\eta} \sum_{s=0}^{\infty} H^{k\kappa}(s) g_{jl}^{k\eta-\kappa}(m+s)$$

$$+ (A_q - \lambda_k) F_{jl}^{k\eta}(m-q) + \sum_{r=1}^{q-1} (A_r - \alpha_r^k) F_{jl}^{k\eta}(m-r) + A_0 F_{jl}^{k\eta}(m)$$

$$+ \sum_{r=1}^{N} B_r F_{jl}^{k\eta}(m+r) + \sum_{\kappa=1}^{\eta} H^{k\kappa-1}(s) g_{jl}^{k\eta+1-\kappa}(m+s)$$

$$= \sum_{r=0}^{q} A_r F_{jl}^{k\eta}(m-r) + \sum_{r=1}^{N} B_r F_{jl}^{k\eta}(m+r) - \sum_{\kappa=1}^{\eta-1} \sum_{s=0}^{\infty} H^{k\kappa}(s) g_{jl}^{k\eta-\kappa}(m+s)$$

$$+ \sum_{\kappa=1}^{\eta-1} \sum_{s=0}^{\infty} H^{k\kappa}(s) g_{jl}^{k\eta-\kappa}(m+s)$$

$$= \sum_{r=0}^{q} A_r F_{jl}^{k\eta}(m-r) + \sum_{r=1}^{N} B_r F_{jl}^{k\eta}(m+r),$$

where we have put $H^{k0}(s) \equiv 0$ and $g_{jl}^{k0}(s) \equiv 0$.

We shall now verify the well-definedness of the functions $F_{jl}^{k\eta}(m)$, the fact of which in turn guarantees the validity of the above calculation. In the paper [11] we have established only a slightly rough result on the growth order of coefficients of formal power series solutions of general canonical systems of linear differential equations by the same consideration as in Section 4, but a refined investigation yields the following result: Consider the canonical system of linear differential equations (6.2), where $N=\infty$ and the power series is assumed to be convergent for sufficiently large $|t|$. There can be derived ν sets of formal power series solutions of the form (6.5–7). Let h_{ik} $(i \neq k;\ i, k=1, 2,\dots, \nu)$ be the largest non-negative integers such that

$$(\not{p}_i(t) + \mu_i) - (\not{p}_k(t) + \mu_k) = \beta_{ik}(h_{ik}) t^{h_{ik}} + \beta_{ik}(h_{ik}-1) t^{h_{ik}-1} + \cdots + \beta_{ik}(0),$$

where $\beta_{ik}(h_{ik}) \neq 0$. For each fixed k $(k=1, 2,\dots, \nu)$ define the positive integer q_k by

$$q_k = \min_{i \neq k} \{h_{ik} \neq 0\}$$

and

$$\beta_k = \min\{|\beta_{ik}(h_{ik})|;\ h_{ik} = q_k\}.$$

Then we have

$$\overline{\lim_{s\to\infty}} \left(\frac{\|H^{k\eta}(s)\|}{|\Gamma(s+1)^{1/q_k}|} \right)^{1/s} \leqq \beta_k^{-1/q_k} \qquad (k = 1, 2,\dots, \nu;\ \eta = 1, 2,\dots, n_k).$$

Applying the above result to our case considered where $\lambda_k \neq \lambda_i$ $(k \neq i)$ $\lambda_k \neq 0$ $(k, i=1, 2,\dots, \nu)$, we have for all k

$$q = q_k = h_{ik} \quad (i \neq k),$$

$$\beta_k = |\hat{\lambda}_k - \lambda_k| = \min_{i \neq k} |\lambda_i - \lambda_k|$$

and hence we obtain

$$\overline{\lim_{s\to\infty}}\left(\frac{\|H^{k\eta}(s)\|}{|\Gamma(s+1)^{1/q}|}\right)^{1/s} \leqq \frac{1}{|\hat{\lambda}_k - \lambda_k|^{1/q}} \tag{6.20}$$

$$(k = 1, 2,\dots, \nu;\ \eta = 1, 2,\dots, n_k).$$

This result is exactly analogous to Theorem 4.1. Combining (6.20) with asymptotic behaviors of $g_{jl}^{k\eta}(m)$, we can finally obtain the required results as stated in the first part of Section 5.

Next we shall show the linear independence of the functions $F_{jl}^{k\eta}(m)(k=1, 2,\dots, \nu;\ \eta=1, 2,\dots, n_k;\ l=1, 2,\dots, q)$ for each fixed j. We first evaluate the initial values of $H^{k\eta}(0)(k=1, 2,\dots, \nu;\ \eta=1, 2,\dots, n_k)$. From (6.10–11) it is easily seen that for each k $(k=1, 2,\dots, \nu)$ they are of the form

$$H^{k\eta}(0) = \begin{pmatrix} 0 \\ \hat{H}^{k\eta}(0) \\ 0 \end{pmatrix}, \quad \hat{H}^{k\eta}(0) = \begin{pmatrix} h_1^{\eta}(0) \\ h_2^{\eta}(0) \\ \vdots \\ h_{n_k}^{\eta}(0) \end{pmatrix} \qquad (\eta = 1, 2,\dots, n_k)$$

and satisfy

$$J_k \hat{H}^{k\eta}(0) + \hat{H}^{k\eta-1}(0) = 0, \text{ i.e.,}$$

$$\begin{cases} -h_j^{\eta}(0) = h_{j+1}^{\eta-1}(0) & (j = 1, 2,\dots, n_k - 1) \\ h_{n_k}^{\eta}(0) = \text{an arbitrary number} & (\eta = 1, 2,\dots, n_k), \end{cases}$$

whence, for instance, we can put

$$\hat{H}^{k1}(0) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{H}^{k2}(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ -1 \end{pmatrix}, \dots, \quad \hat{H}^{kn_k}(0) = (-1)^{n_k-1}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{6.21}$$

In order to prove their linear independence, we investigate the Casorati determinant $\mathscr{C}_{F_j}(m)$ composed of qn vectorial functions $F_{jl}^{k\eta}(m)$, using the asymptotic relations

$$(6.22)\qquad F_{jl}^{k\eta}(m) \sim \sum_{\kappa=1}^{\eta} H^{k\kappa}(0) g_{jl}^{k\eta+1-\kappa}(m)\{1 + O(m^{-1/q})\}$$

for sufficiently large integral values of m. Taking account of (6.21–22), we have the asymptotic relations

$$F_{jl}^{k\eta+1}(m) + F_{jl}^{k\eta}(m) \sim n_k \begin{pmatrix} \mathbf{0} \\ 0 \\ \vdots \\ 0 \\ (-1)^{\eta} g_{jl}^{k1}(m) \\ \vdots \\ (-1) g_{jl}^{k\eta}(m) \\ g_{jl}^{k\eta+1}(m) \\ \mathbf{0} \end{pmatrix} \{1 + O(m^{-1/q})\}$$

$$(\eta = 1, 2, \ldots, n_k - 1)$$

and hence

$$(6.23)\quad \mathscr{F}_j^k(m) \equiv (F_{j1}^{k1}(m), \ldots, F_{jq}^{k1}(m), F_{j1}^{k2}(m) + F_{j1}^{k1}(m), \ldots, F_{jq}^{k2}(m) + F_{jq}^{k1}(m), \ldots,$$
$$F_{j1}^{kn_k}(m) + F_{j1}^{kn_k-1}(m), \ldots, F_{jq}^{kn_k}(m) + F_{jq}^{kn_k-1}(m))$$

$$\sim n_k \begin{pmatrix} & & \mathbf{0} & & \\ & 0 & & (-1)^{n_k-1} g_{j1}^{k1}(m) \cdots (-1)^{n_k-1} g_{jq}^{k1}(m) \\ & & & (-1)^{n_k-2} g_{j1}^{k2}(m) \cdots (-1)^{n_k-2} g_{jq}^{k2}(m) \\ & (-1) g_{j1}^{k1}(m) \cdots (-1) g_{jq}^{k1}(m) \cdots & & \vdots \qquad \vdots \\ g_{j1}^{k1}(m) \cdots g_{jq}^{k1}(m) & g_{j1}^{k2}(m) \cdots & g_{jq}^{k2}(m) \cdots & g_{jl}^{kn_k}(m) \cdots \quad g_{jq}^{kn_k}(m) \\ & & \mathbf{0} & & \end{pmatrix}$$

$$\times \{1 + O(m^{-1/q})\} \qquad (k = 1, 2, \ldots, \nu)$$

for sufficiently large integral values of m. The Casorati determinant $\mathscr{C}_{F_j}(m)$ can be written in the form

$$(6.24)\qquad \mathscr{C}_{F_j}(m) = \begin{vmatrix} \mathscr{F}_j^1(m) & \mathscr{F}_j^2(m) & \cdots \mathscr{F}_j^\nu(m) \\ \mathscr{F}_j^1(m+1) & \mathscr{F}_j^2(m+1) & \cdots \mathscr{F}_j^\nu(m+1) \\ \vdots & \vdots & \vdots \\ \mathscr{F}_j^1(m+1-q) & \mathscr{F}_j^2(m+q-1) & \cdots \mathscr{F}_j^\nu(m+q-1) \end{vmatrix}$$

and, considering (6.23) and interchanging the order of rows, we have the asymptotic relation

$$(6.25)\qquad \mathscr{C}_{F_j}(m)$$

$$\sim \pm \left| qn_k \left\{ \begin{array}{cccccccc} & \mathbf{0} & & & & & (-1)^{n_k-1}g_{j1}^{k1}(m) & \cdots(-1)^{n_k-1}g_{jq}^{k1}(m) \\ & & & & & & \vdots & \vdots \\ & & & & & & (-1)^{n_k-1}g_{j1}^{k1}(m+q-1) & \cdots(-1)^{n_k-1}g_{jq}^{k1}(m+q-1) \\ & 0 & (-1)g_{j1}^{k1}(m) & \cdots(-1)g_{jq}^{k1}(m) & \cdots & & \vdots & \vdots \\ & & \vdots & \vdots & & & & \\ & & (-1)g_{j1}^{k1}(m+q-1) & \cdots(-1)g_{jq}^{k1}(m+q-1) & \cdots & & \vdots & \vdots \\ g_{j1}^{k1}(m) & \cdots g_{jq}^{k1}(m) & g_{j1}^{k2}(m) & \cdots & g_{jq}^{k2}(m) & \cdots g_{j1}^{kn_k}(m) & \cdots & g_{jq}^{kn_k}(m) \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ g_{j1}^{k1}(m+q-1) & \cdots g_{jq}^{k1}(m+q-1) & g_{j1}^{k2}(m+q-1) & \cdots & g_{jq}^{k2}(m+q-1) & \cdots g_{j1}^{kn_k}(m+q-1) & \cdots & g_{jq}^{kn_k}(m+q-1) \end{array} \right. \right|$$

$$\times \{1 + O(m^{-1/q})\}$$

$$\sim \pm \{\mathscr{C}_{g_j^1}(m)\}^{n_1}\{\mathscr{C}_{g_j^2}(m)\}^{n_2}\cdots\{\mathscr{C}_{g_j^\nu}(m)\}^{n_\nu}\{1 + O(m^{-1/q})\}\,.$$

Since $\mathscr{C}_{g_j^k}(m)\neq 0$ for all k, we consequently obtain $\mathscr{C}_{F_j}(m)\neq 0$ for sufficiently large integral values of m.

We here remark that in appearance the order of the system of linear difference equations (6.4) satisfied by the coefficients $G_j(m)$ is higher than q, but $G_j(m)$ are the ones transformed from coefficients of convergent power series solutions of the original system of linear differential equations (1.1), which satisfy just a q-th order system of linear difference equations, by the linear transformation (6.1). In other words, the system of linear difference equation (6.4) is reducible to a q-th order system of linear difference equations under the Jurkat-Lutz conditions. We may therefore admit the validity of the argument to follow.

On the basis of the considerations done so far, we can determine the Stokes multipliers $T_{jl}^{k\eta}$ ($k=1, 2,\ldots, \nu$; $\eta=1, 2,\ldots, n_k$; $l=1, 2,\ldots, q$) by the relations

$$(6.26)\qquad G_j(m) = \sum_{k=1}^{\nu}\sum_{\eta=1}^{n_k}\sum_{l=1}^{q} T_{jl}^{k\eta}F_{jl}^{k\eta}(m) \qquad (j = 1, 2,\ldots, n).$$

Moreover, since there holds

$$\sum_{\eta=1}^{n_k} T_{jl}^{k\eta}F_{jl}^{k\eta}(m) = \sum_{s=0}^{\infty}\sum_{\eta=1}^{n_k}\sum_{\kappa=1}^{\eta} T_{jl}^{k\eta}H^{k\kappa}(s)g_{jl}^{k\eta+1-\kappa}(m+s)$$

$$= \sum_{s=0}^{\infty} \sum_{\kappa=1}^{n_k} H^{k\kappa}(s) \sum_{\eta=\kappa}^{n_k} T_{jl}^{k\eta} g_{jl}^{k\eta+1-\kappa}(m+s)$$

$$= \sum_{s=0}^{\infty} (H^{k1}(s), H^{k2}(s), \ldots, H^{kn_k}(s))$$

$$\times \begin{pmatrix} g_{jl}^{k1}(m+s) g_{jl}^{k2}(m+s) \cdots g_{jl}^{kn_k}(m+s) \\ g_{jl}^{k1}(m+s) \cdots g_{jl}^{kn_k-1}(m+s) \\ 0 \quad \ddots \quad \vdots \\ g_{jl}^{k1}(m+s) \end{pmatrix} \begin{pmatrix} T_j^{k1} \\ T_j^{k2} \\ \vdots \\ T_j^{kn_k} \end{pmatrix},$$

we can rewrite (6.26), using (6.14), in the following form

(6.27) $G_j(m)$

$$= \sum_{s=0}^{\infty} \sum_{l=1}^{q} \sum_{k=1}^{\nu} (H^{k1}(s), H^{k2}(s), \ldots, H^{kn_k}(s)) \mathscr{G}_{jl}^{k}(m+s)_* (T_{jl}^{k1}, T_{jl}^{k2}, \ldots, T_{jl}^{kn_k})_*$$

$$= \sum_{l=1}^{q} \sum_{s=0}^{\infty} (H^{11}(s), \ldots, H^{1n_1}(s), H^{21}(s), \ldots, H^{2n_2}(s), \ldots, H^{\nu 1}(s), \ldots, H^{\nu n_\nu}(s))$$

$$\times \begin{pmatrix} \mathscr{G}_{jl}^{1}(m+s) & & 0 \\ & \mathscr{G}_{jl}^{2}(m+s) & \\ & \ddots & \\ 0 & & \mathscr{G}_{jl}^{\nu}(m+s) \end{pmatrix}_* (T_{jl}^{11}, \ldots, T_{jl}^{1n_1}, T_{jl}^{21}, \ldots, T_{jl}^{2n_2}, \ldots, T_{jl}^{\nu 1}, \ldots, T_{jl}^{\nu n_\nu})_*$$

$$(j = 1, 2, \ldots, n).$$

From this, we derive the expansion formulas of the convergent power series solutions (6.3) in terms of the associated fundamental matrix functions (6.17) as follows:

(6.28) $$X_j(t) = \sum_{m=0}^{\infty} G_j(m) t^{m+\rho_j}$$

$$= \sum_{l=1}^{q} \sum_{s=0}^{\infty} (H^{11}(s), \ldots, H^{1n_1}(s), H^{21}(s), \ldots, H^{2n_2}(s), \ldots, H^{\nu 1}(s), \ldots, H^{\nu n_k}(s))$$

$$\times \begin{pmatrix} Y_{jl}^{1}(t, s) & & 0 \\ & Y_{jl}^{2}(t, s) & \\ & \ddots & \\ 0 & & Y_{jl}^{\nu}(t, s) \end{pmatrix}_* (T_{jl}^{11}, \ldots, T_{jl}^{1n_1}, T_{jl}^{21}, \ldots, T_{jl}^{2n_2}, \ldots, T_{jl}^{\nu 1}, \ldots, T_{jl}^{\nu n_\nu})_*$$

$$(j = 1, 2, \ldots, n).$$

We are now in a position to solve the two point connection problem for the system of linear differential equations (6.2).

Suppose that $\rho_j-\mu_k\neq$ an integer $(j=1, 2,\dots, n;\ k=1, 2,\dots, \nu)$ and $0<|\lambda_k|/|\lambda_i-\lambda_k|<1 (k\neq i;\ k, i=1, 2,\dots, \nu)$. Then, taking account of (6.28) and the results derived in Section 3, we have

$$
\begin{aligned}
(6.29)\quad & X_j(t) \\
&\sim \sum_{s=0}^{\infty}(H^{11}(s),\dots, H^{1n_1}(s), H^{21}(s),\dots, H^{2n_2}(s),\dots, H^{\nu 1}(s),\dots, H^{\nu n_\nu}(s)) \\
&\times \begin{pmatrix} t^{-J_1-s+\mu_1}\exp(p_1(t)) & & 0 \\ & t^{-J_2-s+\mu_2}\exp(p_2(t)) & \\ & \ddots & \\ 0 & & t^{-J_\nu-s+\mu_\nu}\exp(p_\nu(t)) \end{pmatrix}_{*} (T^{11}_{jl_1},\dots,T^{1n_1}_{jl_1},T^{21}_{jl_2},\dots,T^{2n_2}_{jl_2},\dots,T^{\nu 1}_{jl_\nu},\dots,T^{\nu n_\nu}_{jl_\nu})_{*} \\
&\sim (\hat{X}^{11}(t),\dots, \hat{X}^{1n_1}(t), \hat{X}^{21}(t),\dots, \hat{X}^{2n_2}(t),\dots, \hat{X}^{\nu 1}(t),\dots, \hat{X}^{\nu n_\nu}(t)) \\
&\times \begin{pmatrix} t^{-J_{1*}}\exp(p_1(t)) & & 0 \\ & t^{-J_{2*}}\exp(p_2(t)) & \\ & \ddots & \\ 0 & & t^{-J_{\nu *}}\exp(p_\nu(t)) \end{pmatrix} (T^{11}_{jl_1},\dots, T^{1n_1}_{jl_1}, T^{21}_{jl_2},\dots, T^{2n_2}_{jl_2},\dots, T^{\nu 1}_{jl_\nu},\dots, T^{\nu n_\nu}_{jl_\nu})_{*} \\
&\sim (X^{11}(t),\dots, X^{1n_1}(t), X^{21}(t),\dots, X^{2n_2}(t),\dots, X^{\nu 1}(t),\dots, X^{\nu n_\nu}(t)) \\
&\times (T^{11}_{jl_1},\dots, T^{1n_1}_{jl_1}, T^{21}_{jl_2},\dots, T^{2n_2}_{jl_2},\dots, T^{\nu 1}_{jl_\nu},\dots, T^{\nu n_\nu}_{jl_\nu})_{*} \\
&\sim \sum_{k=1}^{\nu}\sum_{\eta=1}^{n_k} T^{k\eta}_{jl_k}X^{k\eta}(t) \qquad (j=1, 2,\dots, n)
\end{aligned}
$$

as $t\to\infty$ in the sector $S(l_1, l_2,\dots, l_\nu)=S_{l_1}(\lambda_1)\cap S_{l_2}(\lambda_2)\cap\cdots\cap S_{l_\nu}(\lambda_\nu) (l_1, l_2,\dots, l_\nu=1, 2,\dots, q)$, where $S_l(\lambda_k) (k=1, 2,\dots, \nu;\ l=1, 2,\dots, q)$ are the sectors of the form (5.37).

§7. Evaluation of the Stokes multipliers

We have established in Section 5–6 that the Stokes multipliers (connection coefficients) can be given by the constant coefficients appearing in the linear combinations, where the coefficients $G_j(m)$ of convergent power series solutions are expressed in terms of the functions $F^k_l(m)$. Regarding the actual determination of the Stokes multipliers, we have only noted that, applying the well-known Cramer formula for the solution of linear equations, they can be evaluated, for instance, by the relations satisfying initial conditions. This is indeed valid if the values of $F^k_l(r)$ for $r=0, -1,\dots, -q+1$ are given in Theorems

5.2–3. Moreover, if we occasionally have the exact values of both $G_j(m)$ and $F_l^k(m)$ for every m, we can immediately apply the Cramer formula to the linear combinations for any q consecutive values of m and obtain the exact values of the Stokes multipliers. Since it, however, is not easy to derive the explicit values of $F_l^k(m)$ which are considered as the modified factorial series, such a direct method of calculation stated above cannot be expected to be applied in general.

In the theory of difference equations, a solution is used to be characterized by its asymptotic behavior near infinity, i.e., by its terminal condition. We here remind of the fact that the coefficients $G_j(m)$ are particular solutions of linear difference equations, which are determined by initial conditions that $G_j(0) \neq 0$, $G_j(r)=0$ $(r<0)$, and hence they have been expressed in terms of linear combinations of an appropriately chosen fundamental set of solutions $F_l^k(m)$ of those linear difference equations. Since we have already obtained the asymptotic behaviors of the solutions $F_l^k(m)$ as $m \to \infty$ in the right half-plane, it therefore follows that to seek the Stokes multipliers is exactly the same as to investigate how the particular solutions $G_j(m)$ behave near infinity in the right half-plane. In other words, if we can know the asymptotic behaviors of the particular solutions $G_j(m)$, then we can immediately determine the Stokes multipliers by a method as follows: For example, in order to determine $T_l^{k1}(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ in (5.10), we apply the Cramer formula to the linear equations

$$G_1(r) = \sum_{k=1}^{n} \sum_{l=1}^{q} T_l^{kl} F_l^{kl}(r) \qquad (r = m,\ m+1,\dots,\ m+q-1)$$

and then let m tend to infinity in the right half-plane.

As will be seen in Section 8, this method of terminal condition is very effective for the evaluation of the Stokes multipliers.

We shall now explain the method in more detail, treating, for simplicity, the single linear differential equation

$$(7.1) \qquad t^n \frac{d^n x}{dt^n} = \sum_{i=1}^{n} \Big(\sum_{r=0}^{qi} a_{i,r} t^r\Big) t^{n-i} \frac{d^{n-i} x}{dt^{n-i}},$$

the connection problem for which has been investigated in the paper [10]. In this case, the Stokes multipliers T_{jl}^k $(j,\ k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ must be determined by the relations

$$(7.2) \qquad G_j(m) = \sum_{k=1}^{n} \sum_{l=1}^{q} T_{jl}^k f_{jl}^k(m) \qquad (j = 1, 2,\dots, n).$$

The coefficients $G_j(m)$ of convergent power series solutions of (7.1) satisfy the linear difference equations

(7.3) $$I(m+\rho_j)G_j(m)=\sum_{i=1}^{n}\sum_{r=1}^{qi}a_{i,r}[m+\rho_j-r]_{n-i}G_j(m-r)$$

with the initial conditions

(7.4) $$G_j(0)\neq 0,\ G_j(r)=0 \quad (r<0) \qquad (j=1,2,\dots,n),$$

where use is made of the notation

$$[\rho]_p=\rho(\rho-1)\cdots(\rho-p+1),\quad [\rho]_0=1,$$

$$I(\rho)=[\rho]_n-\sum_{i=1}^{n}a_{i,0}[\rho]_{n-i},$$

and the functions $f_{jl}^k(m)$ have the asymptotic behaviors

(7.5) $$f_{jl}^k(m)\sim g_{jl}^k(m)\{1+O(m^{-1/q})\} \qquad (j,k=1,2,\dots,n;\ l=1,2,\dots,q)$$

as $m\to\infty$ in the right half-plane, $g_{jl}^k(m)$ being just the same as $g_{jl}^{k1}(m)$ in (6.13). Considering the relations

(7.6) $$\frac{f_{jl}^k(m+r)}{f_{jl}^k(m)}\sim\frac{g_{jl}^k(m+r)}{g_{jl}^k(m)}\{1+O(m^{-1/q})\}$$

$$\sim(\lambda_k^{-1/q}\omega^{l-1})^{-r}m^{-r/q}\{1+O(m^{-1/q})\}$$

$$(\omega=\exp(2\pi i/q);\ j,k=1,2,\dots,n;\ l=1,2,\dots,q),$$

it follows from the Cramer formula that, assuming that $G_j(m)\neq 0$,

(7.7) $$T_{jl}^k=\frac{\begin{vmatrix} f_{j1}^1(m) & \cdots f_{j1}^1(m) & \cdots G_j(m) & \cdots & f_{j1}^n(m) & \cdots f_{jq}^n(m)\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ f_{j1}^1(m+qn-1) & \cdots f_{jq}^1(m+qn-1) & \cdots G_j(m+qn-1) & \cdots & f_{j1}^n(m+qn-1) & \cdots f_{jq}^n(m+qn-1)\end{vmatrix}}{\begin{vmatrix} f_{j1}^1(m) & \cdots f_{jq}^1(m) & \cdots f_{jl}^k(m) & \cdots & f_{jl}^n(m) & \cdots f_{jq}^n(m)\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ f_{j1}^1(m+qn-1) & \cdots f_{jq}^1(m+qn-1) & \cdots f_{jl}^k(m+qn-1) & \cdots & f_{j1}^n(m+qn-1) & \cdots f_{jq}^n(m+qn-1)\end{vmatrix}}$$

$$=\frac{g_{j1}^1(m)g_{j2}^1(m)\cdots G_j(m)\cdots g_{j1}^n(m)\cdots g_{jq}^n(m)}{g_{j1}^1(m)g_{j2}^1(m)\cdots g_{jl}^k(m)\cdots g_{j1}^n(m)\cdots g_{jq}^n(m)}$$

$$\times \frac{\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots \\ (\lambda_1^{-1/q}m^{1/q})^{-1} & \cdots & (\lambda_1^{-1/q}\omega^{q-1}m^{1/q})^{-1} & \cdots & G_j(m+1)/G_j(m) & \cdots \\ \vdots & & \vdots & & \vdots & \\ (\lambda_1^{-1/q}m^{1/q})^{-(qn-1)} & \cdots & (\lambda_1^{-1/q}\omega^{q-1}m^{1/q})^{-(qn-1)} & \cdots & G_j(m+qn-1)/G_j(m) & \cdots \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots \\ (\lambda_1^{-1/q}m^{1/q})^{-1} & \cdots & (\lambda_1^{-1/q}\omega^{q-1}m^{1/q})^{-1} & \cdots & (\lambda_k^{-1/q}\omega^{l-1}m^{1/q})^{-1} & \cdots \\ \vdots & & \vdots & & \vdots & \\ (\lambda_1^{-1/q}m^{1/q})^{-(qn-1)} & \cdots & (\lambda_1^{-1/q}\omega^{q-1}m^{1/q})^{-(qn-1)} & \cdots & (\lambda_k^{-1/q}\omega^{l-1}m^{1/q})^{-(qn-1)} & \cdots \end{vmatrix}}$$

$$\frac{\begin{matrix} 1 & \cdots & 1 \\ (\lambda_n^{-1/q}m^{1/q})^{-1} & \cdots & (\lambda_n^{-1/q}\omega^{q-1}m^{1/q})^{-1} \\ \vdots & & \vdots \\ (\lambda_n^{-1/q}m^{1/q})^{-(qn-1)} & \cdots & (\lambda_n^{-1/q}\omega^{q-1}m^{1/q})^{-(qn-1)} \end{matrix}}{\begin{matrix} 1 & \cdots & 1 \\ (\lambda_n^{-1/q}m^{1/q})^{-1} & \cdots & (\lambda_n^{-1/q}\omega^{q-1}m^{1/q})^{-1} \\ \vdots & & \vdots \\ (\lambda_n^{-1/q}m^{1/q})^{-(qn-1)} & \cdots & (\lambda_n^{-1/q}\omega^{q-1}m^{1/q})^{-(qn-1)} \end{matrix}}$$

$$\times \{1 + O(m^{-1/q})\}$$

$$= \frac{G_j(m)}{g_{jl}^k(m)} \frac{V_{qn}^{(k-1)q+l}(\lambda_1^{1/q},\ldots, \lambda_1^{1/q}\omega^{-(q-1)},\ldots, m^{1/q}G_j(m+r)/G_j(m),\ldots, \lambda_n^{1/q},\ldots, \lambda_n^{1/q}\omega^{-(q-1)})}{V_{qn}(\lambda_1^{1/q},\ldots, \lambda_1^{1/q}\omega^{-(q-1)},\ldots, \lambda_k^{1/q}\omega^{-(l-1)},\ldots, \lambda_n^{1/q},\ldots, \lambda_n^{1/q}\omega^{-(q-1)})}$$

$$\times \{1 + O(m^{-1/q})\} \qquad (j,\ k = 1, 2,\ldots, n;\ l = 1, 2,\ldots, q)$$

for sufficiently large positive integral values of m, where $V_{qn}(x_1, x_2,\ldots, x_{nq})$ denotes the Vandermonde determinant and $V_{qn}^j(x_1, x_2,\ldots, y_r,\ldots, x_{nq})$ denotes a determinant in which the j-th column $(1, x_j,\ldots, x_j^{qn-1})_*$ of the above Vandermonde determinant is replaced by $(1, y_1,\ldots, y_r,\ldots, y_{qn-1})_*$. Since T_{jl}^k are constants not depending on m, letting $m\to\infty$ in (7.7), we have

$$(7.8)\quad T_{jl}^k = \gamma_{jl}^k \frac{V_{qn}^{(k-1)q+l}(\lambda_1^{1/q},\ldots, \lambda_1^{1/q}\omega^{-(q-1)},\ldots, d_j(r),\ldots, \lambda_n^{1/q},\ldots, \lambda_n^{1/q}\omega^{-(q-1)})}{V_{qn}(\lambda_1^{1/q},\ldots, \lambda_1^{1/q}\omega^{-(q-1)},\ldots, \lambda_k^{1/q}\omega^{-(l-1)},\ldots, \lambda_n^{1/q},\ldots, \lambda_n^{1/q}\omega^{-(q-1)})}$$

$$(j,\ k = 1, 2,\ldots, n;\ l = 1, 2,\ldots, q)$$

if we can know the relations

$$(7.9)\qquad \lim_{m\to\infty} \frac{G_j(m)}{g_{jl}^k(m)} = \gamma_{jl}^k$$

and

$$\lim_{m\to\infty} m^{r/q}\frac{G_j(m+r)}{G_j(m)} = d_j(r) \qquad (r=1, 2,\dots, qn-1). \tag{7.10}$$

We can rewrite the above assertion in a more precise manner. Applying O. Perron-H. Poincaré's theorem stated before to (7.3), we easily see that the coefficients $G_j(m)$ have the growth order $C_j^m/\Gamma\left(\frac{m}{q}+1\right)$, i.e., entirely the same growth order as $g_{jl}^k(m)$ for sufficiently large positive integral values of m. From this, if we put

$$G_j(m) = g_{jl}^k(m)G_{jl}^k(m) \qquad (j, k=1, 2,\dots, n;\ l=1, 2,\dots, q), \tag{7.11}$$

the relations (7.9–10) imply that

$$\lim_{m\to\infty} G_{jl}^k(m) = \gamma_{jl}^k \tag{7.12}$$

and

$$\lim_{m\to\infty}\frac{G_{jl}^k(m+r)}{G_{jl}^k(m)} = d_{jl}^k(r) \qquad (G_{jl}^k(m)\neq 0;\ r=1, 2,\dots, qn-1), \tag{7.13}$$

thereby obtaining $d_j(r) = (\lambda_k^{-1/q}\omega^{l-1})^{-r}d_{jl}^k(r)$. From (7.3) and (7.11) we see that the functions $G_{jl}^k(m)(j, k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ satisfy the linear difference equations

$$\begin{aligned} G_{jl}^k(m) &= \sum_{i=1}^{n}\sum_{r=1}^{qi} a_{i,r}\frac{[m+\rho_j-r]_{n-i}}{I(m+\rho_j)}\frac{g_{jl}^k(m-r)}{g_{jl}^k(m)}G_{jl}^k(m-r) \\ &= \sum_{i=1}^{n}\sum_{r=1}^{qi} A_{i,r}^{jkl}(m)G_{jl}^k(m-r) \end{aligned} \tag{7.14}$$

with the initial conditions

$$G_{jl}^k(0)\neq 0,\ G_{jl}^k(r)=0\ (r<0) \qquad (j, k=1, 2,\dots, n;\ l=1, 2,\dots, q). \tag{7.15}$$

Therefore we conclude that if the particular solutions $G_{jl}^k(m)$ of the linear difference equations (7.14) subject to (7.15) have the properties (7.12–13), then the Stokes multipliers T_{jl}^k are given by (7.8). However, we do not insist that we can always verify the relations (7.12–13) for those particular solutions.

We here make a remark on the linear difference equations (7.14). The coefficients of (7.14) for each fixed triple (j, k, l) have the following asymptotic behaviors

$$A_{i,r}^{jkl}(m) \sim a_{i,r}m^{-i}(\lambda_k^{-1/q}\omega^{l-1})^r m^{r/q}\{1+O(m^{-1/q})\},$$

i.e.,

$$\lim_{m\to\infty} A_{i,r}^{\{kl\}}(m) = \begin{cases} a_{i,qi}(\lambda_k^{-1/q}\omega^{l-1})^{qi} & (r = qi), \\ 0 & (r < qi). \end{cases} \tag{7.16}$$

Therefore the linear difference equation (7.14) is of the Poincaré type and its characteristic equation is given by

$$d^{qn} = \sum_{i=1}^{n} a_{i,qi}(\lambda_k^{-1/q}\omega^{l-1})^{qi} d^{q(n-i)}. \tag{7.17}$$

Obviously, qn roots of (7.17) are given by

$$d_{i,l} = \left(\frac{\lambda_i}{\lambda_k}\right)^{1/q}\omega^l \qquad (i = 1, 2,\dots, n;\ l = 1, 2,\dots, q), \tag{7.18}$$

whose q roots $d_{i,l}$ $(l=1, 2,\dots, q)$ necessarily have the same absolute value. As shown, illustrating an example, by O. Perron [18], in such a case that the characteristic equation has roots with the same absolute value, it occurs that $G_{jl}^k(m+1)/G_{jl}^k(m)$ is oscillatory. In practical applications we often encounter cases in which all or some of the ratios in the left hand side of (7.13) for $1 \leqq r \leqq qn-1$ identically vanish and the remaining ratios have the limits.

Lastly we shall mention that even if the exact values of the Stokes multipliers cannot be determined by the method of terminal condition and the like, our theory still shows the usefulness for evaluating approximate values of them.

§8. Applications

In this section, in order to illustrate the effectiveness of our theory established and to make a few remarks on the conditions imposed so far, we shall consider the two point connection problem for the extended Airy equation

$$\frac{d^n y}{dz^n} - z^v y = 0. \tag{8.1}$$

For $n=2$ and $v=1$ this is just the Airy equation. It is well-known that G. G. Stokes [20] first noticed a discontinuous change of coefficients appearing in asymptotic representations of its solutions called Airy functions by a continuous change of sectorial neighborhoods of infinity. Such a fact and cocfficients are named the Stokes phenomenon and the Stokes multipliers, respectively, after the first discoverer. H. L. Turrittin [21] and J. Heading [4] considered (8.1) with an integer v and a rational number v, respectively, and later B. L. J. Braaksma [1] treated (8.1) with an arbitrary complex number v by means of the Barnes integral.

Now we rewrite (8.1) in the form

(8.2) $$z^n \frac{d^n y}{dz^n} - z^q y = 0$$

and consider (8.2), where q is assumed to be a positive integer.***) Putting $z=t^n$ and denoting $t\frac{d}{dt}$ by $\mathscr{D}$, we can write (8.2) in the form

(8.3) $$[\mathscr{D}\{\mathscr{D} - n\}\cdots\{\mathscr{D} - n(n-1)\} - n^n t^{qn}]y = 0.$$

Moreover, if we put

(8.4) $$\begin{cases} y_1 = y, \\ y_p = \{\mathscr{D} - n(n-p+1)\}\{\mathscr{D} - n(n-p+2)\}\cdots\{\mathscr{D} - n(n-1)\}y \end{cases}$$

$$(p = 2, 3, \dots, n),$$

then we have

(8.5) $$\begin{cases} \mathscr{D} y_p = y_{p+1} + n(n-p)y_p \qquad (p = 1, 2, \dots, n-1), \\ \mathscr{D} y_n = \mathscr{D}\{\mathscr{D} - n\}\cdots\{\mathscr{D} - n(n-1)\}y = n^n t^{qn} y_1, \end{cases}$$

whence we obtain the following system of linear differential equations for $Y = (y_1, y_2, \dots, y_n)_*$

(8.6) $$t\frac{dY}{dt} = \begin{pmatrix} n(n-1) & 1 & 0 & & \\ & n(n-2) & 1 & & \\ 0 & & \ddots & \ddots & \\ & & & n & 1 \\ n^n t^{qn} & 0 & \cdots\cdots & & 0 \end{pmatrix} Y.$$

We here apply the so-called shearing transformation

(8.7) $$X = S(t)Y, \; S(t) = \begin{pmatrix} t^{-q(n-1)} & & & 0 \\ & t^{-q(n-2)} & & \\ & & \ddots & \\ & & & t^{-q} & \\ 0 & & & & 1 \end{pmatrix}$$

to (8.6), and obtain the system of linear differential equations of the desired form

(8.8) $$t\frac{dX}{dt} = (A_0 + A_q t^q)X,$$

***) We can treat (8.1) with a rational number v. If $q = \frac{q'}{p}$ in (8.2), then we may only make the change of variables $z = t^{np}$ in the analysis below.

where, putting $\rho_j=(n-j)(n+q)(j=1, 2,\dots, n)$,

$$A_0=\begin{pmatrix} \rho_1 & & & 0 \\ & \rho_2 & & \\ & & \ddots & \\ 0 & & & \rho_n \end{pmatrix}, \qquad A_q=\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & \ddots\ 1 \\ n^n & 0 & \cdots\cdots & 0 \end{pmatrix}.$$

We shall now seek convergent power series solutions of (8.8) in the neighborhood of the origin $t=0$. It is easily seen that some of convergent power series solutions are written in the column vectorial form

$$X_j(t) = t^{\rho_j}\sum_{m=0}^{\infty} \hat{G}_j(m)t^m, \tag{8.9}$$

where the coefficients $\hat{G}_j(m)$ satisfy the system of linear difference equations

$$\begin{cases} (m+\rho_j-A_0)\hat{G}_j(m) = A_q\hat{G}_j(m-q), \\ \hat{G}_j(0)\neq 0,\ \hat{G}_j(r)=0 \quad (r<0). \end{cases} \tag{8.10}$$

From the initial conditions we can see that $\hat{G}_j(m)=0$ $(m\neq qm';\ m'=0, 1, 2,\dots)$ and hence, putting $G_j(m)=\hat{G}_j(qm)$, we have

$$\begin{cases} (qm+\rho_j-A_0)G_j(m) = A_qG_j(m-1), \\ G_j(0)\neq 0,\ G_j(r)=0 \quad (r<0). \end{cases} \tag{8.11}$$

Moreover we put $G_j(m)=(g^{(j,1)}(m), g^{(j,2)}(m),\dots, g^{(j,n)}(m))_*$ and write down (8.11) componentwise as follows:

$$\begin{cases} (qm+\rho_j-\rho_i)g^{(j,i)}(m) = g^{(j,i+1)}(m-1) \qquad (i=1, 2,\dots, n-1), \\ (qm+\rho_j-\rho_n)g^{(j,n)}(m) = n^n g^{(j,1)}(m-1), \\ g^{(j,j)}(0)\neq 0,\ g^{(j,i)}(0)=0 \qquad (i\neq j). \end{cases} \tag{8.12}$$

From (8.12) we can immediately see that for $0\leqq k\leqq n-1$

$$G_j(nm+k) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g^{(j,k')}(nm+k) \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ k' = j-k \pmod n, \tag{8.13}$$

as long as the coefficients in the left hand side of (8.12) never vanish. However, since for $1 \leqq k \leqq j-1$

$$(8.14) \qquad (qnm - kn)g^{(j,j-k)}(nm+k) = g^{(j,j-k+1)}(nm+k-1),$$

we cannot determine $g^{(j,j-k)}(nm+k)$ for such integers m that $m=k/q$ ($1 \leqq k \leqq j-1$). For example, if $q=1, 2,\dots, n-1$, then we can write $k=ql+l'$ ($0 \leqq l' \leqq q-1$) for $1 \leqq k \leqq j-1$, and for $k=q$ and $m=l$ we cannot determine $g^{(j,j-ql)}(nl+ql)$ from (8.14). In these cases logarithmic terms appear in the representation of convergent power series solutions in the neighborhood of the origin $t=0$. We therefore obtain the following:

Case 1. If $q \geqq n$, then there exists a fundamental set of convergent power series solutions of the form

$$(8.15) \qquad X_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^{qm} \qquad (j=1, 2,\dots, n)$$

in the neighborhood of the origin $t=0$, where the coefficients $G_j(m)$ satisfy the system of linear difference equations (8.11).

Case 2. If $1 \leqq q \leqq n-1$, then for $1 \leqq j \leqq q$ we can seek convergent power series solutions $X_j(t)$ of the form (8.15) and for each j ($1 \leqq j \leqq q$) there exist $\nu_j = \max\{\nu \geqq 0;\, j+q\nu \leqq n\}$ linearly independent solutions with logarithmic terms associated with $X_j(t)$. We can write them as follows:

$$(8.16) \qquad X_{j+q\gamma}(t) = \sum_{\nu=0}^{\gamma} \frac{1}{(\gamma-\nu)!} (\log t)^{\gamma-\nu} X_{j,\nu}(t) \qquad (\gamma = 1, 2,\dots, \nu_j)$$

where $X_{j,0}(t) \equiv X_j(t)$ and the functions $X_{j,\nu}(t)$ ($\nu=1, 2,\dots, \nu_j$) are convergent power series solutions of the nonhomogeneous systems of linear differential equations

$$(8.17) \qquad t\frac{d}{dt} X_{j,\nu} = (A_0 + A_q t^q) X_{j,\nu} - X_{j,\nu-1}(t) \qquad (\nu = 1, 2,\dots, \nu_j)$$

with the expressions

$$(8.18) \qquad X_{j,\nu}(t) = t^{\rho_j+q\nu} \sum_{m=0}^{\infty} \hat{G}_{j+q\nu}(m) t^m = t^{\rho_j+q\nu} \sum_{m=0}^{\infty} G_{j+q\nu}(m) t^{qm}$$

$$(\nu = 1, 2,\dots, \nu_j).$$

Since $\rho_{j+q(\nu-1)} - \rho_{j+q\nu} = -q(n+q)$, the coefficients $G_{j+q\nu}(m)$ ($\nu=1, 2,\dots, \nu_j$) satisfy the nonhomogeneous systems of linear difference equations

$$(8.19) \quad (m + \rho_{j+q\nu} - A_0)\hat{G}_{j+q\nu}(m) = A_q \hat{G}_{j+q\nu}(m-q) - \hat{G}_{j+q(\nu-1)}(m - q(n+q)),$$

i.e.,

(8.20) $(qm + \rho_{j+q\nu} - A_0)G_{j+q\nu}(m) = A_q G_{j+q\nu}(m-1) - G_{j+q(\nu-1)}(m-(n+q))$

$$(\nu = 1, 2, \dots, \nu_j).$$

In the next stage we consider formal solutions of (8.8) which are given by

$$X^k(t) = \exp\left(\frac{\lambda_k}{q} t^q\right) t^{\mu_k} \sum_{s=0}^{\infty} \hat{H}^k(s) t^{-s} \qquad (k = 1, 2, \dots, n), \tag{8.21}$$

where the characteristic constants $\lambda_k = n\omega_n^{k-1}$ $(k=1, 2, \dots, n)$, $\omega_n = \exp(2\pi i/n)$. The coefficients $\hat{H}^k(s)$ satisfy the system of linear difference equations

$$\begin{cases} (\lambda_k - A_q)\hat{H}^k(s) = (A_0 - \mu_k + s - q)\hat{H}^k(s-q), \\ \hat{H}^k(0) \neq 0, \ \hat{H}^k(r) = 0 \quad (r < 0). \end{cases} \tag{8.22}$$

From the initial conditions we can see that $\hat{H}^k(s) = 0$ $(s \neq qs';\ s' = 0, 1, 2, \dots)$ and hence, putting $H^k(s) = \hat{H}^k(qs)$, we have

$$\begin{cases} (\lambda_k - A_q)H^k(s) = (A_0 - \mu_k + q(s-1))H^k(s-1), \\ H^k(0) \neq 0, H^k(r) = 0 \quad (r < 0). \end{cases} \tag{8.23}$$

We can here put

$$H^k(0) = \begin{pmatrix} 1 \\ \lambda_k \\ \vdots \\ \lambda_k^{n-1} \end{pmatrix} \qquad (k = 1, 2, \dots, n) \tag{8.24}$$

and from (8.23) for $s=1$ and (8.24) we can immediately obtain

$$\sum_{j=1}^{n} (\rho_j - \mu_k) = 0 \qquad (k = 1, 2, \dots, n),$$

which determine the characteristic constants μ_k as follows:

$$\mu_k = \frac{(n+q)(n-1)}{2} \qquad (k = 1, 2, \dots, n). \tag{8.25}$$

This also implies that

$$\sum_{j=1}^{n} \rho_j - \sum_{k=1}^{n} \mu_k = 0, \tag{8.26}$$

which is an invariant identity called Fuchs' relation. In order to derive the growth order of $H^k(s)$ for sufficiently large values of s, we had better to apply the constant transformation

$$(8.27) \qquad H^k(s) = C\bar{H}^k(s) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \bar{H}^k(s)$$

to (8.23), obtaining the system of linear difference equations

$$(8.28) \qquad \begin{cases} (\lambda_k - \bar{A}_q)\bar{H}^k(s) = (\bar{A}_0 - \mu_k + q(s-1))\bar{H}^k(s-1), \\ \bar{H}^k(0) = (0,\dots, \underset{\hat{k}}{1},\dots, 0)_*, \end{cases}$$

where $\bar{A}_0 = C^{-1}A_0C$ and $\bar{A}_q = C^{-1}A_qC$, i.e.,

$$\bar{A}_q = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

For this system of linear difference equations we can easily derive the estimate of $\bar{H}^k(s)$ in a similar manner to the proof in the paper [8]. We have

$$(8.29) \qquad \|\bar{H}^k(s)\| \leqq \left(\frac{q}{|\hat{\lambda}_k - \lambda_k|}\right)^{s-N} \frac{\Gamma\left(s - 1 + \frac{\alpha}{q}\right)}{\Gamma\left(N - 1 + \frac{\alpha}{q}\right)} \|\bar{H}^k(N)\|,$$

N being a sufficiently large positive integer, where we have put

$$(8.30) \qquad \begin{cases} |\hat{\lambda}_k - \lambda_k| = \min_{j \neq k} |\lambda_j - \lambda_k| = n|1 - \omega_n|, \\ \alpha = \|\bar{A}_0\| + |\mu_k|. \end{cases}$$

Hence we have

$$(8.31) \qquad \|H^k(s)\| \leqq \|C\| \|\bar{H}^k(s)\| \leqq M\left(\frac{q}{|\hat{\lambda}_k - \lambda_k|}\right)^s \Gamma\left(s - 1 + \frac{\alpha}{q}\right) \qquad (k = 1, 2,\dots, n),$$

M being an appropriately chosen constant.

According to our theory, we now have to consider two q-th order linear difference equations

$$(8.32) \qquad \lambda_k \phi_j^k(m+q) = (m + \rho_j - \mu_k)\phi_j^k(m),$$

$$(8.33) \qquad (m + \rho_j - \mu_k)g_j^k(m) = \lambda_k g_j^k(m-q) \qquad (j, k = 1, 2,\dots, n),$$

which correspond to (2.4) and (2.6), respectively, where $\alpha_l=0$ $(l=1, 2,\dots, q-1)$. For each pair (j, k) $(j, k=1, 2,\dots, n)$ we take the fundamental set of solutions of (8.32)

$$(8.34) \quad \phi^k_{jl}(m) = \frac{1}{q}\left\{\left(\frac{q}{\lambda_k}\right)^{1/q}\omega_q^{l-1}\right\}^{m+\rho_j-\mu_k}\Gamma\left(\frac{m+\rho_j-\mu_k}{q}\right)$$

$$(\omega_q = \exp(2\pi i/q);\ l = 1, 2,\dots, q),$$

which are characterized by the asymptotic behaviors (2.5), and then, as a fundamental sets of solutions of (8.33), take the following q solutions defined by (2.13):

$$(8.35) \quad g^k_{jl}(m) = \frac{1}{q}\,\frac{\left\{\left(\frac{\lambda_k}{q}\right)^{1/q}\omega_q^{-(l-1)}\right\}^{m+\rho_j-\mu_k}}{\Gamma\left(\frac{m+\rho_j-\mu_k}{q}+1\right)} \qquad (l = 1, 2,\dots, q).$$

We here make a remark. If $(m+\rho_j-\mu_k)/q+1=-N$ $(N=0, 1, 2,\dots)$, then all $g^k_{jl}(m)$ $(l=1, 2,\dots, q)$ and also the Casorati determinant $\mathscr{C}_{g^k_j}(m)$ vanish, more precisely, have simple isolated zeros at such values of m. On the other hand, $\phi^k_{jl}(m+q)\,(l=1, 2,\dots, q)$ and the Casorati determinant $\mathscr{C}_{\phi^k_j}(m+q)$ have simple poles only at such values of m. Reminding of the definition that no identical vanishing of the Casorati determinant implies the linear independence of solutions of linear difference equations, we can say that $\phi^k_{jl}(m)$ and $g^k_{jl}(m)\,(l=1, 2,\dots, q)$ actually form the respective fundamental sets of solutions of (8.32) and (8.33) in the whole complex m-plane. Moreover we immediately see that the product $g^k_{jl}(m)\phi^k_{jl}(m+q)$ has no singularities in the whole complex m-plane except for infinity, i.e., are entire and the important relations (2.16) necessarily hold everywhere. This is indeed the case in the considerations of Section 2, though we did not explain the above fact explicitly. Only by reason of a concise explanation of our theory we have assumed throughout Sections 2–6 that $\rho_j-\mu_k\neq$ an integer, considering that m ultimately takes integral values. As just shown, the condition that $\rho_j-\mu_k\neq$ an integer is not essential and can be dropped by a slightly detailed observation.

We shall now define the functions $F^k_{jl}(m)\,(j, k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ by

$$(8.36) \quad F^k_{jl}(m) = \sum_{s=0}^{\infty}\hat{H}^k(s)g^k_{jl}(m+s)$$

$$= \sum_{s=0}^{\infty}H^k(s)g^k_{jl}(m+sq) \qquad (j, k = 1, 2,\dots, n;\ l = 1, 2,\dots, q).$$

Taking account of (8.31) and (8.35), we can prove the well-definedness of these functions. For the moment assume that $\operatorname{Re} m > \frac{(n+q)(n-1)}{2}$. Then we have

$$(8.37) \quad \left\| \sum_{s=0}^{\infty} H^k(s) g_{jl}^k(m+sq) \right\|$$

$$\leqq M \left| \frac{1}{q} \left\{ \left(\frac{\lambda_k}{q} \right)^{1/q} \omega_q^{-(l-1)} \right\}^{m+\rho_j-\mu_k} \right|$$

$$\times \sum_{s=0}^{\infty} \left(\frac{q}{|\hat{\lambda}_k - \lambda_k|} \right)^s \left(\frac{q}{|\lambda_k|} \right)^s \frac{\left| \Gamma\left(s - 1 + \frac{\alpha}{q} \right) \right|}{\left| \Gamma\left(s + \frac{m+\rho_j-\mu_k}{q} + 1 \right) \right|}$$

$$\leqq M \left| \frac{1}{q} \left\{ \left(\frac{\lambda_k}{q} \right)^{1/q} \omega_q^{-(l-1)} \right\}^{m+\rho_j-\mu_k} \right| \frac{\left| \Gamma\left(\mathrm{Re}\left(\frac{m+\rho_j-\mu_k}{q} \right) + 1 \right) \right|}{\left| \Gamma\left(\frac{m+\rho_j-\mu_k}{q} + 1 \right) \right| \left| \Gamma\left(\frac{\alpha}{q} - 1 \right) \right|}$$

$$\times \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{q} - 1 \right)_s (1)_s}{\left(\mathrm{Re}\left(\frac{m+\rho_j-\mu_k}{q} \right) + 1 \right)_s s!} \left(\frac{|\lambda_k|}{|\hat{\lambda}_k - \lambda_k|} \right)^s$$

$$= \mathcal{M}(m) F\left(\frac{\alpha}{q} - 1, 1; \mathrm{Re}\left(\frac{m+\rho_j-\mu_k}{q} \right) + 1; \frac{|\lambda_k|}{|\hat{\lambda}_k - \lambda_k|} \right),$$

where we have used the notation

$$(\alpha)_0 = 1, (\alpha)_s = \alpha(\alpha+1)\cdots(\alpha+s-1) \qquad (s = 1, 2, \ldots)$$

and $F(\alpha, \beta; \gamma; z)$ denotes the hypergeometric series. We easily see that the hypergeometric series in (8.37) is convergent for $\left| \frac{\lambda_k}{\hat{\lambda}_k - \lambda_k} \right| < 1$. From (8.30) this condition implies that $1 < |1 - \exp(2\pi i/n)|$, which is satisfied only for $n \leqq 5$. For this reason the condition that $|\lambda_k| < |\lambda_j - \lambda_k|$ $(j \neq k; j, k = 1, 2, \ldots, n)$ was called the pentagonal condition by K. Okubo [14]. However, under the assumption that $\mathrm{Re}\left(\frac{m+\rho_j-\mu_k}{q} \right) - \frac{\alpha}{q} + 1 > 0$ the hypergeometric series is well-defined for $\left| \frac{\lambda_k}{\hat{\lambda}_k - \lambda_k} \right| = 1$ and moreover, even if $\left| \frac{\lambda_k}{\hat{\lambda}_k - \lambda_k} \right| > 1$, the hypergeometric series has meaning through the principle of analytic continuation. From this, it will be seen that the pentagonal condition imposed so far is not essential. In case the pentagonal condition is not necessarily satisfied, it seems that the ε-parameter method originally developed in the paper [14] is more effective for the proof of the well-definedness of the functions $F_{jl}^k(m)$.

We can easily verify, as in the preceeding sections, that for each j the functions $F_{jl}^k(m)$ $(k = 1, 2, \ldots, n; l = 1, 2, \ldots, q)$ form a fundamental set of solutions of the system of linear difference equations (8.10) in the right half-plane $\mathrm{Re}\, m > \frac{(n+q)(n-1)}{2}$. Just by means of (8.10), however, the domain of definition of the functions $F^k_{jl}(m)$ can be extended to the left half-plane $\operatorname{Re} m \leqq \frac{(n+q)(n-1)}{2}$. Moreover we can determine the Stokes multipliers $T^k_{jl}\,(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$, obtaining

$$(8.38) \qquad \hat{G}_j(m) = \sum_{k=1}^{n} \sum_{l=1}^{q} T^k_{jl} F^k_{jl}(m).$$

In Case 2 we introduce the functions $g^k_{jl,\nu}(m)(j=1, 2,\dots, q;\ l=1,\dots, \nu_j)$ defined as (2.20), i.e.,

$$(8.39) \qquad g^k_{jl,\nu}(m) = \frac{1}{(\nu-1)!} \frac{d^{\nu-1}}{dm^{\nu-1}} \{g^k_{jl}(m)\} \qquad (\nu = 1, 2,\dots, \nu_j),$$

which are solutions of the nonhomogeneous linear difference equations

$$(8.40) \qquad (m + \rho_j - \mu_k) g^k_{jl,\nu}(m) = \lambda_k g^k_{jl,\nu}(m - q) - g^k_{jl,\nu-1}(m) \qquad (\nu = 1, 2,\dots, \nu_j),$$

where $g^k_{jl,0}(m) \equiv g^k_{jl}(m)(l=1, 2,\dots, q)$ are solutions of (8.33) for $j=1, 2,\dots, q$. Then we define the functions, the well-definedness of which can be proved in exactly the same manner stated above,

$$(8.41) \qquad F^k_{jl,\nu}(m) = \sum_{s=0}^{\infty} H^k(s) g^k_{jl,\nu}(m + sq)$$

$$(j = 1, 2,\dots, q;\ \nu = 0, 1,\dots, \nu_j;\ k = 1, 2,\dots, n;\ l = 1, 2,\dots, q).$$

We can see that for each j $(j=1, 2,\dots, q)$ the functions $F^k_{jl,\nu}(m)$ satisfy the nonhomogeneous systems of linear difference equations

$$(8.42) \qquad (m + \rho_j - A_0) F^k_{jl,\nu}(m) = A_q F^k_{jl,\nu}(m - q) - F^k_{jl,\nu-1}(m)$$

where we put $F^k_{jl,-1}(m) \equiv 0$. From (8.42), by induction, we can prove the following relations, together with determining the Stokes multipliers:

$$(8.43) \qquad \hat{G}_{j+q\nu}(m) = \sum_{\gamma=0}^{\nu} \sum_{k=1}^{n} \sum_{l=1}^{q} T^k_{jl,\gamma} F^k_{jl,\nu-\gamma}(m - \nu q(n + q)) \qquad (j = 1, 2,\dots, q;\ \nu = 0, 1,\dots, \nu_j).$$

In fact, for each fixed $j\,(j=1, 2,\dots, q)$ let the Stokes multipliers $T^k_{jl,\gamma}(\gamma=0, 1,\dots, \nu;\ k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ be determined. Then, multiplying both sides of

$$(m - (\nu + 1)q(n + q) + \rho_j - A_0) F^k_{jl,\nu+1-\gamma}(m - (\nu + 1)q(n + q))$$
$$= A_q F^k_{jl,\nu+1-\gamma}(m - q - (\nu + 1)q(n + q)) - F^k_{jl,\nu-\gamma}(m - (\nu + 1)q(n + q))$$

by $T^k_{jl,\gamma}$ and summing them over k, l and γ from 1 to n, from 1 to q and from 0 to ν, respectively, we have

$$(m+\rho_{j+(\nu+1)q}-A_0)\sum_{\gamma=0}^{\nu}\sum_{k=1}^{n}\sum_{l=1}^{q}T^k_{jl,\gamma}F^k_{jl,\nu+1-\gamma}(m-(\nu+1)q(n+q))$$

$$=A_q\sum_{\gamma=0}^{\nu}\sum_{k=1}^{n}\sum_{l=1}^{q}T^k_{jl,\gamma}F^k_{jl,\nu+1-\gamma}(m-q-(\nu+1)q(n+q))$$

$$-\hat{G}_{j+q\nu}(m-q(n+q)),$$

where we have used $\rho_j-(\nu+1)q(n+q)=\rho_{j+(\nu+1)q}$, which implies that the sum is a particular solution of (8.19) for $\nu+1$. Therefore we can determine the Stokes multipliers $T^k_{jl,\nu+1}(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ and obtain

$$\hat{G}_{j+(\nu+1)q}(m)=\sum_{k=1}^{n}\sum_{l=1}^{q}T^k_{jl,\nu+1}F^k_{jl,0}(m-(\nu+1)q(n+q))$$

$$+\sum_{\gamma=0}^{\nu}\sum_{k=1}^{n}\sum_{l=1}^{q}T^k_{jl,\gamma}F^k_{jl,\nu+1-\gamma}(m-(\nu+1)q(n+q)).$$

Now we shall calculate the exact values of the Stokes multipliers T^k_{jl}, for simplicity, in Case 1 by the terminal condition method. For that purpose, we first seek the explicit formulas of the coefficients $G_j(m)\,(j=1, 2,\dots, n)$. It follows from (8.12–13) that

$$g^{(j,1)}(j-1)=\frac{g^{(j,j)}(0)}{(-1)^j\Gamma(j)n^{j-1}}, \tag{8.44}$$

$$g^{(j,1)}(mn+j-1)=\frac{g^{(j,1)}((m-1)n+j-1)}{q^n\left(m+\frac{1-j}{q}\right)\left(m+\frac{2-j}{q}\right)\cdots\left(m+\frac{n-j}{q}\right)} \tag{8.45}$$

$$=q^{-mn}\prod_{i=1}^{n}\frac{\Gamma\left(1+\frac{i-j}{q}\right)}{\Gamma\left(m+1+\frac{i-j}{q}\right)}g^{(j,1)}(j-1)$$

$$(j=1, 2,\dots, n).$$

Hence, if we put

$$g^{(j,j)}(0)=\frac{(-1)^j\Gamma(j)n^{j-1}}{\prod_{i=1}^{n}\Gamma\left(1+\frac{i-j}{q}\right)},$$

then we have

$$(8.46)\qquad g^{(j,1)}(mn+j-1)=q^{-mn}\left[\prod_{i=1}^{n}\frac{1}{\Gamma\left(m+1+\frac{i-j}{q}\right)}\right]$$

$$(j=1,2,\ldots,n).$$

Considering the fact that $\hat{G}_j((mn+j-1)q)=(g^{(j,1)}(mn+j-1),0,\ldots,0)_*$ and $\hat{G}_j((mn+j-1)q+r)=0$ $(r=1,2,\ldots,q-1)$, we solve the linear equation

$$\hat{G}_j((mn+j-1)q+r)=\sum_{k=1}^{n}\sum_{l=1}^{q}T^k_{jl}F^k_{jl}((mn+j-1)q+r)$$

$$(r=0,1,\ldots,q-1)$$

by the Cramer formula, thereby obtaining

$$(8.47)\quad T^k_{jl}\sim\frac{\begin{vmatrix}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} & H^1(0) & \cdots H^n(0)\\ 0 & H^1(0)(\lambda_1^{1/q}) & \cdots H^n(0)(\lambda_n^{1/q}\omega_q^{-(q-1)})\\ \vdots & \vdots & \vdots\\ 0 & H^1(0)(\lambda_1^{1/q})^{q-1} & \cdots H^n(0)(\lambda_n^{1/q}\omega_q^{-(q-1)})^{q-1}\end{vmatrix}}{\begin{vmatrix}H^k(0) & H^1(0) & \cdots H^n(0)\\ H^k(0)(\lambda_k^{1/q}\omega_q^{-(l-1)}) & H^1(0)(\lambda_1^{1/q}) & \cdots H^n(0)(\lambda_n^{1/q}\omega_q^{-(q-1)})\\ \vdots & \vdots & \vdots\\ H^k(0)(\lambda_k^{1/q}\omega_q^{-(l-1)})^{q-1} & H^1(0)(\lambda_1^{1/q})^{q-1} & \cdots H^n(0)(\lambda_n^{1/q}\omega_q^{-(q-1)})^{q-1}\end{vmatrix}}$$

$$\times\frac{g^{(j,1)}(mn+j-1)}{g^k_{jl}((mn+j-1)q)}\{1+O(m^{-1})\}$$

$$(j,k=1,2,\ldots,n;\ l=1,2,\ldots,q),$$

where we have used the asymptotic relations

$$F^k_{jl}(m)\sim H^k(0)g^k_{jl}(m)\{1+O(m^{-1})\},$$

$$\frac{g^k_{jl}(m+r)}{g^k_{jl}(m)}\sim\left\{\left(\frac{\lambda_k}{q}\right)^{1/q}\omega_q^{-(l-1)}\right\}^r m^{-r/q}\{1+O(m^{-1/q})\}.$$

Taking account of (8.24) and rearranging rows of the determinants in the order $(1,(\lambda_k^{1/q}\omega_q^{-(l-1)}),\ldots,(\lambda_k^{1/q}\omega_q^{-(l-1)})^{q-1},\lambda_k,\lambda_k(\lambda_k^{1/q}\omega_q^{-(l-1)}),\ldots,\lambda_k(\lambda_k^{1/q}\omega_q^{-(l-1)})^{q-1},\ldots,$ $\lambda_k^{n-1},\lambda_k^{n-1}(\lambda_k^{1/q}\omega_q^{-(l-1)}),\ldots,\lambda_k^{n-1}(\lambda_k^{1/q}\omega_q^{-(l-1)})^{q-1})_*$, we see that the constants in (8.47) are always of the form

$$\frac{\begin{vmatrix} 1 & 1 & \cdots 1 \\ 0 & x_2 & \cdots x_p \\ \vdots & \vdots & \vdots \\ 0 & x_2^{p-1} & \cdots x_p^{p-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots 1 \\ x_1 & x_2 & \cdots x_p \\ \vdots & \vdots & \vdots \\ x_1^{p-1} & x_2^{p-1} & \cdots x_p^{p-1} \end{vmatrix}} = \frac{x_2 x_3 \cdots x_p V_{p-1}(x_2, x_3, \dots, x_p)}{V_p(x_1, x_2, \dots, x_p)}$$

$$= \frac{x_1 x_2 \cdots x_p}{(-1)^{p-1} f'(x_1) x_1}$$

where $x_j\,(j=1, 2,\dots, p)$ are solutions of the equation $f(x)=\prod_{i=1}^{p}(x-x_i)=0$. Since $x_{k,l}=\lambda_k^{1/q}\omega_q^{-(l-1)}(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ are solutions of the equation $f(x)=(x^q-\lambda_1)(x^q-\lambda_2)\cdots(x^q-\lambda_n)=x^{qn}-n^n=0$, we have

$$(-1)^{p-1} x_1 f'(x_1) = (-1)^{qn-1} qn x_1^{qn} = (-1)^{qn-1} qn n^n,$$

$$x_1 x_2 \cdots x_p = (-1)^{qn-1} n^n,$$

and hence we find that the constants in (8.47) are equal to $(1/qn)$. On the other hand, taking account of (8.35) and (8.46), we have

$$\frac{g^{(j,1)}(mn+j-1)}{g_{jl}^{k}((mn+j-1)q)}$$

$$= q^{j+(\rho_j-\mu_k)/q} \lambda_k^{-(mn+j-1)-(\rho_j-\mu_k)/q} \omega_q^{(l-1)(\rho_j-\mu_k)}$$

$$\times \left[\prod_{i=1}^{n} \frac{1}{\Gamma\left(m+1+\frac{i-j}{q}\right)}\right] \Gamma\left(mn+j-1+\frac{\rho_j-\mu_k}{q}+1\right)$$

$$= q^{j+(\rho_j-\mu_k)/q} \omega_q^{(l-1)(\rho_j-\mu_k)} (n\omega_n^{k-1})^{-(mn+j-1)-(\rho_j-\mu_k)/q} \frac{n^{mn+j+(\rho_j-\mu_k)/q-1/2}}{(2\pi)^{(n-1)/2}}$$

$$\times \left[\prod_{i=1}^{n} \frac{\Gamma\left(m+\frac{j}{n}+\frac{\rho_j-\mu_k}{nq}+\frac{i-1}{n}\right)}{\Gamma\left(m+1+\frac{i-j}{q}\right)}\right]$$

$$\sim q^{j+(\rho_j-\mu_k)/q} \omega_q^{(l-1)(\rho_j-\mu_k)} \omega_n^{-(k-1)\{j-1+(\rho_j-\mu_k)/q\}} \left(\frac{n}{(2\pi)^{n-1}}\right)^{1/2} \{1+O(m^{-1})\},$$

where we have used Gauss' multiplication formula

$$\Gamma(nz) = \frac{n^{nz-(1/2)}}{(2\pi)^{(n-1)/2}} \prod_{i=1}^{n} \Gamma\left(z+\frac{i-1}{n}\right),$$

together with Stirling's formula. Consequently, letting $m \to \infty$ in (8.47), we obtain

$$(8.49)\qquad T^k_{jl} = \left(\frac{q}{\omega_n^{k-1}}\right)^{j-1+(\rho_j-\mu_k)/q} \omega_q^{(l-1)(\rho_j-\mu_k)}((2\pi)^{n-1}n)^{-1/2}$$

$$(j,\ k = 1, 2,\dots, n;\ l = 1, 2,\dots, q).$$

Thus we have solved the connection problem for (8.8) as follows:

$$(8.50)\qquad X_j(t) \sim \sum_{k=1}^{n} T^k_{jl_k} X^k(t) \qquad (j = 1, 2,\dots, n)$$

as $t\to\infty$ in the sector $S(l_1, l_2,\dots, l_n) = S_{l_1}(\lambda_1) \cap S_{l_2}(\lambda_2) \cap \cdots \cap S_{l_n}(\lambda_n)$, where

$$(8.51)\qquad S_l(\lambda_k):\ -\frac{3\pi}{q} + \frac{2\pi}{q} l \leqq \arg \lambda_k^{1/q} t < -\frac{\pi}{q} + \frac{2\pi}{q} l.$$

Similarly we can determine the explicit values of the Stokes multipliers $T^k_{jl,\gamma}$ in Case 2 after a more complicated calculation.

Taking account of the change of variables $z=t^n$, (8.4) and (8.7), we shall now return to the original differential equation (8.2). Let $y_j(z)\,(j=1, 2,\dots, n)$ be a fundamental set of solutions of (8.2) which can be written in the form

$$(8.52)\qquad y_j(z) = t^{-(n-1)q} t^{\rho_j} \sum_{m=0}^{\infty} g^{(j,1)}(mn + j - 1) t^{(mn+j-1)q}$$

$$= t^{(n-j)n} \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} \frac{1}{\Gamma\left(m + 1 + \frac{i-j}{q}\right)} \right] (t^{nq} q^{-n})^m$$

$$= z^{n-j} \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} \frac{1}{\Gamma\left(m + 1 + \frac{i-j}{q}\right)} \right] (z^q q^{-n})^m \qquad (j=1, 2,\dots, n).$$

From (8.21) we see that formal solutions of (8.2) are given by

$$(8.53)\qquad y^k(z) = \exp\left(\frac{\lambda_k}{q} z^{q/n}\right) z^{(n-1)(n-q)/2n} \sum_{s=0}^{\infty} h^k(s) z^{-sq/n}$$

$$(h^k(0) = 1;\ k = 1, 2,\dots, n),$$

where the coefficient $h^k(s)$ denotes the first component of the column vectorial coefficient $H^k(s)$. Then (8.50) implies that

$$(8.54)\qquad y_j(z) \sim \sum_{k=1}^{n} T^k_{jl_k} y^k(z) \qquad (j = 1, 2,\dots, n)$$

as $z\to\infty$ in the sector $S(l_1, l_2,\dots, l_n) = S^1_{l_1} \cap S^2_{l_2} \cap \cdots \cap S^n_{l_n}$, where

$$(8.55)\qquad S^k_l:\ (2l - 3)n\pi - 2(k - 1)\pi \leqq \arg z^q < (2l - 1)n\pi - 2(k - 1)\pi.$$

This result has the more explicit expression than derived by H. L. Turrittin and others (see B. L. J. Braaksma [1]).

To see the above result more vividly, we consider the global behavior of $Ai(z)$ called the Airy function of the first kind in the sector $|\arg z|<\frac{4}{3}\pi$. $Ai(z)$ is a particular solution of (8.3), where $n=2$ and $q=3$, and is expressed in terms of (8.52) as follows:

$$(8.56)\qquad Ai(z)=\sum_{m=0}^{\infty}\frac{z^{3m}}{3^{2m+2/3}m!\,\Gamma\left(m+\frac{2}{3}\right)}-\sum_{m=0}^{\infty}\frac{z^{3m+1}}{3^{2m+4/3}m!\,\Gamma\left(m+\frac{4}{3}\right)}$$
$$=3^{-2/3}y_2(z)-3^{-4/3}y_1(z).$$

From the connection formula (8.54) just derived, we have

$$(8.57)\qquad Ai(z)\sim(3^{-2/3}T^1_{2l_1}-3^{-4/3}T^1_{1l_1})y^1(z)+(3^{-2/3}T^2_{2l_2}-3^{-4/3}T^2_{1l_2})y^2(z)$$

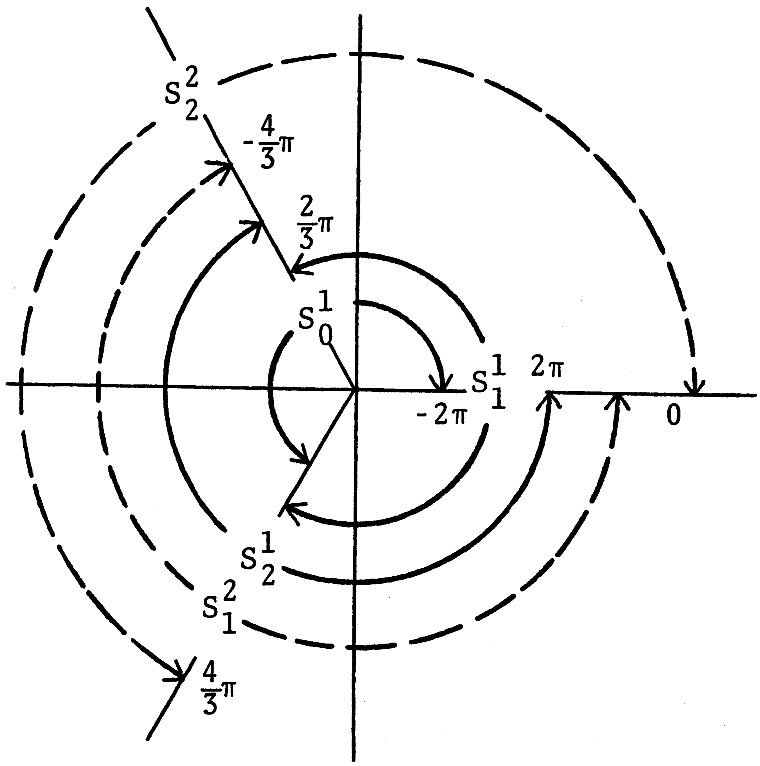
as $z\to\infty$ in the sector $S(l_1, l_2)=S^1_{l_1}\cap S^2_{l_2}$, S^k_l being written in the concise form

$$(8.58)\qquad S^k_l:\ \frac{2}{3}(2l-k-2)\pi\leqq\arg z<\frac{2}{3}(2l-k)\pi\,.$$

In the above the formal solutions $y^k(z)\,(k=1, 2)$ are explicitly written in the form

$$(8.59)\qquad \begin{cases} y^1(z)=\exp\left(\frac{2}{3}z^{3/2}\right)z^{-1/4}\sum_{s=0}^{\infty}h^1(s)z^{-3s/2} \\ \qquad =\exp\left(\frac{2}{3}z^{3/2}\right)z^{-1/4}\sum_{s=0}^{\infty}\left(\frac{3}{4}\right)^s\frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}z^{-3s/2}, \\ y^2(z)=\exp\left(-\frac{2}{3}z^{3/2}\right)z^{-1/4}\sum_{s=0}^{\infty}h^2(s)z^{-3s/2} \\ \qquad =\exp\left(-\frac{2}{3}z^{3/2}\right)z^{-1/4}\sum_{s=0}^{\infty}\left(-\frac{3}{4}\right)^s\frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}z^{-3s/2}. \end{cases}$$

Considering that the sector $|\arg z|<\frac{4}{3}\pi$ is covered by $S(0, 1)\cup S(1, 1)\cup S(1, 2)\cup S(2, 2)$ and the Stokes multipliers corresponding to $S^k_l\,(k=1, 2)$ are given by $T^k_{1l}=\frac{3^{5/6}}{2\sqrt{\pi}}\exp\left\{\frac{5}{6}(2l-k-1)\pi i\right\}$, $T^k_{2l}=\frac{3^{1/6}}{2\sqrt{\pi}}\exp\left\{\frac{1}{6}(-10l-k+11)\pi i\right\}\,(k=1, 2)$, we can calculate the coefficients in the right hand side of (8.57) as below.



		$3^{-2/3}T^1_{2l_1}-3^{-4/3}T^1_{1l_1}$	$3^{-2/3}T^2_{2l_2}-3^{-4/3}T^2_{1l_2}$
$-\frac{4}{3}\pi < \arg z < -\frac{2}{3}\pi$	$S_0^1 \cap S_1^2$	$-\frac{i}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$
$-\frac{2}{3}\pi \leqq \arg z < 0$	$S_1^1 \cap S_1^2$	0	$\frac{1}{2\sqrt{\pi}}$
$0 \leqq \arg z < \frac{2}{3}\pi$	$S_1^1 \cap S_2^2$	0	$\frac{1}{2\sqrt{\pi}}$
$\frac{2}{3}\pi \leqq \arg z < \frac{4}{3}\pi$	$S_2^1 \cap S_2^2$	$\frac{i}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$

From the above table we at last obtain

$$(8.60)\quad Ai(z) \sim \frac{-i}{2\sqrt{\pi}} y^1(z) + \frac{1}{2\sqrt{\pi}} y^2(z)$$

$$\sim \frac{-i}{2\sqrt{\pi}} \exp\left(\frac{2}{3} z^{3/2}\right) z^{-1/4} \sum_{s=0}^{\infty} h^1(s) z^{-3s/2} \quad \text{in } -\frac{4}{3}\pi < \arg z < -\pi,$$

$$(8.61)\quad Ai(z) \sim \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right) z^{-1/4} \sum_{s=0}^{\infty} h^2(s) z^{-3s/2} \quad \text{in } -\pi < \arg z < \pi,$$

$$(8.62)\quad Ai(z) \sim \frac{i}{2\sqrt{\pi}} y^1(z) + \frac{1}{2\sqrt{\pi}} y^2(z)$$

$$\sim \frac{i}{2\sqrt{\pi}} \exp\left(\frac{2}{3} z^{3/2}\right) z^{-1/4} \sum_{s=0}^{\infty} h^1(s) z^{-3s/2} \qquad \text{in } \pi < \arg z < \frac{4}{3}\pi.$$

From this, we see that the Stokes phenomenon of $Ai(z)$ occurs when z goes across the negative real axis $\arg z = \pm\pi$, i.e., the negative real axis is the actual Stokes line of $Ai(z)$. Similarly we can analyze the global behavior of the Airy function of the second kind $Bi(z)$ by exactly the same method as above.

Lastly we mention that our theory can be applied to a variety of problems for linear differential equation involving parameters, e.g., turning problems and eigenvalue problems. In particular, we call attention to the analysis by J. B. Mcleod [12], in which he clarified, treating (8.1), a close relation between the solution of connection problems and the determination of the distribution of eigenvalues for singular boundary value problems (see also Y. Sibuya [19]).

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