HIROSHIMA MATH. J.
9 (1979), 59–60

# *Non-Existence of Torsion Free Flat Connections on a Real Semisimple Lie Group*

Hiroshi MATSUSHIMA and Kiyosato OKAMOTO

(Received August 28, 1978)

In [2], J. Milnor studied curvatures of left invariant riemannian metrics on Lie groups. In particular, he proved that any 3-dimensional Lie group other than $E(2)$ and $\mathbf{R}^3$ has no flat left invariant riemannian metric.

In [3], K. Nomizu has shown that the 3-dimensional unimodular Lie groups $E(2)$, $E(1, 1)$, the Heisenberg group and $\mathbf{R}^3$ admit flat left invariant Lorentz metrics, but $SO(3)$ does not. According to the table on the page 307 in [2], the only case left open is that the Lie group is $SL(2, \mathbf{R})$. We remark that this is a real simple Lie group.

In this paper, we prove the following theorem.

THEOREM. *Let G be a real semisimple Lie group. Then G has no left invariant torsion free flat affine connection.*

PROOF. We denote by $\mathfrak{g}$ the Lie algebra of $G$. Since $\mathfrak{g}$ is semisimple, we obtain $\mathfrak{g}=[\mathfrak{g}, \mathfrak{g}]$. It follows that for any representation $\rho$ of $\mathfrak{g}$ trace $\rho(X)=0$ $(X \in \mathfrak{g})$.

Assume that there exists a left invariant torsion free flat affine connection $\nabla$. Put $V=\mathfrak{g}$. Since $\nabla$ is flat, $\nabla: X \rightarrow \nabla_X$ is a representation of $\mathfrak{g}$ on $V$. We denote by $C^q(\mathfrak{g}, V)$, $H^q(\mathfrak{g}, \nabla)$ the space of $q$-dimensional $V$-cochains, the $q$-dimensional cohomology group associated with the representation $\nabla$, respectively (see [1, Chapter IV]). We define $c \in C^1(\mathfrak{g}, V)$ by $c(X)=X$ $(X \in \mathfrak{g})$. Then we have $dc(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y]$ for all $X, Y \in \mathfrak{g}$. Since $\nabla$ is torsion free, we have $dc=0$. Now, $G$ is semisimple. This implies $H^1(\mathfrak{g}, \nabla)=0$ (cf. [1, Theorem 25.1]). Therefore there exists an element $H$ in $C^0(\mathfrak{g}, V)$ $(=V)$ such that $dH=c$. For any $X \in \mathfrak{g}$, we have $X=c(X)=dH(X)=\nabla_X H$. Since $\nabla$ is torsion free, we have $\nabla_X H=\nabla_H X+[X, H]=\nabla_H X-\mathrm{ad}(H)X$. Hence we have

$$\nabla_H = I + \mathrm{ad}(H),$$

where $I$ is the identity operator on $\mathfrak{g}$, and ad is the adjoint representation of $\mathfrak{g}$. Since $\nabla$ and ad are both representations of $\mathfrak{g}$, we have $0=\mathrm{trace}\,\nabla_H=\mathrm{trace}\,I+\mathrm{trace}\,\mathrm{ad}(H)=\dim \mathfrak{g}$. This is a contradiction. Q.E.D.

## References

[1] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc. **63** (1948), 85–124.
[2] J. Milnor, Curvatures of left invariant metrics on Lie groups, Advances in Math. **21** (1976), 293–329.
[3] K. Nomizu, Left-invariant Lorentz metrics on Lie groups, to appear.

*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*

Added in Proof;

1. During the proof of this paper, we have been acquainted with the fact that our theorem had been obtained by Professor Y. Matsushima (University of Notre Dame) about ten years ago (unpublished).
2. Recently Mr. Doi (Hiroshima University) generalized our theorem to reductive spaces of semisimple Lie groups.