HIROSHIMA MATH. J.
9 (1979), 47–57

On n-Manifolds whose Punctured Manifolds are Imbeddable in $(n+1)$-Spheres and Spherical Manifolds

Akio KAWAUCHI

(Received July 28, 1978)

1. Statements of results

Throughout this paper, spaces and maps are considered in the piecewise linear category unless otherwise stated. Let M^n be a closed connected orientable n-dimensional manifold. By $M^n_\circ$ we denote a compact punctured submanifold of M^n, that is, a submanifold obtained from M^n by removing the interior of an n-ball.

The purpose of this paper is to discuss some properties of M^n such that $M^n_\circ$ can be imbedded in the $(n+1)$-sphere S^{n+1}, or equivalently such that $M^n_\circ$ is homeomorphic to a submanifold in S^{n+1} bounded by some locally flat $(n-1)$-knot $K^{n-1}\subset S^{n+1}$. Since M^n is imbeddable in S^{n+1} for $n\leq 2$, we assume $n\geq 3$.

We shall prove the following complete classification theorem of the homology groups of such manifolds M^n, where an abelian group G is called a *direct double* if $G\simeq A\oplus A$ for some A.

THEOREM I. *Assume that a punctured manifold $M^n_\circ$ of a closed connected orientable n-manifold M^n is imbeddable in S^{n+1}. Then the integral homology groups $G_i=H_i(M^n; Z)$ of M^n satisfy the following properties* (1)–(3):

(1) *If $n=2q+1$ and $q\geq 1$ is odd, then the* 2*-primary component of G_q is a direct double.*

(2) *If $n=2q+1$ and $q\geq 2$ is even, then the torsion part* $\mathrm{Tor}\,G_q$ *of G_q is a direct double.*

(3) *If $n=2q$ and $q\geq 2$, then* $\mathrm{Tor}\,G_{q-1}\simeq\mathrm{Tor}\,G_q$ *and* $G_q/\mathrm{Tor}\,G_q$ *is a direct double.*

Conversely, assume that a series $G_1,\dots, G_q$ of finitely generated abelian groups satisfies the above properties (1)–(3). *Then there exists a closed connected orientable n-manifold M^n such that $H_i(M^n; Z)=G_i$ for $1\leq i\leq q$ and $M^n_\circ$ is imbeddable in S^{n+1}.*

REMARK 1.1. By using the Alexander duality, W. Hantzsche [5, p. 42] obtained the following result which is analogous to the first half of Theorem I: *If a closed manifold M^n is imbedded in S^{n+1}, then* $\mathrm{Tor}\,H_q(M^n; Z)$ *is a direct double for $n=2q+1(\geq 3)$ and* $H_q(M^n; Z)/\mathrm{Tor}\,H_q(M^n; Z)$ *is a direct double for $n=2q(\geq 4)$.* By Lemma 2.2, we see also that this homological classification is complete in an analogous sense to the latter half of Theorem I. A noteworthy difference between imbeddings of punctured and unpunctured manifolds appears in the case that $n=2q+1$ and $q\geq 1$ is odd. As a concrete example, the lens space $L(a, b)$ for odd a is not imbeddable in S^4, though $L(a, b)_\circ$ is imbeddable in S^4 (cf. W. Hantzsche [5, Satz 3], H. Schubert [16, Satz 6] and E. C. Zeeman [20, p. 486]).

For M^n of odd dimension $n=2q+1$ and a field F, consider the semi-characteristic

$$\hat{\chi}(M^n; F) = \textstyle\sum_{i=1}^{q} \dim_F H_i(M^n; F) \qquad (\mathrm{mod}\, 2).$$

Then, the first half of Theorem I implies the following

COROLLARY. *Assume that* $M^n_\circ$ *is imbedded in* S^{n+1}. *Then the difference* $\hat{\chi}(M^n; Z_2)-\hat{\chi}(M^n; Q)$ *of the semi-characteristics is* 0 (mod 2) *for odd* n, *and the Euler characteristic* $\chi(M^n)$ *is* 0 (mod 2) *for even* n.

Now, consider the following notion due to D. Puppe [15]:

DEFINITION. A closed connected orientable n-manifold M^n is *spherical* (*sphärenähnlich*), if there exists a degree one map of the $(n+1)$-sphere S^{n+1} to the suspension ΣM^n of M^n.

Then, we see the following important homotopical property in Proposition 2.2:

ASSERTION. *If a punctured manifold* $M^n_\circ$ *of a closed connected orientable manifold* M^n *is imbedded in* S^{n+1}, *then* M^n *is spherical.*

As is seen in Lemma 2.1, the homological restrications of M^n in Theorem I are deduced from this assertion.

On the other hand, the following are known:

PUPPE'S CRITERION [15, Satz 12]: *M^n is spherical if and only if there is a closed connected orientable n-manifold M' with a degree one map $M'\to M^n$ such that M' is imbedded in S^{n+1} by a locally flat imbedding.* (His arguments, although presented only in the differential category, work equally in the piecewise linear category.)

CAPPELL-SHANESON'S IMBEDDING THEOREM [3, Th. 6.4]: *If M^n is spherical, then M^n is imbeddable in S^{n+2}.*

We can see that the converse of this theorem is not true. In fact, every orientable closed connected 3-manifold M^3 is imbeddable in S^5 by a locally flat imbedding by M. W. Hirsch [6, Cor. 4], but M^3 is not spherical if the 2-primary component of $H_1(M^3; Z)$ is not a direct double (e.g., the lens space $L(a, b)$ for even a) by Lemma 2.1.

Furthermore, we shall show the following

THEOREM II. *For each $n \geq 3$ with $n \neq 4$, there exists a closed connected orientable n-manifold M^n such that M^n is spherical but $M^n_\circ$ is not imbeddable in S^{n+1}.*

In case $n=4$, a corresponding result is not known.

REMARK 1.2. For each $n \geq 3$, there exists a closed connected orientable n-manifold which is not spherical as is seen in Example 3.3.

2. Proof of Theorem I

In the first place, we note the following known result:

PROPOSITION 2.1. *Let V^n be a compact n-manifold and W^{n+1} a (possibly non-compact) manifold without boundary. If V^n is imbeddable in W^{n+1}, then V^n is so in W^{n+1} by a locally flat imbedding.*

PROOF. If $\partial V^n = \emptyset$, then this follows from M. Kato [7, Th. 3.7]. Let $\partial V^n \neq \emptyset$ and assume that V^n is a submanifold of W^{n+1}. Take a regular neighborhood N of ∂V^n in W^{n+1} such that $N \cap V^n$ is a collar neighborhood of ∂V^n in V^n, and set

$$W' = W^{n+1} - \operatorname{Int} N, \qquad V' = V^n - \operatorname{Int}(N \cap V^n).$$

Then V^n is homeomorphic to V', and we obtain a proper imbedding $f\colon V^n \to W'$. By [7, Th. 3.7], f is approximated by a locally flat proper imbedding $f'\colon V^n \to W'$. This completes the proof.

PROPOSITION 2.2 (Assertion in § 1). *Let M^n be a closed connected orientable n-manifold. If a punctured submanifold $M^n_\circ$ of M^n is imbedded in S^{n+1}, then M^n is spherical.*

PROOF. By the above proposition, there is a locally flat imbedding of $M^n_\circ$ in S^{n+1}. Hence there is an imbedding $f\colon M^n_\circ \times [0, 1] \to S^{n+1}$. This induces clearly a homotopy equivalence

$$S^{n+1}/\mathrm{cl}(S^{n+1} - f(M^n_\circ \times [0, 1])) \longrightarrow \Sigma M^n \text{ (the suspension of } M^n)$$

and we have a degree one map $S^{n+1} \to \Sigma M^n$ by composing the projection (cf. D. B. A. Epstein [4]). Thus we have the proposition.

Now, we prove the first half of Theorem I in § 1 by the above proposition and the following

LEMMA 2.1. *Suppose that M^n is spherical.*

(1) *If* $n=2q+1$ *and* $q\geq 1$ *is odd, then the 2-primary component of* $H_q(M^n; Z)$ *is a direct double.*

(2) *If* $n=2q+1$ *and* $q\geq 2$ *is even, then* $\mathrm{Tor}\, H_q(M^n; Z)$ *is a direct double.*

(3) *If* $n=2q$ *and* $q\geq 2$, *then* $H_q(M^n; Z)/\mathrm{Tor}\, H_q(M^n; Z)$ *is a direct double.*

PROOF. (1) Let $n=2q+1$ and $q\geq 1$ be odd. Then D. Puppe [15, Satz 11] proved that the Postnikov square

$$k^i\colon H^q(M^n; Z_{2^i}) \longrightarrow H^n(M^n; Z_{2^{i+1}}) \qquad (i = 1, 2,\dots)$$

vanishes if M^n is spherical. Further, an argument parallel to D. Puppe [15, Satz 16] shows that $k^i=0$ if and only if

$$(*) \qquad 2^{i-1}L(z, z) = 0 \quad \text{for all} \quad z\in H_q(M^n; Z) \quad \text{with} \quad 2^i z = 0,$$

where $L\colon \mathrm{Tor}\, H_q(M^n; Z)\times \mathrm{Tor}\, H_q(M^n; Z)\to Q/Z$ is the dual linking pairing.

Let T be the 2-primary component of Tor $H_q(M^n; Z)$. Then L induces a dual pairing $L\colon T\times T\to Q/Z$.

SUBLEMMA. *T admits an orthogonal splitting* $T^1\oplus\cdots\oplus T^s$ *with respect to L, where* T^i *is isomorphic to a direct sum of some copies of* Z_{2^i}.

By this splitting, L induces also a dual pairing $L\colon T^i\times T^i\to Q/Z$. Define a dual pairing

$$L^i\colon (T^i \otimes Z_2) \times (T^i \otimes Z_2) \longrightarrow Q/Z \qquad (i = 1,\dots, s)$$

by the equality $L^i(a_1\otimes 1, a_2\otimes 1)=2^{i-1}L(a_1, a_2)$ for $a_1, a_2\in T^i$. Then, by translating 1/2 of Q/Z to 1 of Z_2, L^i defines a non-singular form

$$\tilde{L}^i\colon (T^i \otimes Z_2) \times (T^i \otimes Z_2) \longrightarrow Z_2 \qquad (i = 1,\dots, s).$$

Since $2^{i-1}L(a, a)=0$ for all $a\in T^i$ by (*), it follows that $\tilde{L}^i(a\otimes 1, a\otimes 1)=0$ for all $a\in T^i$. Hence the form $\tilde{L}^i$ is symplectic, and we see that $\dim_{Z_2}T^i\otimes Z_2$ is even by taking a symplectic basis. Thus the 2-primary component T of $H_q(M^n; Z)$ must be a direct double.

(2) Let $n=2q+1$ and $q\geq 2$ be even. Then, by W. Browder [1, Th. 1], $\mathrm{Tor}\, H_q(M^n; Z)$ is isomorphic to either $B\oplus B$ or $B\oplus B\oplus Z_2$.

On the other hand, according to G. Lusztig, J. Milnor and F. P. Peterson [13], the difference $\hat{\chi}(M^n; Z_2)-\hat{\chi}(M^n; Q)$ of the semi-characteristics is equal to the Stiefel-Whitney number $w_2w_{2q-1}[M^n]$. Further, since M^n is spherical, the total Stiefel-Whitney class of M^n is trivial by D. Puppe [15, Satz 13]. Thus the above difference is 0.

Moreover, it is easy to see that

$$(**) \qquad \hat{\chi}(M^n; Z_2) - \hat{\chi}(M^n; Q) \equiv \dim_{Z_2}(\mathrm{Tor}\, H_q(M^n; Z) \otimes Z_2) \qquad (\mathrm{mod}\, 2)$$

for $n=2q+1$ (cf. [13, p. 358]). These show that $\mathrm{Tor}\, H_q(M^n; Z)=B\oplus B$.

(3) Let $n=2q$ and $q\geq 2$. Since $Sq^q\colon H^q(M^n; Z_2)\to H^n(M^n; Z_2)$ is trivial by D. Puppe [15, 6.7], the dual pairing

$$\cup : H^q(M^n; Z_2)\times H^q(M^n; Z_2)\longrightarrow H^n(M^n; Z_2)$$

is symplectic. Thus $\dim_{Z_2} H^q(M^n; Z_2)$ is even, and the Euler characteristic $\chi(M^n)$ is even by the Poincaré duality over Z_2. Therefore $\mathrm{rank}_Z H_q(M^n; Z)$ is even by the Poincaré duality over Q, and the conclusion of (3) holds.

These complete the proof of Lemma 2.1 except for the proof of Sublemma.

Proof of Sublemma. Let $T=T^1\oplus\cdots\oplus T^s$ be any splitting, where T^i is isomorphic to a direct sum of copies of Z_{2^i}. Let $x\in T^s$ be any element of order 2^u $(1\leq u\leq s)$. Then we can find $x'\in T^s$ with

$$2^{u-1}x=2^{s-1}x'\neq 0,\quad 2^s x'=0.$$

Since the pairing L is non-singular, $2^{s-1}L(x', y)\neq 0$ for some $y\in T$, and so $2^{s-1}L(x', y')\neq 0$ for some $y'\in T^s$. Hence $2^{u-1}L(x, y')\neq 0$ and $L(x, y')\neq 0$. This shows that $L|T^s\times T^s$ is also non-singular.

Therefore L and $L|T^s\times T^s$ determine the isomorphisms

$$T\simeq \mathrm{Hom}(T, Q/Z),\quad x\longrightarrow L(x,\ \ )\quad (x\in T),$$

$$T^s\simeq \mathrm{Hom}(T^s, Q/Z),\quad x'\longrightarrow L(x',\ \ )\quad (x'\in T^s).$$

Thus, for any $x\in T$, there exists just one element $x'\in T^s$ with

$$L(x, y)=L(x', y)\qquad \text{for all}\quad y\in T^s,$$

and $x-x'$ belongs to the orthogonal complement T' of T^s in T with respect to L. This shows $T=T'\oplus T^s$, and we see the sublemma by the induction on s.

Proof of the First Half of Theorem I. Proposition 2.2 and Lemma 2.1 imply the first half of Theorem I, where $\mathrm{Tor}\, G_{q-1}\simeq \mathrm{Tor}\, G_q$ in (3) is the Poincaré duality.

Proof of Corollary to Theorem I. By the equality (**) in the proof of Lemma 2.1 and the Poincaré duality, the corollary is a direct consequence of the first half of Theorem I.

Remark 2.1. Corollary to Theorem I holds for every spherical manifold M^n (cf. Lemma 2.1).

To prove the latter half of Theorem I, we use the following lemmas.

Lemma 2.2. *For any integers* $n(\geq 3)$, $m(\geq 0)$ *and* p *with* $1\leq p\leq n/2$, *there exists a closed connected orientable n-manifold* $M_p^n(m)$ *which is imbeddable in* S^{n+1} *by a locally flat imbedding and whose homology groups* $H_i(M_p^n(m); Z)$ $(1 \leq i \leq n/2)$ *are given as follows:*

(1) *Let* $n=2q+1$ *and* $q \geq 1$. *Then*

$$H_i(M_p^n(m); Z) = \begin{cases} Z_m & \text{if } i = p, \text{ when } p = q \text{ and } m = 0 \text{ or when } p < q, \\ Z_m \oplus Z_m & \text{if } i = p, \text{ when } p = q \text{ and } m \neq 0, \\ 0 & \text{if } i \neq p \quad (1 \leq i \leq q). \end{cases}$$

(2) *Let* $n=2q$ *and* $q \geq 2$. *Then*

$$H_i(M_p^n(m); Z) = \begin{cases} Z_m \begin{cases} \text{if} & i = p-1 \text{ or } p, \text{ when } p = q \text{ and } m \neq 0, \\ \text{if} & i = p+1, \text{ when } p = q-1 \text{ and } m \neq 0, \\ \text{if} & i = p, \text{ when } p < q, \end{cases} \\ Z_m \oplus Z_m \quad \text{if} \quad i = p, \text{ when } p = q \text{ and } m = 0, \\ 0 \qquad \text{otherwise} \quad (1 \leq i \leq q). \end{cases}$$

PROOF. According as $m=0$ or 1, $M_p^n(m)=S^p \times S^{n-p}$ or S^n is a desired manifold.

Let $m \geq 2$. Let $f_m: S^1 \to S^1$ be a simplicial map of degree m and $C(f_m)$ be the mapping cone of f_m. Then $C(f_m)$ is a simplicial 2-complex, and is imbeddable in S^4. Let $L_1^i(m)$ be an imbedded image of $C(f_m)$ in S^{i+1} $(i \geq 3)$ and set

$$L_p^n(m) = \Sigma^{p-1} L_1^{n-p+1}(m) \subset \Sigma^{p-1} S^{n-p+2} = S^{n+1} \qquad (n-2 \geq p \geq 1),$$

where Σ^{p-1} denotes the $(p-1)$-th suspension. Then, the boundary $M_p^n(m)$ of the regular neighborhood of $L_p^n(m)$ in S^{n+1} is certainly a locally flat submanifold of S^{n+1}, and we see easily that $M_p^n(m)$ has the desired homology groups by using the Alexander duality and the Mayer-Vietoris sequence.

LEMMA 2.3. *Let* $n=2q+1$ *and* $q \geq 1$ *is odd. Then for any odd integer* $a \geq 3$, *there exists a closed connected orientable n-manifold* $M^n(a)$ *such that its punctured submanifold* $M^n(a)_0$ *is imbeddable in* S^{n+1} *by a locally flat imbedding and*

$$H_i(M^n(a); Z) = \begin{cases} Z_a & \text{if} \quad i = q, \\ 0 & \text{if} \quad 1 \leq i < q. \end{cases}$$

PROOF. For $q=1$, the lens space $L(a, b)$ is such a manifold by H. Schubert [16, Satz 6] and E. C. Zeeman [20, p. 486]. For each odd $q \geq 3$, by making use of M. A. Kervaire [11, Th. II.2], [10, Th. 4.3] and W. Browder and J. Levine [2], we can construct a locally flat $2q$-knot $K^{2q} \subset S^{n+1}$ such that $E(K^{2q}) = S^{n+1} - T(K^{2q})$ ($T(K^{2q})$ is an open tubular neighborhood of K^{2q} in S^{n+1}) is a fiber bundle over S^1 and

$$\tilde{H}_i(\tilde{E}(K^{2q}); Z) = \begin{cases} Z<t>/(t+1, a) & \text{if} \quad i = q, \\ 0 & \text{if} \quad i \neq q, \end{cases}$$

where $\tilde{E}(K^{2q})$ is the connected infinite cyclic cover of $E(K^{2q})$ and $Z<t>$ is the integral group ring of the infinite cyclic covering transformation group $<t>$ of $\tilde{E}(K^{2q})$. Let $M^n(a)$ be a manifold such that $M^n(a)_\circ$ is a fiber of this bundle $E(K^{2q})$ over S^1. Then we see easily the lemma.

Now, we are ready to prove Theorem I.

PROOF OF THE LATTER HALF OF THEOREM I. Note that if $M^n_\circ$ and $M'^n_\circ$ are imbeddable in S^{n+1} by locally flat imbeddings, then so is $(M^n \# M'^n)_\circ$. Then, we can realize a desired manifold in the latter half of Theorem I, by making a connected sum of some manifolds in Lemmas 2.2 and 2.3.

3. Proof of Theorem II

First of all, we take notice of simply connected manifolds.

EXAMPLE 3.1. *For a simply connected closed* 4-*manifold* M^4, *the following three conditions* (a)–(c) *are equivalent*:

(a) M^4 *is spherical.*

(b) M^4 *is spin, i.e., the Stiefel-Whitney class* $w_2(M^4)$ *is zero.*

(c) $M^4_\circ$ *is imbeddable in* S^5.

On the other hand, these conditions do not imply that M^4 *is imbeddable in* S^5. *In fact, if* M^4 *is orientable and has non-zero signature* (*e.g., if* M^4 *is a Kummer or* K3 *surface*), *then* M^4 *cannot be imbedded in* S^5.

PROOF. Suppose that M^4 is simply connected and spin. Then the double $D(M) = \partial(M^4_\circ \times [0, 1])$ is also so, and the signature of $D(M)$ is 0. Hence, it follows from J. Milnor [14, Cor. 3] and C. T. C. Wall [19, Th. 3] that a connected sum of $D(M)$ and some copies of $S^2 \times S^2$ is homeomorphic to a connected sum S of some copies of $S^2 \times S^2$. S is clearly imbeddable in S^5. Thus $M^4_\circ$ is also so, and we see (b)$\Rightarrow$(c). Proposition 2.2 and D. Puppe [15, Satz 13] show (c)$\Rightarrow$(a)$\Rightarrow$(b).

If M^4 is imbeddable in S^5, then M^4 is imbedded in S^5 by a locally flat imbedding (cf. Proposition 2.1). Thus M^4 separates S^5 into two compact orientable 5-manifolds whose boundaries are M^4. This implies that the signature of M^4 is 0, and we see the latter half.

EXAMPLE 3.2. *For a simply connected closed 5-manifold M^5, the following three conditions* (a)–(c) *are equivalent*:

(a) *M^5 is spherical.*

(b) *M^5 is spin, i.e., $w_2(M^5)=0$.*

(c) *M^5 is imbeddable in S^6.*

PROOF. Note that $M_2^5(m)$ constructed in the proof of Lemma 2.2 is simply connected and spin. Then Smale's classification of simply connected spin manifolds [17] states that if M^5 is spin, then M^5 is homeomorphic to a connected sum of some copies of $M_2^5(m)$. Thus we see (b)⇒(c).

For $n\geq 6$, we obtain the following

LEMMA 3.1. *For each $n\geq 6$, there is a simply connected closed n-manifold M^n such that M^n is spherical but $M_\circ^n$ is not imbeddable in S^{n+1}.*

PROOF. Consider $S^{4i}\times S^{n-4i}$ for $n-4i\geq 2$ and $i\geq 1$. According to D. Sullivan [18], there exists an n-manifold M^n which is homotopy equivalent to $S^{4i}\times S^{n-4i}$ and whose Hirzebruch-Thom class $L_i(M)\in H^{4i}(M; Q)$ is non-zero. Since $S^{4i}\times S^{n-4i}$ is spherical by D. Puppe [15, Satz 5], M^n is spherical.

Suppose that $M_\circ^n$ is imbedded in S^{n+1} by a locally flat imbedding. Then, by a conic extension, we obtain an imbedding $f: M^n\to S^{n+2}$ such that $A=\{x\in M^n|\ f$ is not locally flat at $x\}$ consists of at most one point. On the other hand, since $n\geq 6$, an argument of S. E. Cappel and J. L. Shaneson [3, Prop. 6.8] shows that $\dim A\geq n-4i(\geq 2)$, which is a contradiction. Thus, by Proposition 2.1, $M_\circ^n$ is not imbeddable in S^{n+1}.

Now, we consider another construction of spherical manifolds.

Let K^{n-2} be a framed knot in S^n, where the framing is assumed to be a null-homologous framing if $n=3$, and set

$$M^n(K) = \partial(D^{n+1}\cup(D^{n-1}\times D^2)),$$

where $(\partial D^{n-1})\times D^2=K^{n-2}\times D^2\subset S^n=\partial D^{n+1}$. Then

LEMMA 3.2. *$M^n(K)$ is a spherical n-manifold.*

PROOF. By the above definition, there is a map

$$f: M^n(K)\longrightarrow S^1\times S^{n-1}$$

which induces an isomorphism $f_*: H_*(M^n(K); Z)\simeq H_*(S^1\times S^{n-1}; Z)$. Then the suspension $\Sigma f: \Sigma M^n(K)\to\Sigma(S^1\times S^{n-1})$ is a homotopy equivalence by the well-known theorem of J. H. C. Whitehead. Since $S^1\times S^{n-1}$ is spherical, we obtain a degree one map $S^{n+1}\to\Sigma M^n(K)$ as desired.

From M. Kato [8, Th. 5.5], it follows that $M^n(K)$ *for any locally flat knot*

$K^{n-2} \subset S^n$ with even $n \geq 4$ is imbeddable in S^{n+1} by a locally flat imbedding.

On the other hand, we see the following

LEMMA 3.3. *For each odd $n \geq 3$, there exists a locally flat knot $K^{n-2} \subset S^n$ such that $M^n(K)_\circ$ is not imbeddable in S^{n+1}.*

PROOF. Let $n=2q+1$ and $q \geq 1$. Consider a locally flat knot $K^{n-2} \subset S^n$ whose qth Alexander polynomial is

$$A(t) = t^2 - t + 1 \quad \text{or} \quad t^4 - t^2 + 1$$

according as q is odd or even. Such a knot exists certainly by an argument of M. A. Kervaire [11, Th. II.2]. (For $q=1$, $A(t)$ is the Alexander polynomial of a trefoil knot.)

Suppose that $M^n(K)_\circ$ is imbeddable in S^{n+1}. Then, by Proposition 2.1, there is a submanifold N in S^{n+1} which is homeomorphic to $M^n(K)_\circ \times [0, 1]$. Thus, we can choose a basis $\{t_1, t_2\}$ for $H_1(\partial N; Z) \simeq Z \oplus Z$ such that $i_{1*}(t_1) = i_{1*}(t_2)$ is a generator of $H_1(N; Z) \simeq Z$, where $i_1 : \partial N \subset N$. Set

$$W = S^{n+1} - \operatorname{Int} N$$

and $i_2 : \partial W = \partial N \subset W$. Since $i_{1*} + i_{2*} : H_1(\partial W; Z) \simeq H_1(N; Z) \oplus H_1(W; Z)$, it follows that

$$i_{2*}(t_1) = ue, \quad i_{2*}(t_2) = ve$$

for some integers u and v with $|u| \neq |v|$, where e is a generator of $H_1(W; Z) \simeq Z$.

Let $\gamma : \pi_1(W) \to \langle t \rangle$ be an epimorphism, and $\tilde{W}$ be the infinite cyclic cover of W associated with γ. Then we can show that the homology exact sequence of $(\tilde{W}, \partial\tilde{W})$ induces the exact sequence

$$(***) \qquad T_{q+1}(\tilde{W}, \partial\tilde{W}) \xrightarrow{\partial} T_q(\partial\tilde{W}) \xrightarrow{i_*} T_q(\tilde{W}),$$

where $T_*(\tilde{X}, \tilde{X}') = \operatorname{Tor}_{Q\langle t \rangle} H_*(\tilde{X}, \tilde{X}'; Q)$.

For $q>1$, we see $H_{q+1}(W, \partial W; Q)=0$ and hence $T_{q+1}(\tilde{W}, \partial\tilde{W}) = H_{q+1}(\tilde{W}, \partial\tilde{W}; Q)$ by using the Wang exact sequence. Thus (***) is exact.

Let $q=1$. Then $\operatorname{rank}_{Q\langle t \rangle} H_1(\tilde{W}; Q)=0$ and $\operatorname{rank}_{Q\langle t \rangle} H_1(\partial\tilde{W}; Q)=1$. Since $H_2(W, \partial W; Z)=Z$, we have $\operatorname{rank}_{Q\langle t \rangle} H_2(\tilde{W}, \partial\tilde{W}; Q) \leq 1$. Thus the exact sequence of $(\tilde{W}, \partial\tilde{W})$ implies that the image of $H_2(\tilde{W}; Q) \to H_2(\tilde{W}, \partial\tilde{W}; Q)$ is contained in $T_2(\tilde{W}, \partial\tilde{W})$. Therefore (***) is exact.

By the exactness of (***) and Fundamental Theorem II in [9], the local signature $\sigma_\omega^{\dot{\gamma}}(\partial W)$ must be 0 at $\omega \in (-1, 1)$, where $\dot{\gamma} : \pi_1(\partial W) \to \langle t \rangle$ is the restriction of γ.

On the other hand, by the construction, the qth Alexander polynomial of ∂W with respect to the epimorphism $\dot{\gamma}$ is equal to $A(t^u)A(t^v)$. Note that a polynomial $t^{2d}-t^d+1$ $(d\geq 1)$ has $2d$ distinct roots of complex numbers of norm 1. Since $|u|\neq|v|$, one finds a real irreducible factor of the form $t^2-2\omega_0 t+1$ $(-1<\omega_0<1)$ of $A(t^u)A(t^v)$ with multiplicity one. Then by [9, Lemma 1.4], we have $\sigma^{\gamma}_{\omega_0}(\partial W)=\pm 2$, which is a contradiction.

Thus $M^n(K)_o$ is not imbeddable in S^{n+1}, and the lemma is proved.

By Lemmas 3.1, 3.2 and 3.3, we see Theorem II in § 1.

Finally, we give some examples of non-spherical manifolds.

EXAMPLE 3.3. *The odd-dimensional real projective space* RP^{2q+1} $(q\geq 1)$, *the* $4m$-*dimensional complex projective space* CP^{2m} $(m\geq 1)$ *and the product* $CP^{2m}\times S^r$ $(m\geq 1,\ r\geq 1)$ *are not spherical.*

PROOF. Since the total Stiefel-Whitney classes of RP^{2q+1} $(q\neq 2^e-1)$, CP^{2m} and $CP^{2m}\times S^r$ are not trivial, none of these manifolds is spherical by D. Puppe [15, Satz 13]. When $q=2^e-1$, $H_q(RP^{2q+1};Z)\simeq Z_2$ and $\chi(RP^{2q+1};Z_2)-\chi(RP^{2q+1};Q)=1$. Thus RP^{2q+1} $(q=2^e-1)$ is not spherical by Remark 2.1.

REMARK 3.1. Note that the analogous product $RP^3\times S^r$ $(r\geq 1)$ or more generally the product $M^3\times S^r$ for every closed connected orientable 3-manifold M^3 is spherical and imbeddable in S^{r+4}. In fact, M^3 is imbeddable in S^{r+4} so that the regular neighborhood of an imbedded image of M^3 in S^{r+4} is homeomorphic to $M^3\times D^{r+1}$ (M. W. Hirsch [6]).

References

[1] W. Browder, *Remark on the Poincaré duality theorem*, Proc. Amer. Math. Soc. **13** (1962), 927–930.

[2] W. Browder and J. Levine, *Fibering manifolds over a circle*, Comment. Math. Helv. **40** (1966), 153–160.

[3] S. E. Cappell and J. L. Shaneson, *Piecewise linear embeddings and their singularities*, Ann. of Math. **103** (1976), 163–228.

[4] D. B. A. Epstein, *Embedding punctured manifolds*, Proc. Amer. Math. Soc. **16** (1965), 175–176.

[5] W. Hantzsche, *Einlagerung von Manigfaltigkeiten in euklidische Räume*, Math. Z. **43** (1938), 38–58.

[6] M. W. Hirsch, *The imbedding of bounded manifolds in Euclidean space*, Ann. of Math. **74** (1961), 1494–1497.

[7] M. Kato, *Embeddings of spheres and balls in codimension* ≤ 2, Invent. Math. **10** (1970), 89–107.

[8] M. Kato, *Higher dimensional PL knots and knot manifolds*, J. Math. Soc. Japan **21** (1969), 458–480.

[9] A. Kawauchi, *On a 4-manifold homology equivalent to a bouquet of surfaces*, to appear.

[10] A. Kawauchi, *A partial Poincaré duality theorem for infinite cyclic coverings*, Quart. J. Math. **26** (1975), 437–458.

[11] M. A. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271.

[12] M. A. Kervaire, *Courbure integral généralisée et homotopie*, Math. Ann. **131** (1956), 219–252.

[13] G. Lusztig, J. Milnor and F. P. Peterson, *Semi-characteristics and cobordism*, Topology **8** (1969), 357–359.

[14] J. Milnor, *On simply connected 4-manifolds*, Symp. Inter. de Topologia Algebrica Mexico (1958), 122–128.

[15] D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen* II, Math. Z. **69** (1958), 395–417.

[16] H. Schubert, *Knoten mit zwei Brücken*, Math. Z. **65** (1950), 133–170.

[17] S. Smale, *On the structure of 5-manifolds*, Ann. of Math. **75** (1962), 38–46.

[18] D. Sullivan, *Triangulating and smoothing homotopy equivalences*, Geometric Topology Seminar Notes, Princeton Univ. 1967.

[19] C. T. C. Wall, *On simply connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141–149.

[20] E. C. Zeeman, *On twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

Department of Mathematics,
Osaka City University