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S^1-Actions on Cohomology Complex Projective Spaces with Three Components of the Fixed Point Sets

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§0. Introduction

A closed oriented smooth $2n$-manifold M is called a *cohomology complex projective n-space* (*cohomology* CP^n), if the integral cohomology ring of M is isomiphic to that of the complex projective n-space CP^n, i.e., if there exists an element $\alpha \in H^2(M; Z)$ such that

(0.1) $$H^*(M; Z) = Z[\alpha]/(\alpha^{n+1}), \quad <\alpha^n, [M]> = 1.$$

Then the conjecture of T. Petrie [4; Intro.] for homotopy complex projective spaces follows immediately from the following statement:

(0.2) *Assume that M is a cohomology CP^n with $\alpha \in H^2(M; Z)$ satisfying* (0.1). *If M admits a non-trivial (smooth) S^1-action, then the total Pontrjagin class of M is given by*

$$p(M) = (1 + \alpha^2)^{n+1}.$$

K. Wang [6] and T. Yoshida [7] have proved independently the conjecture of T. Petrie for semi-free actions by the following

THEOREM ([6; Prop. 2.2–3, Cor. 2.5]). (0.2) *is valid, if M admits an S^1-action whose fixed point set has two (connected) components.*

The purpose of this note is to prove the following

THEOREM 1. (0.2) *and the conjecture of T. Petrie are valid, if M admits an S^1-action whose fixed point set has three components.*

We shall prove this theorem by the following

THEOREM 2. *Let M be a cohomology CP^n. Then any effective S^1-action on M with three components of the fixed point set is of the linear type.*

Here, an S^1-action on M is defined to be of the *linear type*, if its normal representations of the fixed point set are of the same type as those of a linear S^1-action on CP^n (cf. Definition 1.6).

Since a non-trivial S^1-action on M induces an effective S^1-action on M, Theorem 1 follows immediately from Theorem 2 and the following theorem; recently A. Hattori [3] has proved it, and our original proof of it in the special case is thus excluded.

THEOREM ([3; Prop. 4.15]). (0.2) *is valid, if M admits a non-trivial S^1-action of the linear type.*

We note that there exists an exotic S^1-action on cohomology CP^3 which is not of the linear type ([4; II, §4]), but (0.2) is valid for $n=3$ by the results of J. Dejter [2] and T. Yoshida [8].

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§1. Preliminaries

In this section, we recall some results due to G. Bredon [1], T. Petrie [4] and J. C. Su [5].

Let M be a cohomology CP^n, and suppose that M admits a non-trivial (smooth) S^1-action and its fixed point set $F(S^1, M)$ has l (connected) components $F_0, F_1, \ldots, F_{l-1}$. Then

(1.1) (cf. [1; VII, Th. 5.1]) *each S^1-submanifold F_i of M is a cohomology CP^{n_i} and*

$$n + 1 = \textstyle\sum_{i=0}^{l-1} (n_i + 1).$$

Let η be the Hopf bundle over M, i.e., the complex line bundle over M whose first Chern class is equal to $\alpha \in H^2(M; Z)$ in (0.1). Then

(1.2) ([4; II, Def. 1.2, Cor. 1.15]) *there are distinct integers $a_0, a_1, \ldots, a_{l-1}$, which are well-defined up to translation, with the following property: An S^1-action on η, which is a lifting of the given S^1-action on M, acts on the fibre $\eta|_x$ $(x \in F_i)$ via the complex representation $t^{a_i} \in R(S^1)$.*

(1.3) (cf. [1; VII, §5]) *For each $i=0, 1, \ldots, l-1$, let ν_i be the normal bundle of F_i in M. Then there are positive integers $m_{i,u}$ $(u=1, \ldots, n-n_i)$ with the following property: An S^1-action on the complex $(n-n_i)$-plane bundle ν_i, which is induced by the given S^1-action on M, acts on the fibre $\nu_i|_x$ $(x \in F_i)$ via the complex representation*

$$\textstyle\sum_{u=1}^{n-n_i} t^{m_{i,u}} \in R(S^1).$$

This representation is called the *normal representation of F_i in M.*

For any subgroup H of S^1, let $F(H, M)$ denote the fixed point set of the H-action on M which is the restriction of the given S^1-action on M.

LEMMA 1.4. *Let H be a subgroup of S^1 of order h, and Y be the component of $F(H, M)$ containing F_i. Then*

$$\dim Y = \dim F_i + 2\#\{u : h | m_{i,u}\},$$

where $\#A$ is the number of elments of a finite set A.

PROOF. By noticing that Y is an S^1-submanifold of M and by studying the complex dimension of the normal bundle of F_i in Y, we see immediately the lemma by the definition of the normal representation. q.e.d.

(1.5) (cf. [1; VII, Th. 5.5]) *The integers in* (1.2) *and* (1.3) *satisfy*

$$\prod_{u=1}^{n-n_i} m_{i,u} = \prod_{0 \leqq j \leqq l-1, j \neq i} |a_j - a_i|^{n_j+1}$$

for $i=0,\dots, l-1$.

DEFINITION 1.6. The given S^1-action on M is said to be of the *linear type*, if $|a_j - a_i|$ occurs (n_j+1)-times in the integers $m_{i,u}$ $(u=1,\dots, n-n_i)$, i.e., if the normal representation of F_i in M is given by

$$\sum_{0 \leqq j \leqq l-1, j \neq i} (n_j + 1) t^{|a_j - a_i|}$$

for $i=0,\dots, l-1$.

(1.7) ([5; Th. 5.1]) *For any prime p and any positive integer r, each component $F_k(p^r)$ $(k=0, 1,\dots, l(p^r)-1)$ of $F(Z_{p^r}, M)$ is $F_k(p^r) \underset{Z_p}{\sim} CP^{n_k(p^r)}$ and*

$$n + 1 = \sum_{k=0}^{l(p^r)-1} (n_k(p^r) + 1).$$

Here, the notation $N \underset{Z_p}{\sim} CP^m$ means that N is a closed oriented (smooth) $2m$-manifold and the cohomology ring $H^*(N; Z_p)$ is given by

$$H^*(N; Z_p) = Z_p[\alpha]/(\alpha^{m+1}), \qquad \alpha \in H^2(N; Z_p).$$

COROLLARY 1.8. *In* (1.1) *and* (1.7),

$$n_k(p^r) + 1 = \sum_{i \in L_k} (n_i + 1),$$

where $L_k = \{i : i=0, 1,\dots, l-1, F_i \subset F_k(p^r)\}$.

PROOF. Since $F_k(p^r)$ is an S^1-submanifold of M,

$$\chi(F_k(p^r)) = \chi(F(S^1, F_k(p^r)))$$

(χ is the Euler characteristic) by [1; III, Th. 10.9]. Further $F(S^1, F_k(p^r))=\bigcup_{i\in L_k} F_i$. Thus we see the desired equality. q.e.d.

(1.9) ([5; Th. 5.4]) *Two components F_i and F_j of $F(S^1, M)$ are contained in a component of $F(Z_{p^r}, M)$ if and only if $|a_j-a_i|$ is a multiple of p^r.*

(1.10) ([5; Prop. 6.3]) *Let H be a subgroup of S^1 and Y be a component of $F(H, M)$. If $Y\cap F(S^1, M)\neq\emptyset$, then*

$$\dim Y \leqq 2(\chi(Y)-1).$$

COROLLARY 1.11. *In (1.10), if Y contains exactly one component F_i of $F(S^1, M)$, then $Y=F_i$ and* $\dim Y=2(\chi(Y)-1)$.

If the order of H is h in addition, then none of the integer $m_{i,u}$ in (1.3) is a multiple of h.

PROOF. By the assumption, $F(S^1, Y)=F_i$ and $\chi(Y)=\chi(F(S^1, Y))$. Thus

$$\dim Y \geqq \dim F_i = 2(\chi(F_i)-1) = 2(\chi(Y)-1) \geqq \dim Y$$

by (1.1) and (1.10). Thus $\dim Y=\dim F_i$ and $Y=F_i$.

The last half follows immediately from Lemma 1.4. q.e.d.

§2. Proof of Theorem 2

In the rest of this note, we assume that the given S^1-action on M is effective and that the fixed point set $F(S^1, M)$ has three components F_0, F_1 and F_2.

LEMMA 2.1. *For the integers a_0, a_1 and a_2 in (1.2), any two of $|a_0-a_1|$, $|a_0-a_2|$ and $|a_1-a_2|$ are relatively prime.*

PROOF. Assume that p^r (p: prime, $r\geqq 1$) divides $|a_0-a_1|$ and $|a_0-a_2|$. Then some component $F_0(p^r)$ of $F(Z_{p^r}, M)$ contains F_0, F_1 and F_2 by (1.9). Hence $\dim F_0(p^r)=2n_0(p^r)=2n=\dim M$ by (1.7), Corollary 1.8 and (1.1). Thus $F_0(p^r)=M$, which contradicts the effectivity of the S^1-action on M. q.e.d.

LEMMA 2.2. *If $|a_i-a_j|$ ($i\neq j$) is a multiple of p^r (p: prime, $r\geqq 1$), then the fixed point set $F(Z_{p^r}, M)$ has two components $F_0(p^r)$ and F_k, where*

$$F_0(p^r)\supset F_i\cup F_j,\quad \{i, j, k\}=\{0, 1, 2\}.$$

PROOF. By (1.9) and the above proof, $F(Z_{p^r}, M)$ has components $F_0(p^r)$ and $F_1(p^r)$ such that $F_0(p^r)\subset F_i\cup F_j$ and $F_1(p^r)\supset F_k$. Then $n_0(p^r)=n_i+n_j+1$ and $n_1(p^r)=n_k$ by Corollary 1.8, and these equalities imply that

$$F(Z_{p^r}, M) = F_0(p^r)\cup F_1(p^r) \quad\text{and}\quad F_1(p^r)=F_k$$

by (1.1) and (1.7). q.e.d.

LEMMA 2.3. *Let*

$$\sum_{u=1}^{n-n_0} t^{m_{0,u}} \in R(S^1) \qquad (n - n_0 = n_1 + n_2 + 2)$$

be the normal representation of F_0 *in* M (cf. (1.3)).

(i) *If* $p^r \| |a_i - a_0|$ (p: *prime*, $r \geqq 1$), *then* $\{u: p^r | m_{0,u}\}$ *consists of exactly* $n_i + 1$ *elements, where* $i = 1$ *or* 2.

(ii) *If* $p^r \| |a_1 - a_0|$ *and* $q^s \| |a_2 - a_0|$ (p, q: *prime*; r, $s \geqq 1$), *then* $\{u: p^r | m_{0,u}\} \cap \{u: q^s | m_{0,u}\} = \emptyset$.

PROOF. (i) By Lemma 2.2, $F(Z_{p^r}, M)$ has a component $F_0(p^r)$ such that $F_0(p^r) \supset F_0 \cup F_i$ and $F_0(p^r) \cap F_j = \emptyset$ ($\{i, j\} = \{1, 2\}$). Then we see (i) by Lemma 1.4 and Corollary 1.8.

(ii) By Lemma 2.2, there are components $F_0(p^r)$ and $F_0(q^s)$ of $F(Z_{p^r}, M)$ and $F(Z_{q^s}, M)$, respectively, such that

$$F_0(p^r) \supset F_0 \cup F_1,\ F_0(p^r) \cap F_2 = \emptyset;\ F_0(q^s) \supset F_0 \cup F_2,\ F_0(q^s) \cap F_1 = \emptyset.$$

Then, $F(Z_{p^r q^s}, M) = F(Z_{p^r}, M) \cap F(Z_{q^s}, M)$ has a component Y with $Y \supset F_0$, $Y \cap (F_1 \cup F_2) = \emptyset$. Thus, since $(p, q) = 1$ by Lemma 2.1, (ii) follows immediately from the latter half by Corollary 1.11. q.e.d.

Now, we can prove Theorem 2 in §0 by Definition 1.6 and the following

LEMMA 2.4. *The normal representation in the above lemma is given by*

$$(n_1 + 1)t^{|a_1 - a_0|} + (n_2 + 1)t^{|a_2 - a_0|}.$$

PROOF. It is sufficient to prove the lemma by assuming $|a_1 - a_0| \geqq |a_2 - a_0|$.

Case I: $|a_1 - a_0| = 1$ or 2, and $|a_2 - a_0| = 1$.

For this case, the lemma follows immediately from (1.5) and (i) of the above lemma.

Case II: $|a_1 - a_0| \geqq |a_2 - a_0| \geqq 2$.

By the above lemma, we see easily that

$$\{u: p_1^{r_1} | m_{0,u}\} = \{u: p_2^{r_2} | m_{0,u}\}$$

if $p_k^{r_k} \| |a_i - a_0|$ (p_k: prime, $r_k \geqq 1$) for $k = 1, 2$, where $i = 1$ or 2. Thus the lemma follows from (1.5).

Case III: $|a_1 - a_0| \geqq 3$ and $|a_2 - a_0| = 1$.

For this case, $|a_2 - a_1| \geqq 2$ and we see by Case II that

(2.5) the normal representation of F_1 in M is given by

$$(n_0 + 1)t^{|a_0 - a_1|} + (n_2 + 1)t^{|a_2 - a_1|}.$$

Now, suppose $p^r q^s | |a_0 - a_1|$, where p, q are distinct primes and r, $s \geqq 1$, and let Y be the component of $F(Z_{p^r q^s}, M)$ containing F_1. Then Lemmas 1.4, 2.1 and (2.5) imply that

(2.6) $$\dim Y = \dim F_1 + 2(n_0 + 1) = 2(n_0 + n_1 + 1).$$

Furthermore,

(2.7) $$Y \cap F_2 = \emptyset \quad \text{and} \quad Y \supset F_0 \cup F_1.$$

In fact, since $F(Z_{p^r q^s}, M) = F(Z_{p^r}, M) \cap F(Z_{q^s}, M)$, Lemma 2.2 implies $Y \cap F_2 = \emptyset$. By (2.6), $\dim Y > \dim F_1$ and $Y \supsetneqq F_1$. Thus we see $Y \supset F_0$ by the first half of Corollary 1.11.

Therefore, by (2.6–7) and Lemma 1.4, we see that

$$2(n_0 + n_1 + 1) = \dim Y = \dim F_0 + 2\sharp\{u : p^r q^s | m_{0,u}\},$$

which implies $\sharp\{u : p^r q^s | m_{0,u}\} = n_1 + 1$. This and Lemma 2.3 (i) imply that

(2.8) $$\{u : p^r | m_{0,u}\} = \{u : q^s | m_{0,u}\} \qquad \text{if} \quad p^r q^s | |a_0 - a_1|.$$

Thus the lemma follows from (2.8) and (1.5). q.e.d.

References

[1] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Math. Vol. **46**, Academic Press, 1972.

[2] I. J. Dejter, Smooth S^1-manifolds in the homotopy type of CP^3, Michigan Math. J., **23** (1976), 83–95.

[3] A. Hattori, Spin^c-structures and S^1-actions, to appear.

[4] T. Petrie, Smooth S^1-actions on homotopy complex projective spaces and related topics, Bull. Amer. Math. Soc., **78** (1972), 105–153.

[5] J. C. Su, Integral weight system of S^1 actions on cohomology complex projective spaces, Chinese J. Math., **2** (1974), 77–112.

[6] K. Wang, Differentiable circle group actions on homotopy complex projective spaces, Math. Ann., **214** (1975), 73–80.

[7] T. Yoshida, On smooth semifree S^1-actions on complex projective spaces, Publ. Res. Inst. Math. Sci., Kyoto Univ., **11** (1976), 483–496.

[8] ———, On S^1-actions on cohomology complex projective spaces, Sugaku (in Japanese), **29** (1977), 154–164.

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