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Estimates on the Support of Solutions of Elliptic Variational Inequalities in Bounded Domains

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1. Introduction

To the general question proposed in J. L. Lions [6] as to when the solution of a variational inequality has a compact support, H. Brezis [4], A. Bensoussan and J. L. Lions [1], H. Brezis and A. Friedman [5] have given various affirmative answers for solutions of stationary or evolutionary variational inequalities.

In the present note we shall consider the solution u of an elliptic (stationary) variational inequality of the form

$$\text{(VI)}\qquad \begin{aligned} &-\Delta u + \alpha u \geqq f, \qquad u \geqq \Psi, \\ &(u - \Psi)(-\Delta u + \alpha u - f) = 0 \qquad \text{in} \quad \Omega \end{aligned}$$

under various boundary conditions, where Ω is a bounded domain in R^N, Δ denotes the Laplace operator, and α is a positive constant.

By a solution u of (VI), the domain Ω is divided into two subdomains Ω_1 and Ω_2 such that

$$\Omega_1 = \{x \mid u = \Psi\} \qquad \text{(coincidence set)},$$

$$\Omega_2 = \{x \mid -\Delta u + \alpha u = f\} \qquad \text{(continuation set)}.$$

Recently, A. Bensoussan, H. Brezis and A. Friedman [2] obtained an estimate on the size of Ω_1 under the Dirichlet boundary condition.

The purpose of the present note is to obtain some estimates on the size of Ω_1 under other boundary conditions (Neumann, mixed and Signorini). Our main results in this note are stated in section 3 (Theorems 3.2, 3.3 and 3.4). Section 4 is devoted to the study of the behavior of solutions of (VI) near the boundary of Ω. It seems interesting to the author that estimates of the same type can be derived for these different boundary conditions by computing only one comparison function.

2. A comparison theorem

Let Ω be a bounded domain in R^N with smooth boundary Γ. For a maximal monotone graph β in R^2 such that $0 \in \beta(0)$, we use the following notations:

$$\beta^+(r) = \max\{z | z \in \beta(r)\} \qquad \text{if} \quad r \in D(\beta),$$

$$\beta^-(r) = \min\{z | z \in \beta(r)\} \qquad \text{if} \quad r \in D(\beta),$$

$$\beta^+(r) = \beta^-(r) = +\infty \qquad \text{if} \quad r \notin D(\beta) \quad \text{and} \quad r \geqq \sup D(\beta),$$

$$\beta^+(r) = \beta^-(r) = -\infty \qquad \text{if} \quad r \notin D(\beta) \quad \text{and} \quad r \leqq \inf D(\beta).$$

We assume that

$$f \in L^\infty(\Omega),\ \Psi \in W^{2,\infty}(\Omega) \quad \text{and} \quad \psi,\ \phi \in W^{1,\infty}(\Gamma), \tag{2.1}$$

where $W^{k,\infty}(\Omega)$ denotes the usual Sobolev space (which is a subspace of $L^\infty(\Omega)$). Let $\alpha > 0$ and

$$K = \{u \in H^1(\Omega) | u \geqq \Psi \quad \text{a.e. in } \Omega\}.$$

We consider the following elliptic variational inequality with boundary condition:

$$\tag{2.2} \begin{aligned} &\int_\Omega (-\Delta u + \alpha u)(v - u)dx \geqq \int_\Omega f(v - u)dx \qquad \text{for any} \quad v \in K, \\ &-\frac{\partial u}{\partial n} + \phi \in \beta(u|_\Gamma - \psi) \qquad \text{a.e. on} \quad \Gamma. \end{aligned}$$

Here n is the outer normal to Γ.

Existence and uniqueness of the solution of (2.2) are well known. The regularity of the solution is proved by H. Brezis [3, Th. I. 12, p. 55] for more general elliptic operators, but in the case of $\psi = \phi = 0$. By the same method, it is easy to see that in our case the solution u belongs to $H^2(\Omega) \cap W^{1,\infty}(\Omega)$, provided that the corresponding hypothesis

$$\frac{\partial \Psi}{\partial n} + \beta^-(\Psi|_\Gamma - \psi) \leqq \phi \qquad \text{a.e. on} \quad \Gamma \tag{2.3}$$

is satisfied.

Therefore, by (2.1) and (2.3), the solution u of (VI) which we shall consider in the sense of (2.2) exists uniquely and belongs to $H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Thus, in paticular, it is continuous.

The next theorem, although elementary, enables us to estimate the support of the solution u.

THEOREM 2.1 (*Comparison theorem*). *Let* $\hat{f} \in L^\infty(\Omega)$, $\hat{\Psi} \in W^{2,\infty}(\Omega)$, $\hat{\psi}, \hat{\phi} \in W^{1,\infty}(\Gamma)$ *and* $\hat{u} \in H^2(\Omega)$ *satisfy the following differential inequalities*:

$$(2.4)\qquad \begin{aligned} &-\Delta\hat{u}+\alpha\hat{u}\geqq\hat{f}, \quad \hat{u}\geqq\hat{\Psi} && \text{in } \Omega, \\ &-\frac{\partial\hat{u}}{\partial n}+\hat{\phi}\in\hat{\beta}(\hat{u}|_\Gamma-\hat{\psi}) && \text{on } \Gamma, \end{aligned}$$

where $\hat{\beta}$ is a maximal monotone graph in R^2.

Suppose that $f\leqq\hat{f}$, $\Psi\leqq\hat{\Psi}$ a.e. in Ω, $\phi\leqq\hat{\phi}$, $\psi\leqq\hat{\psi}$ a.e. on Γ and $\hat{\beta}^-\leqq\beta^+$. Then we have, for the solution u of (2.2), *$u\leqq\hat{u}$ a.e. in Ω.*

PROOF. Take $v=u-(u-\hat{u})^+=\min(u,\hat{u})$ in (2.2). This is possible since $\Psi\leqq\hat{\Psi}$ and so $v\in K$. By integration by parts, we have

$$(2.5)\qquad \begin{aligned} &-\int_\Omega[\nabla u\nabla(u-\hat{u})^+ + \alpha u(u-\hat{u})^+]dx+\int_\Gamma\frac{\partial u}{\partial n}(u-\hat{u})^+d\Gamma \\ &\geqq-\int_\Omega f(u-\hat{u})^+\mathrm{dx}. \end{aligned}$$

On the other hand, multiplying (2.4) by $(u-\hat{u})^+$ and integrating over Ω, we find, after integration by parts,

$$(2.6)\qquad \begin{aligned} &\int_\Omega[\nabla\hat{u}\nabla(u-\hat{u})^+ + \alpha\hat{u}(u-\hat{u})^+]dx-\int_\Gamma\frac{\partial\hat{u}}{\partial n}(u-\hat{u})^+d\Gamma \\ &\geqq\int_\Omega\hat{f}(u-\hat{u})^+dx. \end{aligned}$$

From (2.5) and (2.6) it follows that

$$\begin{aligned} &-\int_\Omega[\nabla(u-\hat{u})\nabla(u-\hat{u})^+ + \alpha(u-\hat{u})(u-\hat{u})^+]dx \\ &+\int_\Gamma\left(\frac{\partial u}{\partial n}-\frac{\partial\hat{u}}{\partial n}\right)(u-\hat{u})^+d\Gamma\geqq\int_\Omega(\hat{f}-f)(u-\hat{u})^+dx\geqq 0. \end{aligned}$$

We note that

$$\begin{aligned} \left(\frac{\partial u}{\partial n}-\frac{\partial\hat{u}}{\partial n}\right)(u-\hat{u})^+ &\leqq(\hat{\beta}^+(\hat{u}-\hat{\psi})-\beta^-(u-\psi))(u-\hat{u})^+ \\ &\quad-(\hat{\phi}-\phi)(u-\hat{u})^+\leqq 0 \qquad \text{a.e. on} \quad \Gamma. \end{aligned}$$

In fact, if $u>\hat{u}$ at a point of Γ, then there exists a real number ξ such that $u-\psi>\xi>\hat{u}-\hat{\psi}$ and we have

$$\hat{\beta}^+(\hat{u}-\hat{\psi})\leqq\hat{\beta}^-(\xi)\leqq\beta^+(\xi)\leqq\beta^-(u-\psi).$$

Hence we obtain

$$\int_\Omega [\nabla(u-\hat{u})\nabla(u-\hat{u})^+ + \alpha(u-\hat{u})(u-\hat{u})^+]dx \leqq 0.$$

Therefore, the assertion follows from the coerciveness of the bilinear form

$$a(u, v) = \int_\Omega [\nabla u \nabla v + \alpha uv]dx$$

in $H^1(\Omega)$.

3. Estimates on the support of solutions

In the following, we suppose that there exists a positive number γ such that

$$f - (-\Delta\Psi + \alpha\Psi) \leqq -\gamma \quad \text{a.e. in} \quad \Omega. \tag{3.1}$$

Let u be the solution of (2.2). If we set $\tilde{u} = u - \Psi$, then the difference $\tilde{u}$ satisfies the following variational inequality:

$$\begin{aligned} &-\Delta\tilde{u} + \alpha\tilde{u} \geqq \tilde{f}, \quad \tilde{u} \geqq 0, \\ &\tilde{u}(-\Delta\tilde{u} + \alpha\tilde{u} - \tilde{f}) = 0 \qquad \text{in} \quad \Omega, \\ &-\frac{\partial\tilde{u}}{\partial n} + \tilde{\phi} \in \beta(\tilde{u}|_\Gamma - \tilde{\psi}) \qquad \text{on} \quad \Gamma, \end{aligned} \tag{3.2}$$

where $\tilde{f} = f - (-\Delta\Psi + \alpha\Psi)$, $\tilde{\phi} = \phi - \partial\Psi/\partial n$ and $\tilde{\psi} = \psi - \Psi|_\Gamma$.

Throughout this section we choose as a comparison function the function w defined by

$$w(x) = \frac{\gamma}{2N}|x - x_0|^2, \qquad x_0 \in \Omega, \tag{3.3}$$

and compare it with $\tilde{u}$.

3.1. The Dirichlet problem

If we take

$$\beta(r) = \begin{cases}]-\infty, +\infty[& \text{if} \quad r = 0, \\ \phi(\text{empty set}) & \text{if} \quad r \neq 0, \end{cases}$$

then the Dirichlet boundary condition $u|_\Gamma = \psi$ arises.

The following theorem is due to A. Bensoussan, H. Brezis and A. Friedmar [2, Th. 3.1, p. 307].

THEOREM 3.1. *Let* (3.1) *hold and assume that there exists a positive number* δ *such that*

$$\psi(x) - \Psi(x) \leqq \delta \qquad on \quad \Gamma.$$

If $x_0 \in \Omega$ *satisfies* $\operatorname{dist}(x_0, \Gamma) \geqq (2N\delta/\gamma)^{1/2}$, *then we have* $u(x_0) = \Psi(x_0)$.

REMARK 3.1. In [2], this result is proved in the case that Ω is not necessarily bounded.

3.2. The Neumann problem

If we take $\beta(r) = 0$ for any $r \in R$, then the boundary condition becomes $\partial u/\partial n = \phi$, that is, the Neumann problem is considered.

THEOREM 3.2. *Let* Ω *be convex and* (3.1) *hold. Suppose that there exists a positive number* δ *such that*

$$\phi - \frac{\partial \Psi}{\partial n} \leqq \delta \qquad on \quad \Gamma.$$

If $x_0 \in \Omega$ *satisfies* $\theta_0(x_0) \operatorname{dist}(x_0, \Gamma) \geqq N\delta/\gamma$, *then we have* $u(x_0) = \Psi(x_0)$. *Here we have set*

$$\theta_0(x_0) = \inf_{x \in \Gamma} \cos(n(x), x - x_0),$$

and $n(x)$ *is the unit outer normal to* Γ *at* $x \in \Gamma$. *(Since* Ω *is convex and bounded, it is easy to see that* $\theta_0(x_0) > 0$ *for each* $x_0 \in \Omega$.)

PROOF. Take any point $x_0 \in \Omega$ with $\theta_0(x_0) \operatorname{dist}(x_0, \Gamma) \geqq N\delta/\gamma$. We have by (3.1) and (3.3) that

$$w \geqq 0, \qquad -\Delta w + \alpha w \geqq -\gamma \geqq \tilde{f} \qquad \text{in} \quad \Omega. \tag{3.4}$$

If we show

$$\frac{\partial w}{\partial n} \geqq \tilde{\phi} \qquad \text{on} \quad \Gamma, \tag{3.5}$$

we can apply Theorem 2.1 to $\tilde{u}$ and obtain $\tilde{u} \leqq w$. In paticular, we have

$$u(x_0) - \Psi(x_0) \leqq w(x_0) = 0,$$

and the assertion follows.

To show that (3.5) holds at $x \in \Gamma$, we introduce a new coordinate system $e = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_N\}$ with the origin at x_0 such that the direction of $\hat{e}_N$ coincides with the direction from x_0 to x. Let x and $n(x)$ be represented, in terms of e, as $x = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N)$ and $n(x) = (\hat{n}_1, \hat{n}_2, \dots, \hat{n}_N)$ respectively.

We then have

$$w(x) = \frac{\gamma}{2N}\hat{x}_N^2,$$

$$\frac{\partial w}{\partial \hat{x}_N}(x) = \frac{\gamma}{N}\hat{x}_N, \quad \frac{\partial w}{\partial \hat{x}_i}(x) = 0, \qquad i = 1, 2,\dots, N-1,$$

and so,

$$\frac{\partial w}{\partial n}(x) = \hat{n}_N \frac{\partial w}{\partial \hat{x}_N} = \frac{\gamma}{N}|x - x_0|\cos\theta,$$

where θ is the angle between $\hat{e}_N$ and $n(x)$.

Since $\cos\theta \geqq \theta_0(x_0)$, we obtain

$$\frac{\partial w}{\partial n}(x) \geqq \frac{\gamma\theta_0(x_0)}{N}|x - x_0| \geqq \delta \geqq \tilde{\phi} \qquad \text{on} \quad \Gamma.$$

Thus the proof is completed.

REMARK 3.2. The assumption of the convexity of Ω can be relaxed. The assertion is correct for $x_0 \in \Omega$ such that $\theta_0(x_0) > 0$ and $\theta_0(x_0)\operatorname{dist}(x_0, \Gamma) \geqq N\delta/\gamma$. The first condition can be satisfied if, for example, Ω is star-shaped with respect to x_0 in an obvious manner.

3.3. The mixed problem

Consider the case where

$$\beta(r) = kr \qquad (k > 0) \qquad \text{for any} \quad r \in R,$$

and $\psi = 0$. This boundary condition means $\partial u/\partial n + ku|_\Gamma = \phi$, and we are led to the mixed problem.

THEOREM 3.3. *Let Ω be convex and* (3.1) *hold. We assume that there exists a positive number δ such that*

$$\phi - \frac{\partial \Psi}{\partial n} - k\Psi|_\Gamma \leqq \delta \qquad on \quad \Gamma.$$

If $x_0 \in \Omega$ satisfies

$$\operatorname{dist}(x_0, \Gamma) \geqq \left(\frac{\theta_0(x_0)^2}{k^2} + \frac{2N\delta}{\gamma k}\right)^{1/2} - \frac{\theta_0(x_0)}{k}, \tag{3.6}$$

then we have $u(x_0) = \Psi(x_0)$.

PROOF. It is sufficient to show that

$$\frac{\partial w}{\partial n} + kw|_\Gamma \geqq \tilde{\phi} \qquad \text{on} \quad \Gamma. \tag{3.7}$$

By (3.5) we have

$$\frac{\partial w}{\partial n} + kw|_\Gamma \geqq \frac{\gamma\theta_0(x_0)}{N}|x-x_0| + \frac{\gamma k}{2N}|x-x_0|^2.$$

Since the right hand side of (3.6) is the positive root of the equation $(k\gamma/2N)t^2 + (\gamma\theta_0(x_0)/N)t-\delta=0$, we obtain (3.7).

3.4. The Signorini problem

If we choose

$$\beta(r)=\begin{cases} 0 & \text{if } r>0,\\ ]-\infty, 0] & \text{if } r=0,\\ \phi(\text{empty set}) & \text{if } r<0,\end{cases}$$

then the boundary condition is

$$u|_\Gamma \geqq \psi, \quad \partial u/\partial n \geqq \phi, \quad (u|_\Gamma - \psi)(\partial u/\partial n - \phi)=0 \quad \text{on} \quad \Gamma.$$

This condition is called the Signorini condition.

THEOREM 3.4. *Let Ω be convex and* (3.1) *hold. Suppose that there exist two positive numbers δ_1 and δ_2 such that*

$$\psi - \Psi|_\Gamma \leqq \delta_1,\ \phi - \frac{\partial\Psi}{\partial n} \leqq \delta_2 \qquad on \quad \Gamma.$$

If $x_0 \in \Omega$ satisfies

$$\operatorname{dist}(x_0, \Gamma) \geqq \max\{N\delta_2/\gamma\theta_0(x_0), (2N\delta_1/\gamma)^{1/2}\},$$

then $u(x_0)=\Psi(x_0)$.

PROOF. It is sufficient to show that

$$w|_\Gamma \geqq \tilde{\psi},\ \frac{\partial w}{\partial n} \geqq \tilde{\phi} \qquad on \quad \Gamma.$$

But these inequalities are obvious from the proofs of Theorems 3.1 and 3.2.

4. Estimates near the boundary

In this section, we shall study the behavior of solutions of (2.2) near the boundary under suitable conditions.

We suppose that (3.1) holds for f, and choose a comparison function as follows:

$$w_s(x) = \begin{cases} \dfrac{\gamma}{2N}(|x - x_0| - s)^2 & \text{if } |x - x_0| > s, \\ 0 & \text{if } |x - x_0| \leqq s, \end{cases}$$

where $x_0 \in \Gamma$, and s will be suitably chosen according to the boundary assumptions sated below.

From (3.1) we obtain

$$(4.1) \qquad w_s(x) \geqq 0, \quad -\Delta w_s(x) + \alpha w_s(x) \geqq \tilde{f} \quad \text{in} \quad \Omega.$$

Indeed, for any $x \in \Omega$ such that $|x - x_0| > s$, we have

$$\begin{aligned} -\Delta w_s(x) + \alpha w_s(x) &= -\frac{\gamma}{N}\left(N - \frac{(N-1)s}{|x - x_0|}\right) + \alpha w_s(x) \\ &\geqq -\gamma \geqq \tilde{f}, \end{aligned}$$

and, if $|x - x_0| \leqq s$, it is obvious since $w_s(x) = 0$. This fact will be used throughout the following theorems.

THEOREM 4.1 (*The Dirichlet problem*). *Let* (3.1) *hold and* $\psi(x) - \Psi(x) \leqq \delta$ *for some* $\delta > 0$. *Suppose that there exist a point* $x_0 \in \Gamma$ *and a positive number* $r > (2N\delta/\gamma)^{1/2}$ *such that* $\psi(x) = \Psi(x)$ *on* $\Gamma \cap B(x_0, r)$, *where* $B(x_0, r)$ *is the ball with center* x_0 *and radius* r.

Then $u(x) = \Psi(x)$ *in* $\Omega \cap B(x_0, s)$, *where* $s = r - (2N\delta/\gamma)^{1/2}$.

PROOF. It is sufficient to show that $w_s(x) \geqq \tilde{\psi}(x)$ on Γ. If $|x - x_0| \geqq r$, we have

$$w_s(x) \geqq \frac{\gamma}{2N}(r - s)^2 \geqq \delta \geqq \tilde{\psi},$$

and if $|x - x_0| < r$, it is evident. Thus the assertion follows from Theorem 2.1.

THEOREM 4.2 (*The Neumann problem*). *Let* Ω *be strictly convex, i.e.,* Γ *does not contain any line segment. Suppose that* (3.1) *holds and that* $\phi(x) - (\partial\Psi/\partial n)(x) \leqq \delta$ *for some* $\delta > 0$.

If there exist a point $x_0 \in \Gamma$ *and a positive number* r *such that* $\partial\Psi/\partial n = \phi$ *on* $\Gamma \cap B(x_0, r)$, *then* $u(x) = \Psi(x)$ *for* $x \in \Omega \cap B(x_0, s)$, *where we have set*

$$s = r - \frac{N\delta}{\gamma\theta_0(x_0 : r)},$$

$$\theta_0(x_0 : r) = \inf_{x \in \Gamma \cap B(x_0, r)^c} \cos(n(x), x - x_0),$$

and $B(x_0, r)^c$ *denotes the complementary set of* $B(x_0, r)$.

PROOF. First we remark that $\theta_0(x_0: r)>0$ for all $x_0 \in \Omega$ by the strict convexity of Ω. It remains to show (3.5) for w_s. If $|x-x_0|>r$, we have

$$\frac{\partial w_s}{\partial n} = \frac{\gamma \cos\theta}{N}(|x-x_0|-s) \geqq \frac{\gamma\theta_0(x_0: r)}{N}(r-s)$$
$$=\delta \geqq \tilde{\phi},$$

and it follows immediatly for the case of $|x-x_0| \leqq r$. Thus we can apply Theorem 2.1 and the proof is completed.

Corresponding to Theorem 3.3, we have the following theorem.

THEOREM 4.3 (*The mixed problem*). *Let Ω be strictly convex. Suppose that* (3.1) *holds and that $\phi(x)-\partial\Psi/\partial n-k\Psi \leqq \delta$ for some $\delta>0$. We assume that there exist a point $x_0 \in \Gamma$ and a positive number r such that*

$$\frac{\partial\Psi}{\partial n}+k\Psi=\phi(x) \qquad on \quad \Gamma \cap B(x_0, r),$$

and

$$s \equiv r-\left(\frac{\theta_0(x_0: r)^2}{k^2}+\frac{2N\delta}{\gamma k}\right)^{1/2}+\frac{\theta_0(x_0: r)}{k}>0.$$

Then we have $u(x)=\Psi(x)$ for $x \in \Omega \cap B(x_0, s)$.

REMARK 4.1. We can relax the assumption of the strict convexity of Ω (see REMARK 3.2).

The proof of Theorem 4.3 is omitted since it is easy to see (3.7) for w_s.
For the Signorini problem, we immediately have:

THEOREM 4.4 (*The Signorini problem*). *Let Ω be strictly convex. Suppose that* (3.1) *holds and that $\psi-\Psi|_\Gamma \leqq \delta_1$, $\phi-\partial\Psi/\partial n \leqq \delta_2$* for some $\delta_1, \delta_2>0$.
If $\psi(x)=\Psi(x)$ and $\phi(x)=(\partial\Psi/\partial n)(x)$ on $\Gamma \cap B(x_0, r)$ for some positive number r such that

$$s \equiv r-\max\left\{\left(\frac{2N\delta_1}{\gamma}\right)^{1/2}, \frac{N\delta_2}{\gamma\theta_0(x_0: r)}\right\}>0,$$

then we have $u(x)=\Psi(x)$ in $\Omega \cap B(x_0, s)$.

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