HIROSHIMA MATH. J.
9 (1979), 533–545

On the Order at Infinity of Riesz Potentials

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(Received January 18, 1979)

1. Introduction

In this paper we shall be concerned with the behavior at infinity of the Riesz potential U_α^μ which is defined by

$$U_\alpha^\mu(x) = \int |x-y|^{\alpha-n} d\mu(y), \qquad x \in R^n,$$

where $0<\alpha<n$ and μ is a non-negative measure on R^n. The potential U_α^μ may take the value ∞ on a countable dense subset of R^n, but it may occur that $|x|^\beta U_\alpha^\mu(x)$ tends to zero as $|x|\to\infty$ except for x in a set which is thin at infinity in a certain sense, $\beta \geqq 0$ being a number determined by μ.

We shall work on the case of potentials of functions which belong to the Lebesgue class $L^p(R^n)$, $1<p<\infty$. In order to define the thinness at infinity of a set in this case, we shall use the following capacity: The (α, p)-capacity of a set E relative to an open set G is defined by

$$C_{\alpha,p}(E; G) = \inf \|g\|_p^p,$$

where the infimum is taken over all non-negative functions $g \in L^p(R^n)$ such that g vanishes outside G and $U_\alpha^g(x) \geqq 1$ for all $x \in E$. We say that E is (α, p)-thin at infinity if

$$\sum_{k=1}^{\infty} 2^{-k(n-\alpha p)} C_{\alpha,p}(E^{(k)}; G_k) < \infty,$$

where $E^{(k)} = \{x \in E;\ 2^k \leqq |x| < 2^{k+1}\}$ and $G_k = \{x \in R^n;\ 2^{k-1} < |x| < 2^{k+2}\}$. It will be proved in §4 that if $\alpha p \leqq n$ and f is a non-negative function in $L^p(R^n)$ with $U_\alpha^f \not\equiv \infty$, then $|x|^{(n-\alpha p)/p} U_\alpha^f(x)$ tends to zero as $|x| \to \infty$ except for x in a set which is (α, p)-thin at infinity. Further it will be proved that if $\alpha p < n$ and E is (α, p)-thin at infinity, then there exists a non-negative function $f \in L^p(R^n)$ such that $U_\alpha^f \not\equiv \infty$ and

$$\lim_{|x|\to\infty, x\in E} |x|^{(n-\alpha p)/p} U_\alpha^f(x) = \infty.$$

2. Preliminaries

Let R^n be the n-dimensional Euclidean space. Throughout this paper, let $0<\alpha<n$ and $1<p<\infty$. The Riesz potential U_α^μ will be decomposed as $V_\alpha^\mu+W_\alpha^\mu$, where

$$V_\alpha^\mu(x) = \int_{|x-y|<|x|/2} |x-y|^{\alpha-n} d\mu(y),$$

$$W_\alpha^\mu(x) = \int_{|x-y|\geqq |x|/2} |x-y|^{\alpha-n} d\mu(y).$$

If μ has a density f, then we shall write U_α^f, V_α^f, W_α^f for U_α^μ, V_α^μ, W_α^μ respectively.

The Riesz capacity of order α of a set E is defined by

$$C_\alpha(E) = \sup \mu(R^n),$$

where the supremum is taken over all non-negative measures μ such that S_μ (the support of μ)$\subset E$ and $U_\alpha^\mu(x)\leqq 1$ for all $x\in S_\mu$.

LEMMA 2.1 ([1; p. 61]). *Let μ be a non-negative measure on R^n. In order that $U_\alpha^\mu \not\equiv \infty$, it is necessary and sufficient that*

$$\int (1+|y|)^{\alpha-n} d\mu(y) < \infty, \tag{2.1}$$

or equivalently,

$$\int_{R^n - B_{x,r}} |x-y|^{\alpha-n} d\mu(y) < \infty$$

for some, and hence for any, open ball $B_{x,r}$ with center at x and radius r.

COROLLARY 2.2. *If μ satisfies* (2.1), *then $W_\alpha^\mu(x)$ is finite for $x\neq O$.*

LEMMA 2.3 ([2; Lemma 1]). *Let μ be a non-negative measure on R^n. If we set $E=\{x\in R^n;\ U_\alpha^\mu(x)\geqq 1\}$, then $C_\alpha(E)\leqq 2^{n-\alpha}\mu(R^n)$.*

COROLLARY 2.4. *For any non-negative measure μ, we have*

$$C_\alpha(\{x\in R^n;\ V_\alpha^\mu(x)=\infty\}) = 0.$$

For this it suffices to note that if $x\in B_{O,r}$, then

$$\int_{B_{O,2r}} |x-y|^{\alpha-n} d\mu(y) \geqq V_\alpha^\mu(x).$$

By Corollaries 2.2 and 2.4, the following is easily established.

COROLLARY 2.5. *If μ satisfies* (2.1), *then*

$$C_\alpha(\{x \in R^n;\ U_\alpha^\mu(x) = \infty\}) = 0.$$

Denoting by S the boundary of $B_{O,1}$, we define for a set E and a number $r>0$,

$$rE = \{rx;\ x \in E\},\quad \tilde{E} = \bigcup_{r>0} (rE) \cap S,\quad E^{(r)} = E \cap B_{O,2^{r+1}} - B_{O,2^r}.$$

LEMMA 2.6. i) $C_\alpha(rE) = r^{n-\alpha} C_\alpha(E)$.
ii) $C_\alpha(\tilde{E}) \leqq C_\alpha(E)$ *for* $E \subset B_{O,2} - B_{O,1}$.

This follows readily from [1; Theorem 2.9].
As to the capacity $C_{\alpha,p}$, we derive the following result.

LEMMA 2.7. i) $C_{\alpha,p}(rE;\ rG) = r^{n-\alpha p} C_{\alpha,p}(E;\ G)$.
ii) *There is a constant* $M>0$ *such that*

$$C_{\alpha,p}(\tilde{E};\ B_{O,3}) \leqq M C_{\alpha,p}(E;\ B_{O,3})$$

whenever $E \subset B_{O,2} - B_{O,1}$.

The assertion i) can be proved in a way similar to [3; Lemma 4]. The assertion ii) is nothing but [3; Lemma 5].

3. Potentials of measures

Let us begin with the definition of thinness.

DEFINITION 3.1. A set E will be called *α-thin at infinity* if

$$\sum_{k=1}^{\infty} 2^{-k(n-\alpha)} C_\alpha(E^{(k)}) < \infty.$$

We remark here that for every subsequence $\{k_j\}$ of the sequence of natural numbers, $E = \bigcup_{j=1}^{\infty} \{x \in R^n;\ 2^{k_j} < |x| < 2^{k_j+1}\}$ is not α-thin at infinity.

LEMMA 3.2. *Let μ be a non-negative measure such that*

$$\int (1 + |y|)^{\beta-n} d\mu(y) < \infty \tag{3.1}$$

for some β with $\alpha \leqq \beta \leqq n$.
i) *There is a Borel set E which is α-thin at infinity and satisfies*

$$\lim_{|x|\to\infty,\, x\notin E} |x|^{\beta-\alpha} V_\alpha^\mu(x) = 0.$$

ii) *We have*

$$\lim_{|x|\to\infty} |x|^{\beta-\alpha} W_\alpha^\mu(x) = \begin{cases} 0 & \text{in case } \alpha \leqq \beta < n, \\ \mu(R^n) & \text{in case } \beta = n. \end{cases}$$

PROOF. i) Set

$$a_k = \int_{2^{k-1} \leqq |y| < 2^{k+2}} |y|^{\beta-n} d\mu(y)$$

for each positive integer k. Then $\sum_{k=1}^\infty a_k < \infty$ by our assumption (3.1). Hence there is a sequence $\{b_k\}$ of positive numbers such that $\lim_{k\to\infty} b_k = \infty$ but $\sum_{k=1}^\infty a_k b_k < \infty$. Consider the set

$$A_k = \{x \in R^n;\ 2^k \leqq |x| < 2^{k+1},\ V_\alpha^\mu(x) \geqq b_k^{-1} 2^{-k(\beta-\alpha)}\}$$

for each k. If $x \in A_k$, then

$$U_\alpha^{\mu_k}(x) \geqq b_k 2^{k(\beta-\alpha)} V_\alpha^\mu(x) \geqq 1,$$

where μ_k is a non-negative measure defined by

$$\mu_k(A) = b_k 2^{k(\beta-\alpha)} \mu(A \cap B_{O,2^{k+2}} - B_{O,2^{k-1}}) \qquad \text{for a Borel set } A.$$

Therefore it follows from Lemma 2.3 that

$$\begin{aligned} C_\alpha(A_k) &\leqq 2^{n-\alpha} 2^{k(\beta-\alpha)} b_k \int_{2^{k-1} \leqq |y| < 2^{k+2}} d\mu(y) \\ &\leqq 2^{n-\alpha+2(n-\beta)} 2^{k(n-\alpha)} a_k b_k. \end{aligned}$$

We set $E = \cup_{k=1}^\infty A_k$. Obviously $E^{(k)} = A_k$ and hence

$$\sum_{k=1}^\infty 2^{-k(n-\alpha)} C_\alpha(E^{(k)}) \leqq 2^{n-\alpha+2(n-\beta)} \sum_{k=1}^\infty a_k b_k < \infty,$$

which implies that E is α-thin at infinity. Moreover we see that

$$\limsup_{|x|\to\infty, x \notin E} |x|^{\beta-\alpha} V_\alpha^\mu(x) \leqq \limsup_{k\to\infty} 2^{\beta-\alpha} b_k^{-1} = 0.$$

ii) For each fixed y, $|x|^{\beta-\alpha}|x-y|^{\alpha-n} \to 0$ (resp. 1) as $|x|\to\infty$ if $\alpha \leqq \beta < n$ (resp. $\beta = n$). Further, there is a constant $M > 0$ such that

$$|x|^{\beta-\alpha}|x - y|^{\alpha-n} \leqq M(1 + |y|)^{\beta-n}$$

whenever $|x-y| \geqq |x|/2 \geqq 1$. Hence we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{|x|\to\infty} |x|^{\beta-\alpha} W_\alpha^\mu(x) = \begin{cases} 0 & \text{if} \quad \alpha \leqq \beta < n, \\ \mu(R^n) & \text{if} \quad \beta = n. \end{cases}$$

From this lemma we can derive the following theorem:

THEOREM 3.3. *Let μ be a non-negative measure satisfying* (3.1) *for some β with $\alpha \leqq \beta \leqq n$. Then there is a Borel set E which is α-thin at infinity and for which*

$$\lim_{|x|\to\infty, x\notin E} |x|^{\beta-\alpha} U_\alpha^\mu(x) = \begin{cases} 0 & \textit{in case} \quad \alpha \leqq \beta < n, \\ \mu(R^n) & \textit{in case} \quad \beta = n. \end{cases}$$

We state here the existence of limits of potentials along rays issuing from the origin. Our result below is a generalization of [2; Theorem 1].

COROLLARY 3.4. *Let μ be as in Theorem* 3.3. *Then there is a Borel set $E \subset S$ such that $C_\alpha(E)=0$ and*

$$\lim_{r\to\infty} r^{\beta-\alpha} U_\alpha^\mu(r\xi) = \begin{cases} 0 & \textit{in case} \quad \alpha \leqq \beta < n \\ \mu(R^n) & \textit{in case} \quad \beta = n \end{cases}$$

for every $\xi \in S-E$.

This follows readily from Theorem 3.3 and the next lemma.

LEMMA 3.5. *If E is α-thin at infinity, then $C_\alpha(\cap_{k=1}^\infty \widetilde{E-B_{O,k}})=0$.*

PROOF. Since $\widetilde{E-B_{O,2^j}}=\cup_{k=j}^\infty \widetilde{E^{(k)}}$, we have by Lemma 2.6

$$C_\alpha(\widetilde{E-B_{O,2^j}}) \leqq \sum_{k=j}^\infty C_\alpha(\widetilde{E^{(k)}}) \leqq \sum_{k=j}^\infty 2^{-k(n-\alpha)} C_\alpha(E^{(k)}) \longrightarrow 0$$

as $j\to\infty$, which yields the required equality in our lemma.

The rest of this section will be devoted to investigating the best possibility of Theorem 3.3 as to the size of the exceptional set and the order at infinity.

PROPOSITION 3.6. *Let $\alpha \leqq \beta \leqq n$ and let E be a Borel set α-thin at infinity. Then there is a non-negative measure μ satisfying* (3.1) *and*

$$\lim_{|x|\to\infty, x\in E} |x|^{\beta-\alpha} U_\alpha^\mu(x) = \infty.$$

PROOF. Since E is α-thin at infinity, we can find a sequence $\{a_k\}$ of positive numbers such that $\lim_{k\to\infty} a_k=\infty$ and

$$\sum_{k=1}^{\infty} a_k 2^{-k(n-\alpha)} C_\alpha(E^{(k)}) < \infty.$$

For each positive integer k, there is a non-negative measure μ_k such that $U_\alpha^{\mu_k}(x) \geqq a_k$ for $x \in E^{(k)}$, $S_{\mu_k} \subset B_{O,2^{k+2}} - B_{O,2^{k-1}}$ and $\mu_k(R^n) \leqq a_k C_\alpha(E^{(k)}) + 2^{-k}$. Setting $\mu = \sum_{k=1}^{\infty} 2^{-k(\beta-\alpha)} \mu_k$, we have

$$\int |y|^{\beta-n} d\mu(y) = \sum_{k=1}^{\infty} \int |y|^{\beta-n} 2^{-k(\beta-\alpha)} d\mu_k(y)$$

$$\leqq \sum_{k=1}^{\infty} 2^{n-\beta} 2^{-k(n-\alpha)} \{a_k C_\alpha(E^{(k)}) + 2^{-k}\} < \infty,$$

which implies (3.1). For $x \in E^{(k)}$, we have

$$|x|^{\beta-\alpha} U_\alpha^\mu(x) \geqq U_\alpha^{\mu_k}(x) \geqq a_k,$$

or $\lim_{|x| \to \infty, x \in E} |x|^{\beta-\alpha} U_\alpha^\mu(x) = \infty$.

PROPOSITION 3.7. *Let $\alpha \leqq \beta \leqq n$ and let $a(r)$ be a non-decreasing positive function of $r > 0$ such that $\lim_{r \to \infty} a(r) = \infty$. Then there are a non-negative measure μ and a Borel set A with the following properties:*

i) *A is not α-thin at infinity;*

ii) *$\widetilde{A - B_{O,r}} = S$ for $r > 0$;*

iii) *μ satisfies (3.1);*

iv) $\lim_{|x| \to \infty, x \in A} a(|x|) |x|^{\beta-\alpha} V_\alpha^\mu(x) = \infty$.

PROOF. Set $a_k = a(2^k)$ for each positive integer k and choose a sequence $\{k_j\}$ of positive integers such that $k_j > 4$, $2k_j < k_{j+1}$ and $\sum_{j=1}^{\infty} a_{k_j}^{-1/2} < \infty$. Define

$$f(y) = \begin{cases} 2^{-\beta k_j} a_{k_j}^{-1/2} & \text{if } \ 2^{k_j - 1} < |y| < 2^{k_j + 2}, \ \ j = 1, 2, \ldots, \\ 0 & \text{otherwise} \end{cases}$$

and consider the measure $d\mu = f dy$. Then it is easy to prove that μ satisfies iii). For x such that $2^{k_j} < |x| < 2^{k_j+1}$, we have by setting $E_j = \{y \in R^n;\ |x-y| < 2^{k_j-1}\}$

$$a(|x|)|x|^{\beta-\alpha} V_\alpha^\mu(x) \geqq a_{k_j} 2^{k_j(\beta-\alpha)} \int_{E_j} |x-y|^{\alpha-n} f(y) dy$$

$$= a_{k_j}^{1/2} \int_{|z|<1/2} |z|^{\alpha-n} dz \longrightarrow \infty \qquad \text{as } \ j \longrightarrow \infty.$$

Let $A = \bigcup_{j=1}^{\infty} \{x \in R^n;\ 2^{k_j} < |x| < 2^{k_j+1}\}$. Then A is not α-thin at infinity and satisfies ii).

REMARK 3.8. Proposition 3.7 shows that Corollary 3.4 is the best possible as to the order of zero at infinity. Corollary 3.4 is also the best possible as to the size of the exceptional set (cf. [2; Remark 2]).

4. Potentials of functions

We recall the definition of (α, p)-thinness at infinity.

DEFINITION 4.1. A set E is called *(α, p)-thin at infinity* if

$$\sum_{k=1}^{\infty} 2^{-k(n-\alpha p)} C_{\alpha,p}(E^{(k)}; G_k) < \infty,$$

where $G_k = \{x \in R^n;\ 2^{k-1} < |x| < 2^{k+2}\}$.

REMARK 4.2. i) Let $\{k_j\}$ be a subsequence of the sequence of natural numbers. Then $\cup_{j=1}^{\infty}\{x \in R^n;\ 2^{k_j} < |x| < 2^{k_j+1}\}$ is not (α, p)-thin at infinity on account of Lemma 2.7.

ii) If $\alpha p > n$ and E is (α, p)-thin at infinity, then E is a bounded set on account of Lemma 2.7 and [3; Remark 1, ii)].

LEMMA 4.3. *Let $\ell + m + n < 0$. Then there is a constant $M > 0$ such that*

$$\int_{|x-y| \geqq |x|/2} |x-y|^{\ell}(1+|y|)^m dy \leqq M \begin{cases} |x|^{\ell+m+n} & \text{if } m+n>0 \\ |x|^{\ell}\log|x| & \text{if } m+n=0 \\ |x|^{\ell} & \text{if } m+n<0 \end{cases}$$

for any x with $|x| > 2$.

PROOF. We divide the domain of integration into two parts, that is,

$$D_1: |x-y| \geqq |x|/2, \quad |y| \leqq |x|/2;$$

$$D_2: |x-y| \geqq |x|/2, \quad |y| > |x|/2.$$

Since $|x|/2 \leqq |x-y| \leqq 3|x|/2$ in D_1, we have

$$\int_{D_1} |x-y|^{\ell}(1+|y|)^m dy \leqq 2^{|\ell|}|x|^{\ell} \int_{|y| \leqq |x|/2} (1+|y|)^m dy$$

$$\leqq K_1|x|^{\ell} \times \begin{cases} |x|^{m+n} & \text{if } m+n>0 \\ \log|x| & \text{if } m+n=0 \\ 1 & \text{if } m+n<0 \end{cases}$$

with some constant $K_1 > 0$ independent of x with $|x| > 2$. On the other hand, if

$y \in D_2$ and $|x|>2$, then $|x-y|/4 \leqq 1+|y| \leqq 4|x-y|$, so that

$$\int_{D_2} |x-y|^{\ell}(1+|y|)^m dy \leqq 4^{|m|}\int_{|x-y| \geqq |x|/2} |x-y|^{\ell+m} dy$$

$$\leqq 4^{|m|}|x|^{\ell+m+n}\int_{|z| \geqq 1/2} |z|^{\ell+m} dz.$$

Thus we obtain the desired conclusion.

LEMMA 4.4. *Let f be a non-negative function such that*

$$\int (1+|y|)^{\beta-n} f^p(y) dy < \infty, \quad \alpha \leqq \beta/p \leqq n. \tag{4.1}$$

Then the following assertions hold:

i) *There exists a Borel set E such that E is (α, p)-thin at infinity and*

$$\lim_{|x| \to \infty, x \notin E} |x|^{(\beta-\alpha p)/p} V_\alpha^f(x) = 0.$$

ii) *If, in addition, $U_\alpha^f \not\equiv \infty$, then we have*

$$\lim_{|x| \to \infty} |x|^{(\beta-\alpha p)/p} W_\alpha^f(x) = 0 \qquad \textit{in case} \quad \alpha \leqq \beta/p < n, \tag{4.2}$$

$$\lim_{|x| \to \infty} |x|^{n-\alpha}(\log |x|)^{-1/p'} W_\alpha^f(x) = 0 \qquad \textit{in case} \quad \beta/p = n, \tag{4.3}$$

where $1/p+1/p'=1$.

PROOF. i) Set

$$a_k = \int_{2^{k-1}<|y|<2^{k+2}} |y|^{\beta-n} f^p(y) dy$$

for each positive integer k, and note $\sum_{k=1}^{\infty} a_k < \infty$ by assumption (4.1). Hence we can find a sequence $\{b_k\}$ of positive numbers such that $\lim_{k \to \infty} b_k = \infty$ but $\sum_{k=1}^{\infty} a_k b_k < \infty$. We set

$$A_k = \{x \in R^n;\ 2^k \leqq |x| < 2^{k+1},\ V_\alpha^f(x) \geqq b_k^{-1/p} 2^{-k(\beta-\alpha p)/p}\}$$

for each k, and define

$$f_k(x) = \begin{cases} f(x) & \text{if } \ x \in G_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for $x \in A_k$,

$$\int |x-y|^{\alpha-n} f_k(y) dy \geqq V_\alpha^f(x) \geqq b_k^{-1/p} 2^{-k(\beta-\alpha p)/p},$$

so that

$$C_{\alpha,p}(A_k;\, G_k) \leqq b_k 2^{k(\beta-\alpha p)}\int f_k^p(y)dy \leqq 4^{|n-\beta|}2^{k(n-\alpha p)}a_k b_k.$$

Consider the set $E=\cup_{k=1}^{\infty}A_k$. Since $E^{(k)}=A_k$,

$$\sum_{k=1}^{\infty}2^{-k(n-\alpha p)}C_{\alpha,p}(E^{(k)};\, G_k) \leqq 4^{|n-\beta|}\sum_{k=1}^{\infty}a_k b_k < \infty,$$

which implies that E is (α, p)-thin at infinity. Moreover,

$$\limsup_{|x|\to\infty, x\notin E}|x|^{(\beta-\alpha p)/p}V_\alpha^f(x) \leqq \limsup_{k\to\infty}2^{(\beta-\alpha p)/p}b_k^{-1/p} = 0.$$

ii) If $\alpha=\beta/p$, then (4.2) follows from Lemma 3.2, ii), since we assumed $U_\alpha^f \not\equiv \infty$ (cf. Lemma 2.1). Let $\alpha<\beta/p<n$. Then we can find $\gamma>0$ such that $\beta-pn+n<\gamma<n$. By Hölder's inequality,

$$W_\alpha^f(x) \leqq \left\{\int_{|x-y|\geqq|x|/2}|x-y|^{\gamma-n}(1+|y|)^{\beta-\gamma}f^p(y)dy\right\}^{1/p}$$
$$\times\left\{\int_{|x-y|\geqq|x|/2}|x-y|^{p'(\alpha-\gamma/p)-n}(1+|y|)^{p'(\gamma-\beta)/p}dy\right\}^{1/p'}.$$

In view of Lemma 4.3, there exists $M>0$ such that if $|x|\geqq 2$, then

$$|x|^{(\beta-\alpha p)/p}W_\alpha^f(x) \leqq M\left\{\int_{|x-y|\geqq|x|/2}\left(\frac{1+|y|}{|x-y|}\right)^{n-\gamma}(1+|y|)^{\beta-n}f^p(y)dy\right\}^{1/p}.$$

From Lebesgue's dominated convergence theorem it follows that the right-hand side tends to zero as $|x|\to\infty$. Thus (4.2) holds.

To prove (4.3), given $\varepsilon>0$, find $N>0$ such that

$$\left\{\int_{|y|>N}(1+|y|)^{pn-n}f^p(y)dy\right\}^{1/p} < \varepsilon.$$

We define

$$f_N(x) = \begin{cases} f(x) & \text{if } |x| \leqq N, \\ 0 & \text{otherwise,} \end{cases}$$

and $g_N=f-f_N$. Then Lemma 3.2 implies that

$$\lim_{|x|\to\infty}|x|^{n-\alpha}W_\alpha^{f_N}(x) = \|f_N\|_1,$$

which gives

(4.4) $$\lim_{|x|\to\infty}|x|^{n-\alpha}(\log|x|)^{-1/p'}W_\alpha^{f_N}(x) = 0.$$

On the other hand we have by Hölder's inequality

$$W_\alpha^{g_N}(x) \leqq \left\{\int_{|x-y|\geqq|x|/2}(1+|y|)^{np-n}g_N^p(y)dy\right\}^{1/p}$$
$$\times \left\{\int_{|x-y|\geqq|x|/2}|x-y|^{p'(\alpha-n)}(1+|y|)^{-n}dy\right\}^{1/p'}.$$

By Lemma 4.3 there is a constant $M>0$ independent of x and ε such that for x with $|x|\geqq 2$,

$$|x|^{n-\alpha}(\log|x|)^{-1/p'}W_\alpha^{g_N}(x) < M\varepsilon.$$

This together with (4.4) establishes (4.3).

Lemma 4.4 yields the following main theorem:

THEOREM 4.5. *Let $\alpha\leqq\beta/p\leqq n$. If f is a non-negative function satisfying (4.1) and $U_\alpha^f \not\equiv \infty$, then there is a Borel set E such that E is (α, p)-thin at infinity and*

$$\lim_{|x|\to\infty, x\notin E} |x|^{(\beta-\alpha p)/p}U_\alpha^f(x) = 0 \qquad \textit{in case} \quad \alpha\leqq\beta/p<n,$$

$$\lim_{|x|\to\infty, x\notin E} |x|^{n-\alpha}(\log|x|)^{-1/p'}U_\alpha^f(x) = 0 \qquad \textit{in case} \quad \beta/p=n.$$

If, in addition, $\alpha p>n$, then we can take $E=\emptyset$ (the empty set) in the above equalities.

The last statement follows from Remark 4.2, ii).

REMARK 4.6. If $\alpha<\beta/p$, then (4.1) gives

$$\int(1+|y|)^{\alpha-n}f(y)dy<\infty, \tag{4.5}$$

which is equivalent to $U_\alpha^f \not\equiv \infty$ by Lemma 2.1.

From Lemma 2.7 and Theorem 4.5, we obtain the following corollary, which gives an improvement of [2; Theorem 2].

COROLLARY 4.7. *Let $\alpha\leqq\beta/p\leqq n$ and let f be given as in Theorem 4.5. Then there exists a Borel set $E\subset S$ such that $C_{\alpha,p}(E;B_{O,3})=0$ and*

$$\lim_{r\to\infty} r^{(\beta-\alpha p)/p}U_\alpha^f(r\xi) = 0 \qquad \textit{in case} \quad \alpha\leqq\beta/p<n,$$

$$\lim_{r\to\infty} r^{n-\alpha}(\log r)^{-1/p'}U_\alpha^f(r\xi) = 0 \qquad \textit{in case} \quad \beta/p=n$$

for every $\xi \in S-E$.

From now on we shall deal with the best possibility of Theorem 4.5 as to the size of the exceptional set and the order of zero at infinity.

PROPOSITION 4.8. *Let* $\alpha p \leqq n$ *and let* E *be a set* (α, p)*-thin at infinity. Then there exists a non-negative function* f *satisfying* (4.1) *and*

$$\lim_{|x|\to\infty, x\in E} |x|^{(\beta-\alpha p)/p} V_\alpha^f(x) = \infty.$$

PROOF. Since E is (α, p)-thin at infinity, there is a sequence $\{a_k\}$ of positive numbers such that $\lim_{k\to\infty} a_k = \infty$ and

$$\sum_{k=1}^{\infty} a_k^p 2^{-k(n-\alpha p)} C_{\alpha,p}(E^{(k)}; G_k) < \infty.$$

For each k we can find a non-negative function $f_k \in L^p(R^n)$ such that $f_k = 0$ on $R^n - G_k$, $U_\alpha^{f_k}(x) \geqq a_k$ for $x \in E^{(k)}$ and $\|f_k\|_p^p < a_k^p C_{\alpha,p}(E^{(k)}; G_k) + 2^{-k}$. Setting $f = \sum_{k=1}^{\infty} 2^{-k(\beta-\alpha p)/p} f_k$, we have

$$\begin{aligned}\int |y|^{\beta-n} f^p(y)dy &\leqq 3^{p-1} \sum_{k=1}^{\infty} \int |y|^{\beta-n} 2^{-k(\beta-\alpha p)} f_k^p(y)dy \\ &\leqq 3^{p-1} 4^{|n-\beta|} \sum_{k=1}^{\infty} 2^{-k(n-\alpha p)} \{a_k^p C_{\alpha,p}(E^{(k)}; G_k) + 2^{-k}\} < \infty.\end{aligned}$$

For $x \in E^{(k)}$, we obtain

$$\begin{aligned}W_\alpha^{f_k}(x) &\leqq \left\{\int_{2^{k-1}<|x-y|<6\cdot 2^k} |x-y|^{p'(\alpha-n)} dy\right\}^{1/p'} \|f_k\|_p \\ &\leqq (2^{-k(n-\alpha p)} \|f_k\|_p^p)^{1/p} \left(\int_{2^{-1}<|z|<6} |z|^{p'(\alpha-n)} dz\right)^{1/p'},\end{aligned}$$

which tends to zero as $k \to \infty$ because of $\sum_{k=1}^{\infty} 2^{-k(n-\alpha p)} \|f_k\|_p^p < \infty$. Thus there is a positive integer k_0 such that if $k \geqq k_0$ and $x \in E^{(k)}$, then $W_\alpha^{f_k}(x) \leqq a_k/2$ and hence $V_\alpha^{f_k} \geqq a_k/2$ on $E^{(k)}$, which implies that

$$|x|^{(\beta-\alpha p)/p} V_\alpha^f(x) \geqq 2^{k(\beta-\alpha p)/p} 2^{-k(\beta-\alpha p)/p} V_\alpha^{f_k}(x) \geqq a_k/2$$

for any $x \in E^{(k)}$. Therefore $\lim_{|x|\to\infty, x\in E} |x|^{(\beta-\alpha p)/p} V_\alpha^f(x) = \infty$.

REMARK 4.9 Proposition 4.8 shows that Lemma 4.4, i) is the best possible as to the size of the exceptional set. In case $\alpha < \beta/p < n$ and $\alpha p \leqq n$, the function f obtained above satisfies (4.5) on account of Remark 4.6, and hence Theorem 4.5 is also the best possible as to the size of the exceptional set. In case $\alpha p = \beta$, we do not know whether the function f satisfies (4.5) or not. However, if E satisfies

$$\sum_{k=1}^{\infty}[2^{-k(n-\alpha p)}C_{\alpha,p}(E^{(k)};\ G_k)]^{1/p} < \infty,$$

then we can find a non-negative function f which satisfies (4.1), (4.5) and

$$\lim_{|x|\to\infty, x\in E} U_\alpha^f(x) = \infty.$$

Since the proof is similar to that of Proposition 4.8, we omit it.

PROPOSITION 4.10. *Let $\alpha \leqq \beta/p \leqq n$ and let $a(r)$ be a non-decreasing positive function of $r>0$ such that $\lim_{r\to\infty} a(r)=\infty$. Then there are a non-negative function f and a set A with the following properties:*

i) *A is not (α, p)-thin at infinity;*

ii) *$\widetilde{A-B_{O,r}}=S$ for $r>0$;*

iii) *(4.1) and (4.5) are satisfied;*

iv) *we have*

$$\lim_{|x|\to\infty, x\in A} a(|x|)|x|^{(\beta-\alpha p)/p} V_\alpha^f(x) = \infty \qquad \textit{in case} \quad \alpha \leqq \beta/p < n, \tag{4.6}$$

$$\lim_{|x|\to\infty, x\in A} a(|x|)|x|^{n-\alpha}(\log|x|)^{-1/p'} W_\alpha^f(x) = \infty \qquad \textit{in case} \quad \beta/p = n. \tag{4.7}$$

PROOF. First we consider the case $\alpha \leqq \beta/p < n$. Let $\{k_j\}$ be a sequence of positive integers such that $k_j > 4$, $2k_j < k_{j+1}$ and $\sum_{j=1}^{\infty} a(2^{k_j})^{-1/p} < \infty$, and define

$$f(y) = \begin{cases} 2^{-\beta k_j/p} a(2^{k_j})^{-1/p} & \text{if} \quad 2^{k_j-1} < |y| < 2^{k_j+2}, \quad j = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that f satisfies (4.1) and (4.5). As in the proof of Proposition 3.7, we obtain (4.6) with $A = \bigcup_{j=1}^{\infty}\{x \in R^n;\ 2^{k_j} \leqq |x| < 2^{k_j+1}\}$. This A is not (α, p)-thin at infinity on account of Remark 4.2, i).

Next let $\beta/p = n$. Choose a subsequence $\{k_j\}$ of the sequence of natural numbers such that $2k_j < k_{j+1}$ and $\sum_{j=1}^{\infty} a(2^{2k_j})^{-1} < \infty$, and define

$$f(y) = \begin{cases} a(2^{2k_j})^{-1/p}|y|^{-n}(\log|y|)^{-1/p} & \text{if} \quad y \in E_j, \quad j = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where $E_j = \{y \in R^n;\ 2^{k_j} < |y| < 2^{2k_j}\}$. Then (4.1) and (4.5) hold. If $2^{2k_j} \leqq |x| < 2^{2k_j+1}$, then

$$\begin{aligned} W_\alpha^f(x) &\geqq \int_{\{|x-y|>|x|/2\}\cap E_j} |x-y|^{\alpha-n} f(y)\,dy \\ &\geqq (3\cdot 2^{2k_j})^{\alpha-n} a(2^{2k_j})^{-1/p} 2^{-1} \int_{E_j} |y|^{-n}(\log|y|)^{-1/p}\,dy \end{aligned}$$

$$= \text{const.}\, 2^{2k_j(\alpha-n)} a(2^{2k_j})^{-1/p} k_j^{1/p'},$$

so that

$$a(|x|)|x|^{n-\alpha}(\log|x|)^{-1/p'} W_\alpha^f(x) \geqq \text{const.}\, a(2^{2k_j})^{1/p'}.$$

Thus (4.7) is fulfilled with $A=\cup_{j=1}^{\infty}\{x \in R^n;\, 2^{2k_j} \leqq |x| < 2^{2k_j+1}\}$. The proof is now complete.

REMARK 4.11. Proposition 4.10 shows that Theorem 4.5 and Corollary 4.7 are the best possible as to the order of zero at infinity. Corollary 4.7 is also the best possible as to the size of the exceptional set.

To prove the second assertion, let $E \subset S$ satisfy $C_{\alpha,p}(E; B_{O,2})=0$. Then $C_{\alpha,p}(2^k E; G_k)=0$ for each positive integer k. Hence we can find a non-negative function $f_k \in L^p(R^n)$ such that f_k vanishes outside G_k, $U_\alpha^{f_k}(x)=\infty$ for $x \in 2^k E$ and $\|f_k\|_p$ is so small that $U_\alpha^f(O)<\infty$ and f satisfies (4.1) with $\beta=pn$, where $f=\sum_{k=1}^{\infty} f_k$. Clearly,

$$\limsup_{r\to\infty} r^\gamma (\log r)^\delta U_\alpha^f(r\xi) = \infty$$

for any numbers γ, δ and any $\xi \in E$.

REMARK 4.12. We shall end by remarking the following two facts.

i) A set E is α-thin at infinity if and only if there is a non-negative measure μ such that $U_\alpha^\mu \not\equiv \infty$ and $\liminf_{|x|\to\infty, x\in E} U_\alpha^\mu(x)>0$.

ii) In case $\alpha p<n$, E is (α, p)-thin at infinity if and only if there is a non-negative function $f \in L^p(R^n)$ such that $U_\alpha^f \not\equiv \infty$ and $\liminf_{|x|\to\infty, x\in E} |x|^{(n-\alpha p)/p} U_\alpha^f(x)>0$.

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