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Claw-Decomposition and Evenly-Partite-Claw-Decomposition of Complete Multi-Partite Graphs

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1. Introduction

The decomposition problems of a graph arise in the filing theory and the combinatorial theory of design of experiments. These problems have been developed by Bermond and Schönheim [2], Bermond and Sotteau [3], Erdös, Sauer and Schaer [5], Huang and Rosa [7] and so on. Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [16] have completely solved the problem of claw-decomposability of a complete graph. The claw-decomposition of a complete graph yields an optimal binary-valued balanced file organization scheme of order two which is called a Hiroshima University balanced file organization scheme of order two (**HUBFS**$_2$) [17]. A binary-valued balanced file organization scheme is said to be optimal if it has the least redundancy among all possible binary-valued balanced file organization schemes having the same parameters, provided the distribution of records is invariant under the permutation of attributes. A necessary condition and some sufficient conditions for complete graph to be decomposed into a union of subgraphs have also been given by Yamamoto and Tazawa [19]. The subgraph is a generalized graph of a claw which is called a hyperclaw. This hyperclaw decomposition provides us an optimal binary-valued balanced file organization scheme of general order k, which is called an **HUBFS**$_k$, in the above-mentioned sense [20].

Recently, the decomposition problems of other graphs than a complete graph have been investigated by many authors. Myers [9] has investigated the decomposition problems of the product of a complete graph with itself. Sumner [12] has given some theorems on the 1-factorization. Bermond [1], Schönheim [11] and Wilson [15] have investigated the decomposition problems of the directed complete graphs. The decomposition problems of a complete multipartite graph have been developed by Cockayne and Hartnell [4], Tazawa, Ushio and Yamamoto [13], Ushio, Tazawa and Yamamoto [14] and Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [16]. Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [16] have completely solved the problem of claw-decomposability of a complete bipartite graph. Ushio, Tazawa and Yamamoto [14] have given a necessary and sufficient condition for a complete m-partite graph $K_m(n, n,..., n)$ with m sets of n points each to be decomposed into a union of line-disjoint subgraphs which are isomorphic to a claw. This result covers the claw-decomposition theorem for a complete graph in [16]. Tazawa, Ushio and Yamamoto [13] have also given a necessary and sufficient condition for the decomposability of $K_m(n, n,\dots, n)$ into a union of line-disjoint subgraphs each isomorphic to a partite-claw. A partite-claw-decomposition of a complete m-partite graph yields an optimal multiple-valued balanced file organization scheme of order two in that it has the least redundancy among all possible balanced schemes with the same parameters for an equally likely distribution of multiple-valued records. Such an optimal scheme is called a Hiroshima University balanced multiple-valued file organization scheme of order two (**HUBMFS$_2$**) [18].

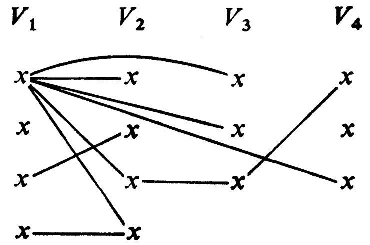
In this paper, we shall, in particular, establish a necessary and sufficient condition for a complete m-partite graph $K_m(n, n,\dots, n)$ to be decomposed into a union of line-disjoint subgraphs, each being isomorphic to a generalized graph of partite-claw.

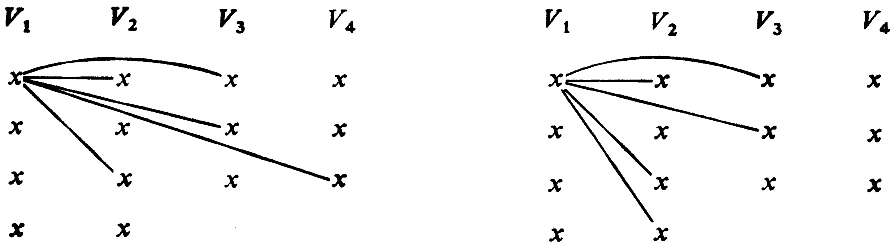
2. Preliminaries

The reader is referred to [6] for any term not defined below. Consider a graph without loops or multiple lines. Let m (≥ 2) be an integer. A graph is said to be m-partite if there exists a partition of its point set into m subsets V_1, $V_2,\dots, V_m$ such that no line joins two points in the same subset. $V_1, V_2,\dots$ and V_m are called its independent sets. An m-partite graph is denoted by $G_m(n_1, n_2,\dots, n_m)$, where n_i is the cardinality $|V_i|$ of V_i $(i=1, 2,\dots, m)$. An m-partite graph is called complete, denoted by $K_m(n_1, n_2,\dots, n_m)$, if it contains every line joining different independent subsets. A complete graph K_m with m points may be considered as a special case of complete m-partite graph where $n_1=n_2=\cdots=n_m=1$. A complete bipartite graph $K_2(1, c)$ with $c+1$ points and c lines is called a claw or star of degree c (≥ 2). A point of degree c is called a root and each point of degree one is called a leaf of the claw.

Consider a claw which is a subgraph of an m-partite graph $G_m(n_1, n_2,\dots, n_m)$ with m independent sets $V_1, V_2,\dots, V_m$. Let $V_{i_1}, V_{i_2},\dots, V_{i_{m-1}}$ be the point sets not containing the root point of the claw and let v_{i_α} be the number of the leaves in V_{i_α} for $\alpha=1, 2,\dots, m-1$. Then the claw is said to be *evenly-partite* in the $G_m(n_1, n_2,\dots, n_m)$ if $|v_{i_\alpha}-v_{i_\beta}|\leq 1$ holds for every $\alpha, \beta=1$ $2,\dots, m-1$. A partite-claw (PC) in [13] is a special case of an evenly-partite-claw (EPC) in which every point set contains at most one leaf. In Fig. 1, a 4-partite graph $G_4(4, 4, 3, 3)$ with four independent sets V_1, V_2, V_3, V_4 of 4, 4, 3, 3 points each is given. Two claws of degree five being subgraphs of the same graph $G_4(4, 4, 3, 3)$ are also given. Fig. 1 (a) shows an EPC since $v_2=v_3=2$ and $v_4=1$, while the claw in Fig. 1 (b) is not evenly-partite since $v_2=3$, $v_3=2$ and $v_4=0$.

$G_4(4, 4, 3, 3)$:





(a) (b)

Fig. 1. Two claws of degree five

DEFINITION 2.1. *Let G be a graph with c lines. A complete m-partite graph* $K_m(n_1, n_2, \ldots, n_m)$ *with m independent sets of* $n_1, n_2, \ldots, n_m$ *points each is said to have a G-decomposition of degree c if it is a union of line-disjoint subgraphs each isomorphic to the graph G.*

3. Adjacency matrix

In this section we shall observe that the G-decomposition problem can be considered by using the property of adjacency matrix associated with a graph. A directed graph obtained by assigning a direction to every line of a graph is called an oriented graph. Let $K_m(n_1, n_2, \ldots, n_m)$ be a labeled complete m-partite graph with m independent sets $V_i = \{v_{ip} | ip = n_1 + n_2 + \cdots + n_{i-1} + p,\ p = 1, 2, \ldots, n_i\}$ $(i = 1, 2, \ldots, m)$ and consider an oriented complete m-partite graph $K_m(n_1, n_2, \ldots, n_m)$. The number of such oriented $K_m(n_1, n_2, \ldots, n_m)$'s is, of course, 2^μ where $\mu = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} n_i n_j$. To an oriented $K_m(n_1, n_2, \ldots, n_m)$ there corresponds a 0–1 adjacency matrix of order $\sum_{i=1}^{m} n_i$

$$M = \|M_{ij}\| \tag{3.1}$$

composed of m^2 submatrices $M_{ij} = \|m_{ip,jq}\|$ of size $n_i \times n_j$ defined by

$$m_{ip,jq} = \begin{cases} 1 & \text{if } v_{ip} \text{ is adjacent to } v_{jq} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$(3.2) \qquad m_{ip,iq} = 0 \quad \text{and} \quad m_{ip,jq} + m_{jq,ip} = 1$$

hold for all p, q, i and j $(\neq i)$, i.e., $M_{ii}=0$ and $M_{ij}+M_{ji}^T=G_{n_i,n_j}$ $(i\neq j)$, where $G_{t,u}$ denotes a $t\times u$ matrix whose elements are all unity.

Conversely, a 0–1 matrix of order $\sum_{i=1}^{m} n_i$ satisfying (3.2) produces an oriented complete m-partite graph $K_m(n_1, n_2,\dots, n_m)$. Thus for a labeled complete m-partite graph $K_m(n_1, n_2,\dots, n_m)$ there is one-to-one correspondence between oriented $K_m(n_1, n_2,\dots, n_m)$'s and $\sum_{i=1}^{m} n_i \times \sum_{i=1}^{m} n_i$ binary matrices satisfying (3.2).

THEOREM 3.1. *A complete m-partite graph $K_m(n_1, n_2,\dots, n_m)$ has a claw-decomposition of degree c if and only if there exists an oriented $K_m(n_1, n_2,\dots, n_m)$ whose adjacency matrix $M=\|M_{ij}\|$ of order $\sum_{i=1}^{m} n_i$ satisfies the following condition*:

(a) *Every row sum of M is an integral multiple of c, i.e.,*

$$\sum_{j=1}^{m}\sum_{q=1}^{n_j} m_{ip,jq} = a_{ip}c \qquad for \quad p = 1, 2,\dots, n_i;\ i = 1, 2,\dots, m.$$

This theorem is proved by the same method as in Ushio, Tazawa and Yamamoto [14] and we omit the proof. In the following theorem we consider an evenly-partite-claw instead of a claw.

THEOREM 3.2. *A complete m-partite graph $K_m(n_1, n_2,\dots, n_m)$ has an* EPC-*decomposition of degree c if and only if there exists an oriented $K_m(n_1, n_2,\dots, n_m)$ whose adjacency matrix $M=\|M_{ij}\|$ of order $\sum_{i=1}^{m} n_i$ satisfies Condition* (a) *of Theorem* 3.1 *and the following condition*:

(b) *The submatrix M_{ij} satisfies the row sum constraints*

$$a_{ip}k \le \sum_{q=1}^{n_j} m_{ip,jq} \le \min\,(a_{ip}(k + 1), n_j) \qquad for \quad p = 1, 2,\dots, n_i$$

for every pair of i and j $(\neq i)$, where k is the greatest integer not exceeding $c/(m-1)$.

PROOF. Suppose $K_m(n_1, n_2,\dots, n_m)$ has an EPC-decomposition of degree c. Consider an oriented $K_m(n_1, n_2,\dots, n_m)$ obtained by assigning a direction to every line in such a manner that the point corresponding to the root of an EPC is adjacent to the other end points corresponding to its leaves. Let M be the adjacency matrix corresponding to the oriented $K_m(n_1, n_2,\dots, n_m)$. If a_{ip} denotes the number of EPC's which have the same root point v_{ip}, then there are exactly $a_{ip}c$ points adjacent from v_{ip}, because the degree of every EPC is c. Thus Condition (a) holds for the adjacency matrix M. Since an EPC with the root v_{ip} has k or $k+1$ leaves in V_j for every j ($\neq i$), there are at least $a_{ip}k$ points and at most $\min(a_{ip}(k+1), n_j)$ points in V_j ($j \neq i$) which are adjacent from v_{ip}. Thus we have Condition (b) for M. Conversely, suppose that there exists an oriented $K_m(n_1, n_2,\dots, n_m)$ whose adjacency matrix $M=\|M_{ij}\|$ of order $\sum_{i=1}^{m} n_i$ satisfies Conditions (a) and (b). Then we shall show by a constructive method that $K_m(n_1, n_2,\dots, n_m)$ has an EPC-decomposition of degree c. Construct an $a_{ip}\times m$ nonnegative integral matrix $B=\|b_{hj}\|$, where b_{hj} satisfies $k\le b_{hj}\le k+1$ for every $h=1, 2,\dots, a_{ip}$ and $j=1, 2,\dots, m$ ($\neq i$). Its row and column sum vectors are denoted by $(c, c,\dots, c)$ and $(s_1, s_2,\dots, s_m)$, respectively, where $s_j=\sum_{q=1}^{n_j} m_{ip,jq}$. Such a matrix can be constructed (cf. Corollary 1.3 and Theorem 1.1 in [16]), since $a_{ip}k\le s_j\le \min(a_{ip}(k+1), n_j)$ by Condition (b). Next, partition the set of s_j 1's standing on the pth row of M_{ij} in M into a_{ip} subsets $S_{1j}, S_{2j},\dots, S_{a_{ip}j}$ of $b_{1j}, b_{2j},\dots, b_{a_{ip}j}$ 1's each. Then $S_h=\bigcup_{j=1}^{m} S_{hj}$ is the set composed of c 1's for every $h=1, 2,\dots, a_{ip}$. Thus if we select c lines corresponding to c 1's of S_h out of $K_m(n_1, n_2,\dots, n_m)$, then a collection of those c lines corresponds to an EPC of degree c. Hence $K_m(n_1, n_2,\dots, n_m)$ has an EPC-decomposition of degree c.

Let a_{ip} ($p=1, 2,\dots, n_i$; $i=1, 2,\dots, m$) be $\sum_{i=1}^{m} n_i$ nonnegative integers satisfying $\sum_{i=1}^{m}\sum_{p=1}^{n_i} a_{ip}=(\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} n_in_j)/c$. Consider the following:

(1) An $m\times m$ nonnegative integral matrix $X=\|x_{ij}\|$ satisfying

$$(3.3)\qquad \sum_{j=1}^{m} x_{ij} = c\sum_{p=1}^{n_i} a_{ip},\quad x_{ii}=0,\quad x_{ij}+x_{ji}=n_in_j \qquad (i\neq j).$$

(2) m nonnegative integral matrices $Y_i=\|y_{ip,j}\|$ ($i=1, 2,\dots, m$) of size $n_i\times m$ satisfying

$$(3.4)\qquad \sum_{j=1}^{m} y_{ip,j} = a_{ip}c,$$

$$(3.5)\qquad \sum_{p=1}^{n_i} y_{ip,j} = x_{ij},$$

$$(3.6)\qquad a_{ip}k \le y_{ip,j} \le \min(a_{ip}(k+1), n_j) \qquad (i\neq j),$$

where k is the greatest integer not exceeding $c/(m-1)$.

(3) $\binom{m}{2}$ 0–1 matrices $M^*_{ij}=\|m^*_{ip,jq}\|$ ($1\le i<j\le m$) of size $n_i\times n_j$ satisfying

$$(3.7)\qquad \sum_{q=1}^{n_j} m^*_{ip,jq} = y_{ip,j} \quad\text{and}\quad \sum_{p=1}^{n_i} m^*_{ip,jq} = n_i - y_{jq,i}.$$

Then we prove

THEOREM 3.3. *If the above-mentioned matrices X, Y_i and M^*_{ij} can be constructed, then $K_m(n_1, n_2,\dots, n_m)$ has an* EPC-*decomposition of degree c.*

PROOF. Consider a $\sum_{i=1}^{m} n_i \times \sum_{i=1}^{m} n_i$ matrix $M=\|M_{ij}\|$ composed of m^2 submatrices $M_{ij}=\|m_{ip,jq}\|$ of size $n_i \times n_j$ defined by

$$M_{ij} = \begin{cases} M^*_{ij} & \text{for } i<j \\ 0 & \text{for } i=j \\ G_{n_i,n_j} - M^{*T}_{ji} & \text{for } i>j. \end{cases}$$

Then M is an adjacency matrix of an oriented $K_m(n_1, n_2,\dots, n_m)$ since M satisfies (3.2). Moreover, since we have $\sum_{q=1}^{n_j} m_{ip,jq} = y_{ip,j}$ for $i, j=1, 2,\dots, m$ by (3.7), it follows from (3.4) and (3.6) that M satisfies Condition (a) of Theorem 3.1 and Condition (b) of Theorem 3.2. Thus $K_m(n_1, n_2,\dots, n_m)$ has an EPC-decomposition of degree c.

Note that Theorems 3.2 and 3.3 are respectively identical with Theorems 4.1 and 4.2 in Tazawa, Ushio and Yamamoto [13] for $n_1=n_2=\cdots=n_m$ and $k=0$.

4. Claw-decomposition

With respect to G-decomposition of a complete m-partite graph $K_m(n_1, n_2,\dots, n_m)$ where G is a claw, we have the following theorem.

THEOREM 4.1. *Let $n_1, n_2,\dots, n_m$ $(m\geq 2)$ be m positive integers. If a complete m-partite graph $K_m(n_1, n_2,\dots, n_m)$ has a claw-decomposition of degree c, then the following two conditions hold:*

(i) *c is a factor of* $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} n_i n_j$.

(ii) $c \leq \dfrac{\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} n_i n_j}{\sum_{i=1}^{m} n_i - \max_j n_j}$.

PROOF. Since the number of lines of $K_m(n_1, n_2,\dots, n_m)$ is $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} n_i n_j$, Condition (i) is obviously necessary. Let $V_1, V_2,\dots, V_m$ be m independent sets of $K_m(n_1, n_2,\dots, n_m)$, where $|V_i|=n_i$ for $i=1, 2,\dots, m$. Let y_i be the number of claws whose roots are points of V_i $(i=1, 2,\dots, m)$. Then we have $y_i \geq n_i$ for all i except at most one, say j_0, since $K_m(n_1, n_2,\dots, n_m)$ has a claw-decomposition of degree c. Therefore, we have

$$(4.1)\qquad \frac{\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} n_i n_j}{c} = \sum_{i=1}^{m} y_i \geq \sum_{\substack{i=1\\ i\neq j_0}}^{m} n_i = \sum_{i=1}^{m} n_i - n_{j_0}.$$

Hence Condition (ii) is necessary since $n_{j_0} \leq \max_j n_j$.

For the case $n_1 = n_2 = \cdots = n_m = n$, Ushio, Tazawa and Yamamoto [14] have shown that a necessary and sufficient condition for $K_m(n, n,\dots, n)$ to have a claw-decomposition of degree c is that (i) and (ii) in Theorem 4.1 hold.

THEOREM 4.2. *Let $n_1, n_2,\dots, n_m$ $(m \geq 2)$ and c (≥ 2) be positive integers satisfying Condition* (i) *of Theorem* 4.1. *Put $b = (\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} n_i n_j)/c$. If b is an integral multiple of $\sum_{i=1}^{m} n_i$, then $K_m(n_1, n_2,\dots, n_m)$ has a claw-decomposition of degree c.*

The following lemma, which has been given by Moon [8], is useful for the proof of Theorem 4.2.

LEMMA 4.3. *There exists an adjacency matrix M of an oriented $K_m(n_1, n_2,\dots, n_m)$ which has a given row sum vector $(\alpha_{11},\dots, \alpha_{1n_1}, \alpha_{21},\dots, \alpha_{2n_2},\dots, \alpha_{m1},\dots, \alpha_{mn_m})$ satisfying $\sum_{i=1}^{m}\sum_{p=1}^{n_i} \alpha_{ip} = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} n_i n_j$ and $\alpha_{i1} \geq \alpha_{i2} \geq \cdots \geq \alpha_{in_i}$ for all i if and only if the inequality*

$$(4.2)\qquad \sum_{i=1}^{m}\sum_{p=1}^{k_i} \alpha_{ip} \leq KN - \sum_{i=1}^{m} k_i n_i - \frac{1}{2}K^2 + \frac{1}{2}\sum_{i=1}^{m} k_i^2$$

holds for every set of m integers k_i satisfying $0 \leq k_i \leq n_i$, where $N = n_1 + n_2 + \cdots + n_m$ and $K = k_1 + k_2 + \cdots + k_m$.

PROOF OF THEOREM 4.2. Put $a_{ip} = b/N$ and $\alpha_{ip} = a_{ip}c$ for $p = 1, 2,\dots, n_i$; $i = 1, 2,\dots, m$. Then $\sum_{i=1}^{m}\sum_{p=1}^{n_i} \alpha_{ip} = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} n_i n_j$. Since $bc = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} n_i n_j = (N^2 - \sum_{i=1}^{m} n_i^2)/2$, we have $\sum_{i=1}^{m}\sum_{p=1}^{k_i} \alpha_{ip} = (N^2 - \sum_{i=1}^{m} n_i^2)K/2N$. Thus

$$(4.3)\qquad KN - \sum_{i=1}^{m} k_i n_i - \frac{1}{2}K^2 + \frac{1}{2}\sum_{i=1}^{m} k_i^2 - \sum_{i=1}^{m}\sum_{p=1}^{k_i} \alpha_{ip} = \frac{S}{2N},$$

where $S = NK(N-K) + K\sum_{i=1}^{m} n_i^2 - N\sum_{i=1}^{m} k_i(2n_i - k_i)$. Let $t_i = n_i - k_i$ $(i = 1, 2,\dots, m)$. Then, substituting into S the following three identities

$$N = \sum t_i + \sum k_i$$

$$\sum n_i^2 = \sum t_i^2 + 2\sum k_i t_i + \sum k_i^2$$

$$\sum k_i(2n_i - k_i) = 2\sum k_i t_i + \sum k_i^2,$$

we have

$$(4.4)\quad S = \sum_{i=1}^{m} k_i \sum_{j=1}^{m} t_j \sum_{l=1}^{m} t_l - 2\sum_{i=1}^{m} k_i t_i \sum_{j=1}^{m} t_j + \sum_{i=1}^{m} k_i \sum_{j=1}^{m} t_j^2 + \sum_{i=1}^{m} t_i\{(\sum_{j=1}^{m} k_j)^2 - \sum_{j=1}^{m} k_j^2\}$$

$$= \sum_{i=1}^{m} k_i\{(t_i + \sum_{\substack{j=1\\ j\neq i}}^{m} t_j)(t_i + \sum_{\substack{l=1\\ l\neq i}}^{m} t_l) - 2t_i(t_i + \sum_{\substack{j=1\\ j\neq i}}^{m} t_j) + (t_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{m} t_j^2)\}$$

$$+ \sum_{i=1}^{m} t_i\{(\sum_{j=1}^{m} k_j)^2 - \sum_{j=1}^{m} k_j^2\}$$

$$= \sum_{i=1}^{m} k_i(\sum_{\substack{j=1\\ j\neq i}}^{m} t_j \sum_{\substack{l=1\\ l\neq i}}^{m} t_l + \sum_{\substack{j=1\\ j\neq i}}^{m} t_j^2) + \sum_{i=1}^{m} t_i\{(\sum_{j=1}^{m} k_j)^2 - \sum_{j=1}^{m} k_j^2\}.$$

Since $t_i \geq 0$ $(i=1, 2,\dots, m)$ and $(\sum_{j=1}^{m} k_j)^2 - \sum_{j=1}^{m} k_j^2 \geq 0$, $S \geq 0$ is obtained. Therefore, the inequality (4.2) holds. It follows from Lemma 4.3 that there exists an adjacency matrix M satisfying Condition (a) in Theorem 3.1. Hence $K_m(n_1, n_2,\dots, n_m)$ has a claw-decomposition of degree c.

5. Evenly-partite-claw decomposition theorem

In the following we shall restrict our attention to the case that $n_1 = n_2 = \cdots = n_m = n$. Let c be a positive integer and put $c = (m-1)k + l$ $(0 < l \leq m-1)$. For $k=0$, Tazawa, Ushio and Yamamoto [13] have given the following theorem:

THEOREM 5.1. *Let m, n and c be three positive integers satisfying $m-1 \geq c \geq 2$. Then a complete m-partite graph $K_m(n, n,\dots, n)$ has an* EPC-*decomposition of degree c if and only if*

(i) *c is a factor of $\binom{m}{2} n^2$, and*

(ii) *$c \leq m-1$ if n is even and $c \leq m-1-\frac{m-2}{n^2+1}$ if n is odd.*

As usual, let $\lfloor r \rfloor$ be the greatest integer not exceeding r and $\lceil r \rceil$ be the smallest integer not less than r. For $k \geq 1$, we have the following theorem which will be proved in Section 6.

THEOREM 5.2. *Let m, n and c be three positive integers satisfying $c \geq m \geq 3$. Then a complete m-partite graph $K_m(n, n,\dots, n)$ has an* EPC-*decomposition of degree c if and only if the following three conditions hold:*

(i) *c is a factor of* $\binom{m}{2}n^2$.

(ii) $c \leq \frac{(m-1)n}{2} + 1$.

(iii) $(m-1)\left\lceil \frac{n^2}{2(k+1)} \right\rceil + v_1 \leq \frac{m(m-1)n^2}{2c} \leq (m-1)\left\lfloor \min\left(\frac{n^2}{2k}, v_2\right) \right\rfloor + v_2$,

where

$$v_1 = \frac{n^2(m-1)(2c-mk-m)}{2c(c-k-1)} \quad \text{and} \quad v_2 = \frac{n^2(m-1)(2c-mk)}{2c(c-k)}.$$

When $l = m-1$, Conditions (i)–(iii) in Theorem 5.2 are simplified as follows:

COROLLARY 5.3. *Let* $c\,(\geq m)$ *be an integral multiple of* $m-1\,(\geq 2)$. *Then a complete m-partite graph* $K_m(n, n,\dots, n)$ *has an* EPC-*decomposition of degree c if and only if the following two conditions hold*:

(1) $2c$ *is a factor of* $(m-1)n^2$.

(2) $c \leq \frac{(m-1)n}{2}$.

PROOF. Note first that $k \geq 1$. It is enough to show that Conditions (1) and (2) hold if and only if Conditions (i)–(iii) in Theorem 5.2 hold. If (i), (ii) and (iii) hold, then (1) is obtained by the first inequality in (iii). (2) is also obtained by (i), (ii) and $m \geq 3$. Conversely, if (1) and (2) hold, then (i) and (ii) hold obviously. It can be shown easily that the first inequality in (iii) is obtained by (1) and that the second inequality of (iii) is obtained by (2).

6. Proof of Theorem 5.2

6.1. Necessity

Suppose that $K_m(n, n,\dots, n)$ has an EPC-decomposition of degree c. Then c is obviously a factor of the number of lines of $K_m(n, n,\dots, n)$ $\left(\text{namely } \binom{m}{2}n^2\right)$. Let $V_1, V_2,\dots, V_m$ be m independent sets, each cardinality being n, of $K_m(n, n,\dots, n)$ and let y_i be the number of EPC's whose roots are points of V_i $(i=1, 2,\dots, m)$. Then we have the following statements which are immediate consequences:

(1) Every line belongs to exactly one EPC.

(2) For each EPC, there are l independent sets such that each set contains $k+1$ leaves and there are $m-1-l$ independent sets such that each set contains k leaves.

(3) $y_i \geq n$ for all i except at most one, say j_0.

Consider now any $m-1$ sets $V_{i_1}, V_{i_2},\dots, V_{i_{m-1}}$ where $\{i_1, i_2,\dots, i_{m-1}\} \subset \{1, 2,\dots, m\}$ and consider an EPC whose root is a point in either of $V_{i_1}, V_{i_2},\dots, V_{i_{m-1}}$. Let X denote the set of all lines joining V_{i_α} and V_{i_β} for all $\alpha, \beta = 1, 2,\dots, m-1$ $(\alpha \neq \beta)$ and let N denote the number of lines contained in X and in the EPC. Then from the above-mentioned statements (1) and (2) it follows that $c-k-1 \leq N \leq c-k$ holds. Thus it is easy to see that the cardinality of X satisfies

$$(6.1) \qquad (\sum_{\alpha=1}^{m-1} y_{i_\alpha})(c-k) \geq \frac{(m-1)(m-2)n^2}{2} \geq (\sum_{\alpha=1}^{m-1} y_{i_\alpha})(c-k-1).$$

When $i_\alpha \neq j_0$ for $\alpha = 1, 2,\dots, m-1$,

$$(6.2) \qquad \frac{(m-1)(m-2)n^2}{2} \geq (m-1)n(c-k-1)$$

since $\sum_{\alpha=1}^{m-1} y_{i_\alpha} \geq (m-1)n$ by the statement (3). Substituting $k=(c-l)/(m-1)$ into (6.2) we have $c \leq \frac{(m-1)n}{2}+1$ that is Condition (ii). We shall show that Condition (iii) is necessary. Consider in (6.1) a set V_j and the remaining $m-1$ sets $V_{i_1}, V_{i_2},\dots, V_{i_{m-1}}$. Then since $c\sum_{\alpha=1}^{m-1} y_{i_\alpha} = \binom{m}{2}n^2 - y_j c$ $(j \neq i_\alpha;\ \alpha = 1, 2,\dots, m-1)$, (6.1) becomes

$$(6.3) \qquad \left\{\frac{m(m-1)n^2}{2c} - y_j\right\}(c-k) \geq \frac{(m-1)(m-2)n^2}{2} \geq \left\{\frac{m(m-1)n^2}{2c} - y_j\right\}(c-k-1).$$

Thus with respect to y_j, we have

$$(6.4) \qquad \frac{n^2(m-1)(2c-mk-m)}{2c(c-k-1)} \leq y_j \leq \frac{n^2(m-1)(2c-mk)}{2c(c-k)} \qquad \text{for} \quad j=1, 2,\dots, m$$

since $c > k+1$. Consider two sets V_i and V_j $(i \neq j)$. Since the number of lines joining V_i and V_j is n^2, it can easily be seen by the statements (1) and (2) that

$$(6.5) \qquad (y_i + y_j)(k+1) \geq n^2 \geq (y_i + y_j)k \qquad \text{for} \quad i \neq j;\ i, j = 1, 2,\dots, m$$

holds. Thus we have

$$(6.6) \qquad y_i \geq \left\lceil \frac{n^2}{2(k+1)} \right\rceil \qquad \textit{for all } i \textit{ except at most one,}$$

by the first inequality of (6.5). Applying (6.6) and the first inequality of (6.4) to $\sum y_i$, we have

$$\frac{m(m-1)n^2}{2c} \geq (m-1)\left\lceil \frac{n^2}{2(k+1)} \right\rceil + \frac{n^2(m-1)(2c-mk-m)}{2c(c-k-1)}. \tag{6.7}$$

The second inequality of (6.5) gives

$$y_i \leq \left\lfloor \frac{n^2}{2k} \right\rfloor \quad \textit{for all } i \textit{ except at most one,} \tag{6.8}$$

since $k>0$. The application of (6.8) and the second inequality of (6.4) to $\sum y_i$ gives

$$\frac{m(m-1)n^2}{2c} \leq (m-1)\left\lfloor \min\left(\frac{n^2}{2k}, \frac{n^2(m-1)(2c-mk)}{2c(c-k)}\right)\right\rfloor + \frac{n^2(m-1)(2c-mk)}{2c(c-k)}. \tag{6.9}$$

Hence combining (6.7) and (6.9) we obtain Condition (iii).

Note that Condition (ii) of Theorem 5.1 is obtained by substituting $k=0$ into the inequality (6.7).

6.2. Sufficiency

For a set of parameters m, n and c satisfying Condition (i), we write in the form

$$\frac{m(m-1)n^2}{2c} = mna + r \qquad (0 \leq r < mn). \tag{6.10}$$

Then we have two cases; $a=0$ and $a \geq 1$ to prove that the remaining conditions (ii) and (iii) are sufficient.

1°) *Case* $a=0$: In this case, we obtain $m=c$ and $n=2$ by Condition (ii). Define m^2 square matrices M_{ij} ($i, j=1, 2,\dots, m$) of order two by

$$M_{ij} = \begin{cases} I_2 & \text{for } \; 1 \leq i < j \leq m-1 \\ G_{2,2} & \text{for } \; 1 \leq i \leq m-1 \text{ and } j=m \\ 0 & \text{for } \; i=j, \end{cases} \tag{6.11}$$

$$M_{ij} = G_{2,2} - M_{ji} \qquad \text{for } \; i > j,$$

where I_t denotes the identity matrix of order t. Then it is easy to check that the

0–1 matrix $M=\|M_{ij}\|$ of order mn composed of these submatrices M_{ij} satisfies Condition (a) in Theorem 3.1 and Condition (b) in Theorem 3.2. Thus $K_m(n, n,\dots, n)$ has an EPC-decomposition of degree $c=2$.

2°) *Case* $a\geq 1$: Write $r=md+s$ $(0\leq s<m)$. Let J_1 and J_2 denote the sets $\{1, 2,\dots, s\}$ and $\{s+1, s+2,\dots, m\}$, respectively. Let

$$a_{ip}=\begin{cases} a+1 & (p=1, 2,\dots, d_\lambda) \\ a & (p=d_\lambda+1, d_\lambda+2,\dots, n) \end{cases} \tag{6.12}$$

for $i\in J_\lambda$ and $\lambda=1, 2$, where $d_\lambda=d+1$ or d according as $\lambda=1$ or 2. Then a_{ip}'s satisfy $\sum_{i=1}^{m}\sum_{p=1}^{n} a_{ip}=\binom{m}{2}n^2/c$. It can be proved that $K_m(n, n,\dots, n)$ has an EPC-decomposition of degree c by the fact that the matrices X, Y_i and M_{ij}^* in Theorem 3.3 can be constructed for the particular set of a_{ip} in (6.12). The constructions of such matrices X, Y_i and M_{ij}^* are given in Sections 7, 8 and 9, in order.

7. Construction of X

As stated in [13], suppose that four nonnegative integral matrices $X_{\lambda\mu}$ $(\lambda, \mu=1, 2)$ satisfying

$$X_{\lambda\mu}+X_{\lambda\mu}^T=\begin{cases} n^2(G_{s_\lambda,s_\lambda}-I_{s_\lambda}) & \text{for } \lambda=\mu \\ n^2G_{s_\lambda,s_\mu} & \text{for } \lambda\neq\mu, \end{cases} \tag{7.1}$$

$$[X_{\lambda 1}\ \ X_{\lambda 2}]\mathbf{j}_m=c(na+d_\lambda)\mathbf{j}_{s_\lambda}, \tag{7.2}$$

can be constructed, where $\mathbf{j}_t$ denotes a t-vector whose components are all unity and $s_t=s$ or $m-s$ according as $t=1$ or 2. Then the matrix

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{7.3}$$

is a required matrix satisfying (3.3). These submatrices $X_{\lambda\mu}$ can be constructed by the same method as in [13]. So we have the following results which are given for the respective cases that n is even and odd.

(1) *Case n is even*. Let $s_\lambda c/m=s_\lambda x+y_\lambda$, $0\leq y_\lambda<s_\lambda$ for $\lambda=1, 2$. Then

$$\begin{aligned} &X_{\lambda\lambda}=\frac{n^2}{2}(G_{s_\lambda,s_\lambda}-I_{s_\lambda}) \qquad \text{for } \lambda=1, 2, \\ &X_{12}=\left(\frac{n^2}{2}+x\right)G_{s_1,s_2}+B, \quad X_{21}=\left(\frac{n^2}{2}-x\right)G_{s_2,s_1}-B^T, \end{aligned} \tag{7.4}$$

where B is a 0–1 matrix of size $s_1 \times s_2$ whose row sums are all y_2 and whose column sums are all y_1.

(2) *Case n is odd.* Let

$$x = \left\lfloor \frac{2c-m}{2m} \right\rfloor,$$

$$y_1 = (m-1)\frac{n^2-1}{2} + \frac{m-s-1}{2} - c(na+d) - sx = s_1\left(\frac{2c-m}{2m} - x\right),$$

$$y_2 = c(na+d+1) - (m-1)\frac{n^2+1}{2} + \frac{s-1}{2} - (m-s)x = s_2\left(\frac{2c-m}{2m} - x\right).$$

Let B be a 0–1 matrix of size $s_1 \times s_2$ whose row sum vector $\boldsymbol{\alpha}$ and column sum vector $\boldsymbol{\beta}$ are respectively

$$\boldsymbol{\alpha}^T = \begin{cases} (y_2, y_2, \ldots, y_2) & \text{for odd } s_1 \\ \Big(\underbrace{y_2 - \tfrac{1}{2}, \ldots, y_2 - \tfrac{1}{2}}_{s_1/2}, \underbrace{y_2 + \tfrac{1}{2}, \ldots, y_2 + \tfrac{1}{2}}_{s_1/2}\Big) & \text{for even } s_1, \text{ and} \end{cases}$$

$$\boldsymbol{\beta}^T = \begin{cases} (y_1, y_1, \ldots, y_1) & \text{for odd } s_2 \\ \Big(\underbrace{y_1 + \tfrac{1}{2}, \ldots, y_1 + \tfrac{1}{2}}_{s_2/2}, \underbrace{y_1 - \tfrac{1}{2}, \ldots, y_1 - \tfrac{1}{2}}_{s_2/2}\Big) & \text{for even } s_2. \end{cases}$$

Then

$$X_{\lambda\lambda} = \frac{n^2-1}{2}(G_{s_\lambda, s_\lambda} - I_{s_\lambda}) + T_{s_\lambda}^{(\lfloor s_\lambda/2 \rfloor, \lceil s_\lambda/2 \rceil - 1)} \qquad \text{for} \quad \lambda = 1, 2,$$

(7.5)

$$X_{12} = \left(\frac{n^2+1}{2} + x\right)G_{s_1, s_2} + B, \quad X_{21} = \left(\frac{n^2-1}{2} - x\right)G_{s_2, s_1} - B^T,$$

where $T_v^{(u_1, u_2)} = \|t_{ij}\|$ is a square matrix of order v defined by

(7.6) $t_{ij} =$

$$= \begin{cases} 1 & \text{if } j-1 \equiv i, i+1, \ldots, i+u_1-1 \bmod v \text{ for } i = 1, 2, \ldots, \left\lfloor \frac{v+1}{2} \right\rfloor \text{ and} \\ & \text{if } j-1 \equiv i, i+1, \ldots, i+u_2-1 \bmod v \text{ for } i = \left\lfloor \frac{v+1}{2} \right\rfloor + 1, \left\lfloor \frac{v+1}{2} \right\rfloor + 2, \ldots, v \\ 0 & \text{otherwise.} \end{cases}$$

8. Construction of Y_i

For the matrix X in the preceding section, partition the partial row sum $\sum_{j\in J_\lambda} x_{ij}$ of X on the submatrix $X_{\lambda\lambda}$ into two parts

$$(8.1) \qquad s^{(1)}_{i,\lambda\lambda} = \left\lfloor \frac{d_\lambda(a+1)}{na+d_\lambda} \sum_{j\in J_\lambda} x_{ij} \right\rfloor, \quad S^{(2)}_{i,\lambda\lambda} = \left\lceil \frac{(n-d_\lambda)a}{na+d_\lambda} \sum_{j\in J_\lambda} x_{ij} \right\rceil$$

for $i\in J_\lambda$ ($\lambda=1, 2$). Further, partition the partial row sum $\sum_{j\in J_\mu} x_{ij}$ of X on $X_{\lambda\mu}$ into two parts

$$(8.2) \qquad S^{(1)}_{i,\lambda\mu} = \left\lceil \frac{d_\lambda(a+1)}{na+d_\lambda} \sum_{j\in J_\mu} x_{ij} \right\rceil, \quad S^{(2)}_{i,\lambda\mu} = \left\lfloor \frac{(n-d_\lambda)a}{na+d_\lambda} \sum_{j\in J_\mu} x_{ij} \right\rfloor$$

for $i\in J_\lambda$ and λ, $\mu=1, 2$ ($\lambda\neq\mu$). Let $W_i=\|w^{(\alpha)}_{ij}\|$ ($i=1, 2,\dots, m$) be m nonnegative integral matrices of size $2\times m$ which satisfy

$$(8.3) \qquad \sum_{j\in J_\mu} w^{(\alpha)}_{ij} = S^{(\alpha)}_{i,\lambda\mu}, \qquad (i\in J_\lambda)$$

for α, λ, $\mu=1, 2$ and which satisfy

$$(8.4) \qquad \sum_{\alpha=1}^{2} w^{(\alpha)}_{ij} = x_{ij} \qquad (i, j = 1, 2,\dots, m).$$

Let $Y_i=\|y_{ip,j}\|$ ($i=1, 2,\dots, m$) be m nonnegative integral matrices of size $n\times m$ satisfying (3.4) and (3.6), and furthermore, satisfying

$$(8.5) \qquad \sum_{p=1}^{d_\lambda} y_{ip,j} = w^{(1)}_{ij} \quad \text{and} \quad \sum_{p=d_\lambda+1}^{n} y_{ip,j} = w^{(2)}_{ij}$$

for $i\in J_\lambda$ ($\lambda=1, 2$) and $j=1, 2,\dots, m$. Then we prove the following lemma.

LEMMA 8.1. *If the above-mentioned matrix Y_i can be constructed for every $i=1, 2,\dots, m$, then the matrices $Y_1, Y_2,\dots, Y_m$ satisfy* (3.4)–(3.6).

PROOF. Since Y_i satisfies (3.4) and (3.6), it remains only to be proved that the matrix Y_i satisfies (3.5). Using (8.4), we have

$$\sum_{p=1}^{n} y_{ip,j} = \sum_{p=1}^{d_\lambda} y_{ip,j} + \sum_{p=d_\lambda+1}^{n} y_{ip,j} = w^{(1)}_{ij} + w^{(2)}_{ij} = x_{ij}.$$

Thus Y_i satisfies (3.5).

As the first step we shall construct in Subsection 8.1 m matrices $W_1, W_2,\dots, W_m$ and then as the second step construct in Subsection 8.2 m matrices $Y_1, Y_2,\dots, Y_m$.

8.1. Construction of W_i

We write $S_{i,\lambda\mu}$ as

$$(8.6)\qquad \begin{aligned} S^{(1)}_{i,\lambda\lambda} &= (s_\lambda - 1)u^{(1)}_{i,\lambda\lambda} + r^{(1)}_{i,\lambda\lambda}, & S^{(2)}_{i,\lambda\lambda} &= (s_\lambda - 1)u^{(2)}_{i,\lambda\lambda} - r^{(2)}_{i,\lambda\lambda}, \\ S^{(1)}_{i,\lambda\mu} &= s_\mu u^{(1)}_{i,\lambda\mu} - r^{(1)}_{i,\lambda\mu}, & S^{(2)}_{i,\lambda\mu} &= s_\mu u^{(2)}_{i,\lambda\mu} + r^{(2)}_{i,\lambda\mu} \end{aligned}$$

for $i \in J_\lambda$ and $\lambda, \mu = 1, 2$ $(\lambda \neq \mu)$, where $0 \leq r^{(\alpha)}_{i,\lambda\lambda} < s_\lambda - 1$ and $0 \leq r^{(\alpha)}_{i,\lambda\mu} < s_\mu$ for $\alpha = 1, 2$. Let $U^{(\alpha)}$ be an $m \times m$ matrix

$$(8.7)\qquad U^{(\alpha)} = \begin{bmatrix} U^{(\alpha)}_{11} & U^{(\alpha)}_{12} \\ U^{(\alpha)}_{21} & U^{(\alpha)}_{22} \end{bmatrix}$$

composed of four submatrices $U^{(\alpha)}_{\lambda\mu}$ $(\lambda, \mu = 1, 2)$ defined by

$$(8.8)\qquad \begin{aligned} U^{(\alpha)}_{11} &= \begin{bmatrix} u^{(\alpha)}_{1,11}\mathbf{j}^{(1)T}_s \\ u^{(\alpha)}_{2,11}\mathbf{j}^{(2)T}_s \\ \vdots \\ u^{(\alpha)}_{s,11}\mathbf{j}^{(s)T}_s \end{bmatrix}, & U^{(\alpha)}_{12} &= \begin{bmatrix} u^{(\alpha)}_{1,12}\mathbf{j}^T_{m-s} \\ u^{(\alpha)}_{2,12}\mathbf{j}^T_{m-s} \\ \vdots \\ u^{(\alpha)}_{s,12}\mathbf{j}^T_{m-s} \end{bmatrix}, \\ U^{(\alpha)}_{21} &= \begin{bmatrix} u^{(\alpha)}_{s+1,21}\mathbf{j}^T_s \\ u^{(\alpha)}_{s+2,21}\mathbf{j}^T_s \\ \vdots \\ u^{(\alpha)}_{m,21}\mathbf{j}^T_s \end{bmatrix}, & U^{(\alpha)}_{22} &= \begin{bmatrix} u^{(\alpha)}_{s+1,22}\mathbf{j}^{(1)T}_{m-s} \\ u^{(\alpha)}_{s+2,22}\mathbf{j}^{(2)T}_{m-s} \\ \vdots \\ u^{(\alpha)}_{m,22}\mathbf{j}^{(m-s)T}_{m-s} \end{bmatrix}, \end{aligned}$$

for each $\alpha = 1, 2$, where $\mathbf{j}^{(i)}_t$ denotes a 0–1 t-vector whose ith component is only zero. Consider 0–1 matrices $N^{(\alpha)}_{\lambda\mu}$ $(\alpha, \lambda, \mu = 1, 2)$ of size $s_\lambda \times s_\mu$ satisfying

$$(8.9)\qquad \begin{aligned} N^{(1)}_{1\mu}\mathbf{j}_{s_\mu} &= (r^{(1)}_{1,1\mu}, r^{(1)}_{2,1\mu}, \ldots, r^{(1)}_{s,1\mu})^T \\ N^{(1)}_{2\mu}\mathbf{j}_{s_\mu} &= (r^{(1)}_{s+1,2\mu}, r^{(1)}_{s+2,2\mu}, \ldots, r^{(1)}_{m,2\mu})^T \end{aligned} \qquad \text{for } \mu = 1, 2,$$

$$(8.10)\qquad \begin{aligned} N^{(1)}_{\lambda\lambda} - N^{(2)}_{\lambda\lambda} &= X_{\lambda\lambda} - U^{(1)}_{\lambda\lambda} - U^{(2)}_{\lambda\lambda} && \text{for } \lambda = 1, 2 \\ -N^{(1)}_{\lambda\mu} + N^{(2)}_{\lambda\mu} &= X_{\lambda\mu} - U^{(1)}_{\lambda\mu} - U^{(2)}_{\lambda\mu} && \text{for } \lambda \neq \mu;\ \lambda, \mu = 1, 2, \text{ and} \end{aligned}$$

$$(8.11)\qquad (N^{(1)})_{ii} = 0 \qquad \text{for } i = 1, 2, \ldots, m,$$

where $(A)_{ij}$ denotes the (i, j)th element of the matrix A and $X_{\lambda\mu}$ is given in Section 7. Then we have the following lemma.

LEMMA 8.2. *Suppose that the matrices of order m*

$$(8.12)\qquad N^{(1)} = \begin{bmatrix} N^{(1)}_{11} & -N^{(1)}_{12} \\ -N^{(1)}_{21} & N^{(1)}_{22} \end{bmatrix} \quad \textit{and} \quad N^{(2)} = \begin{bmatrix} -N^{(2)}_{11} & N^{(2)}_{12} \\ N^{(2)}_{21} & -N^{(2)}_{22} \end{bmatrix}$$

can be constructed. Then the matrices $W_1, W_2, \dots, W_m$ *satisfying* (8.3) *and* (8.4) *can be constructed.*

PROOF. By $\sum_{j\in J_\mu} x_{ij}=S_{i,\lambda\mu}^{(1)}+S_{i,\lambda\mu}^{(2)}$ and by (8.6) and (8.7), a combination of (8.9) and (8.10) gives

$$N_{1\mu}^{(2)}\mathbf{j}_{s_\mu} = (r_{1,1\mu}^{(2)}, r_{2,1\mu}^{(2)}, \dots, r_{s,1\mu}^{(2)})^T$$
$$N_{2\mu}^{(2)}\mathbf{j}_{s_\mu} = (r_{s+1,2\mu}^{(2)}, r_{s+2,2\mu}^{(2)}, \dots, r_{m,2\mu}^{(2)})^T \qquad \text{for } \mu=1, 2. \tag{8.13}$$

Consider a $2\times m$ matrix $W_i=\|w_{ij}^{(\alpha)}\|$ defined by

$$w_{ij}^{(\alpha)} = (U^{(\alpha)})_{ij} + (N^{(\alpha)})_{ij}, \qquad \alpha = 1, 2;\ j = 1, 2, \dots, m \tag{8.14}$$

for every $i=1, 2, \dots, m$. Then it is easy to see that $w_{ij}^{(\alpha)}\geq 0$ for all α, i and j. We have $\sum_{j\in J_\mu} w_{ij}^{(\alpha)}=\sum_{j\in J_\mu}(U^{(\alpha)})_{ij}+\sum_{j\in J_\mu}(N^{(\alpha)})_{ij}=S_{i,\lambda\mu}^{(\alpha)}$ $(i\in J_\lambda)$ from (8.6), (8.9) and (8.13). We also have $\sum_{\alpha=1}^{2} w_{ij}^{(\alpha)}=\sum_{\alpha=1}^{2}(U^{(\alpha)})_{ij}+\sum_{\alpha=1}^{2}(N^{(\alpha)})_{ij}=x_{ij}$ from (8.10). Thus the matrices $W_1, W_2, \dots, W_m$ satisfy (8.3) and (8.4).

From Lemma 8.2 the construction of W_i can reduce to the construction of $N^{(\alpha)}$. The following two lemmas, which are proved easily, are useful for the construction of $N^{(\alpha)}$.

LEMMA 8.3. *Let D be a 0–1 matrix of size $a\times b$ whose αth row sum is Δ_α for $\alpha=1, 2, \dots, a$. Let p_α $(\alpha=1, 2, \dots, a)$ be nonnegative integers satisfying $\Delta_\alpha\leq p_\alpha\leq b$ for every α. Then two 0–1 matrices P and Q of size $a\times b$ each, which satisfy $P\mathbf{j}_b=(p_1, p_2, \dots, p_a)^T$ and $P-Q=D$, can be constructed.*

LEMMA 8.4. *Let D be a 0–1 square matrix of order a with zero diagonal whose αth row sum is Δ_α for $\alpha=1, 2, \dots, a$. Let p_α $(\alpha=1, 2, \dots, a)$ be nonnegative integers satisfying $\Delta_\alpha\leq p_\alpha\leq a-1$ for every α. Then two 0–1 matrices P and Q of size $a\times a$, which have zero in the diagonal positions and which satisfy $P\mathbf{j}_a=(p_1, p_2, \dots, p_a)^T$ and $P-Q=D$, can be constructed.*

Now we proceed to the construction of $N^{(\alpha)}$. Let $i_0, i_0+1, \dots, i_0+s_\lambda-1$ denote all element of J_λ. Let J'_λ and J''_λ be the sets $\left\{i_0, i_0+1, \dots, i_0+\left\lfloor \frac{s_\lambda}{2}\right\rfloor-1\right\}$ and $\left\{i_0+\left\lfloor \frac{s_\lambda}{2}\right\rfloor, i_0+\left\lfloor \frac{s_\lambda}{2}\right\rfloor+1, \dots, i_0+s_\lambda-1\right\}$, respectively. Then note that from the construction of X given in Section 7 $S_{i,\lambda\mu}^{(\alpha)}$ can also be written as follows:

$$S_{i,\lambda\lambda}^{(1)} = (s_\lambda - 1)u_{\lambda\lambda}^{(1)} + r_{\lambda\lambda}^{(1)}, \qquad S_{i,\lambda\lambda}^{(2)} = (s_\lambda - 1)u_{\lambda\lambda}^{(2)} - r_{\lambda\lambda}^{(2)},$$

$$S_{i,\lambda\mu}^{(1)} = s_\mu u_{\lambda\mu}^{(1)} - r_{\lambda\mu}^{(1)}, \qquad S_{i,\lambda\mu}^{(2)} = s_\mu u_{\lambda\mu}^{(2)} + r_{\lambda\mu}^{(2)}$$

for $i \in J'_\lambda$ and

$$S^{(1)}_{i,\lambda\lambda} = (s_\lambda - 1)u'^{(1)}_{\lambda\lambda} + r'^{(1)}_{\lambda\lambda}, \qquad S^{(2)}_{i,\lambda\lambda} = (s_\lambda - 1)u'^{(2)}_{\lambda\lambda} - r'^{(2)}_{\lambda\lambda},$$

$$S^{(1)}_{i,\lambda\mu} = s_\mu u'^{(1)}_{\lambda\mu} - r'^{(1)}_{\lambda\mu}, \qquad S^{(2)}_{i,\lambda\mu} = s_\mu u'^{(2)}_{\lambda\mu} + r'^{(2)}_{\lambda\mu}$$

for $i \in J''_\lambda$, where $0 \le r^{(\alpha)}_{\lambda\lambda}, r'^{(\alpha)}_{\lambda\lambda} < s_\lambda - 1$ and $0 \le r^{(\alpha)}_{\lambda\mu}, r'^{(\alpha)}_{\lambda\mu} < s_\mu$ for $\lambda, \mu, \alpha = 1, 2$ $(\lambda \ne \mu)$. Let $D_{\lambda\mu} = X_{\lambda\mu} - U^{(1)}_{\lambda\mu} - U^{(2)}_{\lambda\mu}$. We consider the construction of $N^{(\alpha)}_{\lambda\mu}$.

Case 1. n is even. We have $u^{(\alpha)}_{\lambda\mu} = u'^{(\alpha)}_{\lambda\mu}$ and $r^{(\alpha)}_{\lambda\mu} = r'^{(\alpha)}_{\lambda\mu}$ for every λ, μ and α, since $S^{(\alpha)}_{i,\lambda\mu} = S^{(\alpha)}_{i',\lambda\mu}$ $(i \in J'_\lambda;\ i' \in J''_\lambda)$ by (7.4). First we construct $N^{(\alpha)}_{\lambda\lambda}$. We have $r^{(1)}_{\lambda\lambda} = r^{(2)}_{\lambda\lambda}$ and $u^{(1)}_{\lambda\lambda} + u^{(2)}_{\lambda\lambda} = \frac{n^2}{2}$, since $S^{(1)}_{i,\lambda\lambda} + S^{(2)}_{i,\lambda\lambda} = \sum_{j \in J_\lambda} x_{ij} = (s_\lambda - 1)\frac{n^2}{2} (i \in J_\lambda)$ by (7.4). Thus $D_{\lambda\lambda} = 0$. Put $N^{(\alpha)}_{\lambda\lambda} = T^{(r^{(1)}_{\lambda\lambda}, r^{(1)}_{\lambda\lambda})}_{s_\lambda}$ for $\lambda, \alpha = 1, 2$ where $T^{(u_1, u_2)}_v$ is defined in (7.6). Then it is easy to check that (8.9), (8.10) and (8.11) hold. We next construct $N^{(\alpha)}_{\lambda\mu}$ $(\lambda \ne \mu)$ and consider $r^{(1)}_{\lambda\mu} - r^{(2)}_{\lambda\mu}$ denoted by $\Delta_{\lambda\mu}$. Since $S^{(1)}_{i,12} + S^{(2)}_{i,12} = \sum_{j \in J_2} x_{ij} = (m-s)\left(\frac{n^2}{2} + x\right) + y_2$ $(i \in J_1)$, Δ_{12} takes either $-y_2$ or $m - s - y_2$. When $\Delta_{12} = -y_2$, we have $D_{12} = B$ since $u^{(1)}_{12} + u^{(2)}_{12} = \frac{n^2}{2} + x$. We also have $y_2 \le r^{(2)}_{12} < m - s$, where y_2 is the row sum of B. Therefore, it is shown easily by Lemma 8.3 that we can construct 0–1 matrices $N^{(1)}_{12}$ and $N^{(2)}_{12}$ which satisfy $N^{(2)}_{12}\mathbf{j}_{m-s} = r^{(2)}_{12}\mathbf{j}_s$ and $N^{(2)}_{12} - N^{(1)}_{12} = B$. Namely, those matrices satisfy (8.9) and (8.10). On the other hand, when $\Delta_{12} = m - s - y_2$, we have $D_{12} = B - G_{s,m-s}$ since $u^{(1)}_{12} + u^{(2)}_{12} = \frac{n^2}{2} + x + 1$. We also have $m - s - y_2 \le r^{(1)}_{12} < m - s$, where $m - s - y_2$ is the row sum of $G_{s,m-s} - B$. Therefore, it is verified by Lemma 8.3 that we can construct 0–1 matrices $N^{(1)}_{12}$ and $N^{(2)}_{12}$ satisfying $N^{(1)}_{12}\mathbf{j}_{m-s} = r^{(1)}_{12}\mathbf{j}_s$ and $N^{(1)}_{12} - N^{(2)}_{12} = G_{s,m-s} - B$, which satisfy (8.9) and (8.10). Since $S^{(1)}_{i,21} + S^{(2)}_{i,21} = \sum_{j \in J_1} x_{ij} = s\left(\frac{n^2}{2} - x\right) - y_1$ for $i \in J_2$ by (7.4), Δ_{21} takes either y_1 or $y_1 - s$. When $\Delta_{21} = y_1$, $u^{(1)}_{21} + u^{(2)}_{21} = \frac{n^2}{2} - x$ holds. Thus we have $D_{21} = -B^T$. We also have $y_1 \le r^{(1)}_{21} < s$, where y_1 is the row sum of B^T. Therefore, it follows from Lemma 8.3 that two 0–1 matrices $N^{(1)}_{21}$ and $N^{(2)}_{21}$ satisfying (8.9) and (8.10) can be constructed. On the other hand, when $\Delta_{21} = y_1 - s$, $u^{(1)}_{21} + u^{(2)}_{21} = \frac{n^2}{2} - x - 1$ holds. Thus we have $D_{21} = -B^T + G_{m-s,s}$. We also have $s - y_1 \le r^{(2)}_{21} < s$, where $s - y_1$ is the row sum of $G_{m-s,s} - B^T$. Therefore, it follows from Lemma 8.3 that we can construct 0–1 matrices $N^{(1)}_{21}$ and $N^{(2)}_{21}$ satisfying (8.13) and (8.10). Note that a combination of (8.13) and (8.10) gives (8.9).

Case 2. n is odd. We shall treat the constructions of $N^{(\alpha)}_{\lambda\lambda}$ and $N^{(\alpha)}_{\lambda\mu}$ $(\lambda \ne \mu)$ separately. First we construct $N^{(\alpha)}_{\lambda\lambda}$ and there are two subcases with respect to s_λ.

Case 2.1. s_λ is odd. We have $u^{(\alpha)}_{\lambda\lambda}=u'^{(\alpha)}_{\lambda\lambda}$ and $r^{(\alpha)}_{\lambda\lambda}=r'^{(\alpha)}_{\lambda\lambda}$, since $S^{(\alpha)}_{i,\lambda\lambda}=S^{(\alpha)}_{i',\lambda\lambda}$ $(i\in J'_\lambda;\ i'\in J''_\lambda)$ by (7.5). Let $\Delta_{\lambda\lambda}=r^{(1)}_{\lambda\lambda}-r^{(2)}_{\lambda\lambda}$. Then since $S^{(1)}_{i,\lambda\lambda}+S^{(2)}_{i,\lambda\lambda}=\sum_{j\in J_\lambda}x_{ij}=(s_\lambda-1)\frac{n^2-1}{2}+\frac{s_\lambda-1}{2}$ $(i\in J_\lambda)$ by (7.5), $\Delta_{\lambda\lambda}$ takes either $\frac{s_\lambda-1}{2}$ or $-\frac{s_\lambda-1}{2}$. When $\Delta_{\lambda\lambda}=\frac{s_\lambda-1}{2}$, we have $D_{\lambda\lambda}=T^{((s_\lambda-1)/2,(s_\lambda-1)/2)}_{s_\lambda}$ since $u^{(1)}_{\lambda\lambda}+u^{(2)}_{\lambda\lambda}=\frac{n^2-1}{2}$. We also have $\frac{s_\lambda-1}{2}\le r^{(1)}_{\lambda\lambda}<s_\lambda-1$, where $\frac{s_\lambda-1}{2}$ is the row sum of $D_{\lambda\lambda}$. Therefore, it is verified from Lemma 8.4 that we can construct two 0–1 matrices $N^{(1)}_{\lambda\lambda}$ and $N^{(2)}_{\lambda\lambda}$ satisfying (8.10), (8.11) and $N^{(1)}_{\lambda\lambda}\mathbf{j}_{s_\lambda}=r^{(1)}_{\lambda\lambda}\mathbf{j}_{s_\lambda}$ which satisfy (8.9). On the other hand, when $\Delta_{\lambda\lambda}=-\frac{s_\lambda-1}{2}$, we have $D_{\lambda\lambda}=-T^{((s_\lambda-1)/2,(s_\lambda-1)/2)T}_{s_\lambda}$ since $u^{(1)}_{\lambda\lambda}+u^{(2)}_{\lambda\lambda}=\frac{n^2+1}{2}$. We also have $\frac{s_\lambda-1}{2}\le r^{(2)}_{\lambda\lambda}<s_\lambda-1$, where $\frac{s_\lambda-1}{2}$ is the row sum of $-D_{\lambda\lambda}$. Therefore, it is shown from Lemma 8.4 that we can construct two 0–1 matrices $N^{(1)}_{\lambda\lambda}$ and $N^{(2)}_{\lambda\lambda}$ which satisfy $N^{(2)}_{\lambda\lambda}\mathbf{j}_{s_\lambda}=r^{(2)}_{\lambda\lambda}\mathbf{j}_{s_\lambda}$, $N^{(2)}_{\lambda\lambda}-N^{(1)}_{\lambda\lambda}=-D_{\lambda\lambda}$ and (8.11). Namely, those matrices satisfy (8.9), (8.10) and (8.11).

Case 2.2. s_λ is even. Let $\Delta_{\lambda\lambda}=r^{(1)}_{\lambda\lambda}-r^{(2)}_{\lambda\lambda}$. Then since $S^{(1)}_{i,\lambda\lambda}+S^{(2)}_{i,\lambda\lambda}=\sum_{j\in J_\lambda}x_{ij}=(s_\lambda-1)\frac{n^2-1}{2}+\frac{s_\lambda}{2}$ $(i\in J'_\lambda)$ by (7.5), $\Delta_{\lambda\lambda}$ takes either $\frac{s_\lambda}{2}\left(\text{then } u^{(1)}_{\lambda\lambda}+u^{(2)}_{\lambda\lambda}=\frac{n^2-1}{2}\right)$ or $-\frac{s_\lambda}{2}+1\left(\text{then } u^{(1)}_{\lambda\lambda}+u^{(2)}_{\lambda\lambda}=\frac{n^2+1}{2}\right)$. Let $\Delta'_{\lambda\lambda}=r'^{(1)}_{\lambda\lambda}-r'^{(2)}_{\lambda\lambda}$. Then since $S^{(1)}_{i,\lambda\lambda}+S^{(2)}_{i,\lambda\lambda}=\sum_{j\in J_\lambda}x_{ij}=(s_\lambda-1)\frac{n^2-1}{2}+\frac{s_\lambda}{2}-1$ $(i\in J''_\lambda)$ by (7.5), $\Delta'_{\lambda\lambda}$ takes either $\frac{s_\lambda-1}{2}\left(\text{then } u'^{(1)}_{\lambda\lambda}+u'^{(2)}_{\lambda\lambda}=\frac{n^2-1}{2}\right)$ or $-\frac{s_\lambda}{2}\left(\text{then } u'^{(1)}_{\lambda\lambda}+u'^{(2)}_{\lambda\lambda}=\frac{n^2+1}{2}\right)$. We have $S^{(\alpha)}_{i,\lambda\lambda}\ge S^{(\alpha)}_{i',\lambda\lambda}$ $(i\in J'_\lambda,\ i'\in J''_\lambda$ and $\alpha=1,\ 2)$ by (8.1), since $\sum_{j\in J_\lambda}x_{ij}\ge\sum_{j\in J_\lambda}x_{i'j}$. Therefore, it follows that $u^{(\alpha)}_{\lambda\lambda}\ge u'^{(\alpha)}_{\lambda\lambda}$. Thus we have the following three possibilities:

$$
\begin{aligned}
&D_{\lambda\lambda}=T^{(\frac{s_\lambda}{2},\frac{s_\lambda}{2}-1)}_{s_\lambda} && \text{if}\quad (\Delta_{\lambda\lambda},\Delta'_{\lambda\lambda})=\left(\frac{s_\lambda}{2},\frac{s_\lambda}{2}-1\right),\\
(8.15)\quad &D_{\lambda\lambda}=T^{(\frac{s_\lambda}{2},\frac{s_\lambda}{2}-1)}_{s_\lambda}-T^{(s_\lambda-1,0)}_{s_\lambda} && \text{if}\quad (\Delta_{\lambda\lambda},\Delta'_{\lambda\lambda})=\left(-\frac{s_\lambda}{2}+1,\frac{s_\lambda}{2}-1\right),\\
&D_{\lambda\lambda}=-T^{(\frac{s_\lambda}{2},\frac{s_\lambda}{2}-1)T}_{s_\lambda} && \text{if}\quad (\Delta_{\lambda\lambda},\Delta'_{\lambda\lambda})=\left(-\frac{s_\lambda}{2}+1,-\frac{s_\lambda}{2}\right).
\end{aligned}
$$

With respect to the respective possibilities in (8.15), it follows from Lemma 8.4 and the method in Case 2.1 that we can construct two 0–1 matrices $N^{(1)}_{\lambda\lambda}$ and $N^{(2)}_{\lambda\lambda}$ satisfying (8.9), (8.10) and (8.11).

We finally construct $N^{(\alpha)}_{\lambda\mu}$ $(\lambda\ne\mu)$ and we consider the following four subcases with respect to s and $m-s$.

Case 2.3. s is odd. This case gives $u^{(\alpha)}_{12}=u'^{(\alpha)}_{12}$ and $r^{(\alpha)}_{12}=r'^{(\alpha)}_{12}$, since $S^{(\alpha)}_{i,12}=$

$S_{i',12}^{(\alpha)}$ $(i \in J_1';\ i' \in J_1'')$ by (7.5). Since $S_{i,12}^{(1)} + S_{i,12}^{(2)} = \sum_{j \in J_2} x_{ij} = (m-s)\left(\frac{n^2+1}{2}+x\right) + y_2$ for $i \in J_1$, $r_{12}^{(1)} - r_{12}^{(2)}$ takes either $-y_2$ or $m-s-y_2$. Along the line similar to Case 1, $N_{12}^{(1)}$ and $N_{12}^{(2)}$ satisfying (8.9) and (8.10) can be constructed for the respective cases of $\Delta_{12} = -y_2$ and $\Delta_{12} = m-s-y_2$ where $\Delta_{12} = r_{12}^{(1)} - r_{12}^{(2)}$.

Case 2.4. s is even. Let $\Delta_{12} = r_{12}^{(1)} - r_{12}^{(2)}$. Then since $S_{i,12}^{(1)} + S_{i,12}^{(2)} = \sum_{j \in J_2} x_{ij} = (m-s)\left(\frac{n^2+1}{2}+x\right) + y_2 - \frac{1}{2}$ $(i \in J_1')$ by (7.5), Δ_{12} takes either $-y_2 + \frac{1}{2}$ $\left(\text{then } u_{12}^{(1)} + u_{12}^{(2)} = \frac{n^2+1}{2}+x\right)$ or $m-s-y_2+\frac{1}{2}$ $\left(\text{then } u_{12}^{(1)} + u_{12}^{(2)} = \frac{n^2+1}{2}+x+1\right)$. Let $\Delta_{12}' = r_{12}'^{(1)} - r_{12}'^{(2)}$. Then since $S_{i,12}^{(1)} + S_{i,12}^{(2)} = \sum_{j \in J_2} x_{ij} = (m-s)\left(\frac{n^2+1}{2}+x\right) + y_2 + \frac{1}{2}$ $(i \in J_1'')$, Δ_{12}' takes either $-y_2 - \frac{1}{2}$ $\left(\text{then } u_{12}'^{(1)} + u_{12}'^{(2)} = \frac{n^2+1}{2}+x\right)$ or $m-s-y_2-\frac{1}{2}$ $\left(\text{then } u_{12}'^{(1)} + u_{12}'^{(2)} = \frac{n^2+1}{2}+x+1\right)$. We have $S_{i',12}^{(\alpha)} \geq S_{i,12}^{(\alpha)}$ for $i \in J_1'$, $i' \in J_1''$ and $\alpha = 1, 2$, since $\sum_{j \in J_2} x_{i'j} \geq \sum_{j \in J_2} x_{ij}$. Therefore, it follows that $u_{12}'^{(\alpha)} \geq u_{12}^{(\alpha)}$. Thus we have the following three possibilities:

$$
\begin{aligned}
&D_{12} = B \qquad \text{if } (\Delta_{12}, \Delta_{12}') = \left(-y_2 + \tfrac{1}{2},\ -y_2 - \tfrac{1}{2}\right),\\
(8.16)\qquad &D_{12} = B - \begin{bmatrix} O_{s/2,m-s} \\ G_{s/2,m-s} \end{bmatrix}\\
&\qquad \text{if } (\Delta_{12}, \Delta_{12}') = \left(-y_2 + \tfrac{1}{2},\ m-s-y_2 - \tfrac{1}{2}\right),\\
&D_{12} = B - G_{s,m-s}\\
&\qquad \text{if } (\Delta_{12}, \Delta_{12}') = \left(m-s-y_2 + \tfrac{1}{2},\ m-s-y_2 - \tfrac{1}{2}\right),
\end{aligned}
$$

where $O_{t,u}$ is the $t \times u$ zero matrix. With respect to the respective possibilities in (8.16), it follows from Lemma 8.3 and the method in Case 1 that we can construct two 0–1 matrices $N_{12}^{(1)}$ and $N_{12}^{(2)}$ satisfying (8.9) and (8.10).

Case 2.5. $m-s$ is odd. This case gives $u_{21}^{(\alpha)} = u_{21}'^{(\alpha)}$ and $r_{21}^{(\alpha)} = r_{21}'^{(\alpha)}$. Since $S_{i,21}^{(1)} + S_{i,21}^{(2)} = \sum_{j \in J_1} x_{ij} = s\left(\frac{n^2-1}{2}-x\right) - y_1$ for $i \in J_2$, $r_{21}^{(1)} - r_{21}^{(2)}$ takes either y_1 or $y_1 - s$. It is verified easily that $N_{21}^{(1)}$ and $N_{21}^{(2)}$ satisfying (8.9) and (8.10) can be constructed for the respective cases of $\Delta_{21} = y_1$ and $\Delta_{21} = y_1 - s$ where $\Delta_{21} = r_{21}^{(1)} - r_{21}^{(2)}$.

Case 2.6. $m-s$ is even. Since $u_{21}'^{(\alpha)} \geq u_{21}^{(\alpha)}$ is obtained by (7.5), it is seen easily that we have the following three possibilities:

$$
D_{21} = -B^T \qquad \text{if } (\Delta_{21}, \Delta_{21}') = \left(y_1 + \tfrac{1}{2},\ y_1 - \tfrac{1}{2}\right),
$$

$$(8.17)\quad D_{21} = -B^T + \begin{bmatrix} G_{(m-s)/2,s} \\ O_{(m-s)/2,s} \end{bmatrix} \quad \text{if} \quad (\Delta_{21}, \Delta'_{21}) = \left(y_1 + \frac{1}{2} - s, y_1 - \frac{1}{2}\right),$$

$$D_{21} = -B^T + G_{m-s,s}$$

$$\text{if} \quad (\Delta_{21}, \Delta'_{21}) = \left(y_1 + \frac{1}{2} - s,\ y_1 - \frac{1}{2} - s\right),$$

where $\Delta_{21} = r_{21}^{(1)} - r_{21}^{(2)}$ and $\Delta'_{21} = r_{21}'^{(1)} - r_{21}'^{(2)}$. In the respective possibilities in (8.17), it follows from Lemma 8.3 and the method in Case 1 that we can construct two 0–1 matrices $N_{21}^{(1)}$ and $N_{21}^{(2)}$ satisfying (8.9) and (8.10).

8.2. Construction of Y_i

We shall construct m nonnegative matrices satisfying (3.4), (3.6) and (8.5) by using m matrices $W_i = \|w_{ij}^{(\alpha)}\|$ $(i=1, 2,\ldots, m)$ of size $2\times m$ which are given in Subsection 8.1. We write $w_{ij}^{(\alpha)}$ as

$$(8.18)\quad \begin{aligned} w_{ij}^{(1)} &= d_\lambda f_{ij}^{(1)} + e_{ij}^{(1)}, & 0 \le e_{ij}^{(1)} &< d_\lambda, \\ w_{ij}^{(2)} &= (n - d_\lambda) f_{ij}^{(2)} + e_{ij}^{(2)}, & 0 \le e_{ij}^{(2)} &< n - d_\lambda \end{aligned}$$

for $i \in J_\lambda$, $j=1, 2,\ldots, m$ and $\lambda = 1, 2$. Let

$$(8.19)\quad f_i^{(1)} = (a+1)c - \sum_{j=1}^m f_{ij}^{(1)} \quad \text{and} \quad f_i^{(2)} = ac - \sum_{j=1}^m f_{ij}^{(2)}$$

for $i=1, 2,\ldots, m$. Then we can construct two 0–1 matrices $Z_i^{(1)}$ of size $d_\lambda \times m$ and $Z_i^{(2)}$ of size $(n-d_\lambda)\times m$ for every $i \in J_\lambda$ and $\lambda = 1, 2$ [10], [16] which satisfy

$$(8.20)\quad \begin{aligned} &Z_i^{(1)}\mathbf{j}_m = f_i^{(1)}\mathbf{j}_{d_\lambda}, \quad Z_i^{(1)T}\mathbf{j}_{d_\lambda} = (e_{i1}^{(1)}, e_{i2}^{(1)},\ldots, e_{im}^{(1)})^T \quad \text{and} \\ &Z_i^{(2)}\mathbf{j}_m = f_i^{(2)}\mathbf{j}_{n-d_\lambda}, \quad Z_i^{(2)T}\mathbf{j}_{n-d_\lambda} = (e_{i1}^{(2)}, e_{i2}^{(2)},\ldots, e_{im}^{(2)})^T. \end{aligned}$$

Define the matrices

$$(8.21)\quad \begin{aligned} &Y_i^{(1)*} = \mathbf{j}_{d_\lambda}(f_{i1}^{(1)}, f_{i2}^{(1)},\ldots, f_{im}^{(1)}) \quad \text{and} \\ &Y_i^{(2)*} = \mathbf{j}_{n-d_\lambda}(f_{i1}^{(2)}, f_{i2}^{(2)},\ldots, f_{im}^{(2)}) \end{aligned}$$

for $i \in J_\lambda$. Further, define a nonnegative integral matrix $Y_i = \|y_{ip,j}\|$ of size $n\times m$ by

$$Y_i = \begin{bmatrix} Y_i^{(1)*} + Z_i^{(1)} \\ Y_i^{(2)*} + Z_i^{(2)} \end{bmatrix} \quad \text{for} \quad i = 1, 2,\ldots, m.$$

Then we have $(Y_i^{(1)*}+Z_i^{(1)})\mathbf{j}_m=(\sum_{j=1}^{m} f_{ij}^{(1)}+f_i^{(1)})\mathbf{j}_{d_\lambda}=(a+1)c\mathbf{j}_{d_\lambda}$ and $(Y_i^{(2)*}+Z_i^{(2)})\mathbf{j}_m=(\sum_{j=1}^{m} f_{ij}^{(2)}+f_i^{(2)})\mathbf{j}_{n-d_\lambda}=ac\mathbf{j}_{n-d_\lambda}$ by (8.19), (8.20) and (8.21). Thus Y_i satisfies (3.4) for every i. It can also be shown easily that (8.5) holds by applying (8.18), (8.19) and (8.20) to $Y_i^T\mathbf{j}_n=Y_i^{(1)*T}\mathbf{j}_{d_\lambda}+Y_i^{(2)*T}\mathbf{j}_{n-d_\lambda}+Z_i^{(1)T}\mathbf{j}_{d_\lambda}+Z_i^{(2)T}\mathbf{j}_{n-d_\lambda}$. We shall prove that Y_i satisfies the remaining condition (3.6). It can be shown easily that $a(k+1)\leq n$. From this fact and the structure of Y_i it is sufficient to show that

$$(8.22)\qquad (a+1)k\leq f_{ij}^{(1)}\leq \min((a+1)(k+1),\, n)-\delta_{ij}^{(1)}\qquad (j\neq i),$$

$$(8.23)\qquad ak\leq f_{ij}^{(2)}\leq a(k+1)-\delta_{ij}^{(2)}\qquad (j\neq i)$$

for $i\in J_\lambda$ where $\delta_{ij}^{(\alpha)}=1$ or 0 according as $e_{ij}^{(\alpha)}$ is a positive integer or zero.

Case 1. $i, j\in J_\lambda$ $(i\neq j)$. In this case $s_\lambda\geq 2$ and $d_\lambda\geq 1$. By the structure of W_i in Subsection 8.1 and by (8.6) and (8.18) we have

$$(8.24)\ f_{ij}^{(1)}=\frac{w_{ij}^{(1)}-e_{ij}^{(1)}}{d_\lambda}\geq\frac{u_{i,\lambda\lambda}^{(1)}-e_{ij}^{(1)}}{d_\lambda}=\frac{S_{i,\lambda\lambda}^{(1)}}{d_\lambda(s_\lambda-1)}-\frac{r_{i,\lambda\lambda}^{(1)}+(s_\lambda-1)e_{ij}^{(1)}}{d_\lambda(s_\lambda-1)}.$$

Substituting $S_{i,\lambda\lambda}^{(1)}$ in (8.1) into (8.24) we obtain

$$(8.25)\quad f_{ij}^{(1)}-(a+1)k>\frac{(a+1)k}{na+d_\lambda}\left\{\frac{1}{(s_\lambda-1)k}\sum_{j'\in J_\lambda}x_{ij'}-(na+d_\lambda)\right\}-R,$$

where $R=\{(r_{i,\lambda\lambda}^{(1)}+1)+(s_\lambda-1)e_{ij}^{(1)}\}/d_\lambda(s_\lambda-1)$. $R\leq 1$ is obvious. Thus from Lemma 8.5 given later it follows that $f_{ij}^{(1)}\geq(a+1)k$.

Put $\mu_0=\min((a+1)(k+1),\, n)$. By considering the structure of W_i we have

$$(8.26)\quad f_{ij}^{(1)}=\frac{w_{ij}^{(1)}-e_{ij}^{(1)}}{d_\lambda}\leq\frac{u_{i,\lambda\lambda}^{(1)}+\varepsilon_i^{(1)}-e_{ij}^{(1)}}{d_\lambda}=\frac{S_{i,\lambda\lambda}^{(1)}}{d_\lambda(s_\lambda-1)}-R_1,$$

where $\varepsilon_i^{(1)}=1$ or 0 according as $r_{i,\lambda\lambda}^{(1)}$ is a positive integer or zero and $R_1=\{r_{i,\lambda\lambda}^{(1)}+(s_\lambda-1)(e_{ij}^{(1)}-\varepsilon_i^{(1)})\}/d_\lambda(s_\lambda-1)$. Substituting $S_{i,\lambda\lambda}^{(1)}$ in (8.1) into (8.26) we obtain

$$(8.27)\qquad \mu_0-\delta_{ij}^{(1)}-f_{ij}^{(1)}\geq\mu_0+R_1-\delta_{ij}^{(1)}-R_2,$$

where $R_2=\{(a+1)\sum_{j'\in J_\lambda}x_{ij'}\}/(na+d_\lambda)(s_\lambda-1)$. Obviously $R_1-\delta_{ij}^{(1)}>-1$. Therefore, (8.27) becomes

$$(8.28)\qquad \mu_0-\delta_{ij}^{(1)}-f_{ij}^{(1)}>\mu_0-R_2-1.$$

Suppose $\mu_0=n$. Apply the inequality $\sum_{j'\in J_\lambda}x_{ij'}\leq(s_\lambda-1)\frac{n^2+1}{2}$ obtained by the structure of X in Section 7 to μ_0-R_2. Then since $a\geq 1$ and $d_\lambda\geq 1$, we have

$$(8.29)\qquad \mu_0 - R_2 = n - R_2 = n - \frac{a+1}{na+d_\lambda}\frac{n^2+1}{2}$$

$$= \frac{1}{na+d_\lambda}\left\{\frac{n^2-1}{2}(a-1) + nd_\lambda - 1\right\} \geq 0.$$

Thus it follows from (8.28) and (8.29) that $f_{ij}^{(1)} \leq n - \delta_{ij}^{(1)}$. If $\mu_0 = (a+1)(k+1)$, then by Lemma 8.8 given later we have

$$(8.30)\quad \mu_0 - R_2 = (a+1)(k+1) - R_2$$

$$= \frac{(a+1)(k+1)}{na+d_\lambda}\left\{(na+d_\lambda) - \frac{1}{(s_\lambda - 1)(k+1)}\sum_{j'\in J_\lambda} x_{ij'}\right\} \geq 0.$$

Thus we find $f_{ij}^{(1)} \leq (a+1)(k+1) - \delta_{ij}^{(1)}$ from (8.28) and (8.30). Hence (8.22) holds for $i, j \in J_\lambda$ $(i \neq j)$. By Lemmas 8.5, 8.8 given later, we can similarly show (8.23).

Case 2. $i \in J_\lambda$ and $j \in J_\mu$ $(\lambda \neq \mu)$. In this case $s \geq 1$. Along the method similar to Case 1, we obtain

$$(8.31)\qquad f_{ij}^{(1)} - (a+1)k \geq \frac{(a+1)k}{na+d_\lambda}\left\{\frac{1}{s_\mu k}\sum_{j'\in J_\mu} x_{ij'} - (na+d_\lambda)\right\} + R_1,$$

$$(8.32)\qquad \min((a+1)(k+1), n) - \delta_{ij}^{(1)} - f_{ij}^{(1)}$$

$$> \min((a+1)(k+1), n) - \frac{a+1}{(na+d_\lambda)s_\mu}\sum_{j'\in J_\mu} x_{ij'} + R_2,$$

where R_1 and R_2 are the numbers satisfying $R_1 > -1$ and $R_2 \geq -1$. Applying Lemmas 8.6 and 8.7 given later to (8.31) we have $f_{ij}^{(1)} \geq (a+1)k$. Let $\mu_0 = \min((a+1)(k+1), n)$. Suppose $\mu_0 = n$. We have the inequality $\sum_{j'\in J_\mu} x_{ij'} \leq s_\mu\left(\frac{n^2+1}{2} + x + 1\right)$ by the structure of X in Section 7. Therefore, by $a \geq 1$, $d_\lambda \geq 1$ and $\frac{n^2-1}{2} - x - 1 \geq 0$, we obtain

$$(8.33)\quad \mu_0 - \frac{a+1}{(na+d_\lambda)s_\mu}\sum_{j'\in J_\mu} x_{ij'} \geq n - \frac{a+1}{na+d_\lambda}\left(\frac{n^2+1}{2} + x + 1\right)$$

$$= \frac{1}{na+d_\lambda}\left\{\left(\frac{n^2-1}{2} - x - 1\right)(a-1) + nd_\lambda - 1\right\} \geq 0.$$

Thus it follows from (8.32) that $f_{ij}^{(1)} \leq n - \delta_{ij}^{(1)}$. If $\mu_0 = (a+1)(k+1)$, then by Lemma 8.9 given later we have

$$\mu_0 - \frac{a+1}{(na+d_\lambda)s_\mu}\sum_{j'\in J_\mu} x_{ij'}$$

$$= \frac{(a+1)(k+1)}{na+d_\lambda}\left\{(na+d_\lambda) - \frac{1}{s_\mu(k+1)}\sum_{j'\in J_\mu} x_{ij'}\right\} \geq 0.$$

Therefore, it follows from (8.32) that $f_{ij}^{(1)} \leq (a+1)(k+1) - \delta_{ij}^{(1)}$. Hence (8.22) holds for $i \in J_\lambda$ and $j \in J_\mu$ ($\lambda \neq \mu$). By Lemmas 8.6, 8.7 and 8.9 given later, we can similarly show (8.23).

Some lemmas used above are given in the following.

Put $n^2 = 2k\alpha + \beta$ $(0 \leq \beta < 2k)$.

LEMMA 8.5. *If $k \geq 1$ and $s_\lambda \geq 2$, then the inequality*

$$\text{(8.34)} \qquad \frac{1}{(s_\lambda - 1)k}\sum_{j\in J_\lambda} x_{ij} \geq na + d_\lambda \qquad \textit{for} \quad i \in J_\lambda$$

holds for the matrix X given in Section 7.

PROOF. It follows from (7.4) and (7.5) that $(\sum_{j\in J_\lambda} x_{ij})/(s_\lambda - 1)k \geq \lfloor n^2/2k \rfloor = \alpha$ for every $i \in J_\lambda$. Therefore, it is enough to prove that $\alpha \geq na + d_\lambda$ holds. Since

$$\text{(8.35)} \qquad \alpha - (na+d) = \left(\frac{n^2}{2k} - \frac{\beta}{2k}\right) - \left(\frac{(m-1)n^2}{2c} - \frac{s}{m}\right)$$

$$= \frac{n^2 l}{2kc} + \frac{s}{m} - \frac{\beta}{2k} > -1,$$

it follows that (8.34) holds for $\lambda = 2$. Consider $\lambda = 1$. We have

$$\text{(8.36)} \qquad l \geq \frac{(m-2)k\beta}{n^2 - \beta}$$

by the inequality $\binom{m}{2} n^2/c \leq (m-1)\left\lfloor \frac{n^2}{2k} \right\rfloor + v_2$ which is obtained by Condition (iii) in Theorem 5.2. By (8.36) and $s \geq 2$, $m(n^2-\beta)l - m(m-1)k\beta + 2cks \geq m(m-2)k\beta - m(m-1)k\beta + 4ck = 4ck - mk\beta$ is obtained. Thus we have

$$\alpha - (na+d+1) = \frac{m(n^2-\beta)l - m(m-1)k\beta + 2cks}{2mck} - 1 \geq \frac{4ck - mk\beta}{2mck} - 1.$$

Therefore, noting $\beta < 2k$ we have $\alpha - (na+d+1) > -1$.

LEMMA 8.6. *If both of k and s are positive integers, then the inequality*

$$\text{(8.37)} \qquad \frac{1}{(m-s)k}\sum_{j=s+1}^{m} x_{ij} \geq na + d + 1 \qquad \textit{for} \quad i = 1, 2, \ldots, s$$

holds for the matrix X given in Section 7.

PROOF. We have from (7.4) and (7.5)

$$\sum_{j=s+1}^{m} x_{ij} \geq (m-s)\left(\frac{n^2}{2}+\frac{c}{m}\right)-\frac{1}{2} \qquad \text{if } n \text{ is odd and } s \text{ is even,}$$

$$= (m-s)\left(\frac{n^2}{2}+\frac{c}{m}\right) \qquad \text{otherwise}$$

for $i=1, 2,\dots, s$. It follows from $s\geq 1$ that $\frac{n^2}{2}+\frac{c}{m}\geq(na+d+1)k$. Put $N=(m-s)\left(\frac{n^2}{2}+\frac{c}{m}\right)-\frac{1}{2}$. Then we have $N\geq(m-s)(na+d+1)k$ for odd n and even s, since N is an integer in (7.5). Hence (8.37) holds.

LEMMA 8.7. *If k, s and a are all positive integers, then*

$$\text{(8.38)} \qquad \frac{1}{sk}\sum_{j=1}^{s} x_{ij} \geq na+d \qquad \text{for } i=s+1, s+2,\dots, m$$

holds for the matrix X given in Section 7.

PROOF. We have from (7.4) and (7.5)

$$\sum_{j=1}^{s} x_{ij} \geq s\left(\frac{n^2}{2}-\frac{c}{m}\right)-\frac{1}{2} \qquad \text{if } n \text{ is odd and } m-s \text{ is even,}$$

$$= s\left(\frac{n^2}{2}-\frac{c}{m}\right) \qquad \text{otherwise}$$

for $i=s+1, s+2,\dots, m$. We shall first prove that

$$\text{(8.39)} \qquad \frac{n^2}{2}-\frac{c}{m}\geq(na+d)k$$

holds, and there are three cases to consider.

Case 1. $\beta=0$ and $l>k$. This case gives $\lfloor v_2\rfloor\leq\frac{n^2}{2k}-1$. Using Condition (iii) in Theorem 5.2 we have $\binom{m}{2}n^2/c\leq m\lfloor v_2\rfloor\leq m\left(\frac{n^2}{2k}-1\right)$. Thus since $\beta=0$ and $s\geq 1$, we obtain $na+d\leq\frac{n^2}{2k}-2$. Therefore, (8.39) holds.

Case 2. $\beta=0$ and $l\leq k$. From $s\geq 1$ and (8.35) it follows that $na+d\leq\alpha-1$. From this fact and by $k\geq l$ we have $\frac{n^2}{2}-\frac{c}{m}-(na+d)k\geq\frac{n^2}{2}-\frac{c}{m}-(\alpha-1)k=\frac{k-l}{m}\geq 0$.

Case 3. $\beta\neq 0$. As seen in (8.35) it is sufficient to examine the following three subcases with respect to $na+d$.

Case 3.1. $na+d\leq\alpha-2$. (8.39) holds obviously, since $k\geq 1$.

Case 3.2. $na+d=\alpha-1$. If $\beta\geq 2$, then (8.39) is obvious. Suppose $\beta=1$. Then using $na+d=\alpha-1$ we have the equation $\frac{(m-1)n^2}{2c}-\frac{s}{m}+1=\frac{n^2-1}{2k}$. Solving for l we obtain $l=(m-1)kr/(n^2-r)$, where $r=1+2k\left(1-\frac{s}{m}\right)$. We also obtain $r\leq 1+2k\leq n$ since n is odd and since $a\geq 1$. Therefore,

$$(8.40)\qquad l=(m-1)k\frac{r}{n^2-r}\leq (m-1)\frac{n-1}{2}\,\frac{n}{n^2-n}=\frac{m-1}{2}.$$

Thus it follows from (8.40) that (8.39) holds.

Case 3.3. $na+d=\alpha$. We have $\frac{sc}{m}=\frac{(m-1)n^2}{2}-\frac{(n^2-\beta)c}{2k}=\frac{(m-1)\beta}{2}-\frac{(n^2-\beta)l}{2k}$ by $na+d=\alpha=(n^2-\beta)/2k$. By using (8.36),

$$\frac{sc}{m}=\frac{(m-1)\beta}{2}-\frac{(n^2-\beta)l}{2k}\leq\frac{(m-1)\beta}{2}-\frac{(m-2)k\beta}{2k}=\frac{\beta}{2}.$$

Thus $\beta\geq\frac{2sc}{m}$. Using this inequality for β and noting $s\geq 1$ we have (8.39). Put $N=s\left(\frac{n^2}{2}-\frac{c}{m}\right)-\frac{1}{2}$. Then from (8.39) it can be seen easily that $N\geq s(na+d)k$ for odd n and even $m-s$, since N is an integer in (7.5). Hence (8.38) holds.

Put $n^2=2(k+1)\alpha'-\beta'$ $(0\leq\beta'<2(k+1))$.

LEMMA 8.8. *If* $s_\lambda\geq 2$, *then*

$$(8.41)\qquad \frac{1}{(s_\lambda-1)(k+1)}\sum_{j\in J_\lambda}x_{ij}\leq na+d_\lambda\qquad \textit{for}\quad i\in J_\lambda$$

holds for the matrix X given in Section 7.

PROOF. It follows from (7.4) and (7.5) that $(\sum_{j\in J_\lambda}x_{ij})/(s_\lambda-1)(k+1)\leq\lceil n^2/2(k+1)\rceil=\alpha'$ for every $i\in J_\lambda$. Therefore, it is enough to prove that

$$(8.42)\qquad na+d_\lambda\geq\alpha'$$

holds. It can be shown easily that (8.42) holds for $\lambda=1$. Consider the case $\lambda=2$. We have by Condition (iii) in Theorem 5.2

$$(8.43)\qquad c\leq\frac{(k+1)\{(m-1)n^2+\beta'\}}{n^2+\beta'}.$$

Using (8.43) and $\beta'<2(k+1)$, we have

$$(8.44)\qquad na+d-\alpha'=\frac{(m-1)n^2}{2c}-\frac{s}{m}-\frac{n^2+\beta'}{2(k+1)}$$

$$\geq \frac{n^2+\beta'}{2(k+1)}\frac{(m-1)n^2}{(m-1)n^2+\beta'}-\frac{n^2+\beta'}{2(k+1)}-\frac{s}{m}$$

$$=-\frac{n^2+\beta'}{(m-1)n^2+\beta'}\frac{\beta'}{2(k+1)}-\frac{s}{m}$$

$$>-\frac{n^2+\beta'}{(m-1)n^2+\beta'}-\frac{s}{m}.$$

Since $\beta'<n^2$ and since $s_2\geq 2$, i.e., $s\leq m-2$,

$$na+d-\alpha'>-\frac{n^2+\beta'}{(m-1)n^2+\beta'}-\frac{s}{m}>-\frac{2}{m}-\frac{s}{m}\geq -1. \tag{8.45}$$

Hence (8.42) holds.

LEMMA 8.9. *If* $s\geq 1$, *then*

$$\frac{1}{(m-s)(k+1)}\sum_{j=s+1}^{m}x_{ij}\leq na+d+1 \qquad \textit{for} \quad i=1,2,\dots,s, \textit{ and} \tag{8.46}$$

$$\frac{1}{s(k+1)}\sum_{j=1}^{s}x_{ij}\leq na+d \qquad \textit{for} \quad i=s+1, s+2,\dots,m \tag{8.47}$$

hold for the matrix X in Section 7.

PROOF. We have from (7.4) and (7.5)

$$\sum_{j=s+1}^{m}x_{ij}\leq (m-s)\left(\frac{n^2}{2}+\frac{c}{m}\right)+\frac{1}{2} \qquad \text{if } n \text{ is odd and } s \text{ is even,}$$

$$=(m-s)\left(\frac{n^2}{2}+\frac{c}{m}\right) \qquad \text{otherwise}$$

for $i=1, 2,\dots, s$. We shall first prove that

$$(na+d+1)(k+1)\geq \frac{n^2}{2}+\frac{c}{m} \tag{8.48}$$

holds. It is enough to examine two cases; $na+d\geq\alpha'$ and $na+d=\alpha'-1$, because $na+d\geq\alpha'-1$ by (8.44).

Case 1. $na+d\geq\alpha'$. (8.48) holds, since

$$(na+d+1)(k+1)-\left(\frac{n^2}{2}+\frac{c}{m}\right)\geq(\alpha'+1)(k+1)-\left(\frac{n^2}{2}+\frac{c}{m}\right)$$

$$=\frac{\beta'}{2}+k+1-\frac{c}{m}>0.$$

Case 2. $na+d=\alpha'-1$. In this case we have $\dfrac{(m-s)c}{m}=-\dfrac{(m-1)n^2}{2}$

$+\frac{(n^2+\beta')c}{2(k+1)}$. Therefore, by (8.43),

$$\frac{(m-s)c}{m} = -\frac{(m-1)n^2}{2} + \frac{(n^2+\beta')c}{2(k+1)}$$

$$\leq -\frac{(m-1)n^2}{2} + \frac{(m-1)n^2+\beta'}{2} = \frac{\beta'}{2}.$$

Thus $\beta' \geq \frac{2(m-s)c}{m} \geq \frac{2c}{m}$. Using this inequality for β' we have (8.48). Let $N=(m-s)\left(\frac{n^2}{2}+\frac{c}{m}\right)+\frac{1}{2}$. Then from (8.48) it can be seen easily that $(m-s)\cdot(k+1)\cdot(na+d+1) \geq N$ holds for odd n and even s, since N is an integer. Hence (8.46) holds. It can also be shown easily that (8.47) holds.

9. Construction of M_{ij}^*

As seen in Subsection 8.2, we know the following fact from the structure of $Y_i = \|y_{ip,j}\|$. $y_{ip_\alpha,j}$ takes either $f_{ij}^{(\alpha)}$ or $f_{ij}^{(\alpha)}+1$ and $n-y_{jq_\alpha,i}$ takes either $n-f_{ji}^{(\alpha)}$ or $n-f_{ji}^{(\alpha)}-1$ for every pair of $i \in J_\lambda$ and $j \in J_\mu$ $(i \neq j)$, where p_α and q_α $(\alpha=1, 2)$ are integers satisfying $1 \leq p_1 \leq d_\lambda$, $d_\lambda+1 \leq p_2 \leq n$, $1 \leq q_1 \leq d_\mu$ and $d_\mu+1 \leq q_2 \leq n$. It can be shown easily from (8.22), (8.23) and [10], [16] that if one of d_λ, $n-d_\lambda$, d_μ and $n-d_\mu$ is zero, then a 0–1 matrix M_{ij}^* of order n satisfying (3.7) can be constructed for every pair of i and j satisfying $i \in J_\lambda$, $j \in J_\mu$ and $i<j$. Consider $(\lambda, \mu)=(1, 1)$, $(2, 2)$ and $(1, 2)$ and suppose that all of d_λ, $n-d_\lambda$, d_μ and $n-d_\mu$ are positive integers. Then we have the following Statements A, B, C and D.

STATEMENT A.

$$f_{ij}^{(1)} \geq \begin{cases} f_{ij}^{(2)}+1 & \textit{if} \quad e_{ij}^{(2)} \geq 1 \\ f_{ij}^{(2)} & \textit{if} \quad e_{ij}^{(2)} = 0 \end{cases}$$

holds for $i \in J_\lambda$ *and* $j \in J_\mu$ $(i \neq j)$.

STATEMENT B.

$$f_{ij}^{(1)} \leq \begin{cases} 2f_{ij}^{(2)}+2 & \textit{if} \quad e_{ij}^{(1)}=0 \quad \textit{and} \quad e_{ij}^{(2)} > \frac{n-d_\lambda-2}{2} \\ 2f_{ij}^{(2)}+1 & \textit{otherwise} \end{cases}$$

holds for $i \in J_\lambda$ *and* $j \in J_\mu$ $(i \neq j)$.

STATEMENT C.

$$n-f_{ji}^{(1)} \geq \begin{cases} 2 & \textit{if} \quad e_{ij}^{(1)} \geq 1 \\ 1 & \textit{if} \quad e_{ji}^{(1)} = 0 \end{cases}$$

holds for each case of (a) *even* n *and* $i, j \in J_\lambda$ $(i \neq j)$, (b) *odd* n, $d_\lambda \geq 2$ *and* $i, j \in J_\lambda$ $(i \neq j)$ *and* (c) $i \in J_1$ *and* $j \in J_2$.

STATEMENT D.

$$n - f_{ji}^{(2)} \geq \begin{cases} \dfrac{n + d_\lambda + 1}{2} & \text{if } n + d_\lambda \text{ is odd and } e_{ji}^{(2)} > \dfrac{n - d_\mu - 2}{2} \\ \dfrac{n + d_\lambda}{2} & \text{otherwise} \end{cases}$$

holds for $i \in J_\lambda$ *and* $j \in J_\mu$ $(i \neq j)$.

It can be verified by $k \geq 1$ and Lemmas 8.5, 8.6 that Statement A holds. It follows from $a \geq 1$ that Statement B holds. Statement D can be shown by the structure of X given in Section 7 and by considering $a \geq 1$. Furthermore, by the structure of $N_{\lambda\lambda}^{(1)}$ given in Subsection 8.1, it can be shown that Statement C holds.

By applying the above Statements A, B, C and D to the existence theorem [10], [16] of 0–1 matrix, though some calculations are needed, we can show that a 0–1 matrix M_{ij}^{*} of order n satisfying (3.7) can be constructed for every pair of i and j satisfying $i \in J_\lambda$, $j \in J_\mu$ and $i < j$.

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