HIROSHIMA MATH. J.
9 (1979), 491–502

Minimal Prime Ideals of a Finitely Generated Ideal

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(Received January 17, 1979)

In this paper, all rings are assumed to be commutative with identity.

In Section 1, we first introduce the notions of K_0-domains and K-domains by making use of the terms of a Krull domain and an integral extension domain, and give some properties of these integral domains. Theorem 3 is our main result on K-domains. By reason that an integral domain R being a K_0-domain (resp. K-domain) does not always imply that its residue domain is a K_0-domain (resp. K-domain), we further give two definitions of a strong K_0-domain and a strong K-domain. We shall investigate the minimal prime ideals of a finitely generated ideal in a strong K_0-domain or a strong K-domain. Some results obtained in this section show that strong K_0-domains and strong K-domains have some properties which noetherian domains have. For example, Theorem 9 asserts that if $\mathfrak{a}$ is an ideal of finite altitude in a strong K_0-domain R, then there exists only a finite number of minimal prime ideals of $\mathfrak{a}$, and Theorem 7 asserts that, in a strong K-domain A, if $\mathfrak{a}$ is an ideal in A generated by n elements, then $ht(\mathfrak{P}) \leq n$ for any minimal prime ideal $\mathfrak{P}$ of $\mathfrak{a}$. In Section 2, for any n ($2 \leq n \leq \infty$), in the same way as in [2], we construct a unique factorization local domain $(A, \mathfrak{M})$ such that $ht(t, u)A = n+1$ for some elements t, u of M.

Throughout this paper, by the word "ideal" we mean an ideal different from the ring itself. We use $\subseteq$ and $\subset$ for weak and strong inclusions respectively.

The author would like to thank Professor M. Nishi for his valuable comments in preparing this paper.

1. Let $R \subseteq A$ be integral domains. Then A is said to be an integral extension domain of R if A is integral over R. We begin with the following definitions.

DEFINITION. We say that an integral domain R is a K_0-*domain* if there exists a Krull domain B such that B is an integral extension domain of R.

DEFINITION. We say that an integral domain A is a K-*domain* if A is an integral extension domain of a K_0-domain.

Let $R \subseteq A$ be integral domains. Then A is said to be almost finite over R if A is integral over R and if the quotient field of A is a finite algebraic extension of the quotient field of R. The following assertions follow easily from the definitions.

PROPOSITION 1. (a) *Let R be an integral domain. Then R is a K_0-domain if and only if its derived normal domain $\bar{R}$ is a Krull domain.*

(b) *A noetherian domain is a K_0-domain.*

(c) *If R is a K_0-domain, then a polynomial ring $R[\dots, X_i, \dots]$, $i \in I$, is a K_0-domain, where I is any set.*

(d) *If R is a K_0-domain and A is an almost finite extension domain of R, then A is a K_0-domain.*

PROOF. (a) If $\bar{R}$ is a Krull domain, then by the definition, R is a K_0-domain. Conversely, suppose that R is a K_0-domain. Let B be a Krull domain such that B is an integral extension domain of R. Let K and L be the quotient fields of R and B respectively. By Theorem 1 in [3] $B_{\mathfrak{P}} \cap K = \bar{R}_{\mathfrak{P} \cap \bar{R}}$ holds for any prime ideal $\mathfrak{P}$ in B. Therefore if $\mathfrak{P}$ is a height one prime ideal in B, then $\bar{R}_{\mathfrak{P} \cap \bar{R}}$ is a principal valuation ring. Let $\mathfrak{P}_i$, $i \in I$, be the height one prime ideals in B. Since B is a Krull domain, $B = \bigcap_{i \in I} B_{\mathfrak{P}_i}$. Hence $\bar{R} = B \cap K = \bigcap_{i \in I} (B_{\mathfrak{P}_i} \cap K) = \bigcap_{i \in I} \bar{R}_{\mathfrak{P}_i \cap \bar{R}}$, so $\bar{R} = \bigcap_{i \in I} \bar{R}_{\mathfrak{P}_i \cap \bar{R}}$. Let a $(\neq 0)$ be a non-unit of $\bar{R}$. Then the number of $\mathfrak{P}_i$ containing a is finite, so $a\bar{R}$ has only a finite number of minimal prime divisors. Thus $\bar{R}$ is a Krull domain.

(b) Let R be a noetherian domain. Since the derived normal domain of R is a Krull domain, R is a K_0-domain.

(c) By the assertion of (a) the derived normal domain $\bar{R}$ of R is a Krull domain. Therefore $\bar{R}[\dots, X_i, \dots]$ is a Krull domain. Thus $R[\dots, X_i, \dots]$ is a K_0-domain.

(d) Let K and L be the quotient fields of R and A respectively. Since the derived normal domain $\bar{R}$ of R is a Krull domain, and since L is a finite algebraic extension of K, the integral closure of $\bar{R}$ in L is a Krull domain by Proposition 4.5 in [8]. As is seen easily, the integral closure of $\bar{R}$ in L is the derived normal domain of A. Thus A is a K_0-domain.

PROPOSITION 2. (a) *If R is a noetherian domain, then any integral extension domain of R is a K-domain.*

(b) *If A is a K-domain, then any integral extension domain of A is a K-domain.*

(c) *If A is a K-domain, then a polynomial ring $A[\dots, X_i, \dots]$, $i \in I$, is a K-domain, where I is any set.*

PROOF. The assertions of (a), (b) and (c) follow immediately from Proposition 1 and the definition of a K-domain.

If $\mathfrak{p}$ is a minimal prime ideal of a non-zero principal ideal in a noetherian domain, then $ht(\mathfrak{p})=1$. The following theorem shows that any K-domain has this property.

THEOREM 3. *Let A be a K-domain and let a ($\neq 0$) be a non-unit of A. Then any minimal prime ideal $\mathfrak{p}$ of aA has height one.*

PROOF. Let R be a K_0-domain such that A is an integral extension domain of R. By (d) of Proposition 1, we may assume that R contains a. Let K and L be the quotient fields of R and A respectively. The derived normal domain $\bar{R}$ of R is a Krull domain by (a) of Proposition 1. It is obvious that the derived normal domain $\bar{A}$ of A is the integral closure of R in L. Let $\mathfrak{P}_i$, $i \in I$, be the prime ideals in $\bar{A}$ such that each $\mathfrak{P}_i$ is lying over $\mathfrak{p}$. Since $\mathfrak{P}_i$ is a minimal prime ideal of $a\bar{A}$, $\mathfrak{P}_i \cap \bar{R}$ is a minimal prime ideal of $a\bar{R}$ by the Going-Down Theorem. Hence $ht(\mathfrak{P}_i \cap \bar{R})=1$ because $\bar{R}$ is a Krull domain. Therefore $ht(\mathfrak{P}_i)=1$ for any $i \in I$. Thus $ht(\mathfrak{p})=1$.

An integral domain A is said to be an S-domain if for any height one prime ideal $\mathfrak{p}$ in A, the height of $\mathfrak{p}A[X]$ in $A[X]$ is one, where X is an indeterminate (see [5], p. 26).

COROLLARY 4. *Let A be a K-domain. Then A is an S-domain. In particular, any integral extension domain of a noetherian domain is an S-domain.*

PROOF. Let $\mathfrak{p}$ be a height one prime ideal of A. Let a be a non-zero element of $\mathfrak{p}$. Since $ht(\mathfrak{p})=1$, $\mathfrak{p}A[X]$ is a minimal prime ideal of $aA[X]$. By the assertion of (c) of Proposition 2, $A[X]$ is a K-domain. Hence $ht(\mathfrak{p}A[X])=1$ by Theorem 3.

In Section 2, for each n ($2 \leq n \leq \infty$), we give an example of a unique factorization local domain $(A, \mathfrak{M})$ such that $ht(t, u)A=n+1$ for some elements t, u of $\mathfrak{M}$. Therefore K-domains do not satisfy the "Altitude Theorem of Krull" that is a theorem concerning noetherian rings. Herein we introduce the following notion.

DEFINITION. An integral domain A is called a *strong K_0-domain* (resp. *strong K-domain*) in case, for each prime ideal $\mathfrak{p}$ in A, $A/\mathfrak{p}$ is a K_0-domain (resp. K-domain).

The following assertions follow immediately from Propositions 1 and 2.

PROPOSITION 5. (a) *Let R be a strong K_0-domain. Then a finite integral extension domain A of R is a strong K_0-domain.*

(b) *Let R be a strong K_0-domain, and let R_0 be a subring of R such that*

R is integral over R_0. Then R_0 is a strong K_0-domain.

(c) *Let R be a strong K-domain. Then any integral extension domain A of R is a strong K-domain.*

(d) *Let R be a noetherian domain. Then the derived normal domain $\bar{R}$ of R is a strong K_0-domain.*

(e) *Let R be a noetherian domain. Then an almost finite extension domain A of R is a strong K_0-domain.*

PROOF. The assertions of (a), (b) and (c) follow immediately from Propositions 1 and 2.

(d) Let $\mathfrak{P}$ be an arbitrary prime ideal in $\bar{R}$. By Theorem (33.10) in [5], $\bar{R}/\mathfrak{P}$ is almost finite over $R/(\mathfrak{P} \cap R)$. Hence $\bar{R}/\mathfrak{P}$ is a K_0-domain by (d) of Proposition 1.

(e) Since the derived normal domain $\bar{A}$ of A is the derived normal one of a noetherian domain, $\bar{A}$ is a strong K_0-domain. Therefore the assertion of (b) implies that A is a strong K_0-domain.

REMARK. Let k be a field, and let $B=k[X, Y, X/Y, X/Y^2,\ldots]$, where X, Y are indeterminates. Then $ht(YB)=2$. Therefore $k[X_1, X_2, X_3,\ldots]$ is not a strong K-domain, where X_i $(i \in \boldsymbol{N})$ are indeterminates.

Let A be a ring, and let $\mathfrak{P}$, $\mathfrak{p}$ be prime ideals in A such that $\mathfrak{P} \supset \mathfrak{p}$ and $ht(\mathfrak{P}/\mathfrak{p})=1$. Then we say that $\mathfrak{p}$ is directly below $\mathfrak{P}$. We next prove that a strong K-domain has the following property: If A is a strong K-domain and $\mathfrak{a}$ is an ideal in A generated by n elements, then for every minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$, $ht(\mathfrak{p}) \leq n$ holds. For the proof of this theorem, we need the following lemma.

LEMMA 6. *Let A be a ring, and let $\mathfrak{a}$ be a finitely generated ideal in A. Let $\mathfrak{P}$ be a minimal prime ideal of $\mathfrak{a}$. Then the following statements hold.*

(a) *For each prime ideal $\mathfrak{q}$ properly contained in $\mathfrak{P}$, there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}$ is directly below $\mathfrak{P}$ and contains $\mathfrak{q}$.*

(b) $ht(\mathfrak{P})=\sup\{ht(\mathfrak{p});$ *$\mathfrak{p}$ is directly below* $\mathfrak{P}\}+1$.

PROOF. (a) Let $E=\{\mathfrak{p} \in \operatorname{Spec}(A); \mathfrak{P} \supset \mathfrak{p} \supseteq \mathfrak{q}\}$. Since $\mathfrak{q}$ is an element of E, E is not empty. E is an ordered set with the inclusion relation. Since $\mathfrak{a}$ is finitely generated, and since $\mathfrak{P}$ is a minimal prime ideal of $\mathfrak{a}$, E is an inductive set. Let $\mathfrak{p}$ be a maximal element of E. Then $ht(\mathfrak{P}/\mathfrak{p})=1$ by the maximality of $\mathfrak{p}$.

(b) It suffices to show that $\sup\{ht(\mathfrak{p}); \mathfrak{p}$ is directly below $\mathfrak{P}\}$ is infinite if $ht(\mathfrak{P})$ is infinite. Suppose that $ht(\mathfrak{P})=\infty$. Then for each positive integer n, there exists a chain of prime ideals $\mathfrak{P} \supset \mathfrak{P}_1 \supset \cdots \supset \mathfrak{P}_n \supset \mathfrak{P}_{n+1}$ in A. By the assertion of (a) we may assume that $\mathfrak{P}_1$ is directly below $\mathfrak{P}$. Since $ht(\mathfrak{P}_1) \geq n$, $\sup\{ht(\mathfrak{p}); \mathfrak{p}$ is directly below $\mathfrak{P}\} \geq n$. Hence $\sup\{ht(\mathfrak{p}); \mathfrak{p}$ is directly below $\mathfrak{P}\}$ is infinite.

THEOREM 7. *Let A be a strong K-domain, and let $\mathfrak{a}$ be an ideal in A generated by n elements. Then $ht(\mathfrak{P}) \leq n$ holds for any minimal prime ideal $\mathfrak{P}$ of $\mathfrak{a}$.*

PROOF. Let $\mathfrak{a}=(a_1,\ldots, a_n)A$. We prove the assertion by induction on n. If $n=1$, then the assertion is obvious by Theorem 3. Assume that $n \geq 2$. Let $\mathfrak{p}$ be any prime ideal in A which is directly below $\mathfrak{P}$. Since $\mathfrak{P}$ is a minimal prime ideal of $\mathfrak{a}$, $\mathfrak{p}$ does not contain $\mathfrak{a}$; so we may assume that $\mathfrak{p} \not\ni a_1$. Then $ht(\mathfrak{P}/\mathfrak{p})=1$ implies that $\mathfrak{P}A_{\mathfrak{P}}$ is a unique prime ideal in $A_{\mathfrak{P}}$ containing $(\mathfrak{p}+a_1A)A_{\mathfrak{P}}$; so the radical of $(\mathfrak{p}+a_1A)A_{\mathfrak{P}}$ is $\mathfrak{P}A_{\mathfrak{P}}$. Therefore $(\mathfrak{p}+a_1A)A_{\mathfrak{P}}$ contains $(a_i)^m$ $(i=2,\ldots, n)$ for some positive integer m. We write $(a_i)^m=(c_i+a_1b_i)/s$ with $c_i \in \mathfrak{p}$, $b_i \in A$ and $s \in A-\mathfrak{P}$. Let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{b}=(c_2,\ldots, c_n)A$ contained in $\mathfrak{p}$. Since the radical of $(\mathfrak{b}+a_1A)A_{\mathfrak{P}}$ contains a_i $(i=1, 2,\ldots, n)$, we have $\sqrt{(\mathfrak{b}+a_1A)A_{\mathfrak{P}}} =\mathfrak{P}A_{\mathfrak{P}}$. Thus $\sqrt{(\mathfrak{q}+a_1A)A_{\mathfrak{P}}}=\mathfrak{P}A_{\mathfrak{P}}$. Therefore $\mathfrak{P}/\mathfrak{q}$ is a minimal prime ideal of $\bar{a}_1(A/\mathfrak{q})$, where $\bar{a}_1=a_1 \bmod \mathfrak{q}$. By hypothesis $A/\mathfrak{q}$ is a K-domain, so $ht(\mathfrak{P}/\mathfrak{q}) =1$ by Theorem 3. Therefore $\mathfrak{P} \supset \mathfrak{p} \supseteq \mathfrak{q}$ implies that $\mathfrak{p}=\mathfrak{q}$. Then by the inductive hypothesis, $ht(\mathfrak{p}) \leq n-1$. Thus $ht(\mathfrak{P}) \leq n$ by Lemma 6.

For an ideal $\mathfrak{a}$ of a ring R, the supremum of heights of minimal prime ideals of $\mathfrak{a}$ is called the altitude of $\mathfrak{a}$. For the sake of convenience we use the following notations: Let R be an integral domain, and let $\mathfrak{a}$ be an ideal in R. Then we denote by $\mathrm{Min}_R(R/\mathfrak{a})$ the set of minimal prime ideals of $\mathfrak{a}$, and denote by $S(\mathfrak{a})$ the set $\{n \in N;\ \mathrm{Min}_R(R/\mathfrak{a})$ contains infinitely many height n prime ideals$\}$. If $S(\mathfrak{a})$ is not empty, we define $t(\mathfrak{a})$ by $t(\mathfrak{a})=\inf S(\mathfrak{a})$. Theorem 9 and Theorem 11 are concerned with strong K_0-domains.

LEMMA 8. *Let R be a K_0-domain. Let a $(\neq 0)$ be a non-unit of R, and let $\mathfrak{a}$ be an ideal such that $S(\mathfrak{a})$ is not empty. Then the following statements hold.*

(a) *There exists only a finite number of minimal prime ideals of aR.*

(b) $t(\mathfrak{a})>1$.

(c) *There exists a height one prime ideal $\mathfrak{p}$ in R such that $S(\mathfrak{b})$ is not empty and $t(\mathfrak{b})<t(\mathfrak{a})$ for some ideal $\mathfrak{b}$ in $R/\mathfrak{p}$.*

PROOF. (a) Since the derived normal domain $\bar{R}$ of R is a Krull domain, the number of minimal prime ideals of $a\bar{R}$ is finite. From this fact the assertion follows immediately.

(b) Let b be a non-zero element of $\mathfrak{a}$. Then $\mathrm{Min}_R(R/bR)$ is a finite set by (a). Hence $\mathfrak{a}$ has only a finite number of height one minimal prime divisors. Therefore $t(\mathfrak{a})>1$.

(c) Let b be a non-zero element of $\mathfrak{a}$. Let $\mathfrak{p}_1,\ldots, \mathfrak{p}_r$ be the minimal prime ideals of bR. Let $n=t(\mathfrak{a})$. We may assume that $\mathfrak{p}_1$ is contained in infinitely many height n minimal prime ideals $\mathfrak{P}_i$ $(i \in I)$ of $\mathfrak{a}$. Set $\mathfrak{c}=\bigcap_{i \in I}\mathfrak{P}_i$. Then, since

$\mathfrak{c}$ contains $\mathfrak{a}$, each $\mathfrak{P}_i$ is a minimal prime ideal of $\mathfrak{c}$. Therefore $\mathfrak{P}_i/\mathfrak{p}_1$ is a minimal prime ideal of $\mathfrak{c}/\mathfrak{p}_1$ and $ht(\mathfrak{P}_i/\mathfrak{p}_1)<n$ for each $i\in I$. Thus $S(\mathfrak{c}/\mathfrak{p}_1)$ is not empty and $t(\mathfrak{c}/\mathfrak{p}_1)<n$.

THEOREM 9. *Let R be a strong K_0-domain. Then the following statements hold.*

(a) *Let $\mathfrak{a}$ be an ideal in R. Then for each positive integer n, there exists only a finite number of height n minimal prime ideals of $\mathfrak{a}$.*

(b) *Let $\mathfrak{a}$ be an ideal in R whose altitude is finite. Then there exists only a finite number of minimal prime ideals of $\mathfrak{a}$.*

(c) *If* $\dim(R)$ *is finite, then* $\mathrm{Spec}(R)$ *is a noetherian space.*

PROOF. (a) Suppose that $S(\mathfrak{a})$ is not empty. Since $R/\mathfrak{p}$ is a K_0-domain for any prime ideal $\mathfrak{p}$ in R, by repeated use of (c) of Lemma 8, we see that there exists a prime ideal $\mathfrak{p}$ in R such that $S(\mathfrak{b})$ is not empty and $t(\mathfrak{b})=1$ for some ideal $\mathfrak{b}$ in $R/\mathfrak{p}$. This contradicts the assertion of (b) of Lemma 8.

(b) This follows immediately from the assertion of (a).

(c) This follows from the assertion of (b) and Proposition 1.1 in [7].

COROLLARY 10. *Let R be a strong K_0-domain, and let $\mathfrak{a}$ be a finitely generated ideal in R. Then there exists only a finite number of minimal prime ideals of $\mathfrak{a}$.*

PROOF. It suffices to show that the altitude of $\mathfrak{a}$ is finite. Let $\mathfrak{a}$ be generated by n elements. Then the altitude of $\mathfrak{a}$ is not greater than n by Theorem 7.

THEOREM 11. *Let R be a strong K_0-domain, and let $\mathfrak{P}$ be a height n prime ideal in R. Then there exist n elements $a_1,\dots, a_n$ of $\mathfrak{P}$ such that $ht(a_1,\dots, a_n)R=n$. In particular, $\mathfrak{P}$ is a minimal prime ideal of a certain ideal generated by n elements of $\mathfrak{P}$.*

PROOF. We prove the assertion by induction on n. If $n=1$, then the assertion is obvious, and we assume that $n\geq 2$. Let $\mathfrak{P}$ be a prime ideal such that $\mathfrak{p}$ is directly below $\mathfrak{P}$ and $ht(\mathfrak{p})=n-1$. By inductive hypothesis there exists $n-1$ elements $a_1,\dots, a_{n-1}$ of $\mathfrak{p}$ such that $ht(a_1,\dots, a_{n-1})R=n-1$. Let $\mathfrak{p}_1,\dots, \mathfrak{p}_r$ be the minimal prime ideals of $(a_1,\dots, a_{n-1})R$. Then $ht(\mathfrak{p}_i)=n-1$ $(i=1,\dots, r)$ by Theorem 7. Therefore $\mathfrak{P}$ is not contained in $\mathfrak{p}_1\cup\cdots\cup\mathfrak{p}_r$. Let a_n be an element of $\mathfrak{P}-\mathfrak{p}_1\cup\cdots\cup\mathfrak{p}_r$. Then it is easy to see that $ht(a_1,\dots, a_{n-1}, a_n)R=n$.

An almost finite extension domain of a noetherian domain is a strong K_0-domain by (e) of Proposition 5. So we have the following corollary.

COROLLARY 12. *Let A be an almost finite extension domain of a noetherian domain R. Then the following statements hold.*

(a) *If* $\mathfrak{a}$ *is an ideal in* A *generated by* n *elements, then* $ht(\mathfrak{p}) \leqq n$ *holds for each minimal prime ideal* $\mathfrak{p}$ *of* $\mathfrak{a}$.

(b) *Let* $\mathfrak{a}$ *be an arbitrary ideal in* A. *Then there exists only a finite number of minimal prime ideals of* $\mathfrak{a}$. (Heinzer [4])

(c) *If* $\mathfrak{P}$ *is a height* n *prime ideal in* A, *then there exist* n *elements* $a_1, \ldots, a_n$ *of* $\mathfrak{P}$ *such that* $ht(a_1, \ldots, a_n)A = n$.

PROOF. The assertions of (a) and (c) follow immediately from Theorem 7 and Theorem 11 respectively.

(b) Let M^* be the set of ideals in A which have an infinite number of minimal prime divisors. Set $M = \{\mathfrak{b} \cap R; \mathfrak{b} \in M^*\}$. It suffices to show that M is empty. Suppose that M is not empty. Since R is noetherian, M has a maximal element with respect to the inclusion relation. Let $\mathfrak{b} \cap R$ be a maximal element of M. Then, as is seen easily, $\mathfrak{b} \cap R$ is a prime ideal in R. Set $\mathfrak{p} = \mathfrak{b} \cap R$. Since A is a strong K_0-domain, there exists only a finite number of minimal prime ideals of $\mathfrak{p}A$ by Corollary 10. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of $\mathfrak{p}A$. Then we may assume that $\mathfrak{p}_1$ is properly contained in infinitely many minimal prime divisors $\mathfrak{P}_i$, $i \in I$, of $\mathfrak{b}$. Set $\mathfrak{c} = \bigcap_{i \in I} \mathfrak{P}_i$. Since $\mathfrak{b}$ is contained in $\mathfrak{c}$, each $\mathfrak{P}_i$ is a minimal prime ideal of $\mathfrak{c}$. Therefore by the maximality of $\mathfrak{p}$ in E, we have $\mathfrak{c} \cap R = \mathfrak{p}$, which implies that $\mathfrak{c}/\mathfrak{p}_1 \cap R/\mathfrak{p} = 0$. Therefore $\mathfrak{c} = \mathfrak{p}_1$. This is a contradiction. Thus M is empty.

REMARK. Since the derived normal domain A of a noetherian domain R is almost finite over R, A satisfies (a), (b) and (c) of Corollary 12.

2. Example 1

Let n be an integer greater than 1. Here we give an example of a unique factorization local domain $(A, \mathfrak{M})$ such that $ht(t, u)A = n+1$ for some elements t, u of $\mathfrak{M}$. For the construction of this example, we need the following lemma.

LEMMA 13. *Let* $(A, \mathfrak{M})$ *be an* $(n+1)$-*dimensional regular local ring, and let* $\{t, t_1, \ldots, t_{n-1}, u\}$ *be a regular system of parameters of* A. *Let* $X_1, \ldots, X_{n-1}$ *be indeterminates. Then the following statements hold.*

(a) *Let* $\mathfrak{P} = (t, uX_1 + t_1, \ldots, uX_{n-1} + t_{n-1})A(X_1, \ldots, X_{n-1})$. *Then* $\mathfrak{P}$ *is a prime ideal in* $A(X_1, \ldots, X_{n-1})$.

(b) $\mathfrak{P} \cap A = tA$.

(c) *Let* $B = A[t_1/t, \ldots, t_{n-1}/t]$, *and let* $\mathfrak{N} = (t, t_1/t, \ldots, t_{n-1}/t, u)B$. *Then* $B_{\mathfrak{N}}$ *is an* $(n+1)$-*dimensional regular local ring. In particular,* $\{t, t_1/t, \ldots, t_{n-1}/t, u\}$ *is a regular system of parameters of* $B_{\mathfrak{N}}$.

(d) $tB_{\mathfrak{N}} \cap A = (t, t_1, \ldots, t_{n-1})A$.

PROOF. (a) Let $A^* = A(X_1, \ldots, X_{n-1})$. Since $(t, uX_1 + t_1, \ldots, uX_{n-1} +$

$t_{n-1}, u)A^*=(t, t_1,\dots, t_{n-1}, u)A^*=\mathfrak{M}A^*$, $\{t, uX_1+t_1,\dots, uX_{n-1}+t_{n-1}, u\}$ is a regular system of parameters of A^*. Therefore $\mathfrak{P}$ is a prime ideal in A^*.

(b) Let $\mathfrak{p}=(t, uX_1+t_1,\dots, uX_{n-1}+t_{n-1})A[X_1,\dots, X_{n-1}]$. By our assumption, A/tA is a regular local ring and $\{\bar{t}_1,\dots, \bar{t}_{n-1}, \bar{u}\}$ is a regular system of parameters of A/tA, where $\bar{t}_i=t_i \bmod (tA)$ and $\bar{u}=u \bmod (tA)$. Hence $\{\bar{u}, -\bar{t}_1,\dots, -\bar{t}_{n-1}\}$ is an (A/tA)-regular sequence, so $(\bar{u}X_1+\bar{t}_1,\dots, \bar{u}X_{n-1}+\bar{t}_{n-1})(A/tA)[X_1,\dots, X_{n-1}]$ is a prime ideal in $(A/tA)[X_1,\dots, X_{n-1}]$ by Proposition 2 in [1]. Therefore $\mathfrak{p}=(t, uX_1+t_1,\dots, uX_{n-1}+t_{n-1})A[X_1,\dots, X_{n-1}]$ is a prime ideal in $A[X_1,\dots, X_{n-1}]$. Since $\mathfrak{P}\cap A[X_1,\dots, X_{n-1}]=\mathfrak{p}$, it suffices to show that $\mathfrak{p}\cap A=tA$. Let c be any element of $\mathfrak{p}\cap A$. We write $c=g_0\cdot t+\sum_{i=1}^{n-1} g_i\cdot(uX_i+t_i)$, where $g_i\in A[X_1,\dots, X_{n-1}]$ $(i=0, 1,\dots, n-1)$. By substituting $-(t_i/u)$ for X_i, we see that c is of the form $(b/u^m)t$. Since A is a unique factorization domain and u is relatively prime to t in A, b is divided by u^m in A. Therefore c belongs to tA. Thus $\mathfrak{p}\cap A=tA$.

(c) Let $C=A[X_1,\dots, X_{n-1}]$, and let $\mathfrak{N}^*=(t, t_1,\dots, t_{n-1}, u, X_1,\dots, X_{n-1})C$. Then $C_{\mathfrak{N}^*}$ is a $2n$-dimensional regular local ring, and $\mathfrak{N}^*C_{\mathfrak{N}^*}=(t, tX_1-t_1,\dots, tX_{n-1}-t_{n-1}, u, X_1,\dots, X_{n-1})C_{\mathfrak{N}^*}$, so $\{t, tX_1-t_1,\dots, tX_{n-1}-t_{n-1}, u, X_1,\dots, X_{n-1}\}$ is a regular system of parameters of $C_{\mathfrak{N}^*}$. Therefore $C_{\mathfrak{N}^*}/(tX_1-t_1,\dots, tX_{n-1}-t_{n-1})C_{\mathfrak{N}^*}$ is an $(n+1)$-dimensional regular local ring. As is seen easily, $C_{\mathfrak{N}^*}/(tX_1-t_1,\dots, tX_{n-1}-t_{n-1})C_{\mathfrak{N}^*}\simeq B_{\mathfrak{N}}$.

(d) $B_{\mathfrak{N}}/tB_{\mathfrak{N}}\simeq C_{\mathfrak{N}^*}/(t, tX_1-t_1,\dots, tX_{n-1}-t_{n-1})C_{\mathfrak{N}^*}=C_{\mathfrak{N}^*}/(t, t_1,\dots, t_{n-1})C_{\mathfrak{N}^*}$ which is isomorphic to a localization of $(A/(t, t_1,\dots, t_{n-1})A)[X_1,\dots, X_{n-1}]$. Therefore $tB_{\mathfrak{N}}\cap A=(t, t_1,\dots, t_{n-1})A$.

Now we construct regular local rings A_m $(m=1, 2, 3,\dots)$ inductively. Let A_0 be an $(n+1)$-dimensional regular local ring, and let $\{t, t_{01}, t_{02},\dots, t_{0(n-1)}, u\}$ be a regular system of parameters of A_0. Let X_{ij} $(i=1, 2, 3,\dots, j=1,\dots, n-1)$ be indeterminates. Set $A_1=A_0(X_{11},\dots, X_{1(n-1)})$, and set $t_{1j}=uX_{1j}+t_{0j}$ for $j=1,\dots, n-1$ Then $\mathfrak{P}_1=(t, t_{11},\dots, t_{1(n-1)})A_1$ is a prime ideal in A_1 by (a) of Lemma 13. Set $A_2=A_1[t_{11}/t,\dots, t_{1(n-1)}/t]_{\mathfrak{N}_1}$, where $\mathfrak{N}_1=(t, t_{11}/t,\dots, t_{1(n-1)}/t, u)A_1[t_{11}/t,\dots, t_{1(n-1)}/t]$. Then by (c) of Lemma 13, A_2 is an $(n+1)$-dimensional regular local ring. Generally for each m $(m\geq 2)$ we set inductively $A_{2m-1}=A_{2m-2}(X_{m1},\dots, X_{m(n-1)})$ and $t_{mj}=uX_{mj}+(t_{(m-1)j}/t)$ for $j=1,\dots, n-1$. Then $\mathfrak{P}_{2m-1}=(t, t_{m1},\dots, t_{m(n-1)})A_{2m-1}$ is a prime ideal in A_{2m-1}. And set $A_{2m}=A_{2m-1}[t_{m1}/t,\dots, t_{m(n-1)}/t]_{\mathfrak{N}_m}$, where $\mathfrak{N}_m=(t, t_{m1}/t,\dots, t_{m(n-1)}/t, u)A_{2m-1}[t_{m1}/t,\dots, t_{m(n-1)}/t]$. Then by repeated use of Lemma 13, we see that for each $m\geq 0$, A_m is an $(n+1)$-dimensional regular local ring.

Let $\mathfrak{M}_m$ be the maximal ideal in A_m. Set $A=\bigcup_{m\geq 0} A_m$, $\mathfrak{M}=\bigcup_{m\geq 0}\mathfrak{M}_m$ and $\mathfrak{P}=\bigcup_{m\geq 0}\mathfrak{P}_{2m+1}$. Then the following assertions hold.

(a) *$tA_{2m+2}\cap A_{2m+1}=\mathfrak{P}_{2m+1}$ and $\mathfrak{P}_{2m+1}\cap A_{2m}=tA_{2m}$ for each $m\geq 0$.*

(b) $\mathfrak{P}=tA$.

(c) $\mathfrak{P}_{2m+1}\not\ni u$ *for each* $m\geq 1$. *In particular* $\mathfrak{P}\not\ni u$.

(d) $\mathfrak{P}\cap A_{2m+1}=\mathfrak{P}_{2m+1}$ *for each* $m\geq 1$.

(e) $ht(\mathfrak{M})=n+1$, *and* $\mathfrak{M}\supset\mathfrak{P}\supset 0$ *is a saturated chain of prime ideals in* A. *In particular* A *is not catenary, and* $ht(t, b)A=n+1$ *for any* $b\in\mathfrak{M}-\mathfrak{P}$.

(f) A *is a unique factorization domain.*

PROOF. (a) This follows immediately from (b) and (d) of Lemma 13.

(b) This is obvious from (a).

(c) Suppose that $\mathfrak{P}_{2m+1}$ contains u. Since $uX_{(m+1)j}+(t_{mj}/t)$, $j=1,\dots, n-1$, are elements of $\mathfrak{P}_{2m+1}$, $\mathfrak{P}_{2m+1}\supseteq(t, t_{m1}/t,\dots, t_{m(n-1)}/t, u)A_{2m+1}=\mathfrak{M}_{2m+1}$, which is a contradiction.

(d) Since $\mathfrak{P}$ does not contain u, we have $\mathfrak{P}_{2m+1}\subseteq\mathfrak{P}\cap A_{2m+1}\subset\mathfrak{M}_{2m+1}$. Therefore $ht(\mathfrak{M}_{2m+1}/\mathfrak{P}_{2m+1})=1$ implies that $\mathfrak{P}_{2m+1}=\mathfrak{P}\cap A_{2m+1}$.

(e), (f) Since $A_P=\varinjlim_m (A_{2m})_{tA_{2m}}$ holds, we see that $ht(\mathfrak{P})=1$. Let $\mathfrak{Q}$ be a prime ideal in A such that $\mathfrak{M}\supseteq\mathfrak{Q}\supseteq\mathfrak{P}$. Then by (d) $\mathfrak{M}_{2m+1}\supseteq\mathfrak{Q}\cap A_{2m+1}\supseteq\mathfrak{P}_{2m+1}$ for each $m\geq 0$. If there exists a positive integer m_0 such that $\mathfrak{Q}\cap A_{2m+1}=\mathfrak{P}_{2m+1}$ for any $m\geq m_0$, then $\mathfrak{Q}=\mathfrak{P}$. On the other hand, if there exist integers $m_1<m_2<m_3<\cdots$ such that $\mathfrak{Q}\cap A_{2m_i+1}=\mathfrak{M}_{2m_i+1}$, then $\mathfrak{Q}=\mathfrak{M}$. Thus $\mathfrak{M}\supset\mathfrak{P}\supset 0$ is saturated. Set $B_0=A_0$, and set inductively $B_m=B_{m-1}(X_{m1},\dots, X_{m(n-1)})$ for each positive integer m. Then B_m is an $(n+1)$-dimensional regular local ring for each $m\geq 0$. Set $B=\bigcup_{m\geq 0} B_m$. Then $B=A_0[X_{11},\dots, X_{1(n-1)},\dots, X_{m1},\dots, X_{m(n-1)},\dots]_{\mathfrak{M}^*}$, where $\mathfrak{M}^*=\mathfrak{M}_0A_0[X_{11},\dots, X_{1(n-1)},\dots, X_{m1},\dots, X_{m(n-1)},\dots]$. Therefore B is a unique factorization domain. Since $A_{2m-1}[1/t]=A_{2m}[1/t]=B_m[1/t]$ holds, we have $A[1/t]=B[1/t]$. Therefore $\dim(A[1/t])=n$. Hence $\dim(A)\geq n+1$. On the other hand $A=\varinjlim_m (A_m)$ implies that $\dim(A)\leq n+1$. Thus $\dim(A)=n+1$. Finally we show that A is a unique factorization domain. Since $ht(tA)=1$, A_{tA} is a principal valuation ring. For each $m\geq 0$, A_{2m} is a Krull domain, so $A_{2m}[1/t]\cap(A_{2m})_{tA_{2m}}=A_{2m}$. Hence $A[1/t]\cap A_{tA}=A$. Thus the facts that $A[1/t]$ is a Krull domain and A_{tA} is a principal valuation ring imply that A is a Krull domain. Since t is a prime element in A, and since $A[1/t]$ is a unique factorization domain, by Nagata's theorem ([7], p. 21) we see that A is a unique factorization domain. Thus the proof is complete.

Now we obtain the following conclusion.

THEOREM 14. $(A, \mathfrak{M})$ *is an* $(n+1)$*-dimensional unique factorization local domain and* $ht(t, u)A=n+1$.

Example 2

We here give an example of a unique factorization local domain $(A, \mathfrak{M})$

such that $ht(t, u)A=\infty$ for some elements t, u of $\mathfrak{M}$. For the construction of this example we need the following lemma.

LEMMA 15. *Let k be a field, and let $t, t_1,\dots, t_n,\dots, u$ be algebraically independent elements over k. Let $X_1, X_2,\dots, X_n,\dots$ be indeterminates. Set $R=k[t, t_1, t_2,\dots, u]$, $\mathfrak{m}=(t, t_1, t_2,\dots, u)R$, $B=R[t_1/t, t_2/t,\dots]$ and $\mathfrak{N}=(t, t_1/t, t_2/t,\dots, u)B$. Then the following statements hold.*

(a) *Let $K = k(X_1, X_2,\dots)$. Then* $R_{\mathfrak{m}}(X_1, X_2,\dots) = K[t, t_1, t_2,\dots, u]_{\mathfrak{m}K[t,t_1,t_2,\dots,u]}$.

(b) *$t, t_1/t, t_2/t,\dots, u$ are algebraically independent over k.*

(c) $B_{\mathfrak{N}}=k[t, t_1/t, t_2/t,\dots, u]_{(t,t_1/t,t_2/t,\dots,u)}$.

(d) $tB_{\mathfrak{N}} \cap R_{\mathfrak{m}}=(t, t_1, t_2,\dots)R_{\mathfrak{m}}$.

(e) *Let $\mathfrak{P}=(t, uX_1+t_1, uX_2+t_2,\dots)R_{\mathfrak{m}}(X_1, X_2,\dots)$. Then $\mathfrak{P}$ is a prime ideal in $R_{\mathfrak{m}}(X_1, X_2,\dots)$ and $\mathfrak{P} \cap R_{\mathfrak{m}}=tR_{\mathfrak{m}}$.*

(f) *Set $t_i^*=uX_i+t_i$ $(i=1, 2,\dots)$. Then $t, t_1^*/t, t_2^*/t,\dots, u$ are algebraically independent over $K=k(X_1, X_2,\dots)$.*

PROOF. It is easy to see that the assertions of (a), (b) and (c) hold.

(d) We see that the assertion of (d) is proved immediately by the same proof as in (d) of Lemma 13.

(e) Since $(t, uX_1+t_1,\dots, uX_n+t_n)R_{\mathfrak{m}}[X_1 \dots, X_n]$ is a prime ideal in $R_{\mathfrak{m}}[X_1,\dots, X_n]$ for each positive integer n, $(t, uX_1+t_1, uX_2+t_2,\dots)R_{\mathfrak{m}}[X_1, X_2,\dots]$ is a prime ideal in $R_{\mathfrak{m}}[X_1, X_2,\dots]$. Hence $\mathfrak{P}$ is a prime ideal in $R_{\mathfrak{m}}(X_1, X_2,\dots)$. The equality $\mathfrak{P} \cap R_{\mathfrak{m}}=tR_{\mathfrak{m}}$ can be proved by the same way as the proof of (b) of Lemma 13.

(f) By the assertion of (b) it is enough to prove that $t, t_1^*, t_2^*,\dots, u$ are algebraically independent over K. Let $K_n=k(X_1,\dots, X_n)$. Then $K_n(t, uX_1+t_1,\dots, uX_n+t_n, u)=K_n(t, t_1,\dots, t_n, u)$. The transcendental degree of $K_n(t, t_1,\dots, t_n, u)$ over K_n is $n+2$, and so $t, uX_1+t_1,\dots, uX_n+t_n, u$ are algebraically independent over K_n. Therefore $t, uX_1+t_1, uX_2+t_2,\dots, u$ are algebraically independent over K.

Now we construct infinite dimensional unique factorization local domains A_n $(n=1, 2,\dots)$ inductively. Let k be a field, and let $t, t_{01}, t_{02},\dots, t_{0n},\dots, u$ be algebraically independent elements over k. Let X_{mn} $(m, n=1, 2,\dots)$ be indeterminates. Set $R=k[t, t_{01}, t_{02},\dots, u]$, $\mathfrak{m}=(t, t_{01}, t_{02},\dots, u)R$. Then we set $A_0, A_1, \mathfrak{P}_1, A_2, A_3, \mathfrak{P}_3, A_4,\dots$ as follows:

$A_0 = R_{\mathfrak{m}}$,

$A_1 = A_0(X_{11}, X_{12},\dots)$, $\mathfrak{P}_1 = (t, t_{11}, t_{12},\dots)A_1$,

where $t_{1j} = uX_{1j} + t_{0j}$ $(j = 1, 2,\dots)$,

$A_2 = A_1[t_{11}/t, t_{12}/t,\dots]_{\mathfrak{N}_1}$,

where $\mathfrak{N}_1 = (t, t_{11}/t, t_{12}/t,\dots, u)A_1[t_{11}/t, t_{12}/t \dots]$,

$A_3 = A_2(X_{21}, X_{22}, \dots)$, $\mathfrak{P}_3 = (t, t_{21}, t_{22}, \dots)A_3$,
where $t_{2j} = uX_{2j} + (t_{1j}/t) \quad (j = 1, 2, \dots)$,
⋮
$A_{2n-1} = A_{2n-2}(X_{n1}, X_{n2}, \dots)$, $\mathfrak{P}_{2n-1} = (t, t_{n1}, t_{n2}, \dots)A_{2n-1}$,
where $t_{nj} = uX_{nj} + (t_{(n-1)j}/t) \quad (j = 1, 2, \dots)$,
$A_{2n} = A_{2n-1}[t_{n1}/t, t_{n2}/t, \dots]_{\mathfrak{N}_n}$,
where $\mathfrak{N}_n = (t, t_{n1}/t, t_{n2}/t, \dots, u)A_{2n-1}[t_{n1}/t, t_{n2}/t, \dots]$,
⋮

Let $\mathfrak{M}_n$ be the maximal ideal in the local domain A_n. Set $A = \bigcup_{n\geq 0} A_n$, $\mathfrak{M} = \bigcup_{n\geq 0} \mathfrak{M}_n$ and $\mathfrak{P} = \bigcup_{n\geq 0} \mathfrak{P}_{2n+1}$. Let k_n be the field over k generated by X_{ij} ($i=1, 2, \dots, n, j=1, 2, \dots$). Then the following assertions hold.

(a) *$t, t_{(n-1)1}/t, t_{(n-1)2}/t, \dots, u$ are algebraically independent over k_n.*
(b) $A_{2n-1} = k_n[t, t_{(n-1)1}/t, t_{(n-1)2}/t, \dots, u]_{(t, t_{(n-1)1}/t, t_{(n-1)2}/t, \dots, u)}$,
$A_{2n} = k_n[t, t_{n1}/t, t_{n2}/t, \dots, u]_{(t, t_{n1}/t, t_{n2}/t, \dots, u)}$.
(c) *$\mathfrak{P}$ does not contain u.*
(d) $\mathfrak{P} \cap A_{2n+1} = \mathfrak{P}_{2n+1}$ *for each* $n \geq 0$.
(e) $tA_{2n} \cap A_{2n-1} = \mathfrak{P}_{2n-1}$ *and* $\mathfrak{P}_{2n+1} \cap A_{2n} = tA_{2n}$ *for each* n.
(f) $\mathfrak{P} = tA$.
(g) $\dim(A) = \infty$, *and $\mathfrak{M} \supset \mathfrak{P} \supset 0$ is a saturated chain of prime ideals in A. In particular A is not catenary and $ht(t, b)A = \infty$ for any $b \in \mathfrak{M} - \mathfrak{P}$.*
(h) *A is a unique factorization domain.*

PROOF. (a) and (b) follow from Lemma 15, and (c)∼(h) are proved by the same way as in case Example 1.

Thus we obtain the following conclusion.

THEOREM 16. *$(A, \mathfrak{M})$ is an infinite dimensional unique factorization local domain and $ht(t, u)A$ is infinite.*

REMARK. The local domain $(A, \mathfrak{M})$ in Example 1 is a finite dimensional unique factorization domain which is not a strong K_0-domain.

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