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Maximal Conditions for Ideals in Lie Algebras

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1.

For a class $\mathfrak{X}$ of Lie algebras over a field $\mathfrak{k}$, let Max-$\lhd\mathfrak{X}$ be the class of Lie algebras which satisfy the maximal condition for $\mathfrak{X}$-ideals. Let $\mathfrak{A}$, $\mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$ be respectively the classes of abelian, nilpotent and solvable Lie algebras over $\mathfrak{k}$. Then it holds that

$$\text{Max-}\lhd\mathfrak{A} \gneqq \text{Max-}\lhd\mathfrak{N} \gneqq \text{Max-}\lhd\mathrm{E}\mathfrak{A}.$$

The first inequality was shown by Kubo [3] over any formally real field and the second inequality was shown by Ikeda [2] over any field. As a matter of fact, it is shown in [2, 3] that

$$\text{Max-}\lhd\mathfrak{A} \gneqq \text{Max-}\lhd\mathfrak{N}_2 \quad \text{and} \quad \text{Max-}\lhd\mathfrak{N} \nleqq \text{Max-}\lhd\mathfrak{A}^2.$$

Therefore it is desirable to see whether the classes Max-$\lhd\mathfrak{N}_2$ and Max-$\lhd\mathfrak{N}$ (resp. Max-$\lhd\mathfrak{A}^2$ and Max-$\lhd\mathrm{E}\mathfrak{A}$) coincide or not.

In this paper, we shall show that

$$\text{Max-}\lhd\mathfrak{N}_2 \gneqq \text{Max-}\lhd\mathfrak{N} \quad \text{and} \quad \text{Max-}\lhd\mathfrak{A}^2 \gneqq \text{Max-}\lhd\mathrm{E}\mathfrak{A};$$

more precisely

$$\bigcap_{k=1}^{\infty} \text{Max-}\lhd\mathfrak{N}_k \gneqq \text{Max-}\lhd\mathfrak{N} \quad \text{and} \quad \bigcap_{k=1}^{\infty} \text{Max-}\lhd\mathfrak{A}^k \gneqq \text{Max-}\lhd\mathrm{E}\mathfrak{A}.$$

Throughout the paper, we shall employ the notations and terminology in [1].

2.

Let L be the Lie algebra over a field $\mathfrak{k}$ with basis $\{e_{ij} \mid i<j;\ i, j=1, 2, \cdots\}$ and multiplication

$$[e_{ij}, e_{mn}] = \delta_{jm}e_{in} - \delta_{in}e_{mj}.$$

This is a special type of McLain Lie algebras considered in Section 3 of [2].

For $1 \leqq m \leqq n$, we put

$$I_{mn} = <e_{mn}{}^L> = <e_{ij} | i \leqq m < n \leqq j>,$$

$$I_m = I_{12} + I_{23} + \cdots + I_{m\,m+1}.$$

For $x = \sum_{i<j} \alpha_{ij} e_{ij} \in L$, we denote α_{ij} by $\alpha_{ij}(x)$ at our convenience and put

$$\ell(x) = \max\{i | \alpha_{ij}(x) \neq 0 \text{ for some } j\},$$

$$\ell(0) = 0.$$

LEMMA 1. *Let H be an ideal of L and let x be an element of H such that $\ell(x) = k > 0$. Then for any positive integer $m \leqq k$ there exists an element y of H such that $\ell(y) = m$.*

PROOF. If $m < k$, put $y = [e_{mk}, x]$. Then $y \in H$ and $\ell(y) = m$.

LEMMA 2. *Let x_1 and x_2 be elements of L such that $\ell(x_1) = m_1$ and $\ell(x_2) = m_2$. If*

$$\min\{j | \alpha_{m_1 j}(x_1) \neq 0\} = m_2,$$

then $\ell([x_1, x_2]) = m_1$.

PROOF. $[x_1, x_2]$ is the sum of

$$\sum_j \alpha_{m_1 m_2}(x_1) \alpha_{m_2 j}(x_2) e_{m_1 j} \tag{$*$}$$

and lower terms. The sum $(*)$ is not zero and $\ell([x_1, x_2]) = m_1$.

LEMMA 3. *Let H be an ideal of L. If $H \nleqq I_n$ for any n, then $H^{(1)} \nleqq I_n$ for any n.*

PROOF. Let n be any positive integer. Then by assumption there exists $x_1 \in H$ such that $\ell(x_1) > n$. Put $m_1 = \ell(x_1)$ and

$$m_2 = \min\{j | \alpha_{m_1 j}(x_1) \neq 0\}.$$

Then again by assumption there exists $y \in H$ such that $\ell(y) \geqq m_2$. Owing to Lemma 1, we can take $x_2 \in H$ such that $\ell(x_2) = m_2$. It follows from Lemma 2 that

$$\ell([x_1, x_2]) = m_1.$$

Hence $[x_1, x_2] \notin I_n$ and therefore $H^{(1)} \nleqq I_n$.

LEMMA 4. *Every solvable ideal of L is nilpotent and contained in I_n for some n.*

PROOF. Let H be an ideal of L and assume that $H \nleqq I_n$ for any n. Then by repeated use of Lemma 3 we see that

$$H^{(m)} \nleqq I_n \quad \text{for any } m \text{ and } n.$$

Therefore $H^{(m)} \neq 0$ for any m. Thus H is not solvable.

3.

We are now in a position to show the following results stated in Section 1.

THEOREM. *Over any field*

(a) $\bigcap_{k=1}^{\infty} \text{Max-}\lhd\mathfrak{N}_k \gneqq \text{Max-}\lhd\mathfrak{N}$,

(b) $\bigcap_{k=1}^{\infty} \text{Max-}\lhd\mathfrak{A}^k \gneqq \text{Max-}\lhd\text{E}\mathfrak{A}$.

PROOF. Let L be the Lie algebra in Section 2 and let $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ be any ascending chain of $\mathfrak{A}^k$-ideals of L. Put $H = \bigcup_{i=1}^{\infty} H_i$. Then H is an $\mathfrak{A}^k$-ideal of L. By Lemma 4 $H \leqq I_n$ for some n. Since $I_n \in \text{Max-}L$ by Lemma 2 in [2], there exists $m > 0$ such that $H_m = H_{m+1} = \cdots$. Therefore $L \in \text{Max-}\lhd\mathfrak{A}^k$. Since $\mathfrak{N}_k \leqq \mathfrak{A}^k$, it follows that $L \in \text{Max-}\lhd\mathfrak{N}_k$.

On the other hand, $\{I_n\}$ is a strictly ascending chain of nilpotent ideals of L. Therefore $L \notin \text{Max-}\lhd\mathfrak{N}$ and a priori $L \notin \text{Max-}\lhd\text{E}\mathfrak{A}$.

COROLLARY. *Over any field*

(a) $\text{Max-}\lhd\mathfrak{N}_2 \gneqq \text{Max-}\lhd\mathfrak{N}$,

(b) $\text{Max-}\lhd\mathfrak{A}^2 \gneqq \text{Max-}\lhd\text{E}\mathfrak{A}$.

References

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