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Chain Conditions for Abelian, Nilpotent and Soluble Ideals in Lie Algebras

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1. Introduction

Let $\mathfrak{X}$ be a class of Lie algebras over a field $\mathfrak{k}$, and let Max-$\lhd\mathfrak{X}$ (resp. Min-$\lhd\mathfrak{X}$) be the class of Lie algebras which satisfy the maximal (resp. minimal) condition for $\mathfrak{X}$-ideals. Amayo and Stewart have asked the following among "Some open questions" in [1]: *Are there any inclusions between* Max-$\lhd\mathfrak{A}$, Max-$\lhd\mathfrak{N}$, Max-$\lhd$ E$\mathfrak{A}$; Min-$\lhd\mathfrak{A}$, Min-$\lhd\mathfrak{N}$, Min-$\lhd$ E$\mathfrak{A}$?

Recently it was shown by Kubo [2] that Max-$\lhd\mathfrak{A}$ and Max-$\lhd\mathfrak{N}$ (resp. Min-$\lhd\mathfrak{A}$ and Min-$\lhd\mathfrak{N}$) do not necessarily coincide with each other. He showed these facts by considering a certain Lie algebra over the rational number field.

The purpose of this paper is to show the following theorems.

THEOREM 1. *Over any field*

$$\text{Max-}\lhd\mathfrak{N} \gneqq \text{Max-}\lhd \mathrm{E}\mathfrak{A} \quad \textit{and} \quad \text{Min-}\lhd\mathfrak{N} \gneqq \text{Min-}\lhd \mathrm{E}\mathfrak{A}.$$

THEOREM 2. *Over any field*

$$\text{Max-}\lhd\mathfrak{A} \gneqq \text{Max-}\lhd\mathfrak{N}.$$

Throughout the paper, we shall employ the notations and terminology in [1].

2. Proof of Theorem 1

Let $\mathfrak{k}$ be an arbitrary field and A an infinite extension field of $\mathfrak{k}$. Let ρ be the regular representation of A. Consider A as an abelian Lie algebra over $\mathfrak{k}$, so that ρ becomes a Lie homomorphism of A into Der(A). Thus we can form the split extension

$$L = A \dotplus \rho(A),$$

where $A \lhd L$ and $[a, \rho(b)] = ab$ for any $a, b \in A$.

We first show that any non-zero ideal of L contains A. Suppose $0 \neq I \lhd L$. Then $0 \neq I \cap A \lhd L$. In fact, if $I \cap A = 0$, then there exist $a, b \in A$ with $b \neq 0$ such that $a+\rho(b)\in I$. Hence $I\cap A\ni[1, a+\rho(b)]=b\neq 0$. This is a contradiction. Observing that the Lie ideals of L contained in A are the associative ideals of A and that A is a field, we obtain $I\cap A=A$. Therefore $I\geqq A$.

Now let I be an ideal of L such that $I\gneqq A$. Then there is a non-zero $x\in A$ such that $\rho(x)\in I$. For any positive integer n, $0\neq x^n=[x, {}_{n-1}\rho(x)]\in I^n$. Hence $I\notin\mathfrak{N}$. Consequently A is the only non-zero nilpotent ideal of L. Thus $L\in\text{Max-}\lhd\mathfrak{N}\cap\text{Min-}\lhd\mathfrak{N}$.

Finally we choose a $\mathfrak{k}$-free subset $\{e_i|i=1, 2,\cdots\}$ of A. Since ρ is injective, $\{\rho(e_i)|i=1, 2,\cdots\}$ is $\mathfrak{k}$-free. For any n put

$$B_n = A + <\rho(e_1), \rho(e_2),\cdots, \rho(e_n)>,$$

$$C_n = A + <\rho(e_n), \rho(e_{n+1}),\cdots>.$$

Then $\{B_n\}$ and $\{C_n\}$ are respectively strictly ascending and strictly descending chains of soluble ideals of L. Therefore $L\notin\text{Max-}\lhd\text{E}\mathfrak{A}\cup\text{Min-}\lhd\text{E}\mathfrak{A}$.

3. Proof of Theorem 2

Let L be a Lie algebra over $\mathfrak{k}$ with basis $\{e_{ij}|i<j;\ i, j=1, 2,\cdots\}$ and multiplication

$$[e_{ij}, e_{mn}] = \delta_{jm}e_{in} - \delta_{in}e_{mj}.$$

This is one of the McLain Lie algebras ([1], p. 111). Put

$$I_{0n} = 0 \qquad \text{for } n\geqq 1,$$

$$I_{mn} = <e_{ij}|i\leqq m<n\leqq j> \qquad \text{for } 1\leqq m<n$$

and furthermore

$$I_m = I_{12} + I_{23} + \cdots + I_{m\,m+1}.$$

We prepare two lemmas.

LEMMA 1. *If I is a non-zero ideal of L, then there is a positive integer n such that $I_{1\,n+1}\leqq I$.*

PROOF. Let $0\neq x=\sum_{i<j}\alpha_{ij}e_{ij}\in I$. Put $n=\max\{j|\alpha_{ij}\neq 0 \text{ for some } i\}$ and $m=\max\{i|\alpha_{in}\neq 0\}$. Then we have $I\ni[e_{1m}, [x, e_{n\,n+1}]]=[e_{1m}, \sum_i\alpha_{in}e_{i\,n+1}]=\alpha_{mn}e_{1\,n+1}$. Thus $I_{1\,n+1}\leqq I$.

LEMMA 2. $I_n\in\text{Max-}L$ *for any* $n\geqq 1$.

PROOF. Since Max-L is E-closed and $I_n/I_{n\,n+1}\in\mathfrak{F}\leqq\text{Max-}L$, it is sufficient to show that $I_{i+1\,n+1}/I_{i\,n+1} \in$ Max-L for $i=0, 1, \cdots, n-1$. Let J be an ideal of L such that $I_{i\,n+1} < J \leqq I_{i+1\,n+1}$. We can find $x \in J$ such that $x = \sum_j \alpha_j e_{i+1\,j} \neq 0$. Put $m = \max\{j | \alpha_j \neq 0\}$. Then we have $J \ni [x, e_{m\,m+1}] = \alpha_m e_{i+1\,m+1}$. Hence $I_{i+1\,m+1} \leqq J$ and $I_{i+1\,n+1}/J \in \mathfrak{F}$. Therefore $I_{i+1\,n+1}/I_{i\,n+1} \in$ Max-L.

By making use of these lemmas we can now establish Theorem 2. Let $0 < A_1 \leqq A_2 \leqq \cdots$ be an ascending chain of abelian ideals of L. Put $A = \bigcup_{i=1}^{\infty} A_i$. Then A is an abelian ideal of L. By Lemma 1, there is a positive integer n such that $I_{1\,n+1} \leqq A$. We first claim that $A \leqq I_n$. For any non-zero $a = \sum_{i<j} \alpha_{ij} e_{ij} \in A$, put $k = \max\{i | \alpha_{ij} \neq 0$ for some $j\}$. Then we have $[e_{1k}, a] = \sum_j \alpha_{kj} e_{1j} \neq 0$. If $k \geqq n+1$, we have $[e_{1k}, a] = 0$ since $e_{1k} \in I_{1\,n+1} \leqq A \in \mathfrak{A}$. This is a contradiction. Hence $k \leqq n$. Thus $A \leqq I_n$, as claimed.

By Lemma 2, $I_n \in$ Max-L. Since $A_i \leqq A \leqq I_n$ for $i = 1, 2, \cdots$, there is a positive integer m such that $A_m = A$. Thus $L \in$ Max-$\lhd \mathfrak{A}$. However $L \notin$ Max-$\lhd \mathfrak{N}$, since $\{I_i\}$ is obviously a strictly ascending chain of nilpotent ideals of L.

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References

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[2] F. Kubo, Finiteness conditions for abelian ideals and nilpotent ideals in Lie algebras, Hiroshima Math. J., **8** (1978), 301–303.

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