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Incomparability in Ring Extensions

Hirohumi UDA

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Introduction

Throughout this paper rings will be all commutative rings with units and morphisms will mean unitary ring-homomorphisms.

The purpose of this paper is to study some properties of *an incomparable morphism* (cf. [3]) and to introduce the notion of *universally incomparable morphisms* which will play an important role in this paper.

We shall discuss in § 1 some basic properties of an incomparable morphism. In § 2, we shall define a universally incomparable morphism and shall examine its properties. Let k be a field. For a k-algebra A, we shall prove in Theorem 2.9 that $k \to A$ is a universally incomparable morphism if and only if A is integral over k, and also if and only if $k[X] \to A[X]$ is an incomparable morphism. We shall also give in Theorem 2.11 and in Theorem 2.12 some necessary and sufficient conditions for a morphism f: $A \to B$ to be a universally incomparable one. Moreover, in Proposition 2.17, we shall show that if a morphism f of finite type is incomparable, then f is a universally incomparable morphism.

In § 3, we shall discuss incomparability for some special ring extensions. In Corollary 3.2, we shall give some necessary and sufficient conditions for a morphism: $A \to A[X]/I$ to be an incomparable one, where I is an ideal of $A[X]$. In Corollary 3.6, we shall also give two necessary and sufficient conditions for incomparability to hold for $A \to \bigotimes_{i=1}^{n} A[X]/I_i$, where I_i is an ideal of $A[X]$ for each i. In Proposition 3.11, we shall show that $A \to A[\alpha]$ is an incomparable morphism for each $\alpha \in \Omega$, where A is a Prüfer domain and Ω is the algebraic closure of the quotient field of A.

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Notation and terminology

Let A be a ring. We let Spec A, Max A and Min A stand for the set of all prime ideals of A, that of all maximal ideals of A and that of all minimal prime ideals of A respectively. For $P \in \operatorname{Spec} A$, we denote by $\kappa(P)$ the quotient field of A/P. Let $f: A \to B$ be a morphism. For an ideal J of B, we understand that $J \cap A$ means $f^{-1}(J)$ and we say that J *lies over* the ideal $J \cap A$ in B and that $J \cap A$ is *the contraction* of J into A. Moreover, we define three properties that a morphism: $A \to B$ might satisfy (cf. [3]).

(LO) For any $P \in \operatorname{Spec} A$ there exists a prime ideal $Q \in \operatorname{Spec} B$ with $Q \cap A = P$.

(GU) Given prime ideals $P \subset P_0$ in A and $Q \in \operatorname{Spec} B$ with $Q \cap A = P$, there exists a prime ideal $Q_0 \in \operatorname{Spec} B$ satisfying $Q \subset Q_0$ and $Q_0 \cap A = P_0$.

(GD) The same with $\subset$ replaced by $\supset$.

For a ring A, we denote the Krull dimension of A by $\dim A$. Moreover, we put $\dim A = 0$ even if $A = 0$.

§1. Basic properties of an incomparable morphism

Let $f: A \to B$ be a morphism. We say that $f: A \to B$ is *an incomparable morphism* if two different prime ideals of B with the same contraction into A can not be comparable. Then it follows easily from the definition that f is an incomparable morphism if and only if $\dim(B \otimes_A \kappa(P)) = 0$ for each $P \in \operatorname{Spec} A$. In this section, we examine basic properties of an incomparable morphism. Although the following Proposition 1.1, 1.2 and Corollary 1.3 can be proved easily, these are very useful.

PROPOSITION 1.1. *Let $f: A \to B$ be a morphism. Then we have the following statements.*

(1) *If f is integral, then f is an incomparable morphism.*

(2) *If f is surjective, then f is an incomparable morphism.*

(3) *If f is a localization, then f is an incomparable morphism.*

PROPOSITION 1.2. *Let $f: A \to B$ and $g: B \to C$ be two morphisms. Then we have the following statements.*

(1) *If both f and g are incomparable morphisms, then so is $g \circ f$.*

(2) *If $g \circ f$ is an incomparable morphism, then so is g.*

(3) *Assume that $g \circ f$ is an incomparable morphism. If g satisfies* GU *or if g satisfies* GD *and* LO, *then f is an incomparable morphism.*

COROLLARY 1.3. *Let $f: A \to B$ be an incomparable morphism. Then we* *have the following statements.*

(1) *If J is an ideal of B with $J \cap A = I$, then $A/I \to B/J$ is an incomparable morphism.*

(2) *If S and T are two multiplicatively closed subsets of A and B respectively with $f(S) \subset T$, then $A_S \to B_T$ is an incomparable morphism.*

We now give characterizations of an incomparable morphism which follow immediately from the above results.

PROPOSITION 1.4. *Let $f: A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *f is an incomparable morphism.*

(2) *For each $M \in \operatorname{Max} A$, $f_M: A_M \to B_M$ is an incomparable morphism.*

(3) *For each $P \in \operatorname{Spec} A$, $f_P: A_P \to B_P$ is an incomparable morphism.*

(4) *For each $Q \in \operatorname{Max} B$ with $Q \cap A = P$, $A_P \to B_Q$ is an incomparable morphism.*

(5) *For each $Q \in \operatorname{Min} B$ with $Q \cap A = P$, $A/P \to B/Q$ is an incomparable morphism.*

As for the change of rings, we have the following

PROPOSITION 1.5. *Let $f: A \to B$ and $g: A \to C$ be two morphisms. Then the following statements hold.*

(1) *If f is an incomparable morphism and the contraction map: $\operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} B$ is injective, then $C \to B \otimes_A C$ is an incomparable morphism.*

(2) *If f is an incomparable morphism and if g is surjective or a localization, then $C \to B \otimes_A C$ is an incomparable morphism.*

(3) *If $C \to B \otimes_A C$ is an incomparable morphism and g satisfies* LO, *then f is an incomparable morphism.*

PROOF. The assertion (1) is obvious.

(2) If g is surjective (resp. a localization), then $B \to B \otimes_A C$ is surjective (resp. a localization). Therefore, (2) follows immediately from (1).

(3) Let $P \in \operatorname{Spec} A$. Since g satisfies LO, there exists a prime ideal $Q \in \operatorname{Spec} C$ such that $Q \cap A = P$. It follows from Proposition 5 of (1.3.3) in [1] that $B \otimes_A \kappa(P) \to B \otimes_A \kappa(Q)$ is faithfully flat, and hence $B \otimes_A \kappa(P) \to B \otimes_A \kappa(Q)$ satisfies GD and LO. Since $\dim(B \otimes_A \kappa(Q)) = 0$ by the assumption, $\dim(B \otimes_A \kappa(P)) = 0$, which implies that f is an incomparable morphism.

In the next proposition, we give a characterization of incomparability in the category of A-algebras.

PROPOSITION 1.6. *Let A be a ring, and B, C be two A-algebras. Let*

$f: B \to C$ be a morphism of A-algebras. Then f is an incomparable morphism if and only if $B \otimes_A \kappa(P) \to C \otimes_A \kappa(P)$ is an incomparable morphism for each $P \in \operatorname{Spec} A$.

In particular, $A \to B$ is an incomparable morphism if and only if $\kappa(P) \to B \otimes_A \kappa(P)$ is an incomparable morphism for each $P \in \operatorname{Spec} A$.

PROOF. Assume that $B \otimes_A \kappa(P) \to C \otimes_A \kappa(P)$ is an incomparable morphism for each $P \in \operatorname{Spec} A$. Let $Q_1, Q_2 \in \operatorname{Spec} C$ with $Q_1 \subset Q_2$ and $Q_1 \cap B = Q_2 \cap B$. We put $Q_1 \cap A = Q_2 \cap A = P$, and denote $B \otimes_A \kappa(P)$ and $C \otimes_A \kappa(P)$ by $\bar{B}$ and $\bar{C}$ respectively. Since $Q_1\bar{C}, Q_2\bar{C} \in \operatorname{Spec} \bar{C}$ and $Q_1\bar{C} \cap \bar{B} = Q_2\bar{C} \cap \bar{B}$, we have $Q_1\bar{C} = Q_2\bar{C}$ by the assumption. Thus, $Q_1 = Q_2$. This implies that f is an incomparable morphism.

The converse follows immediately from Corollary 1.3.

Here we give some properties of an incomparable morphism of finite type.

PROPOSITION 1.7. *Let A be a finitely generated k-algebra with k a field. Then the following statements are equivalent.*

(1) *$k \to A$ is an incomparable morphism.*

(2) *A is integral over k.*

(3) *$\operatorname{Spec} A$ is a finite set.*

PROOF. (2)⇒(1). It is well known.

(1)⇒(3). We can readily see that $\dim A = 0$, and hence $\operatorname{Spec} A$ is a finite set since A is a Noetherian ring.

(3)⇒(2). By virtue of Theorem 147 in [3], any prime ideal of A is maximal. Let $A = k[\alpha_1, \alpha_2, \ldots, \alpha_n]$ and $M \in \operatorname{Spec} A$. Put $\beta_i = \alpha_i$ modulo M. Then $A/M = k[\beta_1, \beta_2, \ldots, \beta_n]$ is a field. Therefore, $\beta_1, \beta_2, \ldots, \beta_n$ are all integral over k from Theorem 23 in [3]. On the other hand, $\operatorname{Spec} A$ is a finite set. Thus, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all integral over k. This completes the proof.

COROLLARY 1.8 (cf. (6.11.5) in [2]). *Let $f: A \to B$ be a morphism of finite type. Then the following statements are equivalent.*

(1) *f is an incomparable morphism.*

(2) *For each $P \in \operatorname{Spec} A$, $B \otimes_A \kappa(P)$ is a finite dimensional vector space over $\kappa(P)$.*

(3) *For each $P \in \operatorname{Spec} A$, $\operatorname{Spec}(B \otimes_A \kappa(P))$ is a finite set.*

PROOF. This corollary follows easily from Proposition 1.6 and Proposition 1.7.

REMARK 1.9. The condition that $\operatorname{Spec}(B \otimes_A \kappa(M))$ is a finite set for each $M \in \operatorname{Max} A$ does not necessarily imply that $A \to B$ is an incomparable morphism;

in fact the morphism: $\boldsymbol{Z}\to\boldsymbol{Q}[X]$, where $\boldsymbol{Z}$ is *the integers* and $\boldsymbol{Q}$ is *the rational number field*, is such an example.

§2. Universally incomparable morphisms

Let $f: A\to B$ be a morphism. We say that $f: A\to B$ is *a universally incomparable morphism* if for each morphism $A\to C$, $C\to B\otimes_A C$ is an incomparable morphism. If f is a universally incomparable morphism, then f is obviously an incomparable morphism. In this section, we examine some properties of a universally incomparable morphism and give some characterizations.

Throughout this section we shall denote by X an indeterminate. We begin with the following

PROPOSITION 2.1. *Let $f: A\to B$ and $g: B\to C$ be two morphisms. Then we have the following statements.*

(1) *If both f and g are universally incomparable morphisms, then so is $g\circ f$.*

(2) *If $g\circ f$ is a universally incomparable morphism, then so is g.*

PROOF. These assertions follow immediately from definitions and Proposition 1.2.

As for the change of rings, we have the following

PROPOSITION 2.2. *Let $f: A\to B$ and $g: A\to C$ be two morphisms. Then we have the following statements.*

(1) *If f is a universally incomparable morphism and g is an incomparable morphism, then $A\to B\otimes_A C$ is an incomparable morphism.*

(2) *If both f and g are universally incomparable morphisms, then so is $A\to B\otimes_A C$.*

(3) *If $C\to B\otimes_A C$ is a universally incomparable morphism and g satisfies* LO, *then f is a universally incomparable morphism.*

PROOF. The assertion (1) follows immediately from definitions and (1) of Proposition 1.2.

(2) Let $A\to D$ be a morphism. Since g is a universally incomparable morphism, $D\to D\otimes_A C$ is an incomparable morphism. Since f is a universally incomparable morphism, $D\otimes_A C\to(D\otimes_A C)\otimes_A B$ is an incomparable morphism. Therefore, $D\to D\otimes_A(B\otimes_A C)$ is an incomparable morphism by (1) of Proposition 1.2. Thus, $A\to B\otimes_A C$ is a universally incomparable morphism.

(3) Let $A\to D$ be a morphism, and let $C\to C\otimes_A D$ be the change of rings for it. Then our assumption means that $C\otimes_A D\to(C\otimes_A D)\otimes_C(B\otimes_A C)$ is an incomparable morphism. Thus, $C\otimes_A D\to(C\otimes_A D)\otimes_D(B\otimes_A D)$ is an incomparable

morphism. Since g satisfies LO, $D \to C \otimes_A D$ satisfies LO. Therefore, $D \to B \otimes_A D$ is an incomparable morphism from (3) of Proposition 1.5. Thus, f is a universally incomparable morphism.

REMARK 2.3. With the notation of Proposition 2.2, assume that both f and g are incomparable morphisms. In this case, $A \to B \otimes_A C$ is not necessarily an incomparable morphism (cf. (1) in Proposition 2.2). For example, let X and Y be two indeterminates and k be a field. Then both $k \subset k(X)$ and $k \subset k(Y)$ are incomparable morphisms, but $k \to k(X) \otimes_k k(Y)$ is not an incomparable morphism.

For k-algebras with k a field, we give a characterization of a universally incomparable morphism.

PROPOSITION 2.4. *Let A be a k-algebra with k a field. Then $k \to A$ is a universally incomparable morphism if and only if for each field extension L of k, $L \to L \otimes_k A$ is an incomparable morphism.*

PROOF. We have only to prove the 'if' part. Let B be a k-algebra and $P \in \operatorname{Spec} B$. From the assumption, $\kappa(P) \to A \otimes_k \kappa(P)$ is an incomparable morphism, hence $\dim (A \otimes_k \kappa(P)) = 0$. Since $(B \otimes_k A) \otimes_B \kappa(P) = A \otimes_k \kappa(P)$, $B \to B \otimes_k A$ is an incomparable morphism by Proposition 1.6. This implies that $k \to A$ is a universally incomparable morphism.

To characterize universally incomparable morphisms, we need the following lemmas.

LEMMA 2.5. *For a field extension $F \to K$, the following statements are equivalent.*

(1) *$F[X] \subset K[X]$ is an incomparable morphism.*

(2) *K is algebraic over F.*

(3) *$K[X]$ is integral over $F[X]$.*

PROOF. (2)$\Rightarrow$(3)$\Rightarrow$(1). These implications are well known.
(1)$\Rightarrow$(2). If K is not algebraic over F, then there exists an element α of K which is not algebraic over F. Since $(X-\alpha)K[X] \cap F[X] = 0$, $F[X] \subset K[X]$ is not an incomparable morphism. This is a contradiction. Thus, K is algebraic over F.

COROLLARY 2.6 (cf. THEOREM 2 in [4]). *Let $A \subset B$ be integral domains. Then there is no non-zero prime ideal of $B[X]$ lying over 0 in $A[X]$ if and only if the quotient field of B is algebraic over that of A.*

PROOF. Let F and K be the quotient fields of A and B respectively. Assume that there is no non-zero prime ideal of $B[X]$ lying over 0 in $A[X]$. Then this implies that $F[X]\subset K[X]$ is an incomparable morphism. By Lemma 2.5, K is algebraic over F.

Conversely, assume that K is algebraic over F. Let $Q\in \operatorname{Spec} B[X]$. Suppose that $Q\cap A[X]=0$. We put $Q\cap B=P$. Assume that $P\neq 0$. Since $P\cap A=0$, there exists an element α of P such that $\alpha\notin A$. On the other hand, K is algebraic over F. Therefore, there are elements $a_0, a_1,\dots, a_n$ of A such that $\sum_{i=0}^{n} a_i\alpha^i=0$ and $a_0a_n\neq 0$. Then $a_0=-\sum_{i=1}^{n} a_i\alpha^i\in \alpha B\cap A\subset P\cap A=0$. This is a contradiction. Thus, $P=0$, and hence we have $QK[X]\in \operatorname{Spec} K[X]$. Since $F[X]\subset K[X]$ is an incomparable morphism by Lemma 2.5, we have $QK[X]=0$. Thus, $Q=0$, which completes the proof.

Lemma 2.7. *Let A be an integral domain and B be a ring containing A. Then there exists a prime ideal $P\in \operatorname{Min} B$ such that $P\cap A=0$.*

Proof. Let $S=A-\{0\}$. Then $A_S\subset B_S$. Since $B_S\neq 0$ and A_S is a field, there exists a prime ideal $Q\in \operatorname{Spec} B_S$ with $Q\cap A_S=0$. The assertion follows immediately from the above result.

Corollary 2.8. *Let A be a k-algebra with k a field. If A/P is integral over k for each $P\in \operatorname{Min} A$, then A is integral over k.*

Proof. Assume that there exists an element t of A which is transcendental over k. Since $k[t]$ is an integral domain, there exists a prime ideal $P\in \operatorname{Min} A$ with $P\cap k[t]=0$ from Lemma 2.7, hence $k\subset k[t]\subset A/P$. On the other hand, A/P is integral over k. This is a contradiction, which settles the proof.

With these preparations, we give two more characterizations of a universally incomparable morphism of k-algebras, where k is a field.

Theorem 2.9. *Let A be a k-algebra with k a field. Then the following statements are equivalent.*

(1) *$k\to A$ is a universally incomparable morphism.*
(2) *A is integral over k.*
(3) *$k[X]\to A[X]$ is an incomparable morphism.*

Proof. The implications (2)⇒(1)⇒(3) are obvious.
(3)⇒(2). Let $P\in \operatorname{Min} A$. By (5) of Proposition 1.4, $k[X]\to A/P[X]$ is an incomparable morphism, and hence $k[X]\to \kappa(P)[X]$ is an incomparable morphism by (3) of Proposition 1.1 and (1) of Proposition 1.2. Therefore, $k\to A/P$ is algebraic by Lemma 2.5. Thus, A is integral over k by Corollary 2.8.

Corollary 2.10. *Let D be an integral domain which contains a field k. Then $k\to D$ is a universally incomparable morphism if and only if D is a field algebraic over k.*

PROOF. The assertion follows easily from Theorem 2.9.

We will now proceed to the general case.

THEOREM 2.11. *Let $A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *$A \to B$ is a universally incomparable morphism.*

(2) *For each morphism $A \to C$, $\dim(B \otimes_A C) \leqq \dim C$.*

(3) *For each morphism $A \to K$ with K a field, $K \to B \otimes_A K$ is an incomparable morphism.*

(4) *For each $P \in \operatorname{Spec} A$, $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism.*

PROOF. The implications (1)⇒(2)⇒(3) are obvious.

(3)⇒(4). Let $P \in \operatorname{Spec} A$ and L be a field extension of $\kappa(P)$. By the assumption, $L \to L \otimes_A B$ is an incomparable morphism. On the other hand, $L \otimes_A B = L \otimes_{\kappa(P)} (B \otimes_A \kappa(P))$. Therefore, $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism by Proposition 2.4.

(4)⇒(1). Let $A \to C$ be a morphism. Let $Q \in \operatorname{Spec} C$ and put $Q \cap A = P$. Since $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism, $\kappa(Q) \to (B \otimes_A \kappa(P)) \otimes_{\kappa(P)} \kappa(Q)$ is an incomparable morphism. That is, $\kappa(Q) \to B \otimes_A \kappa(Q)$ is an incomparable morphism. Therefore, $\dim((B \otimes_A C) \otimes_C \kappa(Q)) = \dim(B \otimes_A \kappa(Q)) = 0$. Thus, $C \to B \otimes_A C$ is an incomparable morphism, and hence $A \to B$ is a universally incomparable morphism.

The following theorem gives two further necessary and sufficient conditions for $A \to B$ to be a universally incomparable morphism.

THEOREM 2.12. *Let $A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *$A \to B$ is a universally incomparable morphism.*

(2) *For each $Q \in \operatorname{Spec} B$ with $Q \cap A = P$, $\kappa(Q)$ is algebraic over $\kappa(P)$.*

(3) *$A[X] \to B[X]$ is an incomparable morphism.*

PROOF. The implication (1)⇒(3) is obvious.

(3)⇒(2). Let $Q \in \operatorname{Spec} B$ and put $Q \cap A = P$. Since $QB[X] \cap A[X] = PA[X]$, $\kappa(P)[X] \to \kappa(Q)[X]$ is an incomparable morphism by Corollary 1.3. Therefore, $\kappa(Q)$ is algebraic over $\kappa(P)$ by Lemma 2.5.

(2)⇒(1). Let $P \in \operatorname{Spec} A$. We shall prove that $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism. To do this we may assume that $B \otimes_A \kappa(P) \neq 0$. Then there exists a prime ideal $Q \in \operatorname{Spec} B$ such that $Q \cap A = P$. The assumption of (2) means that $(B \otimes_A \kappa(P))/M$ is algebraic over $\kappa(P)$ for each $M \in \operatorname{Spec}(B \otimes_A \kappa(P))$, and hence $B \otimes_A \kappa(P)$ is integral over $\kappa(P)$ by Corollary 2.8. Therefore, $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism by Theorem 2.9. Thus, $A \to B$

is a universally incomparable morphism by Theorem 2.11.

REMARK 2.13. Let $f: A \to B$ be a morphism. Then it is obvious from Theorem 2.12 that the condition (2) in Theorem 2.12 implies that f is an incomparable morphism. This fact can also be proved directly by Corollary 2.6 in the following way. Assume that $\kappa(Q)$ is algebraic over $\kappa(P)$ for any $Q \in \operatorname{Spec} B$ with $Q \cap A = P$. Then there is no non-zero prime ideal of B/Q lying over 0 in A/P by Corollary 2.6. This implies that f is an incomparable morphism.

On the other hand, it is obvious that an incomparable morphism does not necessarily imply the condition (2) in Theorem 2.12.

COROLLARY 2.14. *Let $f: A \to B$ and $g: B \to C$ be two morphisms. Assume that $g \circ f$ is a universally incomparable morphism. If $B[X] \to C[X]$ satisfies* GU *or if $B[X] \to C[X]$ satisfies* GD *and* LO, *then $A \to B$ is a universally incomparable morphism.*

PROOF. This corollary follows immediately from (3) of Proposition 1.2 and Theorem 2.12.

COROLLARY 2.15. *Let $f: A \to B$ be a morphism. Then f is a universally incomparable morphism if and only if so is $A[X] \to B[X]$.*

PROOF. The assertion follows easily from Theorem 2.12.

COROLLARY 2.16. *Let $X_1, X_2, \ldots, X_n$ be indeterminates. Let $f: A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *$A[X_1] \to B[X_1]$ is an incomparable morphism.*

(2) *$A[X_1, X_2, \ldots, X_n] \to B[X_1, X_2, \ldots, X_n]$ is an incomparable morphism for some $n \geqq 1$.*

(3) *$A[X_1, X_2, \ldots, X_n] \to B[X_1, X_2, \ldots, X_n]$ is an incomparable morphism for all $n \geqq 0$. Here, $A[X_1, X_2, \ldots, X_n] = A$, if $n = 0$.*

PROOF. The assertion follows immediately from Corollary 2.15.

Finally, we prove that the notion of incomparability coincides with that of universal incomparability for any morphism of finite type (cf. [2]).

PROPOSITION 2.17. *Let $f: A \to B$ be a morphism of finite type. Then f is an incomparable morphism if and only if f is a universally incomparable morphism.*

PROOF. It is sufficient to prove the 'only if' part. Assume that f is an incomparable morphism. Let $P \in \operatorname{Spec} A$. Then $B \otimes_A \kappa(P)$ is a finitely generated $\kappa(P)$-algebra, and hence by Proposition 1.6, $\kappa(P) \to B \otimes_A \kappa(P)$ is an incomparable morphism. Therefore, $B \otimes_A \kappa(P)$ is integral over $\kappa(P)$ by Proposition 1.7, and hence $\kappa(P)\to B\otimes_A \kappa(P)$ is a universally incomparable morphism by Theorem 2.9. Thus, $A\to B$ is a universally incomparable morphism by Theorem 2.11.

COROLLARY 2.18. *Let $f: A\to B$ and $g: A\to C$ be two morphisms. If both f and g are incomparable morphisms and if f is of finite type, then $A\to B\otimes_A C$ is an incomparable morphism.*

PROOF. The assertion follows immediately from (1) of Proposition 2.2 and Proposition 2.17.

REMARK 2.19. In general, an incomparable morphism is not necessarily a universally incomparable morphism (cf. Corollary 2.10).

§3. Incomparability for certain ring extensions

In this section, we give some necessary and sufficient conditions for a morphism: $A\to \bigotimes_{i=1}^{n} A[X]/I_i$ to be an incomparable one, where I_i is an ideal of $A[X]$ for each i. We also show a result on incomparability for simple extensions of Prüfer domains.

Throughout this section, A will be a ring and $X, X_1, X_2,\dots, X_n$ will be indeterminates. Let I be an ideal of $A[X_1, X_2,\dots, X_n]$. We denote by $c(I)$ the ideal generated by all coefficients of all polynomials in I and we call it *the content* of I. In particular, if $I=(f)$, then $c(I)$ will be denoted by $c(f)$.

THEOREM 3.1. *Let I be an ideal of $A[X]$ and put $B=A[X]/I$. Let $P\in$ Spec A. Then Spec$(B\otimes_A \kappa(P))$ is a finite set if and only if $c(I)\not\subset P$.*

PROOF. There is a one-to-one correspondence between prime ideals of B lying over P and prime ideals of $\kappa(P)[X]$ containing $\bar{I}$, where $\bar{I}$ is the ideal generated by the homomorphic image of I in $\kappa(P)[X]$. Assume that $c(I)\subset P$. Then $I\subset PA[X]$, hence $\bar{I}=0$. Thus, Spec$(B\otimes_A \kappa(P))$ is an infinite set.

Conversely, assume that $c(I)\not\subset P$. Then $\bar{I}\neq 0$. Therefore, since $\kappa(P)[X]$ is a 1-dimensional Noetherian domain, Spec$(B\otimes_A \kappa(P))$ is a finite set.

COROLLARY 3.2. *With the notation of Theorem 3.1, the following statements are equivalent.*

(1) *$A\to B$ is an incomparable morphism.*

(2) $c(I)=A$.

(3) *For each $M\in$ Max A,* Spec$(B\otimes_A \kappa(M))$ *is a finite set.*

PROOF. The assertion follows easily from Theorem 3.1 and Corollary 1.8.

COROLLARY 3.3. *Let $f(X)\in A[X]$. Then $A\to A[X]/(f(X))$ is an incomparable morphism if and only if $c(f)=A$.*

REMARK 3.4. Let I be an ideal of $A[X_1, X_2,\dots, X_n]$ and put $B=A[X_1, X_2,\dots, X_n]/I$. If $A\to B$ is an incomparable morphism, we have obviously $c(I)=A$. Again, if $A\to B$ is an incomparable morphism, then by Corollary 1.8, $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set for each $M\in \mathrm{Max}\, A$. However, the converse of each statement is false as is seen in the following example.

EXAMPLE 3.5. Let (A, M) be a local domain which is not a field. Let a be a non-zero element of M, and put $B=A[X, Y]/(aY-1)$. We have obviously $c(aY-1)=A$. Since $MB=B$, $\mathrm{Spec}(B\otimes_A \kappa(M))$ is an empty set. On the other hand, $A\to B$ is not an incomparable morphism obviously.

COROLLARY 3.6. *Let* $I_1, I_2,\dots, I_n$ *be ideals of* $A[X]$ *and put* $B=\bigotimes_{i=1}^{n} A[X]/I_i$. *Then the following statements are equivalent.*

(1) $A\to B$ *is an incomparable morphism.*

(2) *Let* $Q\in \mathrm{Spec}\, B$. *If* $Q\cap A[X]$ *contains all* I_i, *then* $c(I_i)\not\subset Q\cap A$ *for each* i.

(3) *Let* $P\in \mathrm{Spec}\, A$. *If there exists a prime ideal of* B *lying over* P, *then* $\mathrm{Spec}(A[X]/I_i\otimes_A\kappa(P))$ *is a finite set for each* i.

PROOF. The equivalence between (2) and (3) follows immediately from Theorem 3.1.

Let $P\in \mathrm{Spec}\, A$. We put $B_i=A[X]/I_i\otimes_A \kappa(P)$.

(1)⇒(3). Let $P\in$ Spec A and assume that there exists a prime ideal of B lying over P. Since $B\otimes_A \kappa(P)\neq 0$, $\bigotimes_{j\neq i\ \kappa(P)} B_j\neq 0$ for each i. That is, $\kappa(P)\to \bigotimes_{j\neq i\ \kappa(P)} B_j$ satisfies LO. On the other hand, since $\kappa(P)\to \bigotimes_{j=1\ \kappa(P)}^{n} B_j$ is an incomparable morphism by Proposition 1.6, $\bigotimes_{j\neq i\ \kappa(P)} B_j\to \bigotimes_{j=1\ \kappa(P)}^{n} B_j$ is an incomparable morphism by (2) of Proposition 1.2. Therefore, $\kappa(P)\to B_i$ is an incomparable morphism by (3) of Proposition 1.5. Thus, $\mathrm{Spec}\, B_i$ is a finite set by Proposition 1.7.

(3)⇒(1). Let $P\in \mathrm{Spec}\, A$ and assume that $\mathrm{Spec}(B\otimes_A \kappa(P))\neq \phi$. By the assumption and Proposition 1.7, B_i is integral over $\kappa(P)$, and hence $B\otimes_A \kappa(P)$ is integral over $\kappa(P)$. Therefore, $\kappa(P)\to B\otimes_A \kappa(P)$ is a universally incomparable morphism by Theorem 2.9. Thus, $A\to B$ is a universally incomparable morphism by Theorem 2.11.

REMARK 3.7 (cf. (6.11.5) in [2]). Let $A\to B_i$ be a morphism of finite type for $i=1, 2,\dots, n$. If every $A\to B_i$ is an incomparable morphism, then $A\to \bigotimes_{i=1}^{n} B_i$ is an incomparable morphism by Corollary 2.18. In particular, if $f_1(X), f_2(X),\dots, f_n(X)$ are polynomials of $A[X]$ with $c(f_i)=A$ for all i, then $A\to \bigotimes_{i=1}^{n} A[X]/(f_i(X))$ is an incomparable morphism. However, the converse is not true as is seen in the following example.

EXAMPLE 3.8. Let k be an algebraically closed field and let X, Y, Z be three indeterminates. Let $A=k[X]$ and $B=A[Y]/(XY-1)\otimes_A A[Z]/(XZ)$. By Corollary 3.2, $A\to A[Y]/(XY-1)$ is an incomparable morphism, but $A\to A[Z]/(XZ)$ is not an incomparable morphism. On the other hand, $B=k[X, Y, Z]/(XY-1, XZ)$. Since k is algebraically closed, we can readily see that $A\to B$ is an incomparable morphism.

Let $A\to B$ be a morphism. We consider a condition (*) that $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set for each $M\in \mathrm{Max}\,A$. In Remark 3.4, we pointed out the following fact: (*) does not necessarily imply that $A\to B$ is an incomparable morphism. In the following proposition, we give a condition for (*) to imply that $A\to B$ is an incomparable morphism.

PROPOSITION 3.9. *With the notation of Corollary* 3.6, *assume that for each $P\in \mathrm{Spec}\,A$ which is the contraction of a prime ideal of B into A, there exists a maximal ideal $M\in \mathrm{Max}\,A$ containing P such that M is the contraction of a prime ideal of B into A. Then the following statements are equivalent.*

(1) *$A\to B$ is an incomparable morphism.*

(2) *Let $Q\in \mathrm{Spec}\,B$. If $Q\cap A[X]$ contains all I_i, then $c(I_i)\not\subset Q\cap A$ for each i.*

(3) *For each $M\in \mathrm{Max}\,A$, $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set.*

PROOF. (1)$\Leftrightarrow$(2) and (1)$\Rightarrow$(3). These implications follow from Corollary 3.6.
(3)$\Rightarrow$(2). Let $P\in \mathrm{Spec}\,A$ and assume that $\mathrm{Spec}(B\otimes_A \kappa(P))\neq\phi$. Then there exists a maximal ideal $M\in \mathrm{Max}\,A$ such that $P\subset M$ and $\mathrm{Spec}(B\otimes_A \kappa(M))\neq\phi$. By the assumption (3), $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set. In the same manner as (1)$\Rightarrow$(3) in Corollary 3.6, $\mathrm{Spec}(A[X]/I_i\otimes_A \kappa(M))$ is a finite set for each i. By Theorem 3.1, $c(I_i)\not\subset M$, hence $c(I_i)\not\subset P$. This completes the proof.

REMARK 3.10. If $A\to B$ satisfies LO, then the assumption of Proposition 3.9 is satisfied.

PROPOSITION 3.11. *Let A be a Prüfer domain and let Ω be the algebraic closure of the quotient field F of A. Then for each $\alpha\in\Omega$, $A\to A[\alpha]$ is an incomparable morphism.*

PROOF. Let $P\in \mathrm{Spec}\,A$. Since A_P is a valuation ring, there is a polynomial $f(X)$ in $A_P[X]$ such that $f(\alpha)=0$, $c(f)=A_P$ and $f(X)$ is irreducible over F. By Theorem A in [5], $f(X)A_P[X]$ is a prime ideal, hence $A_P[\alpha]=A_P[X]/(f(X))$. Therefore, $A_P\to A_P[\alpha]$ is an incomparable morphism by Corollary 3.3. Thus, $A\to A[\alpha]$ is an incomparable morphism by Proposition 1.4.

COROLLARY 3.12. *With the notation of Proposition* 3.11, *let $\alpha_1, \alpha_2,\dots, \alpha_n$*

$\in \Omega$. *Then* $A \to A[\alpha_1, \alpha_2, \ldots, \alpha_n]$ *is an incomparable morphism.*

PROOF. By Proposition 3.11, $A \to A[\alpha_i]$ is an incomparable morphism for each i, and hence $A \to \bigotimes_{i=1}^{n} {}_A A[\alpha_i]$ is an incomparable morphism by Remark 3.7. Thus, $A \to A[\alpha_1, \alpha_2, \ldots, \alpha_n]$ is an incomparable morphism by (2) of Proposition 1.1 and (1) of Proposition 1.2.

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Faculty of Education,
Miyazaki University