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Sufficient Conditions for Duality Theorems in Infinite Linear Programming Problems

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§1. Introduction with problem setting

There are many sufficient conditions for duality theorems in infinite linear programming problems. In this paper, we shall investigate sufficient conditions for a general duality theorem due to K. S. Kretschmer by the aid of the closedness of the sum of two convex cones and find some relations among well-known sufficient conditions in [3]–[11].

More precisely, let X and Y be real linear spaces which are in duality with respect to the bilinear functional $((\ ,\ ))_1$ and let Z and W be real linear spaces which are in duality with respect to the bilinear functional $((\ ,\ ))_2$. Throughout this paper, we always assume that each space of the paired spaces is assigned the weak topology which is compatible with the duality, so that every topological notion is used without any specifying adjective unless otherwise stated. Let A be a continuous linear transformation from X into Z, P and Q be closed convex cones in X and Z respectively and $y_0 \in Y$ and $z_0 \in Z$ be fixed elements. Denote by A^* the adjoint of A and by P^+ and Q^+ the dual cones of P and Q respectively. Let us consider the following infinite linear programming problem (1.1) and its dual problem (1.2):

(1.1) Find $M = \inf\{((x, y_0))_1;\ x \in S\}$,

where $S = \{x \in P;\ Ax - z_0 \in Q\}$.

(1.2) Find $M^* = \sup\{((z_0, w))_2;\ w \in S^*\}$,

where $S^* = \{w \in Q^+;\ y_0 - A^*w \in P^+\}$.
Here we use the convention that the infimum of a real function on the empty set ϕ is equal to ∞. We say that problem (1.1) has an optimal solution if there exists an $x \in S$ such that $M = ((x, y_0))_1$. A result which assures the equality $M = M^*$ is called a duality theorem.

Let R be the set of real numbers and R^+ be the set of non-negative real numbers. When R is considered as a topological space, the topology is the usual one. Product spaces $Z \times R$ and $W \times R$ are in duality with respect to the bilinear functional $[\ ,\ ]$ defined by $[(z, r), (w, s)] = ((z, w))_2 + rs$ for every $(z, r) \in Z \times R$ and $(w, s) \in W \times R$.

Let us define the following two sets:

$$H = \{(Ax - z, r + ((x, y_0))_1); x \in P, z \in Q, r \in R^+\},$$

$$H^* = \{(A^*w + y, r - ((z_0, w))_2); w \in Q^+, y \in P^+, r \in R^+\}.$$

We have by [7; Theorem 3] and [11; Theorem 8]

THEOREM 1.1. *Assume that H is closed. If M or M^* is finite, then $M = M^*$ holds and problem* (1.1) *has an optimal solution.*

As a dual statement of this theorem, we have

COROLLARY. *Assume that H^* is closed. If M or M^* is finite, then $M = M^*$ holds and problem* (1.2) *has an optimal solution.*

In this paper, we shall be concerned with the following problems which seem to be very important in view of several applications of the above duality theorems.
(H) When is the set H closed?
(H*) When is the set H^* closed?
By the duality of our problems, we have only to study problem (H). This problem was studied so far in [4], [5], [7], [8] and [11]. To study problem (H), we shall prepare in § 2 several main results concerning the closedness of the sum of two closed convex cones by using the idea due to [2], [4] and [5]. Several answers to problem (H) will be given in § 3 by the aid of the results in § 2 and the following representations of the set H:

(I) $H = F_1 + G_1$, where $F_1 = \{(Ax - z, ((x, y_0))_1); x \in P, z \in Q\}$ and $G_1 = \{0\} \times R^+$.

(II) $H = F_2 + G_2$, where $F_2 = \{(Ax, ((x, y_0))_1); x \in P\}$ and $G_2 = (-Q) \times R^+$.

(III) $H = F_3 + G_3$, where $F_3 = \{(Ax, r + ((x, y_0))_1); x \in P, r \in R^+\}$ and $G_3 = (-Q) \times \{0\}$.

W. Krabs [8] showed that Theorem 1.1 still holds even if the set H is replaced by the set F_1 in representation (I). Some superconsistency conditions will be discussed in § 4 in connection with condition (F) in § 3.

§ 2. Main results

Let E and F be real linear spaces which are in duality with respect to the bilinear functional $((\ ,\ ))$, let U and V be real linear spaces which are in duality with respect to the bilinear functional $<\ ,\ >$ and let T be a continuous linear transformation from E into U. Denote by T^* the adjoint of T, i.e., $<Tx, v>

$=((x, T^*v))$ for all $x \in E$ and $v \in V$.

Let C and D be convex cones in E and U respectively. We always assume that C and D contain 0 and that D is closed. First we shall study the closedness of $T(C)-D$ under some additional conditions for C and T.

We have

LEMMA 2.1. *Assume that C is locally compact. Let $\{z_\alpha\}$ be a net in $T(C)-D$ which converges to z. Then either* (a) $z \in T(C)-D$ *or* (b) $C \cap T^{-1}(D) \neq \{0\}$.

PROOF. In case $C=\{0\}$, our assertion is clear by the closedness of D. Assume that $C \neq \{0\}$. Since $z_\alpha \in T(C)-D$, there is $x_\alpha \in C$ such that $Tx_\alpha - z_\alpha \in D$. We show that there is a compact subset K of C such that $0 \notin K$ and $C=\{tx;\ x \in K$ and $t \in R^+\}$. In fact, since C is locally compact, there is an open convex neighborhood B of 0 in E for which $C \cap \bar{B}$ is compact, where $\bar{B}$ denotes the closure of B. We may take K as the intersection of $C \cap \bar{B}$ and the complement of $(1/2)B$. Thus we can find $u_\alpha \in K$ and $t_\alpha \in R^+$ such that $x_\alpha = t_\alpha u_\alpha$. First we consider the case where there exists α_0 such that $\{t_\alpha;\ \alpha \geq \alpha_0\}$ is bounded. Since K is compact, we can find a convergent subnet of $\{x_\alpha\}$. Let x be the limit. Then $x \in C$ and $Tx - z \in D$, which shows (a). Next we consider the case where there is no α_0 such that $\{t_\alpha;\ \alpha \geq \alpha_0\}$ is bounded. We may assume that $\{u_\alpha\}$ converges to $u \in K$ and $\{t_\alpha\}$ converges to ∞, by choosing subnets if necessary. Since D is a cone, $Tu_\alpha - z_\alpha/t_\alpha \in D$. It follows that $u \in C$, $u \neq 0$ and $Tu \in D$, which shows (b).

REMARK 2.1. In the above proof, we see easily that C is closed if C is locally compact.

By the above lemma, we have

THEOREM 2.1. *If C is locally compact and if $C \cap T^{-1}(D)=\{0\}$, then $T(C)-D$ is closed.*

COROLLARY 1. *If C is locally compact and if $C \cap T^{-1}(\{0\})=\{0\}$, then $T(C)$ is closed.*

COROLLARY 2. *Let C_1 and C_2 be closed convex cones in E. If $C_1 \cap C_2 =\{0\}$ and if at least one of C_1 and C_2 is locally compact, then $C_1 - C_2$ is closed.*

This corollary is a special version of Dieudonné's result [2; Théorème 1].

Next we shall study the case where $C \cap T^{-1}(D)=\{0\}$. Denote by $s(E, F)$ the Mackey topology on E which is compatible with the duality and by C^+ and D^+ the dual cones of C and D respectively, i.e., $C^+=\{y \in F;\ ((x, y)) \geq 0$ for all $x \in C\}$. For a subset B of E, denote by B^i the $s(E, F)$-interior of B.

THEOREM 2.2. *Assume that C is closed and that the $s(F, E)$-interior $(C^+)^i$ of C^+ is nonempty. Then $C \cap T^{-1}(D)=\{0\}$ if and only if $(C^+)^i \cap T^*(-D^+) \neq \phi$.*

PROOF. Assume that $(C^+)^i \cap T^*(-D^+) \neq \phi$. There exists $\bar{v} \in -D^+$ such that $\bar{y}=T^*\bar{v} \in (C^+)^i$. Let $x \in C \cap T^{-1}(D)$ and suppose that $x \neq 0$. If $\bar{y} \neq 0$, then

$$0 < ((x, \bar{y})) = <Tx, \bar{v}> \leq 0,$$

which is a contradiction. If $\bar{y}=0$, then $C^+=F$ and $C=\{0\}$, so that $C \cap T^{-1}(D) =\{0\}$. Next assume that $(C^+)^i \cap T^*(-D^+)=\phi$. Then there exists $x \in E$ such that $x \neq 0$ and $((x, -T^*v)) \leq 0 \leq ((x, y))$ for all $v \in D^+$ and $y \in C^+$ (cf. [1]; p. 71, Proposition 1). It follows that $x \in C$ and $Tx \in D$, i.e., $C \cap T^{-1}(D) \neq \{0\}$.

COROLLARY. *Let C_1 and C_2 be closed convex cones in E such that $(C_1^+)^i \neq \phi$. Then $C_1 \cap C_2=\{0\}$ if and only if $(C_1^+)^i \cap (-C_2^+) \neq \phi$.*

For a subset B of E, denote by B° the polar of B, i.e.,

$$B^\circ = \{y \in F;\ ((x, y)) \leq 1 \text{ for all } x \in B\}.$$

Let us recall the following fundamental result due to Fan [4; Theorem 1]:

PROPOSITION 2.1. *Let B be a closed subset of E. If its polar B° has a nonempty $s(F, E)$-interior, then B is locally compact.*

Noting that $C^+=-C^\circ$, we obtain another proof of the following result due to Kretschmer [7; Lemma 5] by Theorems 2.1 and 2.2 and Proposition 2.1:

PROPOSITION 2.2. *If $(C^+)^i \cap T^*(D^+) \neq \phi$, then $T(C)+D$ is closed.*

COROLLARY. *Let C_1 and C_2 be closed convex cones in E. If $(C_1^+)^i \cap C_2^+ \neq \phi$, then C_1+C_2 is closed.*

This corollary is a special version of Fan's result [4; Theorem 2].

For later use, we further prepare

LEMMA 2.2. *Let K be a compact set in E which does not contain 0. Then the cone $\mathrm{co}(K)=\{tx;\ x \in K,\ t \in R^+\}$ generated by K is locally compact.*

PROOF. Since K is compact and does not contain 0, there is a closed convex neighborhood B of 0 such that $K \cap B=\phi$. Let $x \in B \cap \mathrm{co}(K)$. Then there exist $u \in K$ and $t \in R^+$ such that $x=tu$. Since B is convex and contains 0, we have $t \leq 1$, so that $B \cap \mathrm{co}(K) \subset \tilde{K}=\{tu;\ u \in K,\ 0 \leq t \leq 1\}$. Since $\tilde{K}$ is compact, $B \cap \mathrm{co}(K)$ is also compact. It follows that $\mathrm{co}(K)$ is locally compact.

LEMMA 2.3. *Let C be locally compact. If $C \cap T^{-1}(\{0\})=\{0\}$, then $T(C)$ is locally compact.*

PROOF. We may assume that $C \neq \{0\}$. Since C is locally compact, there is a compact subset K of C such that $0 \notin K$ and $C = \mathrm{co}(K)$ (cf. the proof of Lemma 2.1). Then $T(C) = \{tz;\ z \in T(K),\ t \in R^+\}$ and $T(K)$ is compact. Since $C \cap T^{-1}(\{0\}) = \{0\}$, $T(K)$ does not contain 0. Therefore $T(C)$ is locally compact by Lemma 2.2.

§3. The closedness of the set *H*

In order to study problem (H) in §1, let us define conditions (K) and (F):

(K) $\qquad (P^+)^i \neq \phi$ and $y_0 \in A^*(Q^+) + (P^+)^i$,

where $(P^+)^i$ denotes the $s(Y, X)$-interior of P^+.

(F) $\qquad x \in P \cap A^{-1}(Q)$ and $((x, y_0))_1 \leq 0$ imply $x = 0$.

To study problem (H*), we need their dual conditions (K*) and (F*):

(K*) $\qquad Q^i \neq \phi$ and $z_0 \in A(P) - Q^i$,

where Q^i denotes the $s(Z, W)$-interior of Q.

(F*) $\qquad w \in Q^+ \cap A^{*-1}(-P^+)$ and $((z_0, w))_2 \geq 0$ imply $w = 0$.

Conditions (K) and (K*) were called superconsistency conditions in [3]. Conditions (F) and (F*) were introduced in [4].

We have

THEOREM 3.1. *Assume that P is locally compact. If condition* (F) *is fulfilled, then H is closed.*

PROOF. Let us take

$$\begin{aligned} &E = X,\ F = Y,\ U = Z \times R,\ V = W \times R, \\ &C = P,\ D = Q \times (-R^+),\ Tx = (Ax, ((x, y_0))_1) \end{aligned} \tag{3.1}$$

in §2. Then $H = T(C) - D$. Condition (F) is equivalent to $C \cap T^{-1}(D) = \{0\}$. Thus H is closed by Theorem 2.1.

Fan proved this theorem under the additional condition that Q is also locally compact [5; Theorem 7]. We obtain another proof of the following result due to Kretschmer [7; Corollary 3.1].

COROLLARY. *If condition* (K) *is fulfilled, then H is closed.*

PROOF. Since $(P^+)^i \neq \phi$, P is locally compact by Proposition 2.1. With the setting (3.1), we have $D^+=Q^+\times(-R^+)$ and $T^*(w, s)=A^*w+sy_0$ for $(w, s) \in W\times R$. Condition (K) implies $(P^+)^i \cap T^*(-D^+)\neq\phi$, so that condition (F) is fulfilled by Theorem 2.2. Thus H is closed by Theorem 3.1.

The reasoning in the proof of Theorem 3.1 is efficient to the study of the closedness of the sets F_1, F_2 and F_3 in representations (I), (II) and (III). Let us define convex cones in $Z\times R$ by

$$D_1 = Q \times \{0\},\ D_2 = \{0\} \times \{0\},\ D_3 = \{0\} \times (-R^+).$$

With the setting (3.1), we have $F_i=T(C)-D_i$ for each i. The condition $C\cap T^{-1}(D_i)=\{0\}$ is equivalent to each of the following conditions (D_i) $(i=1, 2, 3)$:

(D_1) $x\in P \cap A^{-1}(Q)$ and $((x, y_0))_1 = 0$ imply $x = 0$.

(D_2) $x\in P \cap A^{-1}(\{0\})$ and $((x, y_0))_1 = 0$ imply $x = 0$.

(D_3) $x\in P \cap A^{-1}(\{0\})$ and $((x, y_0))_1 \leq 0$ imply $x = 0$.

We have

THEOREM 3.2. *Assume that P is locally compact. If condition* (D_1) *is fulfilled, then F_1 is closed. If condition* (D_i) *is fulfilled, then F_i is locally compact for $i=2$, 3.*

PROOF. The closedness of F_1 follows from Theorem 2.1. Since $F_2=T(P)$ and $P\cap T^{-1}(\{0\})=\{0\}$, $T(P)$ is locally compact by Lemma 2.3. To prove that F_3 is locally compact, let us take

$$E = X \times R,\ F = Y\times R,\ U = Z \times R,\ V= W\times R,$$

$$C = P \times R^+,\ T_3(x, r) = (Ax, r + ((x, y_0))_1).$$

Then $F_3=T_3(P\times R^+)$ and $P\times R^+$ is locally compact. We have $(P\times R^+)\cap T_3^{-1}(\{0\}) =\{0\}$ by condition (D_3), so that $T_3(P\times R^+)$ is locally compact by Lemma 2.3.

Notice that condition (F) implies any one of conditions (D_1), (D_2) and (D_3). Any one of conditions (D_1) and (D_3) implies condition (D_2). If $P\cap A^{-1}(\{0\}) =\{0\}$, then conditions ($D_2$) and ($D_3$) are equivalent.

Theorems 3.1 and 3.2 do not hold in general if we omit the assumption that P is locally compact. This is shown by

EXAMPLE 3.1. Let $L_2[0, 1]$ be the Hilbert space of Lebesgue measurable functions on the unit interval in R which are square integrable, and $L_2^+[0, 1]$ be the subset of $L_2[0, 1]$ which consists of non-negative functions. Put

$$<u, v> = \int_0^1 u(t)v(t)dt$$

for $u, v \in L_2[0, 1]$. Let us take

$$X = Y = L_2[0, 1] \times R, \ P = L_2^+[0, 1] \times R^+,$$

$$Z = W = L_2[0, 1], \ Q = L_2^+[0, 1],$$

$$((x, y))_1 = <f, g> + rs \quad \text{for} \quad x = (f, r) \quad \text{and} \quad y = (g, s),$$

$$((z, w))_2 = <z, w>,$$

$$(Ax)(t) = \int_t^1 f(s)ds + r \quad \text{for} \quad x = (f, r),$$

$$z_0(t) = 1, \ y_0(t) = (t, 2).$$

It is easily seen that condition (F) is fulfilled. It is proved in [7] that $M^* = 1 < 2 = M$. We see by Theorem 1.1 that H is not closed. To prove that none of F_i ($i = 1, 2, 3$) is closed, let us define $f_n(t)$ by $f_n(t) = 0$ if $0 \le t < 1 - 1/n$ and $f_n(t) = n$ if $1 - 1/n \le t \le 1$. Then

$$\int_t^1 f_n(s)ds = 1 \quad \text{if} \quad 0 \le t \le 1 - 1/n,$$

$$\int_t^1 f_n(s)ds = n(1 - t) \quad \text{if} \quad 1 - 1/n < t \le 1,$$

$$\int_0^1 t f_n(t)dt = 1 - 1/(2n).$$

Let $x_n = (f_n, 0)$. Then $x_n \in P$ and $(Ax_n, ((x_n, y_0))_1)$ belongs to each F_i. It is easily seen that $(z_0, 1)$ is a limit of the sequence $\{(Ax_n, ((x_n, y_0))_1)\}$ and that $(z_0, 1)$ belongs to none of F_i. Therefore none of F_i is closed. Notice that P is not locally compact.

Next we shall apply Corollary 2 of Theorem 2.1 to our representations (I), (II) and (III) in §1. Notice that the condition $F_i \cap (-G_i) = \{0\}$ is equivalent to the following condition (F_i) for each $i = 1, 2, 3$:

(F_1) $x \in P \cap A^{-1}(Q)$ and $((x, y_0))_1 \le 0$ imply $((x, y_0))_1 = 0$.

(F_2) $x \in P \cap A^{-1}(Q)$ and $((x, y_0))_1 \le 0$ imply $((x, y_0))_1 = 0$ and $Ax = 0$.

(F_3) $x \in P \cap A^{-1}(Q)$ and $((x, y_0))_1 \le 0$ imply $Ax = 0$.

By Corollary 2 of Theorem 2.1 and Remark 2.1, we have

THEOREM 3.3. *The set H is closed if any one of the following conditions is fulfilled*:

(C.1) *F_1 is closed and condition* (F_1) *is fulfilled.*

(C.2) *F_2 is closed, condition* (F_2) *is fulfilled and Q is locally compact.*

(C.3) *F_2 is locally compact and condition* (F_2) *is fulfilled.*

(C.4) *F_3 is closed, condition* (F_3) *is fulfilled and Q is locally compact.*

(C.5) *F_3 is locally compact and condition* (F_3) *is fulfilled.*

Clearly condition (F) implies any one of conditions (F_1), (F_2) and (F_3). Condition (F_2) implies any one of conditions (F_1) and (F_3). In case $y_0 \in A^*(Q^+)+P^+$, condition (F_3) implies condition (F_2). By taking $A=0$, $y_0=0$ and $P \neq \{0\}$, we see easily that there is no relation between conditions (F_i) and (D_i) in general.

We can easily prove

THEOREM 3.4. *Condition* (F) *is equivalent to the pair of conditions* (D_i) *and* (F_i) *for each $i=1, 2, 3$.*

Now we shall rewrite conditions (C.1), (C.2) and (C.4) by the aid of the corollary of Theorem 2.2. We can easily verify that the dual cones F_i^+ and G_i^+ of F_i and G_i can be written as follows:

$$F_1^+ = \{(w, s) \in W \times R;\ -w \in Q^+,\ sy_0 + A^*w \in P^+\},$$

$$G_1^+ = W \times R^+,$$

$$F_2^+ = \{(w, s) \in W \times R;\ sy_0 + A^*w \in P^+\},$$

$$G_2^+ = (-Q^+) \times R^+,$$

$$F_3^+ = \{(w, s) \in W \times R;\ s \in R^+,\ sy_0 + A^*w \in P^+\},$$

$$G_3^+ = (-Q^+) \times R.$$

As for condition (C.1), we have

PROPOSITION 3.1. *If F_1 is closed and $y_0 \in A^*(Q^+)+P^+$, then H is closed.*

PROOF. Since $(G_1^+)^i \neq \phi$ and F_1 is closed, we see by the corollary of Theorem 2.2 that condition (F_1) is equivalent to $(G_1^+)^i \cap F_1^+ \neq \phi$, which is equivalent to $y_0 \in A^*(Q^+)+P^+$. Thus condition (C.1) is fulfilled.

This result was stated in [8; Théorème 4.3.1] in case $Q=\{0\}$. We show by an example that the converse of Proposition 3.1 does not hold in general.

EXAMPLE 3.2. Let $M(0, 1)$ be the set of all real Radon measures of any sign on the unit interval $[0, 1]$ in the real line, $M^+(0, 1)$ be the subset of $M(0, 1)$ which consists of non-negative measures, $C(0, 1)$ be the set of all finite real-valued continuous functions on $[0, 1]$ and $C^+(0, 1)$ be the subset of $C(0, 1)$ which consists of non-negative functions. Let us take

$$X = M(0, 1),\ Y = C(0, 1),\ P = M^+(0, 1),\ y_0(t) = t,$$

$$Z = W = R,\ Q = \{0\},\ z_0 \in R,$$

$$Av = \int_0^1 t^2 dv = a(v).$$

Then X and Y are in duality with respect to the bilinear functional $((\ ,\ ))_1$ defined by

$$((v, y))_1 = \int_0^1 y(t) dv \qquad \text{for} \quad v \in X \quad \text{and} \quad y \in Y,$$

and Z and W are in duality with respect to the bilinear functional $((\ ,\ ))_2$ defined by $((r, s))_2 = rs$ for $r \in Z$ and $s \in W$. Let us put $b(v) = ((v, y_0))_1$. Then

$$H = \{(a(v), b(v) + r);\ v \in M^+(0, 1),\ r \in R^+\}$$

$$F_1 = \{(a(v), b(v));\ v \in M^+(0, 1)\}.$$

Denote by ε_t the unit point measure at $t \in [0, 1]$ and put $v_n = n\varepsilon_{1/n}$. Then $(1/n, 1) = (a(v_n), b(v_n)) \in F_1$. However $(0, 1)$ does not belong to F_1, so that F_1 is not closed. On the other hand, since $(1, 1) = (a(\varepsilon_1), b(\varepsilon_1))$ and $(0, 1)$ belong to H and $0 \leq a(v) \leq b(v)$ for every $v \in M^+(0, 1)$, we have

$$H = \{(r_1, r_2) \in R^2;\ 0 \leq r_1 \leq r_2\},$$

so that H is closed.

REMARK 3.1. $H = F_1$ if and only if $(0, 1) \in F_1$.

As for condition (C.2), we have

PROPOSITION 3.2. *If F_2 is closed and if there exists $w \in (Q^+)^i$ such that $y_0 - A^*w \in P^+$, then H is closed.*

PROOF. Our assumption implies $(G_2^+)^i \neq \phi$ and $F_2^+ \cap (G_2^+)^i \neq \phi$. Since F_2 and G_2 are closed, condition (F_2) is fulfilled by the corollary of Theorem 2.2. Since $(Q^+)^i \neq \phi$, Q is locally compact by Proposition 2.1. Thus condition (C.2) is fulfilled.

Fan [4] proved this proposition under the assumption that $(P^+)^i \neq \phi$, $(Q^+)^i \neq \phi$ and $y_0 \in A^*((Q^+)^i) + (P^+)^i$.

As for condition (C.4), we can prove

PROPOSITION 3.3. *If F_3 is closed and if there exist $w \in (Q^+)^i$ and $s \in R^+$ such that $sy_0 - A^*w \in P^+$, then H is closed.*

REMARK 3.2. Since $F_1 = F_2 + (-Q) \times \{0\}$, F_1 is closed if F_2 is closed and if there exist $w \in (Q^+)^i$ and $s \in R$ such that $sy_0 - A^*w \in P^+$. Since $F_3 = F_2 + \{0\} \times R^+$, F_3 is closed if F_2 is closed and if $y_0 \in A^*(W) + P^+$. These facts are proved like Propositions 3.2 and 3.3.

In case $P = X$, we have as for condition (C.2)

PROPOSITION 3.4. *Assume that $P = X$ and $A(X)$ is closed and Q is locally compact. If $y_0 \in A^*(Q^+)$ and if condition* (F$_3$) *is fulfilled, then H is closed.*

PROOF. Since $y_0 \in A^*(Q^+)$, conditions (F$_2$) and (F$_3$) are equivalent. To verify condition (C.2), we have only to prove that F_2 is closed. There exists $w \in Q^+$ such that $A^*w = y_0$ by our assumption. Thus $F_2 = \{(Ax, ((Ax, w))_2); x \in X\}$. Since F_2 is the kernel of the continuous linear functional: $(z, r) \rightarrow r - ((z, w))_2$ defined on $A(X) \times R$, F_2 is closed in $A(X) \times R$. Since $A(X)$ is closed in Z, F_2 is closed.

Finally we give an improvement of [11; Proposition 6], which shows that a condition related to condition (K*) implies the closedness of H.

THEOREM 3.5. *The set H is closed if the following condition is fulfilled:*

(C.6) *P is locally compact, $Ax \in Q^i$ for all $x \in P$, $x \neq 0$ and M is finite.*

PROOF. In case $P = \{0\}$, $H = (-Q) \times R^+$ is closed. We consider the case where $P \neq \{0\}$. Let $\{(z_\alpha, r_\alpha)\}$ be a net in H which converges to (z, r). There exists $x_\alpha \in P$ such that $Ax_\alpha - z_\alpha \in Q$ and $((x_\alpha, y_0))_1 \leq r_\alpha$. By Lemma 2.1 and our representation (II), either (a) $(z, r) \in H$ or (b) there exists $u \in P$ such that $u \neq 0$, $Au \in Q$ and $((u, y_0))_1 \leq 0$. Since $P = P \cap A^{-1}(Q)$ and M is finite, $y_0 \in [P \cap A^{-1}(Q)]^+ = P^+$. Therefore $((u, y_0))_1 = 0$ and $r_\alpha \geq 0$, and hence $r \geq 0$. Since $Au \in Q^i$, there exists $t > 0$ such that $Au - tz \in Q^i$. Taking $x = u/t$, we see that $Ax - z \in Q$ and $((x, y_0))_1 = 0 \leq r$, and hence $(z, r) \in H$. Therefore H is closed.

§4. Superconsistency conditions

We shall study some relations between condition (F) and superconsistency conditions similar to condition (K). Let us consider the following interior conditions:

(G) $\qquad y_0 \in (A^*(Q^+) + P^+)^i$.

(L) $\qquad A^*(Q^+)^i \neq \phi$ and $y_0 \in A^*(Q^+)^i + P^+$.

Clearly any one of conditions (K) and (L) implies condition (G). We have

THEOREM 4.1. *Assume that* $(A^*(Q^+)+P^+)^i \neq \phi$. *Then conditions* (F) *and* (G) *are equivalent.*

PROOF. Let us take

$$E = X,\ F = Y,\ C = P \cap A^{-1}(Q),$$

$$U = V = R,\ D = -R^+,\ Tx = ((x, y_0))_1$$

in Theorem 2.2. Then C^+ is equal to the closure of $A^*(Q^+)+P^+$ and $(C^+)^i \neq \phi$ by our assumption. Condition (F) is equivalent to $C \cap T^{-1}(D) = \{0\}$. Notice that $T^*s = sy_0$ and that $(C^+)^i \cap T^*(-D^+) \neq \phi$ holds if and only if there exists $s \in R^+$ such that $sy_0 \in (C^+)^i = (A^*(Q^+)+P^+)^i$, which is equivalent to condition (G).

COROLLARY 1. *If* $(P^+)^i \neq \phi$, *then conditions* (F) *and* (K) *are equivalent.*

This was proved in [7; Corollary 7.1].

COROLLARY 2. *If* $A^*(Q^+)^i \neq \phi$, *then conditions* (F) *and* (L) *are equivalent.*

The dual statements of conditions (G) and (L) can be written as follows:

(G*) $\qquad z_0 \in (A(P) - Q)^i$.

(L*) $\qquad A(P)^i \neq \phi$ and $z_0 \in A(P)^i - Q$.

We can state sufficient conditions for the closedness of H^* defined in § 1 by the aid of conditions (F*), (G*), (K*) and (L*) and the condition that Q^+ is locally compact.

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