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Normal Functions Bounded on Arcs and a Proof of the Gross Cluster-Value Theorem

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Abstract

A differential form of the two-constants theorem that is valid for meromorphic functions is given. Treated as a (differential) maximum principle, this two-constants estimate is used to give a simple proof of the Gross cluster-value theorem.

Introduction

In their 1957 paper on normal functions, O. Lehto and K. I. Virtanen [4, Theorem 6] gave an improved (differential) form for normal meromorphic functions of the classical two-constants theorem. In this paper we show that a more general form of this result is true for arbitrary meromorphic functions f, although in most applications the most useful estimates on f' in the hypothesis will imply f is a normal function. We recast the two-constants estimate as a (differential) maximum principle.

We conclude with a reasonably short proof of the Gross cluster-value theorem [3] which has not had an easily digested proof, although J. L. Doob [2] gave an accessible proof.

§1. The differential two-constants theorem

A domain G in the finite plane W bounded by a finite number of disjoint Jordan curves is called a Jordan domain. A nonempty subset γ of ∂G (the boundary of G) is an admissible set if it is the union of a finite number of open arcs in ∂G and boundary curves of ∂G. The harmonic measure at $z \in G$ of γ relative to G is denoted by $\omega(z)=\omega(z, \gamma, G)$. For $z \in G$, let $f_{\omega}(z)=\omega(z)+i\omega^*(z)$, where

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$\omega^*(z)$ is a conjugate to $\omega(z)$ defined in some neighborhood of z. (In the following analysis we use the quantity $|f'_\omega(z)|$ so neither the constant nor the neighborhood chosen matters.)

If $A \subseteq W$, $\bar{A}$ denotes the closure of A in W.

THEOREM 1. *Let f be meromorphic in a domain $G^* \subseteq W$. Let G be a Jordan subdomain of G^* in which f is holomorphic and bounded by K. Suppose there is an admissible subset γ_1 of ∂G for which*

$$\limsup_{z \to \tau \in \gamma_1} |f(z)| \leq k < K \qquad (z \in G),$$

and an open analytic arc $\gamma_2 \subseteq G^* \cap \partial G$ such that at some point $q \in \gamma_2$, $|f(q)| = K$. Then, with $\omega(z, \gamma_1, G) = \omega(z)$,

$$|f'(q)| \geq |f'_\omega(q)| K \log \frac{K}{k}. \tag{1.1}$$

PROOF. According to the two-constants theorem, for $z \in G$,

$$|f(z)| \leq K\left(\frac{k}{K}\right)^{\omega(z)},$$

with equality occurring at $z = q$. The level line $|f(z)| = K$ is tangent to γ_2 at $z = q$, and $f'(q) \neq 0$. If n is the inner normal to γ_2 at q, then

$$\left.\frac{\partial |f(z)|}{\partial n}\right|_{z=q} \leq \left.\frac{\partial \omega(z)}{\partial n}\right|_{z=q} K \log \frac{k}{K}. \tag{1.2}$$

With f_ω defined in a neighborhood of q, it is simple to calculate that $\partial|f(z)|/\partial n = -|f'(q)|$ and $\partial\omega(z)/\partial n = |f'_\omega(q)|$ at $z = q$. If these are substituted in (1.2), the theorem is proved.

By dividing both sides of (1.1) by $(1 + |f(q)|^2)$ we obtain the Lehto-Virtanen result [4, Theorem 6]. We have been unable to obtain Theorem 1 from the Lehto-Virtanen result.

As Lehto and Virtanen noted [4, Theorem 7], $|f'_\omega(q)|$ assumes a simple form if G^* is the unit disk D: $|z| < 1$, and G is defined as follows. Let l be an open subarc of the unit circle C: $|z| = 1$, and, for $0 < \alpha < \pi$, let

$$L_\alpha(l) = \left\{ z \in D \colon \omega(z, l, D) = \frac{\pi - \alpha}{\pi} \right\},$$

$$T_\alpha(l) = \left\{ z \in D \colon \omega(z, l, D) > \frac{\pi - \alpha}{\pi} \right\}.$$

Note that the lens $T_\alpha(l)$ has interior angle α at each cusp. If we let $G = T_\alpha(l)$, $\gamma_1 = l$ and $\gamma_2 = L_\alpha(l)$, then, for $z \in L_\alpha(l)$,

$$(1.3) \qquad |f'_\omega(z)| = \frac{2\sin\alpha}{\alpha}\frac{1}{1-|z|^2}.$$

The important feature of (1.3), aside from its simple form, is its independence from l. In this situation, under the assumptions of Theorem 1, (1.1) becomes

$$(1.4) \qquad (1-|q|^2)|f'(q)| \geq \frac{2\sin\alpha}{\alpha}|f'_\omega(q)|\log\frac{|f(q)|}{k}.$$

This suggests seeking upper estimates for $(1-|q|^2)|f'(q)|$ in the form displayed by the right-hand side of (1.4). We begin by letting

$$(1.5) \qquad (1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} = N(z;f).$$

If we define

$$I(x, s) = xe^{-\frac{s}{2}(x+\frac{1}{x})}, \quad 0 \leq x < \infty, \quad 0 \leq s < \infty,$$

then (1.5) can be written as

$$(1.6) \qquad (1-|z|^2)|f'(z)| = 2|f(z)|\log\frac{|f(z)|}{I(|f(z)|, N(z;f))}.$$

Note the convenient identity, valid for $0<t<\infty$,

$$(1.7) \qquad t\log\frac{|f(z)|}{I(|f(z)|, N(z;f))} = \log\frac{|f(z)|}{I(|f(z)|, tN(z;f))}.$$

A few other properties of $I(x, s)$ will be needed. First, observe that for any s, $0<s<\infty$, $I(\cdot, s)$ has a single maximum value

$$B^*(s) = \frac{1+\sqrt{1+s^2}}{s}e^{-(1+s^2)}$$

which occurs at

$$B(s) = \frac{1+\sqrt{1+s^2}}{s}.$$

Thus $I(\cdot, s)$ is increasing in the interval $[0, B(s)]$ and we let $I^{-1}(\cdot, s)$ denote the inverse function relative to this interval. As either variable tends to ∞ — the other fixed and finite — I tends to zero and so we define $I(\infty, s)=I(x, \infty)=I(\infty, \infty)=0$. For any x, $0<x<\infty$, $I(x, \cdot)$ is a decreasing function on $[0, \infty]$. From this and an investigation of the graph of $I(\cdot, s)$, one can easily see the following. If $m<B^*(s_0)$ and

$$I(K, s_0) \leq m, \qquad m < K < M,$$

then, for each $s \geq s_0$,

$$M \leq I^{-1}(m, s). \tag{1.8}$$

§2. The maximum principle

We give now the Lehto-Virtanen maximum principle formulated for meromorphic functions via Theorem 1 (see also [1], p. 30). For $\tau \in C$, let

$$N(\tau; f) = \limsup_{z \to \tau} N(z; f), \qquad z \in D;$$

and for $A \subseteq D$ set

$$M(A; f) = \sup_{z \in A} |f(z)|,$$

and

$$N(A; f) = \sup_{z \in A} N(z; f).$$

All of the above quantities are allowed to be infinite. We suppress f if no confusion results.

THEOREM 2. *Let f be meromorphic in D, and let G be a Jordan domain in D contained in some $T_\alpha(l)$, $0 < \alpha < \pi$. Suppose for $\tau \in \partial G - \bar{L}_\alpha(l)$ and $z \in G$,*

$$\limsup_{z \to \tau} |f(z)| \leq m < \infty,$$

and set $M = M(G; f)$. Then

$$I(K, \frac{\alpha}{\sin \alpha} N(G; f)) \leq m \qquad (m \leq K \leq M). \tag{2.1}$$

If $N(G; f) < \infty$ and $m < B^*\left(\frac{\alpha}{\sin \alpha} N(G; f)\right)$, then

$$M \leq I^{-1}\left(m, \frac{\alpha}{\sin \alpha} N(G; f)\right). \tag{2.2}$$

PROOF. We first prove the following: for each K, $m < K < M$, there is a point $q = q(K) \in \bar{G} \cap D$ such that

$$I\left(K, \frac{\alpha}{\sin \alpha} N(q; f)\right) \leq m; \tag{2.3}$$

then, (2.1) holds for each (fixed) K with $m < K < M$ because $I(K, \cdot)$ is decreasing. Finally (2.1) holds for $m \leq K \leq M$ because $I(\cdot, s)$ is continuous. Also, (2.2) follows from (2.1) and (1.8).

Since $K<M$, there exists an arc $l(K)\subset l$ such that $|f(z)|<K$ in $T_\alpha(l(K))\cap G \equiv G^*(K)$ while $|f(q)|=K$ for some point $q\in L_\alpha(l(K))\cap\partial G^*(K)$. Let $G(K)$ be the component of $G^*(K)$ containing q on its boundary, and set $\gamma(K)=\partial G(K)-L_\alpha(l(K))$.

We now apply Theorem 1 with $\omega(z)=\omega(z,\gamma(K),G(K))$ to obtain

$$|f'(q)|\geq|f'_\omega(q)|K\log\frac{K}{m}. \tag{2.4}$$

(If there are "holes" of G in $G(K)$ that intersect $L_\alpha(l(K))$ at a point so as to prevent $G(K)$ from being a Jordan domain, we shrink them slightly to produce a Jordan domain and use $m+\varepsilon<K$ instead of m in (2.4); we then let $\varepsilon\to 0$.) Because

$$\omega(z,\gamma(K),G(K))\geq\omega(z,l(K),T_\alpha(l(K))),\qquad z\in G(K),$$

with equality at $z=q$, the same inequality applies to the normal derivatives at q; thus from (1.3) and (2.4) it follows that

$$(1-|q|^2)|f'(q)|\geq\frac{2\sin\alpha}{\alpha}K\log\frac{K}{m}. \tag{2.5}$$

This inequality, together with (1.6) and (1.7), leads to

$$2K\log\frac{K}{I\left(K,\frac{\alpha}{\sin\alpha}N(q;f)\right)}\geq 2K\log\frac{K}{m}$$

and so (2.3) is verified.

In the sequel we will be concerned with domains G of a fairly simple type, namely those cut from a lens $T_\alpha(l)$ by a Jordan arc γ on which f is bounded. We say that a Jordan arc γ *properly intersects* $T_\alpha(l)$ if there is a subarc $\gamma^*\subseteq\gamma$ such that $\gamma^*\subset T_\alpha(l)$ except for its endpoints which lie on $L_\alpha(l)$. The (simply connected) domain in $T_\alpha(l)$ bounded by γ^* and an arc of $L_\alpha(l)$ we denote generically by $H_\alpha(\gamma,l)$ even though there may be more than one possible subarc γ^*. If $|f(z)|\leq m$ on γ, Theorem 2 is applicable to any $H_\alpha(\gamma,l)$.

§3. Functions bounded on arcs ending at points

To prepare for the proof of the Gross theorem, we investigate the local behavior of functions bounded on an arc ending at a point on C.

For $\tau\in C$ and $0<\beta<\pi$, we define a β-angle at τ, $S(\beta,\tau)$, in the usual fashion; that is, if l_τ is the arc on C from τ to $-\tau$ in the clockwise sense, then

$$S(\beta) = S(\beta, \tau) = \left\{z \in \mathrm{D}: \frac{\pi - \beta}{2\pi} < \omega(z, l_\tau, D) < \frac{\pi + \beta}{2\pi}\right\}.$$

We set

$$|f(\tau)|_{S(\beta)} = \limsup_{z \to \tau} |f(z)|, \qquad z \in S(\beta, \tau),$$

and say that f is bounded in angles at τ if $|f(\tau)|_{S(\beta)}$ is finite for each β, $0<\beta<\pi$.

If γ is a Jordan arc in D except for one endpoint at $\tau \in C$, let

$$|f(\tau)|_\gamma = \limsup_{z \to \tau} |f(z)|, \qquad z \in \gamma, \quad z \neq \tau.$$

THEOREM 3. *Let f be meromorphic in D, and suppose for some $\tau \in C$, $N(\tau; f) < \infty$, and there is an arc γ ending at τ for which $|f(\tau)|_\gamma$ is finite.*

If for some β, $0<\beta<\pi$,

$$|f(\tau)|_\gamma < \min\left\{|f(\tau)|_{S(\beta)}, \ B^*\left(\frac{\pi + \beta}{2\cos\beta/2} N(\tau; f)\right)\right\}, \tag{3.1}$$

then

$$I\left(K, \frac{\pi + \beta}{2\cos\beta/2} N(\tau; f)\right) \leq |f(\tau)|_\gamma, \quad |f(\tau)|_\gamma \leq K \leq |f(\tau)|_{S(\beta)} \tag{3.2}$$

and so

$$|f(\tau)|_{S(\beta)} \leq I^{-1}\left(|f(\tau)|_\gamma, \ \frac{\pi + \beta}{2\cos\beta/2} N(\tau; f)\right). \tag{3.3}$$

PROOF. First choose a sequence of arcs $\{l_n\}$, $l_1 = l_\tau$, $l_{n+1} \subset l_n$, with one endpoint at τ and the other endpoint approaching τ as $n \to \infty$; let $\{l_n^*\}$ denote the sequence obtained from $\{l_n\}$ by reflection in the diameter from τ to $-\tau$. Let $\alpha = (\pi + \beta)/2$.

We assume that γ has its initial point at $-\tau$. Then γ properly intersects either $T_\alpha(l_n)$ or $T_\alpha(l_n^*)$, or both, for each $n \geq 1$. If we set

$$T_n = T_\alpha(l_n) \cap T_\alpha(l_n^*),$$

then the union of all the closed domains $\bar{H}_\alpha(\gamma, l_n)$ and $\bar{H}_\alpha(\gamma, l_n^*)$ covers T_n. Because f is normal in a neighborhood of τ and thus is uniformly continuous with respect to the non-Euclidean metric close to τ, we infer that

$$\limsup_{z \to \tau} |f(z)| = |f(\tau)|_{S(\beta)}, \qquad z \in T_n, \quad n = 1, 2, \ldots.$$

Select a sequence z_k in T_n (n fixed) tending to τ for which

$$\lim_{k \to \infty} |f(z_k)| = |f(\tau)|_{S(\beta)}.$$

For each $k=1, 2, \ldots$, there is a domain H_k with either $H_k=H_\alpha(\gamma, l_n)$ or $H_k=H_\alpha(\gamma, l_n^*)$ such that $z_k \in \bar{H}_k$. For the moment, we suppose

$$H_k = H_\alpha(\gamma, l_n)$$

for all values of k; the analysis is similar if the other case holds for all values of k. From (2.1) of Theorem 2, we have, after suppressing f from the notation,

$$I\left(M(H_k), \frac{\alpha}{\sin\alpha} N(H_k)\right) \le M(\gamma \cap H_k). \tag{3.4}$$

Since $I(x, \cdot)$ is decreasing and $H_k \subset T_\alpha(l_n)$,

$$I\left(M(H_k), \frac{\alpha}{\sin\alpha} N(T_\alpha(l_n))\right) \le M(\gamma \cap T_\alpha(l_n))$$

and if we let $A_n = \liminf M(H_k)$, then

$$I\left(A_n, \frac{\alpha}{\sin\alpha} N(T_\alpha(l_n))\right) \le M(\gamma \cap T_\alpha(l_n)).$$

Finally, using

$$\limsup N(T_\alpha(l_n)) \le N(\tau)$$

and

$$\limsup M(\gamma \cap T_\alpha(l_n)) \le |f(\tau)|_\gamma,$$

we obtain

$$I\left(\limsup A_n, \frac{\alpha}{\sin\alpha} N(\tau)\right) \le |f(\tau)|_\gamma. \tag{3.5}$$

Since (3.4) holds for each K with $M(\gamma \cap H_k) \le K \le M(H_k)$, (3.5) holds for each K with $\limsup M(\gamma \cap H_k) < K < \liminf M(H_k) = A_n$.

Since $|f(z_k)| \to |f(\tau)|_{S(\beta)}$,

$$|f(\tau)|_{S(\beta)} \le A_n. \tag{3.6}$$

If $|f(\tau)|_\gamma$ satisfies (3.1), then from (3.6) we see that (3.5) holds for each K with $|f(\tau)|_\gamma \le K \le |f(\tau)|_{S(\beta)}$ which gives (3.2). Then (3.2) and (1.8) give (3.3). This completes the proof.

REMARK. If $|f(\tau)|_\gamma = 0$, then Theorem 3 shows that f has angular limit 0 at τ which is the Lehto-Virtanen result [4] on asymptotic values for normal functions.

§4. The Gross cluster-value theorem

A value w in the extended plane Ω is said to be a principal value of f at τ if w is in the cluster set of f on every arc γ ending at τ; w is said to be an angular value of f at τ if w is in the cluster set of f in some angle $S(\beta, \tau)$, $0<\beta<\pi$. The range of f at τ is the set

$$R(f, \tau) = \{w \in \Omega : f(z_n) = w \quad \text{for some sequence} \quad z_n \longrightarrow \tau,\ z_n \in D\}.$$

GROSS CLUSTER-VALUE THEOREM. *Let f be meromorphic in D and suppose w_0 is an angular value of f at τ and also w_0 is an accumulation point of $\Omega - R(f, \tau)$. Then w_0 is a principal value of f at τ.*

REMARK. The original proof of Gross is involved and Doob [2] gave a more direct proof using one-sided cluster values and the non-Euclidean-based properties of normal functions. Difficulties arise in using the non-Euclidean geometry with normal functions; curves must lie in D and sometimes estimates on $|f|$ are needed in terms of the boundary cluster sets, and the journey between angular and tangential approach via non-Euclidean geometry requires skillful navigation. The differential two-constants theorem in the form of Theorem 3 allows an integrated approach by avoiding much of the non-Euclidean geometry.

PROOF OF THE GROSS THEOREM. We can assume that $w_0 = \infty$. Take any arc γ ending at τ and suppose $|f(\tau)|_\gamma$ is finite; we assume that for $z \in \gamma$,

$$|f(z)| \leq k.$$

We will show that f is bounded in each angle $S(\beta, \tau)$ for z sufficiently close to τ; then ∞ is not an angular value of f at τ and the proof of the theorem will be complete.

Fix β, $0<\beta<\pi$, and set $\alpha = (\pi+\beta)/2$. Let $\mu(z)$ denote the classical elliptic modular function in D. Choose any finite point w_1 in $\Omega - R(f, \tau)$ and then select w_2 in $\Omega - R(f, \tau)$ with $|w_2| > k$ so that first

$$2\frac{|w_1| + k}{|w_2| - k} < B^*\left(\frac{\pi + \beta}{2\cos\beta/2} N(D; \mu)\right)$$

and then also

$$I^{-1}\left(2\frac{|w_1| + k}{|w_2| - k}, \frac{\pi + \beta}{2\cos\beta/2} N(D; \mu)\right) < \frac{1}{3}.$$

Now choose w_3 in $\Omega - R(f, \tau)$ so that

$$\frac{1}{2} < \left|\frac{w_3 - w_2}{w_3 - w_1}\right| < 2. \tag{4.1}$$

By using a conformal mapping if necessary, we can assume that f omits w_1, w_2, w_3 in D. Then the function

$$g(z) = \frac{w_3 - w_2}{w_3 - w_1} \frac{f(z) - w_1}{f(z) - w_2}$$

omits 0, 1, ∞ and is therefore subordinate to μ; hence for $z \in D$,

$$N(z; g) \leq N(D; \mu).$$

Also, for $z \in \gamma$,

$$|g(z)| \leq 2 \frac{|w_1| + k}{|w_2| - k}$$

so that

$$|g(\tau)|_\gamma \leq 2 \frac{|w_1| + k}{|w_2| - k} < B^*\left(\frac{\pi + \beta}{2 \cos \beta/2} N(D; \mu)\right) \leq B^*\left(\frac{\pi + \beta}{2 \cos \beta/2} N(\tau; g)\right).$$

Now either

$$|g(\tau)|_{S(\beta)} \leq |g(\tau)|_\gamma \leq 2 \frac{|w_1| + k}{|w_2| - k} < \frac{1}{3}$$

or $|g(\tau)|_\gamma$ satisfies (3.1) and hence (3.2); in the latter case we use (1.8) — or we could use (3.3) and the fact that $I^{-1}(|g(\tau)|_\gamma, \cdot)$ is increasing — to deduce that

$$|g(\tau)|_{S(\beta)} \leq I^{-1}\left(|g(\tau)|_\gamma, \frac{\pi + \beta}{2 \cos \beta/2} N(D; \mu)\right)$$

and thus

$$|g(\tau)|_{S(\beta)} \leq I^{-1}\left(2 \frac{|w_1| + k}{|w_2| - k}, \frac{\pi + \beta}{2 \cos \beta/2} N(D; \mu)\right) < \frac{1}{3}.$$

In either case we have

$$|g(\tau)|_{S(\beta)} < \frac{1}{3}. \tag{4.2}$$

If f were not bounded in $S(\beta, \tau)$, then from (4.1) we see that

$$|g(\tau)|_{S(\beta)} \geq \frac{1}{2},$$

which contradicts (4.2). This completes the proof of the theorem.

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