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Asymptotic Expansions for the Distributions of Statistics Based on the Sample Correlation Matrix in Principal Component Analysis

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Contents

0. Introduction

An important problem in multivariate analysis is to reduce the dimension of multivariate data with as little loss of information as possible. Principal component analysis is a method useful for this problem. This method, which originated with Pearson [72] and was developed by Hotelling [36], provides reduction of a large set of correlated variables to a smaller number of uncorrelated new variables called principal components. Principal component analysis is algebraically based on latent roots and vectors of a covariance matrix or a correlation matrix. In particular, latent roots play an important role in considering how much information is condensed into a smaller number of new variables. On the basis of a normal sample, tests of hypotheses concerning latent roots of a covariance matrix may be constructed and the likelihood ratio principle leads to suitable test statistics, which are expressed as functions of latent roots of the sample covariance matrix (cf. Anderson [6]).

The problem of deriving the distributions of statistics based on sample covariance matrices in normal samples has been studied by many authors. Some works have been done in the area of asymptotic distributions, and others in that of exact distributions. Furthermore, some asymptotic expansions have been derived by inverting asymptotic formulae of characteristic functions obtained by the Taylor series expansion. Surveys of the area of asymptotic distributions are given by Muirhead [63] and Siotani [82, 83].

It may be emphasized that the sample covariance matrix is not invariant under a change of scale and so neither are principal components derived from the sample covariance matrix. In practice, there are many situations in which variables are measured on different units. The problem of units can be avoided by employing the sample correlation matrix, since principal components are then invariant under the different units of the original variables. Very little work has been done in the problem of deriving the distributions of statistics based on the sample correlation matrix in normal sample. This may mainly be due to the fact that an explicit expression for the distribution of the sample correlation matrix has not been obtained yet and hence the distribution problem associated with the sample correlation matrix is intractable.

The purpose of this paper is to derive asymptotic expansions for the distributions of statistics based on the sample correlation matrix in principal component analysis. This paper is divided into two parts. Part I contains asymptotic results concerning the latent roots and vectors of the sample correlation matrix. Section 1 is devoted to prepare tools employed in Part I. In Section 2, asymptotic expansions are given for the distributions of certain functions of the latent roots of the sample correlation matrix both in the case when

population roots are all simple and in the case when population roots have multiplicity. As special cases of the resulting expansions, asymptotic expansions for the distributions of statistics used as a measure of the importance of principal components are given in Section 3.1, whereas asymptotic nonnull distributions of test statistics for testing the equality of the last several latent roots of a correlation matrix are given in Section 3.2. Some numerical comparisons are shown in Section 3.3. In Section 4, asymptotic expansions are given for the distributions of latent vectors of the sample correlation matrix. Section 5 contains some tests for latent vectors of a correlation matrix. The likelihood ratio tests have been successful in finding appropriate statistics for testing hypotheses concerning covariance matrices. It is however difficult to obtain the likelihood ratio statistics for testing hypotheses concerning correlation matrices in closed form and so approximate tests are proposed in this section.

Part II deals with asymptotic expansions for the distributions of statistics expressed as functions of the sample correlation matrix. In Section 6.1, an asymptotic expansion is given for the joint density of certain functions of the sample correlation matrix. As an example of the use of the results, an asymptotic expansion for the joint density of elements of the sample correlation matrix is presented. In Section 6.2, an asymptotic expansion for the distribution of a function of the sample correlation matrix is derived up to the term of order of the reciprocal of sample size. Using the resulting expansion, asymptotic nonnull distributions of the statistics proposed in Section 5 are given in Section 7. Finally, in the bivariate case approximations to the distribution of the sample correlation coefficient are discussed and some numerical comparisons of approximate distributions are made in Section 8.

Part I. Latent Roots and Vectors of the Sample Correlation Matrix

1. Preliminaries

1.1. *Correlation matrices.* Let $\boldsymbol{X}_\alpha'=(X_{1\alpha},\ldots, X_{p\alpha})$, $\alpha=1,\ldots, N$, be a random sample of size $N=n+1$ $(n\geqq p)$ from a p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and positive definite covariance matrix $\Sigma=(\sigma_{ij})$, and let

$$\overline{\boldsymbol{X}}=\frac{1}{N}\sum_{\alpha=1}^{N}\boldsymbol{X}_\alpha,\quad S=(s_{ij})=\sum_{\alpha=1}^{N}(\boldsymbol{X}_\alpha-\overline{\boldsymbol{X}})(\boldsymbol{X}_\alpha-\overline{\boldsymbol{X}})'.$$

It is well known that S/n is an unbiased estimate of Σ and that S is distributed according to the Wishart distribution $W_p(n, \Sigma)$ (see, e.g., Anderson [5, p. 157]). The population correlation coefficient between the ith and jth components of the random vector is defined as

$$\rho_{ij}=\sigma_{ij}/(\sigma_{ii}\sigma_{jj})^{1/2}.$$

The $p \times p$ matrix $P=(\rho_{ij})$ is called the population correlation matrix. On the basis of normal sample $\{\boldsymbol{X}_\alpha\}$, $\alpha=1,\ldots, N$, the maximum likelihood estimate of ρ_{ij} is

$$r_{ij} = s_{ij}/(s_{ii}s_{jj})^{1/2}. \tag{1.1}$$

The $p \times p$ matrix $R=(r_{ij})$ with $r_{ii}=1$ is called the sample correlation matrix.

Fisher [20] has given the distribution of R in the form

$$c_{p\cdot n}\{|P|\prod_{i=1}^{p}\rho^{ii}\}^{-n/2}|R|^{(n-p-1)/2}\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{p}x_i^{n-1}\exp\left(-\frac{1}{2}\boldsymbol{x}'\Gamma\boldsymbol{x}\right)d\boldsymbol{x}$$

for R positive definite, where $c_{p\cdot n}=2^{-p(n-2)/2}\left\{\Gamma_p\left(\frac{1}{2}n\right)\right\}^{-1}$, $P^{-1}=(\rho^{ij})$, $\boldsymbol{x}'=(x_1, \ldots, x_p)$ and Γ is a $p \times p$ symmetric matrix with $\rho^{ij}r_{ij}/(\rho^{ii}\rho^{jj})^{1/2}$ as its (i, j)th element. This expression, however, contains the multiple integral, for which the explicit form has not been derived yet except for the cases $P=I$ and $p=2$ where I is the identity matrix of order p. In view of these facts, it is difficult to obtain exact distributions of statistics based on the sample correlation matrix. We are thus led to consideration of the problem of deriving asymptotic distributions.

1.2. *Perturbation method.* In order to find approximations to the latent roots and vectors of the sample correlation matrix, we use the technique known as the perturbation method.

Consider a diagonal matrix Λ with the ordered latent roots $\lambda_1 \geqq \cdots \geqq \lambda_p$ and assume that the perturbation of Λ can be expressed as a power series in ε as follows:

$$M = \Lambda + \varepsilon V^{(1)} + \varepsilon^2 V^{(2)} + \varepsilon^3 V^{(3)} + O(\varepsilon^4), \tag{1.2}$$

where $V^{(j)}$ $(j=1, 2,\ldots)$ are symmetric matrices of order p and ε is a small real number.

We shall first discuss the case when λ_α is distinct from other $p-1$ latent roots. Let l_α be the αth largest latent root of M and $\boldsymbol{c}'_\alpha=(c_{1\alpha},\ldots, c_{p\alpha})$ the corresponding latent vector with $\boldsymbol{c}'_\alpha\boldsymbol{c}_\alpha=1$ and $c_{\alpha\alpha}>0$. The l_α and $\boldsymbol{c}_\alpha$ can be assumed to take the form

$$\begin{aligned} l_\alpha &= \lambda_\alpha + \varepsilon\lambda_\alpha^{(1)} + \varepsilon^2\lambda_\alpha^{(2)} + \varepsilon^3\lambda_\alpha^{(3)} + \cdots, \\ \boldsymbol{c}_\alpha &= \boldsymbol{e}_\alpha + \varepsilon\sum_{i=1}^{p} a_{i\alpha}^{(1)}\boldsymbol{e}_i + \varepsilon^2\sum_{i=1}^{p} a_{i\alpha}^{(2)}\boldsymbol{e}_i + \varepsilon^3\sum_{i=1}^{p} a_{i\alpha}^{(3)}\boldsymbol{e}_i + \cdots, \end{aligned} \tag{1.3}$$

where $\boldsymbol{e}'_\alpha=(0,\ldots, 0, \overset{\alpha}{1}, 0,\ldots, 0)$. To determine the unknown coefficients $\lambda_\alpha^{(1)}$, $\lambda_\alpha^{(2)},\ldots$ and $a_{i\alpha}^{(1)}$, $a_{i\alpha}^{(2)},\ldots$ $(i=1,\ldots, p)$, we substitute (1.3) into the characteristic equation $M\boldsymbol{c}_\alpha=l_\alpha\boldsymbol{c}_\alpha$ and equate the coefficients in the both sides under the con-

dition $\boldsymbol{c}_\alpha'\boldsymbol{c}_\alpha=1$. For details we refer to Bellman [9, pp. 60–63] or Wigner [98, pp. 40–46]. Results are given in the following:

LEMMA 1.1. *Let l_α be the αth largest latent root of the $p\times p$ symmetric matrix M defined by* (1.2) *and $\boldsymbol{c}_\alpha'=(c_{1\alpha},\ldots,c_{p\alpha})$ the corresponding normalized ($\boldsymbol{c}_\alpha'\boldsymbol{c}_\alpha=1$ and $c_{\alpha\alpha}>0$) latent vector. If λ_α is simple, that is, of multiplicity* 1, *the perturbation expansions of l_α and $\boldsymbol{c}_\alpha$ are given by*

$$
\begin{aligned}
(1.4)\qquad l_\alpha &= \lambda_\alpha + \varepsilon v_{\alpha\alpha}^{(1)} + \varepsilon^2\Big(v_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)2}\Big)\\
&\quad + \varepsilon^3\Big(v_{\alpha\alpha}^{(3)} + 2\sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)}v_{\alpha\beta}^{(2)} - \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}^2 v_{\alpha\alpha}^{(1)}v_{\alpha\beta}^{(1)2}\\
&\quad + \sum_{\beta\neq\alpha}^{p}\sum_{\gamma\neq\alpha}^{p}\lambda_{\alpha\beta}\lambda_{\alpha\gamma}v_{\alpha\beta}^{(1)}v_{\beta\gamma}^{(1)}v_{\gamma\alpha}^{(1)}\Big) + O(\varepsilon^4),
\end{aligned}
$$

$$
\begin{aligned}
(1.5)\qquad c_{i\alpha} &= -\lambda_{i\alpha}\Big[\varepsilon v_{i\alpha}^{(1)} + \varepsilon^2\Big(\lambda_{i\alpha}v_{i\alpha}^{(1)}v_{\alpha\alpha}^{(1)} + \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}v_{i\beta}^{(1)}v_{\beta\alpha}^{(1)}\\
&\quad + v_{i\alpha}^{(2)}\Big) + \varepsilon^3\Big\{\sum_{\beta\neq\alpha}^{p}\sum_{\gamma\neq\alpha}^{p}\lambda_{\alpha\beta}\lambda_{\alpha\gamma}v_{i\beta}^{(1)}v_{\beta\gamma}^{(1)}v_{\gamma\alpha}^{(1)}\\
&\quad + \lambda_{i\alpha}^2 v_{i\alpha}^{(1)}v_{\alpha\alpha}^{(1)2} - \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}^2\Big(v_{i\beta}^{(1)}v_{\beta\alpha}^{(1)}v_{\alpha\alpha}^{(1)} + \frac{1}{2}v_{i\alpha}^{(1)}v_{\alpha\beta}^{(1)2}\Big)\\
&\quad + \lambda_{i\alpha}\sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}\big(v_{i\alpha}^{(1)}v_{\alpha\beta}^{(1)2} + v_{i\beta}^{(1)}v_{\beta\alpha}^{(1)}v_{\alpha\alpha}^{(1)}\big)\\
&\quad + \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}\big(v_{i\beta}^{(1)}v_{\beta\alpha}^{(2)} + v_{i\beta}^{(2)}v_{\beta\alpha}^{(1)}\big)\\
&\quad + \lambda_{i\alpha}\big(v_{i\alpha}^{(1)}v_{\alpha\alpha}^{(2)} + v_{i\alpha}^{(2)}v_{\alpha\alpha}^{(1)}\big) + v_{i\alpha}^{(3)}\Big\}\Big] + O(\varepsilon^4) \qquad \text{if } i\neq\alpha,
\end{aligned}
$$

$$
\begin{aligned}
(1.6)\qquad c_{\alpha\alpha} &= 1 + \varepsilon^2\Big(-\frac{1}{2}\sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}^2 v_{\alpha\beta}^{(1)2}\Big) + \varepsilon^3\Big(-\sum_{\beta\neq\alpha}^{p}\sum_{\gamma\neq\alpha}^{p}\lambda_{\alpha\beta}^2\lambda_{\alpha\gamma}v_{\alpha\beta}^{(1)}v_{\beta\gamma}^{(1)}v_{\gamma\alpha}^{(1)}\\
&\quad + \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}^3 v_{\alpha\alpha}^{(1)}v_{\alpha\beta}^{(1)2} - \sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}^2 v_{\alpha\beta}^{(1)}v_{\alpha\beta}^{(2)}\Big) + O(\varepsilon^4),
\end{aligned}
$$

where $V^{(j)}=(v_{\alpha\beta}^{(j)})$ and $\lambda_{\alpha\beta}=(\lambda_\alpha-\lambda_\beta)^{-1}$ $(\alpha\neq\beta)$.

The expansion (1.4) with $v_{\alpha\beta}^{(j)}=0$, $j=2, 3,\ldots$, gives an expansion of the αth latent root of the sample covariance matrix S/n, which was first obtained by Lawley [57]. Sugiura [87] has subsequently derived the expansion, based on the Taylor series expansion. Sugiura [91] used the perturbation method to obtain asymptotic expansions for the distributions of the latent roots and vectors of a Wishart matrix S and a multivariate F matrix $S_1S_2^{-1}$ where S_j $(j=1, 2)$ are

independently distributed according to $W_p(n_j, \Sigma_j)$.

We now proceed to the case when the latent roots of Λ in (1.2) have multiplicity. It is assumed that Λ has r distinct latent roots $\theta_1, \theta_2, \ldots, \theta_r$ with multiplicities $q_1, q_2, \ldots, q_r$, that is,

$$
\begin{aligned}
\lambda_1 = \cdots = \lambda_{q_1} &= \theta_1, \\
\lambda_{q_1+1} = \cdots = \lambda_{q_1+q_2} &= \theta_2, \\
&\cdots \\
\lambda_{p-q_r+1} = \cdots = \lambda_p &= \theta_r,
\end{aligned} \tag{1.7}
$$

where $\theta_1 > \cdots > \theta_r$, $\sum_{\alpha=1}^{r} q_\alpha = p$ and $q_0 \equiv 0$. Partition the matrices Λ and $V^{(j)}$ $(j=1, 2, \ldots)$ into submatrices with $q_1, \ldots, q_r$ rows and columns as follows:

$$
\Lambda = \begin{bmatrix} \theta_1 I_{q_1} & 0 & \cdots & 0 \\ 0 & \theta_2 I_{q_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_r I_{q_r} \end{bmatrix}, \quad V^{(j)} = \begin{bmatrix} V_{11}^{(j)} & V_{12}^{(j)} & \cdots & V_{1r}^{(j)} \\ V_{21}^{(j)} & V_{22}^{(j)} & \cdots & V_{2r}^{(j)} \\ \vdots & \vdots & \cdots & \vdots \\ V_{r1}^{(j)} & V_{r2}^{(j)} & \cdots & V_{rr}^{(j)} \end{bmatrix},
$$

where I_{q_α} are identity matrices of order q_α and $V_{\alpha\beta}^{(j)}$ are $q_\alpha \times q_\beta$ submatrices. The following lemma, due to Fujikoshi [24], is useful in deriving asymptotic expansions for the distributions of statistics based on the latent roots of the sample correlation matrix in the case when the corresponding population roots have multiplicity.

LEMMA 1.2. *Let $l_1 \geqq \cdots \geqq l_p$ be the ordered latent roots of a $p \times p$ symmetric matrix M defined by* (1.2). *Then, for $j=1, \ldots, q_\alpha$ the $(q_1 + \cdots + q_{\alpha-1} + j)$th latent root $l_{q_1+\cdots+q_{\alpha-1}+j}$ is equal to the jth latent root of*

$$
W_\alpha = \theta_\alpha I_{q_\alpha} + \varepsilon W_\alpha^{(1)} + \varepsilon^2 W_\alpha^{(2)} + \varepsilon^3 W_\alpha^{(3)} + O(\varepsilon^4),
$$

where

$$
W_\alpha^{(1)} = V_{\alpha\alpha}^{(1)}, \quad W_\alpha^{(2)} = V_{\alpha\alpha}^{(2)} + \sum_{\beta \neq \alpha}^{r} \theta_{\alpha\beta} V_{\alpha\beta}^{(1)} V_{\beta\alpha}^{(1)},
$$

$$
\begin{aligned}
W_\alpha^{(3)} = V_{\alpha\alpha}^{(3)} &+ \sum_{\beta \neq \alpha}^{r} \theta_{\alpha\beta} (V_{\alpha\beta}^{(1)} V_{\beta\alpha}^{(2)} + V_{\alpha\beta}^{(2)} V_{\beta\alpha}^{(1)}) \\
&- \sum_{\beta \neq \alpha}^{r} \theta_{\alpha\beta}^2 V_{\alpha\alpha}^{(1)} V_{\alpha\beta}^{(1)} V_{\beta\alpha}^{(1)} + \sum_{\beta \neq \alpha}^{r} \sum_{\gamma \neq \alpha}^{r} \theta_{\alpha\beta} \theta_{\alpha\gamma} V_{\alpha\beta}^{(1)} V_{\beta\gamma}^{(1)} V_{\gamma\alpha}^{(1)}
\end{aligned}
$$

with $\theta_{\alpha\beta} = (\theta_\alpha - \theta_\beta)^{-1}$ $(\alpha \neq \beta)$.

Fujikoshi [24, 25, 26] has obtained, using Lemma 1.2, asymptotic expansions for the distributions of certain test statistics based on the latent roots of multivariate F, multivariate beta and other random matrices. Konishi [44] has

discussed the perturbation expansions of latent vectors of a symmetric matrix in the case when the initial matrix Λ in (1.2) has multiple latent roots.

2. Generalized asymptotic expansions

In this section, asymptotic expansions are given for the distributions of certain functions of the latent roots of the sample correlation matrix R both in the case when the population roots are all simple and in the case when the population roots have multiplicity. The results are used to obtain asymptotic expansions for the distributions of some statistics expressed as functions of the latent roots of R.

2.1. *An asymptotic expansion when population roots are simple.* Let $S=(s_{ij})$ be the matrix of the corrected sum of squares and sum of products of observations in a sample of size $n+1$ from a p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and positive definite covariance matrix Σ. As defined in Section 1.1, let P be the population correlation matrix. Since P is positive definite, there exists an orthogonal matrix $H=(h_{ij})$ such that

$$H'PH = \Lambda \quad \text{or} \quad PH = H\Lambda, \tag{2.1}$$

where Λ is a diagonal matrix with the ordered latent roots $\lambda_1 \geqq \cdots \geqq \lambda_p > 0$. The sample correlation matrix R defined in Section 1.1 can be expressed as

$$R = S_0^{-1/2} S S_0^{-1/2}, \tag{2.2}$$

where $S_0 = \operatorname{diag}(s_{11},\ldots, s_{pp})$. Let

$$V = (v_{ij}) = \sqrt{n}\left(\frac{1}{n}\Sigma_0^{-1/2} S \Sigma_0^{-1/2} - P\right), \tag{2.3}$$

where $\Sigma_0 = \operatorname{diag}(\sigma_{11},\ldots, \sigma_{pp})$.

It is known (see, e.g., Anderson [5] or Kshirsagar [54]) that $\frac{1}{n}\Sigma_0^{-1/2} S \Sigma_0^{-1/2}$ converges to P in probability as n tends to infinity and that the limiting distribution of V is normal with means 0 and covariances $\operatorname{cov}(v_{ij}, v_{kl}) = \rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk}$.

From (2.1), (2.2) and (2.3), it follows that

$$H'RH = \Lambda + \frac{1}{\sqrt{n}} V^{(1)} + \frac{1}{n} V^{(2)} + \frac{1}{n\sqrt{n}} V^{(3)} + O_p(n^{-2}), \tag{2.4}$$

where

$$V^{(1)} = H'\left(V - \tfrac{1}{2} V_0 P - \tfrac{1}{2} P V_0\right) H,$$

$$V^{(2)} = H'\left(\tfrac{1}{4} V_0 P V_0 - \tfrac{1}{2} V V_0 - \tfrac{1}{2} V_0 V + \tfrac{3}{8} V_0^2 P + \tfrac{3}{8} P V_0^2\right) H,$$

$$V^{(3)} = H'\Big(\frac{1}{4}V_0VV_0 - \frac{3}{16}V_0^2PV_0 - \frac{3}{16}V_0PV_0^2 + \frac{3}{8}V_0^2V + \frac{3}{8}VV_0^2 - \frac{5}{16}V_0^3P - \frac{5}{16}PV_0^3\Big)H$$

with $V_0 = \mathrm{diag}\,(v_{11},\ldots, v_{pp})$.

Let l_α be the αth largest latent root of R. If the latent root λ_α of P is simple, then it follows from (2.4) and Lemma 1.1 that the perturbation expansion of l_α is

$$\begin{aligned}(2.5)\qquad l_\alpha &= \lambda_\alpha + \frac{1}{\sqrt{n}}v_{\alpha\alpha}^{(1)} + \frac{1}{n}(v_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)2}) \\ &\quad + \frac{1}{n\sqrt{n}}(v_{\alpha\alpha}^{(3)} + 2\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)}v_{\alpha\beta}^{(2)} - \sum_{\beta\neq\alpha}\lambda_{\alpha\beta}^2v_{\alpha\alpha}^{(1)}v_{\alpha\beta}^{(1)2} \\ &\quad + \sum_{\beta\neq\alpha}\sum_{\gamma\neq\alpha}\lambda_{\alpha\beta}\lambda_{\alpha\gamma}v_{\alpha\beta}^{(1)}v_{\beta\gamma}^{(1)}v_{\gamma\alpha}^{(1)}) + O_p(n^{-2}),\end{aligned}$$

where $V^{(j)} = (v_{\alpha\beta}^{(j)})$, $\lambda_{\alpha\beta} = (\lambda_\alpha - \lambda_\beta)^{-1}$ and

$$\begin{aligned}v_{\alpha\beta}^{(1)} &= \sum_j\sum_k h_{j\alpha}h_{k\beta}v_{jk} - \frac{1}{2}(\lambda_\alpha + \lambda_\beta)\sum_j h_{j\alpha}h_{j\beta}v_{jj}, \\ v_{\alpha\beta}^{(2)} &= \frac{1}{4}\sum_j\sum_k \rho_{jk}h_{j\alpha}h_{k\beta}v_{jj}v_{kk} - \frac{1}{2}\sum_j\sum_k h_{j\alpha}h_{k\beta}v_{jk}(v_{jj} + v_{kk}) \\ &\quad + \frac{3}{8}(\lambda_\alpha + \lambda_\beta)\sum_j h_{j\alpha}h_{j\beta}v_{jj}^2, \qquad (2.6) \\ v_{\alpha\beta}^{(3)} &= \frac{1}{4}\sum_j\sum_k h_{j\alpha}h_{k\beta}v_{jj}v_{kk}v_{jk} - \frac{3}{16}\sum_j\sum_k \rho_{jk}h_{j\alpha}h_{k\beta}v_{jj}v_{kk}(v_{jj} + v_{kk}) \\ &\quad + \frac{3}{8}\sum_j\sum_k h_{j\alpha}h_{k\beta}v_{jk}(v_{jj}^2 + v_{kk}^2) - \frac{5}{16}(\lambda_\alpha + \lambda_\beta)\sum_j h_{j\alpha}h_{j\beta}v_{jj}^3.\end{aligned}$$

Here the summations $\sum_{\beta\neq\alpha}$ and $\sum_j$ stand for $\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{p}$ and $\sum_{j=1}^{p}$, respectively. It may be remarked that the latent roots of R are invariant under the transformation $R \to H'RH$. Let $f(l_1,\ldots, l_p)$ be a real-valued function defined on a domain D in the p-dimensional Euclidean space. We assume that the function f is analytic in a neighborhood of $(\lambda_1,\ldots, \lambda_p)$ $(\in D)$. Expanding the function $f(l_1,\ldots, l_p)$ in Taylor's series about $(l_1,\ldots, l_p) = (\lambda_1,\ldots, \lambda_p)$ and substituting (2.5) in the resulting expansion gives

$$\begin{aligned}(2.7)\qquad f(l_1,\ldots, l_p) &= f(\lambda_1,\ldots, \lambda_p) + \frac{1}{\sqrt{n}}\sum_{\alpha=1}^{p}v_{\alpha\alpha}^{(1)}f_\alpha + \frac{1}{n}\Big\{\sum_{\alpha=1}^{p}v_{\alpha\alpha}^{(2)}f_\alpha \\ &\quad + \sum_{\alpha=1}^{p}\sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)2}f_\alpha + \frac{1}{2}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{p}v_{\alpha\alpha}^{(1)}v_{\beta\beta}^{(1)}f_{\alpha\beta}\Big\} + O_p(n^{-3/2}),\end{aligned}$$

where $f_{\alpha\beta}=\dfrac{\partial^2}{\partial l_\alpha \partial l_\beta} f(l_1,\ldots, l_p)\Big|_{(l_1,\ldots, l_p)=(\lambda_1,\ldots, \lambda_p)}$ and $v_{\alpha\beta}^{(1)}$, $v_{\alpha\alpha}^{(2)}$ are given by (2.6). To derive an asymptotic expansion for the distribution of $f(l_1,\ldots, l_p)$ up to and including the term of order n^{-1}, the term of order $n^{-3/2}$ in (2.7) is needed. It is however very lengthy and hence omitted here.

From (2.7), the characteristic function of $\sqrt{n}\{f(l_1,\ldots, l_p)-f(\lambda_1,\ldots, \lambda_p)\}$ can be written as

$$E\Big[\operatorname{etr}(itAV)\Big\{1+\frac{(it)}{\sqrt{n}}\Big(\sum_{\alpha=1}^{p} v_{\alpha\alpha}^{(2)} f_\alpha+\sum_{\alpha=1}^{p}\sum_{\beta\neq\alpha}^{p}\lambda_{\alpha\beta}v_{\alpha\beta}^{(1)2}f_\alpha + \frac{1}{2}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{p} v_{\alpha\alpha}^{(1)}v_{\beta\beta}^{(1)}f_{\alpha\beta}\Big)+O_p(n^{-1})\Big\}\Big], \tag{2.8}$$

where $\operatorname{etr}(\cdot)$ stands for $\exp\{\operatorname{tr}(\cdot)\}$ and A is a $p\times p$ symmetric matrix having

$$\sum_{\alpha=1}^{p} f_\alpha(1-\delta_{jk}\lambda_\alpha)h_{j\alpha}h_{k\alpha}$$

as its (j, k)th element with the Kronecker delta δ_{jk}.

To calculate each expectation in (2.8), we use Lemma 5.1 in Sugiura [87]. Putting $f(\Gamma)\equiv 1$ and substituting A defined in (2.8) into Sugiura's lemma, we obtain after some calculations

$$\begin{aligned} E[\operatorname{etr}(itAV)] &= \exp\Big(-\frac{1}{2}\tau^2t^2\Big)\Big[1+\frac{(it)^3}{\sqrt{n}}\Big\{\frac{4}{3}\sum_\alpha\lambda_\alpha^3f_\alpha^3-4\sum_\alpha\sum_\beta\lambda_\alpha^3\lambda_\beta f_\alpha^2f_\beta\sum_j h_{j\alpha}^2h_{j\beta}^2 \\ &\quad+\frac{4}{3}\sum_\alpha\sum_\beta\sum_\gamma\lambda_\alpha\lambda_\beta\lambda_\gamma f_\alpha f_\beta f_\gamma\sum_j\sum_k\rho_{jk}h_{j\beta}^2h_{k\gamma}^2(3\lambda_\alpha h_{j\alpha}h_{k\alpha} \\ &\quad-\sum_l\rho_{jl}\rho_{kl}h_{l\alpha}^2)\Big\}+O(n^{-1})\Big], \end{aligned} \tag{2.9}$$

where

$$\tau^2=2\sum_\alpha\sum_\beta\lambda_\alpha\lambda_\beta\{\delta_{\alpha\beta}-(\lambda_\alpha+\lambda_\beta)\sum_j h_{j\alpha}^2h_{j\beta}^2+\sum_j\sum_k\rho_{jk}^2h_{j\alpha}^2h_{k\beta}^2\}f_\alpha f_\beta. \tag{2.10}$$

Similarly, putting $f(\Gamma)=n(\gamma_{jk}-\rho_{jk})(\gamma_{lm}-\rho_{lm})$ and taking A defined in (2.8) gives

$$\begin{aligned} E[\operatorname{etr}(itAV)v_{jk}v_{lm}] &= \exp\Big(-\frac{1}{2}\tau^2t^2\Big)\{\rho_{jl}\rho_{km}+\rho_{jm}\rho_{kl} \\ &\quad+4(it)^2\sum_\beta\sum_\gamma\psi(\beta,j,k)\psi(\gamma,l,m)f_\beta f_\gamma+O(n^{-1/2})\}, \end{aligned} \tag{2.11}$$

where

$$\psi(\beta, j, k) = \lambda_\beta(\lambda_\beta h_{j\beta}h_{k\beta} - \sum_{l=1}^{p} \rho_{jl}\rho_{kl}h_{l\beta}^2). \tag{2.12}$$

It is now relatively easy to calculate each expectation of the term of order $n^{-1/2}$ in (2.8) by using (2.11) and the relation (2.1). Thus we have the following form for the characteristic function of $\sqrt{n}\{f(l_1,\ldots, l_p)-f(\lambda_1,\ldots, \lambda_p)\}$:

$$\exp\Big(-\frac{1}{2}\tau^2t^2\Big)\Big[1 + \frac{1}{\sqrt{n}}\{(it)g_1 + (it)^3g_3\} + O(n^{-1})\Big], \tag{2.13}$$

where τ^2 is given by (2.10) and the coefficients g_1, g_3 are

$$\begin{aligned}
g_1 = &-\frac{1}{2}\sum_\alpha[\lambda_\alpha - \sum_j\sum_k \rho_{jk}^3 h_{j\alpha}h_{k\alpha} \\
&- \sum_{\beta\neq\alpha}\lambda_{\alpha\beta}\{2\lambda_\alpha\lambda_\beta - 4\lambda_\alpha\lambda_\beta(\lambda_\alpha+\lambda_\beta)\sum_j h_{j\alpha}^2h_{j\beta}^2 \\
&+ (\lambda_\alpha+\lambda_\beta)^2\sum_j\sum_k \rho_{jk}^2h_{j\alpha}h_{j\beta}h_{k\alpha}h_{k\beta}\}]f_\alpha \\
&+ \sum_\alpha \lambda_\alpha^2 f_{\alpha\alpha} - \sum_\alpha\sum_\beta \lambda_\alpha\lambda_\beta(2\lambda_\beta\sum_j h_{j\alpha}^2h_{j\beta}^2 - \sum_j\sum_k \rho_{jk}^2h_{j\alpha}^2h_{k\beta}^2)f_{\alpha\beta}, \\
g_3 = &\frac{4}{3}\sum_\alpha \lambda_\alpha^3 f_\alpha^3 - 4\sum_\alpha\sum_\beta \lambda_\alpha^3\lambda_\beta f_\alpha^2 f_\beta \sum_j h_{j\alpha}^2h_{j\beta}^2 \\
&+ \sum_\alpha\sum_\beta\sum_\gamma f_\alpha f_\beta f_\gamma[\frac{4}{3}\lambda_\alpha\lambda_\beta\lambda_\gamma\sum_j\sum_k \rho_{jk}h_{j\beta}^2h_{k\gamma}^2 \\
&\times (3\lambda_\alpha h_{j\alpha}h_{k\alpha} - \sum_l \rho_{jl}\rho_{kl}h_{l\alpha}^2) \\
&+ \sum_j\sum_k h_{j\alpha}h_{k\alpha}\psi(\beta,j,j)\{\rho_{jk}\psi(\gamma,k,k) - 4\psi(\gamma,j,k)\} \\
&+ 3\lambda_\alpha\sum_j h_{j\alpha}^2\psi(\beta,j,j)\psi(\gamma,j,j)] \\
&+ \sum_\alpha\sum_{s\neq\alpha}\lambda_{\alpha s}f_\alpha[\sum_\beta f_\beta\{2\sum_j\sum_k h_{j\alpha}h_{ks}\psi(\beta,j,k) \\
&- (\lambda_\alpha+\lambda_s)\sum_j h_{j\alpha}h_{js}\psi(\beta,j,j)\}]^2 \\
&+ 2\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta f_{\alpha\beta}f_\gamma f_\delta[\sum_j\sum_k\sum_l h_{j\alpha}h_{k\beta}h_{l\beta}\psi(\gamma,k,l) \\
&\times\{\sum_m h_{m\alpha}\psi(\delta,j,m) - 2\lambda_\alpha h_{j\alpha}\psi(\delta,j,j)\} \\
&+ \lambda_\alpha\lambda_\beta\sum_j\sum_k h_{j\alpha}^2h_{k\beta}^2\psi(\gamma,j,j)\psi(\delta,k,k)]
\end{aligned} \tag{2.14}$$

with $\psi(\beta,j,k)$ defined by (2.12).

Inverting this characteristic function, we have the following theorem.

THEOREM 2.1. *Let* $l_1 \geqq \cdots \geqq l_p > 0$ *be the latent roots of the sample correlation matrix* R *and let* $H=(h_{ij})$ *be an orthogonal matrix such that* $H'PH = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, *where* P *is the population correlation matrix and* $\lambda_1 \geqq \cdots \geqq \lambda_p > 0$ *are the ordered latent roots of* P. *Let* $f(l_1,\ldots,l_p)$ *be an analytic function in a neighborhood of* $(l_1,\ldots,l_p)=(\lambda_1,\ldots,\lambda_p)$. *If the latent roots of* P *are all simple and the* τ^2 *given by* (2.10) *is not zero, then the distribution function of*

$$f_1^* = \sqrt{n}\{f(l_1,\ldots,l_p) - f(\lambda_1,\ldots,\lambda_p)\}/\tau$$

can be expanded for large n *as*

$$\Pr(f_1^* < x) = \Phi(x) - \frac{1}{\sqrt{n}}\{g_1\Phi^{(1)}(x)/\tau + g_3\Phi^{(3)}(x)/\tau^3\} + O(n^{-1}), \tag{2.15}$$

where $\Phi^{(j)}(x)$ *denote the* j*th derivatives of the standard normal distribution function* $\Phi(x)$ *and* g_1, g_3 *are given by* (2.14).

From the form of the leading term in (2.13), we have the following

COROLLARY 2.1. *Let* $f(l_1,\ldots,l_p)$ *be a continuously differentiable function in a neighborhood of* $(l_1,\ldots,l_p)=(\lambda_1,\ldots,\lambda_p)$. *If the population roots* λ_α *are all simple, then the limiting distribution of* $\sqrt{n}\{f(l_1,\ldots,l_p)-f(\lambda_1,\ldots,\lambda_p)\}$ *is normal with mean* 0 *and variance* τ^2 *given by* (2.10).

2.2. *An asymptotic expansion when population roots have multiplicity.* We now proceed to the case when the latent roots of the population correlation matrix P have any multiplicities. Let H be an orthogonal matrix such that $H'PH=\mathrm{diag}(\lambda_1,\ldots,\lambda_p)$ where $\lambda_1 \geqq \cdots \geqq \lambda_p > 0$ are the ordered latent roots of P. Assume that P has r distinct latent roots $\theta_1 > \cdots > \theta_r$ with multiplicities $q_1,\ldots,q_r$ as indicated by (1.7). Let L_α $(\alpha=1,\ldots,r)$ be the set of integers $q_1+\cdots+q_{\alpha-1}+1$, $\ldots$, $q_1+\cdots+q_\alpha$ $(q_0 \equiv 0)$.

We make the following assumptions for the real-valued function $f(l_1,\ldots,l_p)$ where $l_1 \geqq \cdots \geqq l_p$ are the ordered latent roots of R:

(i) f is analytic in a neighborhood of $(l_1,\ldots,l_p)=(\lambda_1,\ldots,\lambda_p)$,

(ii) For $j \in L_\alpha$, $k \in L_\beta$

$$\left.\frac{\partial f}{\partial l_j}\right|_{(l_1,\ldots,l_p)=\Theta} = f_\alpha, \quad \left.\frac{\partial^2 f}{\partial l_j \partial l_k}\right|_{(l_1,\ldots,l_p)=\Theta} = f_{\alpha\beta}, \tag{2.16}$$

where $\Theta=(\theta_1,\ldots,\theta_1,\ \theta_2,\ldots,\theta_2,\ldots,\theta_r,\ldots,\theta_r)$ with θ_1 repeated q_1 times, θ_2 repeated q_2 times, and θ_r repeated q_r times.

Under the assumptions (i) and (ii) in (2.16), the Taylor series expansion of $f(l_1,\ldots,l_p)$ about $(l_1,\ldots,l_p)=(\lambda_1,\ldots,\lambda_p)$ can be written in the form

$$f(l_1,\ldots,l_p) = f(\lambda_1,\ldots,\lambda_p) + \sum_{\alpha=1}^{r}\sum_{j\in L_\alpha}(l_j-\lambda_j)f_\alpha$$

$$+\frac{1}{2}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{j\in L_\alpha}\sum_{k\in L_\beta}(l_j-\lambda_j)(l_k-\lambda_k)f_{\alpha\beta}+\cdots.$$

Then, from (2.4) and Lemma 1.2, it follows that

$$\begin{aligned}f(l_1,\ldots,l_p)=f(\lambda_1,\ldots,\lambda_p)&+\frac{1}{\sqrt{n}}\sum_{\alpha=1}^{r}f_\alpha\operatorname{tr}V_{\alpha\alpha}^{(1)}\\&+\frac{1}{n}\Big(\sum_{\alpha=1}^{r}f_\alpha\operatorname{tr}V_{\alpha\alpha}^{(2)}+\sum_{\alpha=1}^{r}\sum_{\beta\neq\alpha}^{r}f_\alpha\theta_{\alpha\beta}\operatorname{tr}V_{\alpha\beta}^{(1)}V_{\beta\alpha}^{(1)}\\&+\frac{1}{2}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}f_{\alpha\beta}\operatorname{tr}V_{\alpha\alpha}^{(1)}\operatorname{tr}V_{\beta\beta}^{(1)}\Big)+O_p(n^{-3/2}),\end{aligned}$$

where $\theta_{\alpha\beta}=(\theta_\alpha-\theta_\beta)^{-1}$ and $V_{\alpha\beta}^{(j)}$ are the submatrices of $V^{(j)}$ partitioned into $q_1,\ldots,q_r$ rows and columns. Here the (α',β')th elements of $V_{\alpha\beta}^{(j)}$, say $v_{\alpha'\beta'}^{(j)}$, are

$$v_{\alpha'\beta'}^{(1)}=\sum_{j=1}^{p}\sum_{k=1}^{p}h_{j\alpha'}h_{k\beta'}v_{jk}-\frac{1}{2}(\theta_\alpha+\theta_\beta)\sum_{j=1}^{p}h_{j\alpha'}h_{j\beta'}v_{jj},$$

$$\begin{aligned}v_{\alpha'\beta'}^{(2)}=&\frac{1}{4}\sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}h_{j\alpha'}h_{k\beta'}v_{jj}v_{kk}-\frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{p}h_{j\alpha'}h_{k\beta'}v_{jk}(v_{jj}+v_{kk})\\&+\frac{3}{8}(\theta_\alpha+\theta_\beta)\sum_{j=1}^{p}h_{j\alpha'}h_{j\beta'}v_{jj}^2.\end{aligned}$$

The characteristic function of $\sqrt{n}\{f(l_1,\ldots,l_p)-f(\lambda_1,\ldots,\lambda_p)\}$ can therefore be expressed in the form

$$\begin{aligned}(2.17)\quad E\Big[\operatorname{etr}(itAV)\Big\{1+\frac{(it)}{\sqrt{n}}\Big(&\sum_{\alpha=1}^{r}f_\alpha\operatorname{tr}V_{\alpha\alpha}^{(2)}+\sum_{\alpha=1}^{r}\sum_{\beta\neq\alpha}^{r}f_\alpha\theta_{\alpha\beta}\operatorname{tr}V_{\alpha\beta}^{(1)}V_{\beta\alpha}^{(1)}\\&+\frac{1}{2}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}f_{\alpha\beta}\operatorname{tr}V_{\alpha\alpha}^{(1)}\operatorname{tr}V_{\beta\beta}^{(1)}\Big)+O_p(n^{-1})\Big\}\Big],\end{aligned}$$

where A is a symmetric matrix of order p having

$$\sum_{\alpha=1}^{r}\sum_{\alpha'\in L_\alpha}f_\alpha(1-\theta_\alpha\delta_{jk})h_{j\alpha'}h_{k\alpha'}$$

as its (j,k)th element. Each expectation in (2.17) can be calculated by an argument similar to that discussed in Section 2.1. The result is of the form

$$\begin{aligned}(2.18)\quad &E[\exp(it\sqrt{n}\{f(l_1,\ldots,l_p)-f(\lambda_1,\ldots,\lambda_p)\})]\\&=\exp\Big(-\frac{1}{2}\tau^2t^2\Big)\Big[1+\frac{1}{\sqrt{n}}\{(it)g_1+(it)^3g_3\}+O(n^{-1})\Big],\end{aligned}$$

where

$$\tau^2 = 2\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{\alpha'\in L_\alpha}\sum_{\beta'\in L_\beta}\theta_\alpha\theta_\beta\{\delta_{\alpha'\beta'} - (\theta_\alpha+\theta_\beta)\sum_{j=1}^{p}h_{j\alpha'}^2h_{j\beta'}^2 + \sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^2h_{j\alpha'}^2h_{k\beta'}^2\}f_\alpha f_\beta \tag{2.19}$$

and

$$\begin{aligned}
g_1 = &-\frac{1}{2}\sum_{\alpha=1}^{r}\sum_{\alpha'\in L_\alpha}[\theta_\alpha - \sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^3h_{j\alpha'}h_{k\alpha'} \\
&- \sum_{\beta\neq\alpha}^{r}\sum_{\beta'\in L_\beta}\theta_{\alpha\beta}\{2\theta_\alpha\theta_\beta - 4\theta_\alpha\theta_\beta(\theta_\alpha+\theta_\beta)\sum_{j=1}^{p}h_{j\alpha'}^2h_{j\beta'}^2 \\
&+ (\theta_\alpha+\theta_\beta)^2\sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^2h_{j\alpha'}h_{j\beta'}h_{k\alpha'}h_{k\beta'}\}]f_\alpha \\
&+ \sum_{\alpha=1}^{r}q_\alpha\theta_\alpha^2f_{\alpha\alpha} - \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{\alpha'\in L_\alpha}\sum_{\beta'\in L_\beta}\theta_\alpha\theta_\beta \\
&\times (2\theta_\alpha\sum_{j=1}^{p}h_{j\alpha'}^2h_{j\beta'}^2 - \sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^2h_{j\alpha'}^2h_{k\beta'}^2)f_{\alpha\beta}, \\
g_3 = &\frac{4}{3}\sum_{\alpha=1}^{r}q_\alpha\theta_\alpha^3f_\alpha^3 - 4\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{\alpha'\in L_\alpha}\sum_{\beta'\in L_\beta}\theta_\alpha^3\theta_\beta f_\alpha^2f_\beta\sum_{j=1}^{p}h_{j\alpha'}^2h_{j\beta'}^2 \\
&+ \frac{4}{3}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{r}\sum_{\alpha'\in L_\alpha}\sum_{\beta'\in L_\beta}\sum_{\gamma'\in L_\gamma}\theta_\alpha\theta_\beta\theta_\gamma f_\alpha f_\beta f_\gamma \\
&\times \sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}h_{j\beta'}^2h_{k\gamma'}^2(3\theta_\alpha h_{j\alpha'}h_{k\alpha'} - \sum_{l=1}^{p}\rho_{jl}\rho_{kl}h_{l\alpha'}^2) \\
&+ \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{r}\sum_{\alpha'\in L_\alpha}f_\alpha f_\beta f_\gamma[\sum_{j=1}^{p}\sum_{k=1}^{p}h_{j\alpha'}h_{k\alpha'}\psi(\beta,j,j) \\
&\times \{\rho_{jk}\psi(\gamma,k,k) - 4\psi(\gamma,j,k)\} \\
&+ 3\theta_\alpha\sum_{j=1}^{p}h_{j\alpha'}^2\psi(\beta,j,j)\psi(\gamma,j,j)] \\
&+ \sum_{\alpha=1}^{r}\sum_{s\neq\alpha}^{r}\theta_{\alpha s}f_\alpha\sum_{\alpha'\in L_\alpha}\sum_{s'\in L_s}[\sum_{\beta=1}^{r}f_\beta\{2\sum_{j=1}^{p}\sum_{k=1}^{p}h_{j\alpha'}h_{ks'}\psi(\beta,j,k) \\
&- (\theta_\alpha+\theta_s)\sum_{j=1}^{p}h_{j\alpha'}h_{js'}\psi(\beta,j,j)\}]^2 \\
&+ 2\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{r}\sum_{\delta=1}^{r}\sum_{\alpha'\in L_\alpha}\sum_{\beta'\in L_\beta}f_{\alpha\beta}f_\gamma f_\delta[\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}h_{j\alpha'}h_{k\beta'}h_{l\beta'} \\
&\times \psi(\gamma,k,l)\{\sum_{m=1}^{p}h_{m\alpha'}\psi(\delta,j,m) - 2\theta_\alpha h_{j\alpha'}\psi(\gamma,j,j)\} \\
&+ \theta_\alpha\theta_\beta\sum_{j=1}^{p}\sum_{k=1}^{p}h_{j\alpha'}^2h_{k\beta'}^2\psi(\gamma,j,j)\psi(\delta,k,k)]
\end{aligned} \tag{2.20}$$

with

$$\psi(\alpha, j, k) = \sum_{\alpha' \in L_\alpha} \theta_\alpha(\theta_\alpha h_{j\alpha'} h_{k\alpha'} - \sum_{l=1}^{p} \rho_{jl}\rho_{kl} h_{l\alpha'}^2). \tag{2.21}$$

Inversion of the characteristic function (2.18) yields

THEOREM 2.2. *Let* $l_1 \geqq \cdots \geqq l_p > 0$ *be the latent roots of the sample correlation matrix* R, *and let* $H=(h_{ij})$ *be an orthogonal matrix such that* $H'PH = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ *where* $\lambda_1 \geqq \cdots \geqq \lambda_p > 0$. *Assume that the ordered latent roots* $\lambda_1 \geqq \cdots \geqq \lambda_p$ *have the multiplicities indicated by* (1.7) *and that the* τ^2 *given by* (2.19) *is not zero. Then, under the assumptions in* (2.16), *the distribution function of*

$$f_2^* = \sqrt{n}\{f(l_1, \ldots, l_p) - f(\lambda_1, \ldots, \lambda_p)\}/\tau$$

can be expanded for large n *as*

$$\Pr(f_2^* < x) = \Phi(x) - \frac{1}{\sqrt{n}}\{g_1\Phi^{(1)}(x)/\tau + g_3\Phi^{(3)}(x)/\tau^3\} + O(n^{-1}), \tag{2.22}$$

where $\Phi^{(j)}(x)$ *are the* j*th derivatives of the standard normal distribution function* $\Phi(x)$ *and* g_1, g_3 *are given by* (2.20).

From the form of the leading term in (2.18), we have

COROLLARY 2.2. *Let* $f(l_1, \ldots, l_p)$ *be a continuously differentiable function in a neighborhood of* $(l_1, \ldots, l_p) = (\lambda_1, \ldots, \lambda_p)$. *Under the condition* (1.7) *for the population roots and the assumption* (ii) *in* (2.16) *for the first derivatives of* f, *the limiting distribution of* $\sqrt{n}\{f(l_1, \ldots, l_p) - f(\lambda_1, \ldots, \lambda_p)\}$ *is normal with mean* 0 *and variance* τ^2 *given by* (2.19).

Lee and Krishnaiah [59] have recently obtained asymptotic expressions for the joint densities of the ratios of linear combinations of the latent roots of the sample correlation matrices in real and complex multivariate normal samples, when the population roots are all simple. Konishi [45] has obtained an asymptotic expansion for the distribution of a function of the latent roots of the sample covariance matrix in the case when the population roots are simple and derived, as special cases of the results, asymptotic expansions for the distributions of various ratios of latent roots. Recently Fujikoshi [26] has obtained asymptotic expansions for the distributions of some functions of the latent roots of the random matrices associated with principal component analysis, MANOVA model and canonical correlation analysis, when the population roots have any multiplicities.

3. Asymptotic expansions for the distributions of statistics associated with reduction of dimensionality in principal component analysis

3.1. *Certain functions of latent roots.* Principal component analysis was first introduced by Pearson [72] for the case of nonstochastic variables, and later Hotelling [36] generalized this concept to the case of stochastic variables. Briefly principal component analysis based on a correlation matrix is as follows:

Consider a p-dimensional random vector $\boldsymbol{X}'=(X_1,\ldots, X_p)$ with correlation matrix P. Let $\lambda_1 \geqq \cdots \geqq \lambda_p$ be the ordered latent roots and $\boldsymbol{h}_1,\ldots, \boldsymbol{h}_p$ be the corresponding orthonormal latent vectors of P. Then, from the spectral decomposition theorem it follows that

$$
\begin{aligned}
&P=\lambda_1\boldsymbol{h}_1\boldsymbol{h}_1'+\cdots+\lambda_p\boldsymbol{h}_p\boldsymbol{h}_p', \\
&\boldsymbol{h}_\alpha' P\boldsymbol{h}_\alpha=\lambda_\alpha, \quad \boldsymbol{h}_\alpha' P\boldsymbol{h}_\beta=0 \qquad \text{if} \quad \alpha\neq\beta.
\end{aligned} \tag{3.1}
$$

The linear combination $y_1=\boldsymbol{h}_1'\boldsymbol{X}$ is called the first principal component of $\boldsymbol{X}$. Generally, the αth principal component of $\boldsymbol{X}$ is given by $y_\alpha=\boldsymbol{h}_\alpha'\boldsymbol{X}$. From (3.1) it is easy to see that the principal components have the properties

$$
\begin{aligned}
&\operatorname{Var}(y_\alpha) = \boldsymbol{h}_\alpha' P\boldsymbol{h}_\alpha = \lambda_\alpha, \\
&\operatorname{Cov}(y_\alpha, y_\beta) = \boldsymbol{h}_\alpha' P\boldsymbol{h}_\beta = 0.
\end{aligned} \tag{3.2}
$$

It may be found that the principal component analysis is algebraically based on the latent roots and vectors of a correlation matrix or a covariance matrix. For further details of principal component analysis we may refer to Anderson [5, Chapter 11], Kshirsagar [54, Chapter 11], Morrison [61, Chapter 8] or Rao [77, Chapter 8].

In practice, it is necessary to know how much information is condensed into the principal components. From (3.2) the total variance of the p principal components is $\sum_{\alpha=1}^{p}\lambda_\alpha=\operatorname{tr} P=p$. The importance of the αth principal component in accounting for the total variance is measured by λ_α/p. Furthermore, the fraction of the total variance accounted for by the first q principal components is measured by $\sum_{\alpha=1}^{q}\lambda_\alpha/p$ $(q<p)$, which was suggested by Rao [76].

Considering the inference problems based on the sample data which are drawn from a p-variate normal population with covariance matrix Σ, we use the maximum likelihood estimate of P, namely, the sample correlation matrix R. When the population roots are simple, the λ_α/p and $\sum_{\alpha=1}^{q}\lambda_\alpha/p$ are estimated by

$$
l_\alpha/p \quad \text{and} \quad \sum_{\alpha=1}^{q} l_\alpha/p \qquad (q<p),
$$

respectively, where $l_1 \geqq \cdots \geqq l_p$ are the latent roots of R. Konishi [46] has ob-

tained asymptotic expansions for the distributions of l_α and $\sum_{\alpha=1}^{q} l_\alpha/p$ when the corresponding population roots are simple.

We now give, using Theorem 2.1, an asymptotic expansion for the distribution of l_α/p. Taking $f(l_1,\ldots, l_p)=l_\alpha/p$ in Theorem 2.1, we obtain the following

THEOREM 3.1. *Let l_α be the αth largest latent root of the sample correlation matrix R and let $H=(h_{ij})$ be an orthogonal matrix such that $H'PH=\mathrm{diag}(\lambda_1,\ldots, \lambda_p)$ with the ordered latent roots $\lambda_1 \geqq \cdots \geqq \lambda_p > 0$. If λ_α is simple, then an asymptotic expansion for the distribution of l_α/p is given by*

$$\Pr\{\sqrt{n}(l_\alpha/p - \lambda_\alpha/p)/\tau_\alpha < x\}$$

$$= \Phi(x) - \frac{1}{\sqrt{n}}\{g_1\Phi^{(1)}(x)/(p\tau_\alpha) + g_3\Phi^{(3)}(x)/(p\tau_\alpha)^3\} + O(n^{-1}),$$

where

$$\tau_\alpha^2 = 2\lambda_\alpha^2(1 - 2\lambda_\alpha\sum_j h_{j\alpha}^4 + \sum_j\sum_k \rho_{jk}^2 h_{j\alpha}^2 h_{k\alpha}^2)/p^2$$

and

$$g_1 = -\frac{1}{2}[\lambda_\alpha - \sum_j\sum_k \rho_{jk}^3 h_{j\alpha}h_{k\alpha} - \sum_{\beta\neq\alpha}\lambda_{\alpha\beta}\{2\lambda_\alpha\lambda_\beta$$

$$- 4\lambda_\alpha\lambda_\beta(\lambda_\alpha+\lambda_\beta)\sum_j h_{j\alpha}^2 h_{j\beta}^2 + (\lambda_\alpha+\lambda_\beta)^2\sum_j\sum_k \rho_{jk}^2 h_{j\alpha}h_{j\beta}h_{k\alpha}h_{k\beta}\}],$$

$$\text{(3.3)}\quad g_3 = \frac{4}{3}\lambda_\alpha^3 - 4\lambda_\alpha^4\sum_j h_{j\alpha}^4 + \frac{4}{3}\lambda_\alpha^3\sum_j\sum_k \rho_{jk}h_{j\alpha}^2 h_{k\alpha}^2(3\lambda_\alpha h_{j\alpha}h_{k\alpha} - \sum_l \rho_{jl}\rho_{kl}h_{l\alpha}^2)$$

$$+ \sum_j\sum_k h_{j\alpha}h_{k\alpha}\psi(\alpha,j,j)\{\rho_{jk}\psi(\alpha,k,k) - 4\psi(\alpha,j,k)\}$$

$$+ 3\lambda_\alpha\sum_j h_{j\alpha}^2\psi(\alpha,j,j)^2 + \sum_{s\neq\alpha}\lambda_{\alpha s}\{2\sum_j\sum_k h_{j\alpha}h_{ks}\psi(\alpha,j,k)$$

$$- (\lambda_\alpha+\lambda_s)\sum_j h_{j\alpha}h_{js}\psi(\alpha,j,j)\}^2,$$

where the summations $\sum_j$ and $\sum_{\beta\neq\alpha}$ stand for $\sum_{j=1}^{p}$ and $\sum_{\beta\neq\alpha}^{p}$, respectively.

It may be remarked that this result holds if only the λ_α is distinct from other $p-1$ latent roots which may have any multiplicities.

Now recall that the formula (2.13) gives the expansion of the characteristic function of $\sqrt{n}\{f(l_1,\ldots, l_p)-f(\lambda_1,\ldots, \lambda_p)\}$. Taking $f(l_1,\ldots, l_p)=\sum_{\alpha=1}^{p} t_\alpha l_\alpha$ and putting $t=1$ in (2.13), we obtain the expansion of the joint characteristic function of $\sqrt{n}(l_\alpha-\lambda_\alpha)$, $\alpha=1,\ldots, p$, in the form

$$\text{(3.4)}\quad \exp\Big(-\frac{1}{2}\boldsymbol{t}'\Omega\boldsymbol{t}\Big)\Big\{1 + \frac{1}{\sqrt{n}}(i\sum_{\alpha=1}^{p} b_\alpha t_\alpha + i^3\sum_{\alpha=1}^{p}\sum_{\beta=1}^{p}\sum_{\gamma=1}^{p} b_{\alpha\beta\gamma}t_\alpha t_\beta t_\gamma) + O(n^{-1})\Big\},$$

where $\boldsymbol{t}'=(t_1,\ldots, t_p)$, $\Omega=(\omega_{\alpha\beta})$ is a $p\times p$ symmetric matrix with

$$\omega_{\alpha\beta} = 2\lambda_\alpha\lambda_\beta\{\delta_{\alpha\beta} - (\lambda_\alpha + \lambda_\beta)\sum_j h_{j\alpha}^2 h_{j\beta}^2 + \sum_j\sum_k \rho_{jk}^2 h_{j\alpha}^2 h_{k\beta}^2\}, \tag{3.5}$$

and b_α is given by g_1 in (3.3) and $b_{\alpha\beta\gamma}$ is

$$\begin{aligned} b_{\alpha\beta\gamma} = {} & \frac{4}{3}\lambda_\alpha^3\delta_{\alpha\beta}\delta_{\beta\gamma} - 4\lambda_\alpha^3\lambda_\beta\delta_{\alpha\gamma}\sum_j h_{j\alpha}^2 h_{j\beta}^2 \\ & + \frac{4}{3}\lambda_\alpha\lambda_\beta\lambda_\gamma\sum_j\sum_k \rho_{jk} h_{j\beta}^2 h_{k\gamma}^2 (3\lambda_\alpha h_{j\alpha}h_{k\alpha} - \sum_l \rho_{jl}\rho_{kl} h_{l\alpha}^2) \\ & + \sum_j\sum_k h_{j\alpha}h_{k\alpha}\psi(\beta,j,j)\{\rho_{jk}\psi(\gamma,k,k) - 4\psi(\gamma,j,k)\} \\ & + 3\lambda_\alpha\sum_j h_{j\alpha}^2\psi(\beta,j,j)\psi(\gamma,j,j) \\ & + \sum_{s\neq\alpha}\lambda_{\alpha s}\{2\sum_j\sum_k h_{j\alpha}h_{ks}\psi(\beta,j,k) - (\lambda_\alpha+\lambda_s)\sum_j h_{j\alpha}h_{js}\psi(\beta,j,j)\} \\ & \times\{2\sum_j\sum_k h_{j\alpha}h_{ks}\psi(\gamma,j,k) - (\lambda_\alpha+\lambda_s)\sum_j h_{j\alpha}h_{js}\psi(\gamma,j,j)\}. \end{aligned}$$

The limiting joint distribution of $\sqrt{n}(l_\alpha-\lambda_\alpha)$, $\alpha=1,\ldots, p$, is singular normal with mean vector $(0,\ldots, 0)'$ and covariance matrix $\Omega=(\omega_{\alpha\beta})$ given by (3.5), since $\sum_{\alpha=1}^p l_\alpha = \operatorname{tr} R = p$, and so the inversion of (3.4) is very complicated. An asymptotic expansion for the joint density function of any q $(q<p)$ set of $\{\sqrt{n}(l_\alpha-\lambda_\alpha);$ $\alpha=1,\ldots, p\}$ is obtainable, using the formula (3.4). For the sake of simplicity we give an asymptotic expansion for the joint density function of $\sqrt{n}(l_\alpha-\lambda_\alpha)$, $\alpha=1,\ldots, q$, in the following:

THEOREM 3.2. *Let $\Omega=(\omega_{\alpha\beta})$ be a $q\times q$ symmetric matrix with $\omega_{\alpha\beta}$ given by* (3.5). *If $\lambda_1,\ldots,\lambda_q$ are simple, then an asymptotic expansion for the joint density function of*

$$\boldsymbol{x}' = \{\sqrt{n}(l_1-\lambda_1),\ldots,\sqrt{n}(l_q-\lambda_q)\} \qquad (q<p)$$

is given by

$$\phi(\boldsymbol{x};\Omega)\Big[1 + \frac{1}{\sqrt{n}}\{\sum_{\alpha=1}^q b_\alpha H_\alpha(\boldsymbol{x};\Omega) + \sum_{\alpha=1}^q\sum_{\beta=1}^q\sum_{\gamma=1}^q b_{\alpha\beta\gamma}H_{\alpha\beta\gamma}(\boldsymbol{x};\Omega)\} + O(n^{-1})\Big],$$

where

$$\phi(\boldsymbol{x};\Omega) = \frac{1}{(2\pi)^{q/2}|\Omega|^{1/2}}\exp\Big(-\frac{1}{2}\boldsymbol{x}'\Omega^{-1}\boldsymbol{x}\Big),$$

for $1\leqq i_\alpha\leqq q$ and $\boldsymbol{x}'=(x_1,\ldots, x_q)$

$$H_{i_1\ldots i_r}(\boldsymbol{x};\Omega) = \frac{(-1)^r}{\phi(\boldsymbol{x};\Omega)}\frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}}\phi(\boldsymbol{x};\Omega)$$

and the coefficients b_α, $b_{\alpha\beta\gamma}$ *are given in* (3.4).

When the population roots have any multiplicities as in (1.7), the θ_d and the fraction $\sum_{\alpha=1}^{q} q_\alpha\theta_\alpha/p$ $(q<r)$ of the total variance accounted for by the first $q_1+\cdots+q_q$ principal components are estimated by

$$\bar{l}_d = \sum_{\alpha\in L_d} l_\alpha/q_d \quad \text{and} \quad \sum_{\alpha=1}^{q} \sum_{\alpha'\in L_\alpha} l_{\alpha'}/p,$$

respectively, where L_α is the set of integers $q_1+\cdots+q_{\alpha-1}+1,\ldots, q_1+\cdots+q_\alpha$ $(q_0\equiv 0)$.

Applying Theorem 2.2 to the case of $\bar{l}_d$, we have the following

THEOREM 3.3. *Under the same notations as in Theorem* 2.2, *an asymptotic expansion for the distribution of* $\bar{l}_d = \sum_{\alpha\in L_d} l_\alpha/q_d$ *is given by*

$$\Pr\{\sqrt{n}(\bar{l}_d-\theta_d)/\tau_d < x\}$$
$$= \Phi(x) - \frac{1}{\sqrt{n}}\{g_1\Phi^{(1)}(x)/(q_d\tau_d) + g_3\Phi^{(3)}(x)/(q_d\tau_d)^3\} + O(n^{-1}),$$

where

$$\tau_d^2 = 2\sum_{\alpha\in L_d}\sum_{\beta\in L_d}\theta_d^2\{\delta_{\alpha\beta} - 2\theta_d\sum_j h_{j\alpha}^2h_{j\beta}^2 + \sum_j\sum_k \rho_{jk}^2h_{j\alpha}^2h_{k\beta}^2\}/q_d^2 \tag{3.6}$$

and

$$g_1 = -\frac{1}{2}\sum_{\alpha\in L_d}[\theta_d - \sum_j\sum_k \rho_{jk}^3h_{j\alpha}h_{k\alpha}$$
$$- \sum_{\beta\neq d}^{r}\sum_{\beta'\in L_\beta}\theta_{d\beta}\{2\theta_d\theta_\beta - 4\theta_d\theta_\beta(\theta_d+\theta_\beta)\sum_j h_{j\alpha}^2h_{j\beta'}^2$$
$$+ (\theta_d+\theta_\beta)^2\sum_j\sum_k \rho_{jk}^2h_{j\alpha}h_{j\beta'}h_{k\alpha}h_{k\beta'}\}],$$

$$g_3 = \frac{4}{3}q_d\theta_d^3 - 4\sum_{\alpha\in L_d}\sum_{\beta\in L_d}\theta_d^4\sum_j h_{j\alpha}^2h_{j\beta}^2$$
$$+ \frac{4}{3}\sum_{\alpha\in L_d}\sum_{\beta\in L_d}\sum_{\gamma\in L_d}\theta_d^3\sum_j\sum_k \rho_{jk}h_{j\beta}^2h_{k\gamma}^2(3\theta_d h_{j\alpha}h_{k\alpha} - \sum_l \rho_{jl}\rho_{kl}h_{l\alpha}^2)$$
$$+ \sum_{\alpha\in L_d}[\sum_j\sum_k h_{j\alpha}h_{k\alpha}\psi(d,j,j)\{\rho_{jk}\psi(d,k,k) - 4\psi(d,j,k)\}$$
$$+ 3\theta_d\sum_j h_{j\alpha}^2\psi(d,j,j)^2]$$
$$+ \sum_{s\neq d}^{r}\theta_{ds}\sum_{\alpha\in L_d}\sum_{s'\in L_s}\{2\sum_j\sum_k h_{j\alpha}h_{ks'}\psi(d,j,k)$$
$$- (\theta_d+\theta_s)\sum_j h_{j\alpha}h_{js'}\psi(d,j,j)\}^2$$

with

$$\psi(d,j,k) = \sum_{\alpha\in L_d} \theta_d(\theta_d h_{j\alpha}h_{k\alpha} - \sum_l \rho_{jl}\rho_{kl}h_{l\alpha}^2).$$

From Theorem 3.3 it follows that the limiting distribution of $\sqrt{n}(\bar{l}_d-\theta_d)$ is normal with mean 0 and variance τ_d^2 given by (3.6) and so in general the variance depends on H and P. In particular case of P with the two distinct latent roots θ_1 and θ_2 of respective multiplicities q_1 and q_2, Anderson [6] has obtained the confidence interval for θ_2, showing that

$$\frac{\sqrt{n}(\bar{l}_2 - \theta_2)}{\sqrt{2}\theta_2(p - q_2\theta_2)(pq_1q_2)^{-1/2}}$$

is asymptotically normally distributed with mean 0 and variance 1. It may be remarked that the term of order $1/\sqrt{n}$ in an asymptotic expansion formula for $\bar{l}_2$ contains H and the latent roots other that θ_2.

An asymptotic expansion for the distribution of l_α is obtainable, provided that the corresponding population root is simple (cf. Konishi [46]). Unfortunately, a similar result for the l_α corresponding to the same multiple population root can not be obtained, since the limiting distribution of l_α is no longer normal.

Now, putting $f(l_1,\ldots, l_p)=\sum_{\alpha=1}^{q}\sum_{\alpha'\in L_\alpha} l_{\alpha'}/p$ in Theorem 2.2 and differentiating f with respect to l_α at $(l_1,\ldots, l_p)=(\lambda_1,\ldots, \lambda_p)$, we have

THEOREM 3.4. *Under the same notations as in Theorem* 2.2, *an asymptotic expansion for the distribution of* $\sum_{\alpha=1}^{q}\sum_{\alpha'\in L_\alpha} l_{\alpha'}/p$ $(q<r)$ *is given by*

$$\Pr\{\sqrt{n}(\sum_{\alpha=1}^{q}\sum_{\alpha'\in L_\alpha} l_{\alpha'}/p - \sum_{\alpha=1}^{q} q_\alpha\theta_\alpha/p)/\tau_{(q)} < x\}$$

$$= \Phi(x) - \frac{1}{\sqrt{n}}\{g_1\Phi^{(1)}(x)/(p\tau_{(q)}) + g_3\Phi^{(3)}(x)/(p\tau_{(q)})^3\} + O(n^{-1}),$$

where

$$\tau_{(q)}^2 = 2\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q}\sum_{\alpha'\in L_\alpha}\sum_{\beta'\in L_\beta}\theta_\alpha\theta_\beta\{\delta_{\alpha'\beta'} - (\theta_\alpha+\theta_\beta)\sum_{j=1}^{p} h_{j\alpha'}^2h_{j\beta'}^2$$

$$+ \sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^2h_{j\alpha'}^2h_{k\beta'}^2\}/p^2,$$

and the coefficients g_1, g_3 *are given by* (2.20) *with* $f_\alpha=1$ *for* $\alpha=1,\ldots, q$, $f_\alpha=0$ *for* $\alpha=q+1,\ldots, r$ *and* $f_{\alpha\beta}=0$ *for* $\alpha, \beta=1,\ldots, r$.

The problem of deriving the distributions of latent roots of the sample covariance matrix S/n has been studied by many authors. James [39] has obtained the exact joint density function of the latent roots of S/n in a form involving

a hypergeometric function of matrix argument (see Constantine [14] and Herz [35]). Some work has been done on approximations to the hypergeometric function. Asymptotic results in this area have been given by G. A. Anderson [3] and Muirhead and Chikuse [64] in the case when the population roots are simple, and by Chattopadhyay and Pillai [10], Chikuse [13], Constantine and Muirhead [15] and James [41] in the case when the population roots have multiplicity. It is however difficult to obtain results concerning the latent roots of the sample correlation matrix by arguments similar to those discussed in the literatures.

Anderson [6] has obtained the asymptotic distributions of the latent roots and vectors of S, when the population roots have any multiplicities. The method discussed in Anderson [4, 6] is useful in obtaining the asymptotic distributions of the latent roots and vectors of Wishart matrices except for the case of the sample correlation matrix. Sugiura [87] has obtained an asymptotic expansion for the distribution of the αth largest latent root of S, when the corresponding population root is simple. His result includes the limiting distribution given by Girshick [29]. In the case of multiple population roots, Chikuse [12] and Sugiura [91] have derived the asymptotic expansion of the joint density function of the latent roots of S from the results given by Chattopadhyay and Pillai [10]. The expansion of this type has also been derived by Fujikoshi [24] based on the perturbation method with multiple population roots.

3.2. *Tests for latent roots of a correlation matrix.* On the basis of $N=n+1$ observations drawn from a p-variate normal population $N_p(\boldsymbol{\mu}, \Sigma)$, we consider first a test of the hypothesis

$$H_0: \lambda_1 = \lambda_2 = \cdots = \lambda_p$$

that all of the latent roots of P are equal. It is easy to see that H_0 is equivalent to the hypothesis $H_0^*: P=I$ and that the likelihood ratio criterion is given by $\Lambda_0 = |R|^{N/2}$. Bartlett [7] has shown that the statistic

$$-\left\{n - \frac{1}{6}(2p+5)\right\} \log |R|$$

is asymptotically distributed as a chi-square distribution with $p(p-1)/2$ degrees of freedom when H_0 is true. The asymptotic nonnull distribution of $|R|$ will be discussed in Section 6.2.

If the hypothesis H_0 is rejected, then it is of interest to test the hypothesis that the last $q=p-a$ latent roots of P are equal, namely,

$$H_1: \lambda_{a+1} = \lambda_{a+2} = \cdots = \lambda_p,$$

which is one of the most commonly used tests in reduction of dimensionality in

principal component analysis. Anderson [6] has considered a test of the equality of any subset of the latent roots of P, which includes the hypothesis H_1. The hypothesis H_1 is tested by using the statistic

$$\Lambda_1 = [\prod_{j=a+1}^{p} l_j/(q^{-1}\sum_{j=a+1}^{p} l_j)^q]^{N/2}. \tag{3.7}$$

Bartlett [7] has shown that

$$-\frac{2}{N}\left\{n - \frac{1}{6}(2p+5) - \frac{2}{3}a\right\}\log \Lambda_1$$

can be approximated as a chi-square distribution with $(q-1)(q+2)/2$ degrees of freedom when H_1 is true.

We now give, using Theorem 2.1, asymptotic nonnull distribution of the statistic Λ_1 when the population roots $\lambda_{a+1},\ldots,\lambda_p$ are all simple. Put

$$f(l_1,\ldots,l_p) = -\frac{2}{N}\log \Lambda_1 = q\log(q^{-1}\sum_{j=a+1}^{p} l_j) - \sum_{j=a+1}^{p}\log l_j$$

in Theorem 2.1. Partial derivatives of $f(l_1,\ldots,l_p)$ at $(l_1,\ldots,l_p)=(\lambda_1,\ldots,\lambda_p)$ are

$$\begin{aligned} &f_\alpha = q(\sum_{j=a+1}^{p}\lambda_j)^{-1} - \lambda_\alpha^{-1}, \quad f_{\alpha\alpha} = -q(\sum_{j=a+1}^{p}\lambda_j)^{-2} + \lambda_\alpha^{-2}, \\ &f_{\alpha\beta} = -q(\sum_{j=a+1}^{p}\lambda_j)^{-2} \ (\alpha \neq \beta) \qquad \text{for} \quad \alpha, \beta = a+1,\ldots,p \end{aligned} \tag{3.8}$$

and other first and second partial derivatives are all zero. Then we have the following

THEOREM 3.5. *If the population roots $\lambda_{a+1},\ldots,\lambda_p$ are all simple, then an asymptotic expansion for the nonnull distribution of the statistic Λ_1 for testing the equality of the last $q=p-a$ latent roots of P are given by*

$$\begin{aligned} &\Pr\left[\sqrt{N}\left\{-\frac{2}{N}\log \Lambda_1 + \sum_{j=a+1}^{p}\log\lambda_j - q\log(q^{-1}\sum_{j=a+1}^{p}\lambda_j)\right\}\Big/\tau_{1q} < x\right] \\ &= \Phi(x) - \frac{1}{\sqrt{n}}\{g_1\Phi^{(1)}(x)/\tau_{1q} + g_3\Phi^{(3)}(x)/\tau_{1q}^3\} + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \tau_{1q}^2 = 2\sum_{\alpha=a+1}^{p}\sum_{\beta=a+1}^{p}&(q\lambda_\alpha - \sum_{j=a+1}^{p}\lambda_j)(q\lambda_\beta - \sum_{j=a+1}^{p}\lambda_j)(\sum_{j=a+1}^{p}\lambda_j)^{-2} \\ &\times\{\delta_{\alpha\beta} - (\lambda_\alpha+\lambda_\beta)\sum_{j=1}^{p}h_{j\alpha}^2h_{j\beta}^2 + \sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^2h_{j\alpha}^2h_{k\beta}^2\} \end{aligned}$$

and the coefficients g_1, g_3 are given by (2.14) *with the partial derivatives* (3.8).

We now consider the hypothesis

$$H_2: \lambda_{a+1} = \lambda_{a+2} = \cdots = \lambda_p = \lambda_0 \qquad (\lambda_0 > 0)$$

that the last $q=p-a$ latent roots of P are equal to a specified value λ_0 (>0). This hypothesis may be tested by using the statistic

$$\Lambda_2 = \{ \prod_{j=a+1}^{p} l_j/\lambda_0^q \}^{N/2} \exp\left\{ -\frac{1}{2} N(\sum_{j=a+1}^{p} l_j/\lambda_0 - q) \right\}, \tag{3.9}$$

which is based on the likelihood ratio criterion for testing the hypothesis H_2 concerning the latent roots of the covariance matrix Σ, due to Anderson [6]. Taking $f(l_1,\ldots, l_p) = -(2/N)\log \Lambda_2$ in Theorem 2.1, we obtain the following

THEOREM 3.6. *If the population roots $\lambda_{a+1},\ldots, \lambda_p$ are all simple, then an asymptotic expansion for the nonnull distribution of the statistic Λ_2 for testing the equality of the last $q=p-a$ latent roots of P to a specified value λ_0 is given by*

$$\Pr\left[\sqrt{n} \left\{ -\frac{2}{N}\log \Lambda_2 + \sum_{j=a+1}^{p} \log \lambda_j - \sum_{j=a+1}^{p} \lambda_j/\lambda_0 + q(1 - \log \lambda_0) \right\} \Big/ \tau_{2q} < x \right]$$

$$= \Phi(x) - \frac{1}{\sqrt{n}} \{ g_1 \Phi^{(1)}(x)/\tau_{2q} + g_3 \Phi^{(3)}(x)/\tau_{2q}^3 \} + O(n^{-1}),$$

where

$$\tau_{2q}^2 = 2 \sum_{\alpha=a+1}^{p} \sum_{\beta=a+1}^{p} (\lambda_\alpha - \lambda_0)(\lambda_\beta - \lambda_0)\lambda_0^{-2} \{ \delta_{\alpha\beta} - (\lambda_\alpha + \lambda_\beta) \sum_{j=1}^{p} h_{j\alpha}^2 h_{j\beta}^2$$

$$+ \sum_{j=1}^{p} \sum_{k=1}^{p} \rho_{jk}^2 h_{j\alpha}^2 h_{k\beta}^2 \}$$

and the coefficients g_1, g_3 are given by (2.14) *with the partial derivatives $f_\alpha = 1/\lambda_0 - 1/\lambda_\alpha$, $f_{\alpha\alpha} = 1/\lambda_\alpha^2$ $(\alpha = a+1,\ldots, p)$ and other first and second derivatives being zero.*

The asymptotic distributions of the likelihood ratio criteria for the hypotheses H_0, H_1 and H_2 on the latent roots of the covariance matrix Σ have been studied by many authors (cf. [6], [7], [15], [25], [27], [41], [43], [66], [68], [69], [85], [89]).

Further, in multivariate analysis various functions of latent roots of some random matrices including individual latent roots have been proposed as estimates or test statistics. The problem of deriving the sampling distributions in the null and nonnull cases has been individually considered by many authors. Examples are found in [11], [15], [21], [23], [25], [33], [37], [38], [51], [62], [79], [80], [81], [84], [86], [88], [90], [91], [92] and others for asymptotic distributions, and in [14], [17], [18], [32], [39], [40], [52], [53], [75], [95] and

others for exact distributions. Recently a survey of the area of exact distributions is given by Krishnaiah [50]. A number of references on multivariate distribution theory are found in Pillai [73, 74].

3.3. *Numerical comparisons.* Our asymptotic formula (2.15) in the special case of $f(l_1,\ldots, l_p)=l_\alpha$ is examined by numerical comparisons. In the case of $p=2$, the latent root l_1 of R can be written as $l_1=1+|r|$ where r is the sample correlation coefficient in a sample of size $N=n+1$ from a bivariate normal distribution with population correlation coefficient ρ. An asymptotic expansion for the distribution of l_1 in the bivariate case is, assuming that $\rho>0$, given by

$$\Pr\{\sqrt{n}(l_1-1-\rho)/(1-\rho^2)<x\} \tag{3.10}$$

$$=\Phi(x)+\frac{1}{\sqrt{n}}\rho\left\{\frac{1}{2}\Phi^{(1)}(x)+\Phi^{(3)}(x)\right\}+O(n^{-1})$$

(see Konishi [46]). Konishi [46] has checked the accuracy of the asymptotic formula (3.10), based on exact values of the probability integral of r due to David [16].

Further comparisons are given in the following:

Table 3.1. *Comparison of exact and approximate values of* $\Pr(|r|<r_0)$

r_0	term of $O(1)$	term of $O(1/\sqrt{n})$	total	exact
		$N=25,\ \rho=0.7$		
.50	.02736	.02873	.05609	.05744
.55	.07481	.03182	.10663	.10494
.60	.16838	.01519	.18357	.18366
.65	.31551	−.01368	.30183	.30445
.70	.50000	−.02850	.47150	.47078
.75	.68449	−.01368	.67081	.66647
.80	.83162	.01519	.84681	.84739
.85	.92519	.03182	.95701	.96073
		$N=25,\ \rho=0.9$		
.83	.03555	.03965	.07519	.07423
.85	.09866	.03710	.13576	.13041
.87	.21961	.00534	.22495	.22387
.89	.39827	−.03073	.36753	.36883
.90	.50000	−.03665	.46335	.46244
.91	.60173	−.03073	.57100	.56762
.92	.69696	−.01502	.68194	.67872
.93	.78039	.00534	.78574	.78645
.94	.84881	.02427	.87309	.87911
.95	.90134	.03710	.93844	.94612
.96	.93907	.04193	.98101	.98350

Table 3.2. *Comparison of exact and approximate values of* $\Pr(|r|<r_0)$ *for* $N=50$ *and* $\rho=0.9$

r_0	term of $O(1)$	term of $O(1/\sqrt{n})$	total	exact
.85	.03273	.02720	.05993	.05998
.86	.07028	.02895	.09923	.09681
.87	.13452	.02009	.15462	.15229
.88	.23061	.00168	.23229	.23202
.89	.35628	−.01746	.33882	.33974
.90	.50000	−.02565	.47435	.47403
.91	.64372	−.01746	.62626	.62459
.92	.76939	.00168	.77107	.77108
.93	.86548	.02009	.88557	.88871
.94	.92972	.02895	.95867	.96114

Table 3.3. *Comparison of exact and approximate values of* $\Pr(|r|<r_0)$ *for* $N=200$ *and* $\rho=0.7$

r_0	term of $O(1)$	term of $O(1/\sqrt{n})$	total	exact
.64	.04850	.01126	.05976	.05963
.66	.13427	.00777	.14205	.14190
.68	.29006	−.00329	.28677	.28713
.70	.50000	−.00990	.49010	.49007
.71	.60896	−.00807	.60089	.60053
.72	.70994	−.00329	.70664	.70622
.73	.79668	.00265	.79932	.79917
.74	.86573	.00777	.87350	.87372
.75	.91667	.01075	.92742	.92779
.76	.95150	.01126	.96276	.96291
.77	.97358	.00987	.98345	.98312

In the case of $p=4$, asymptotic formulae for the distributions of the individual latent roots of R are examined by simulation, which was done with the help of Sugiyama [96].

Four independent standard normal deviates are generated and are transformed to a four dimensional normal variate with the specified correlation matrix

$$P=\begin{bmatrix} 1.00 & -0.19 & -0.45 & 0.24 \\ -0.19 & 1.00 & 0.41 & 0.16 \\ -0.45 & 0.41 & 1.00 & -0.51 \\ 0.24 & 0.16 & -0.51 & 1.00 \end{bmatrix}. \tag{3.11}$$

This process is repeated 100 times and the latent roots of the sample correlation matrix based on the generated observations are calculated. Each simulation is independently repeated 1000 times and the sample variances of $l_\alpha - \lambda_\alpha$ ($\alpha=1, 2, 3, 4$) are calculated. The percentage points of the latent roots of R are also obtained.

The latent roots and vectors of the population correlation matrix (3.11) are given by

$$\lambda_1 = 1.90223, \quad \lambda_2 = 1.16044, \quad \lambda_3 = 0.68671, \quad \lambda_4 = 0.25062,$$

$$H = \begin{pmatrix} 0.51280 & 0.02702 & 0.84909 & 0.12391 \\ -0.32585 & -0.76858 & 0.28959 & -0.46824 \\ -0.65575 & -0.03320 & 0.29584 & 0.69380 \\ 0.44816 & -0.63832 & -0.32813 & 0.53295 \end{pmatrix}.$$

Then asymptotic variances of $l_\alpha - \lambda_\alpha$ are obtainable by using (2.10) with $f_\alpha=1$ and $f_\beta=0$ ($\beta\neq\alpha$). The approximate values of the probability integral $\Pr(l_\alpha < x_\alpha)$ ($\alpha=1, 2, 3, 4$) can be obtained by using Theorem 2.1, where x_α are the upper 10 percentage points of l_α calculated by simulation. Comparisons are shown in the following:

Table 3.4. *Comparison of approximate values of* $\mathrm{Var}(l_\alpha - \lambda_\alpha)$ *with simulation results*

	l_1	l_2	l_3	l_4
approximate value	0.0188	0.0090	0.0080	0.0016
simulation result	0.0182	0.0086	0.0078	0.0016

Table 3.5. *Comparison of approximate values of* $\Pr(l_\alpha < x_\alpha)$ *with simulation results*

	x_1	x_2	x_3	x_4
percentage point	1.38310	1.22556	1.12164	1.11638
the term of $O(1)$	0.91668	0.88982	0.86899	0.86787
the term of $O(1/\sqrt{n})$	0.04438	0.02542	0.03065	0.03664
total	0.96106	0.91524	0.89965	0.90451
simulation result	0.90	0.90	0.90	0.90

From these tables it may be seen that the agreements are good. We can also see the efficacy of the term of order $1/\sqrt{n}$ in asymptotic formulae.

4. Asymptotic expansions for the distributions of latent vectors of the sample correlation matrix

This section contains asymptotic results concerning the latent vectors of the sample correlation matrix R based on a sample of size $n+1$ observations from a p-variate normal distribution with positive definite covariance matrix Σ.

Let $\lambda_1 \geqq \cdots \geqq \lambda_p > 0$ be the ordered latent roots of the population correlation matrix P and let $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_p$ be the corresponding orthonormal latent vectors of P, so that

$$H'PH = \Lambda \quad \text{and} \quad H'H = I, \tag{4.1}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $H = (\boldsymbol{h}_1, \ldots, \boldsymbol{h}_p)$.

We consider first the characteristic function of the latent vectors of the transformed matrix $H'RH$. Let S/n be the sample covariance matrix. It is easy to check that if the random matrix S is distributed according to the Wishart distribution $W_p(n, \Sigma)$, then

$$\Lambda^{-1/2}H'\Sigma_0^{-1/2}S\Sigma_0^{-1/2}H\Lambda^{-1/2}$$

is distributed according to $W_p(n, I)$ where $\Sigma_0 = \mathrm{diag}(\sigma_{11}, \ldots, \sigma_{pp})$. Let

$$Y = \left(\frac{n}{2}\right)^{1/2}\left(\frac{1}{n}\Lambda^{-1/2}H'\Sigma_0^{-1/2}S\Sigma_0^{-1/2}H\Lambda^{-1/2} - I\right). \tag{4.2}$$

Noting that V in (2.4) can be written in the form

$$V = \sqrt{2}H\Lambda^{1/2}Y\Lambda^{1/2}H',$$

we can expand $H'RH$ as

$$H'RH = \Lambda + \frac{1}{\sqrt{n}}V^{(1)} + \frac{1}{n}V^{(2)} + O_p(n^{-3/2}), \tag{4.3}$$

where

$$V^{(1)} = \sqrt{2}\,\Lambda^{1/2}Y\Lambda^{1/2} - \frac{1}{2}H'(Y_0^{(1)}P + PY_0^{(1)})H,$$

$$V^{(2)} = H'\left(\frac{1}{4}Y_0^{(1)}PY_0^{(1)} + \frac{3}{8}Y_0^{(1)2}P + \frac{3}{8}PY_0^{(1)2}\right)H$$

$$- \frac{\sqrt{2}}{2}(\Lambda^{1/2}Y\Lambda^{1/2}H'Y_0^{(1)}H + H'Y_0^{(1)}H\Lambda^{1/2}Y\Lambda^{1/2}).$$

Here $Y_0^{(1)}$ is a diagonal matrix with jth diagonal elements

$$\sqrt{2}\sum_{l=1}^{p}\sum_{m=1}^{p}(\lambda_l\lambda_m)^{1/2}h_{jl}h_{jm}y_{lm} \qquad \text{for} \quad Y = (y_{ij}).$$

Let $\boldsymbol{c}_g'=(c_{1g},\ldots, c_{pg})$ be the latent vector such that $\boldsymbol{c}_g'\boldsymbol{c}_g=1$ and $c_{gg}>0$, corresponding to the gth latent root of $H'RH$. If λ_g is simple, then it follows from (4.3) and Lemma 1.1 that the perturbation expansion of $\boldsymbol{c}_g$ is

$$
\begin{aligned}
c_{\alpha g} &= -\lambda_{\alpha g}\Big\{\frac{1}{\sqrt{n}}v_{\alpha g}^{(1)}+\frac{1}{n}(\lambda_{\alpha g}v_{\alpha g}^{(1)}v_{gg}^{(1)} \\
&\qquad -\sum_{\beta\neq g}^{p}\lambda_{\beta g}v_{\alpha\beta}^{(1)}v_{\beta g}^{(1)}+v_{\alpha g}^{(2)})\Big\}+O_p(n^{-3/2}) \qquad \text{if} \quad \alpha\neq g, \\
c_{gg} &= 1+\frac{1}{n}\Big(-\frac{1}{2}\sum_{\alpha\neq g}^{p}\lambda_{\alpha g}^2v_{\alpha g}^{(1)2}\Big)+O_p(n^{-3/2}),
\end{aligned}
\tag{4.4}
$$

where $\lambda_{\alpha g}=(\lambda_\alpha-\lambda_g)^{-1}$ and

$$
\begin{aligned}
v_{\alpha g}^{(1)} &= (2\lambda_\alpha\lambda_g)^{1/2}y_{\alpha g}-\frac{\sqrt{2}}{2}(\lambda_\alpha+\lambda_g)\sum_{j=1}^{p}\sum_{k=1}^{p}(\lambda_j\lambda_k)^{1/2}b_{jk\alpha g}y_{jk}, \\
v_{\alpha g}^{(2)} &= \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{m=1}^{p}(\lambda_j\lambda_k\lambda_l\lambda_m)^{1/2}\Big\{\frac{1}{2}\sum_{s=1}^{p}\lambda_s b_{jks\alpha}b_{lmsg} \\
&\qquad -\delta_{j\alpha}b_{klmg}-\delta_{kg}b_{jlm\alpha}+\frac{3}{4}(\lambda_\alpha+\lambda_g)\sum_{s=1}^{p}h_{s\alpha}h_{sg}h_{sj}h_{sk}h_{sl}h_{sm}\Big\}y_{jk}y_{lm}
\end{aligned}
$$

with

$$
b_{jk\alpha g}=\sum_{i=1}^{p}h_{ij}h_{ik}h_{i\alpha}h_{ig}. \tag{4.5}
$$

The expansion of the probability density function of Y defined by (4.2) is of the form (cf. Fujikoshi [24])

$$
\mathrm{c}\cdot\exp\Big(-\frac{1}{2}\operatorname{tr}Y^2\Big)\Big[1+\frac{1}{\sqrt{n}}\Big\{-\frac{\sqrt{2}}{2}(p+1)\operatorname{tr}Y+\frac{\sqrt{2}}{3}\operatorname{tr}Y^3\Big\}+O(n^{-1})\Big],
$$

where $\mathrm{c}=\pi^{-p(p+1)/4}2^{-p/2}$.

Then we may write the joint characteristic function of $\sqrt{n}\,c_{\alpha g}$ ($\alpha=1,\ldots, p$; $\alpha\neq g$) and $\sqrt{n}\,(c_{gg}-1)$ as

$$
\begin{aligned}
&E[\exp\{i\sum_{\alpha\neq g}^{p}t_\alpha\sqrt{n}\,c_{\alpha g}+it_g\sqrt{n}\,(c_{gg}-1)\}] \\
&\quad=\mathrm{c}\int\exp\Big(i\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}v_{\alpha g}^{(1)}-\frac{1}{2}\operatorname{tr}Y^2\Big) \\
&\quad\quad\times\Big[1+\frac{i}{\sqrt{n}}\Big\{\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}(\lambda_{\alpha g}v_{\alpha g}^{(1)}v_{gg}^{(1)}-\sum_{\beta\neq g}^{p}\lambda_{\beta g}v_{\alpha\beta}^{(1)}v_{\beta g}^{(1)}+v_{\alpha g}^{(2)}) \\
&\quad\quad -\frac{1}{2}t_g\sum_{\alpha\neq g}^{p}\lambda_{\alpha g}^2v_{\alpha g}^{(1)2}\Big\}-\frac{\sqrt{2}}{2\sqrt{n}}(p+1)\operatorname{tr}Y+\frac{\sqrt{2}}{3\sqrt{n}}\operatorname{tr}Y^3\Big]dY+O(n^{-1}),
\end{aligned}
\tag{4.6}
$$

where $dY=\prod_{j\leqq k}^{p} dy_{jk}$ and the integration is over the space of a real symmetric matrix Y. Put

$$a_{jk\cdot\alpha g}=(2\lambda_j\lambda_k)^{1/2}\left\{\delta_{j\alpha}\delta_{kg}-\frac{1}{2}(\lambda_\alpha+\lambda_g)\sum_{s=1}^{p}h_{sj}h_{sk}h_{s\alpha}h_{sg}\right\} \tag{4.7}$$

for simplicity. Then the integral of the limiting term in (4.6) can be expressed as

$$\begin{aligned}
&\mathrm{c}\int\exp\left(i\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}v_{\alpha g}^{(1)}-\frac{1}{2}\operatorname{tr}Y^2\right)dY \qquad (4.8)\\
&\quad=\mathrm{c}\int\exp\Bigg[i\Big\{\sum_{j=1}^{p}\Big(\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}a_{jj\cdot\alpha g}\Big)y_{jj}\\
&\qquad+\sum_{j<k}^{p}\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}(a_{jk\cdot\alpha g}+a_{kj\cdot\alpha g})y_{jk}\Big\}-\frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{p}y_{jk}^2\Bigg]dY\\
&\quad=\exp\left(-\frac{1}{2}\boldsymbol{t}_g'\Sigma_g\boldsymbol{t}_g\right),
\end{aligned}$$

where $\boldsymbol{t}_g'=(t_1,\ldots,t_{g-1},t_{g+1},\ldots,t_p)$ and $\Sigma_g=(\sigma_{\alpha\beta\cdot g})$ $(\alpha,\beta\neq g)$ is a symmetric matrix of order $p-1$ with

$$\begin{aligned}
\sigma_{\alpha\beta\cdot g}&=\frac{1}{4}\lambda_{\alpha g}\lambda_{\beta g}\sum_{j=1}^{p}\sum_{k=1}^{p}(a_{jk\cdot\alpha g}+a_{kj\cdot\alpha g})(a_{jk\cdot\beta g}+a_{kj\cdot\beta g}) \qquad (4.9)\\
&=\lambda_{\alpha g}\lambda_{\beta g}\Bigg[\delta_{\alpha\beta}\lambda_\alpha\lambda_g-\{2\lambda_\alpha\lambda_\beta\lambda_g+\lambda_g^2(\lambda_\alpha+\lambda_\beta)\}\sum_{j=1}^{p}h_{j\alpha}h_{j\beta}h_{jg}^2\\
&\qquad+\frac{1}{2}(\lambda_\alpha+\lambda_g)(\lambda_\beta+\lambda_g)\sum_{j=1}^{p}\sum_{k=1}^{p}\rho_{jk}^2h_{j\alpha}h_{k\beta}h_{jg}h_{kg}\Bigg].
\end{aligned}$$

The integrals of other terms in (4.6) are of the form

$$\begin{aligned}
&\mathrm{c}\int f(Y)\exp\left(i\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}v_{\alpha g}^{(1)}-\frac{1}{2}\operatorname{tr}Y^2\right)dY\\
&\quad\equiv G[f(Y)]\exp\left(-\frac{1}{2}\boldsymbol{t}_g'\Sigma_g\boldsymbol{t}_g\right),
\end{aligned}$$

where $f(Y)$ denotes a polynomial in the elements of Y. Putting

$$\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}(a_{jk\cdot\alpha g}+a_{kj\cdot\alpha g})=t_{jk}\ (j<k)\quad\text{and}\quad\sum_{\alpha\neq g}^{p}t_\alpha\lambda_{g\alpha}a_{jj\cdot\alpha g}=t_{jj}$$

in (4.8) and differentiating the resulting formula with respect to t_{jk} $(1\leqq j\leqq k\leqq p)$, we obtain

$$(4.10)\quad G[\sum_{\alpha\neq g}^{p} t_\alpha\lambda_{g\alpha}(\lambda_{\alpha g}v_{\alpha g}^{(1)}v_{gg}^{(1)} - \sum_{\beta\neq g}^{p}\lambda_{\beta g}v_{\alpha\beta}^{(1)}v_{\beta g}^{(1)} + v_{\alpha g}^{(2)})]$$

$$= \frac{1}{2}\sum_{\alpha\neq g}\sum_{j}\sum_{k}\lambda_{\alpha g}\{-2\lambda_{\alpha g}a_{jk\cdot\alpha g}a_{jk\cdot gg}$$

$$+ \sum_{\beta\neq g}\lambda_{\beta g}a_{jk\cdot\alpha\beta}(a_{jk\cdot\beta g} + a_{kj\cdot\beta g}) - \rho_{jk}^{3}h_{j\alpha}h_{kg}\}t_\alpha$$

$$+ i^2\frac{1}{4}\sum_{\alpha\neq g}\sum_{\beta\neq g}\sum_{\gamma\neq g}\lambda_{\alpha g}\lambda_{\beta g}\lambda_{\gamma g}\sum_{j}\sum_{k}\sum_{l}\sum_{m}\Big[-\lambda_{\alpha g}a_{jk\cdot\alpha g}a_{lm\cdot gg}$$

$$+ \sum_{\nu\neq g}\lambda_{\nu g}a_{jk\cdot\alpha\nu}a_{lm\cdot\nu g}$$

$$- (\lambda_j\lambda_k\lambda_l\lambda_m)^{1/2}\Big\{\frac{1}{2}\sum_{s}\lambda_s b_{jks\alpha}b_{lmsg} - \delta_{j\alpha}b_{klmg} - \delta_{kg}b_{jlm\alpha}$$

$$+ \frac{3}{4}(\lambda_\alpha + \lambda_g)\sum_{s}h_{s\alpha}h_{sg}h_{sj}h_{sk}h_{sl}h_{sm}\Big\}\Big]$$

$$\times (a_{jk\cdot\beta g} + a_{kj\cdot\beta g})(a_{lm\cdot\gamma g} + a_{ml\cdot\gamma g})t_\alpha t_\beta t_\gamma,$$

$$(4.11)\quad G\Big[-\frac{1}{2}t_g\sum_{\alpha\neq g}^{p}\lambda_{\alpha g}^2 v_{\alpha g}^{(1)2}\Big]$$

$$= -\frac{1}{4}\sum_{\alpha\neq g}\lambda_{\alpha g}^2\sum_{j}\sum_{k}(a_{jk\cdot\alpha g} + a_{kj\cdot\alpha g})a_{jk\cdot\alpha g}t_g$$

$$- \frac{i^2}{8}\sum_{\alpha\neq g}\sum_{\beta\neq g}\sum_{\gamma\neq g}\lambda_{\alpha g}^2\lambda_{\beta g}\lambda_{\gamma g}\sum_{j}\sum_{k}\sum_{l}\sum_{m}a_{jk\cdot\alpha g}a_{lm\cdot\alpha g}$$

$$\times (a_{jk\cdot\beta g} + a_{kj\cdot\beta g})(a_{lm\cdot\gamma g} + a_{ml\cdot\gamma g})t_\beta t_\gamma t_g,$$

and

$$(4.12)\quad G[\operatorname{tr} Y] = 0,$$

$$(4.13)\quad G[\operatorname{tr} Y^3] = -\frac{i^3}{4}\sum_{\alpha\neq g}\sum_{\beta\neq g}\sum_{\gamma\neq g}\lambda_{\alpha g}\lambda_{\beta g}\lambda_{\gamma g}\sum_{j}\sum_{k}\sum_{l}a_{jk\cdot\alpha g}$$

$$\times (a_{kl\cdot\beta g} + a_{lk\cdot\beta g})(a_{jl\cdot\gamma g} + a_{lj\cdot\gamma g})t_\alpha t_\beta t_\gamma,$$

where $b_{jk\alpha g}$ and $a_{jk\cdot\alpha g}$ are given by (4.5) and (4.7), respectively.

Combining these results with (4.8), we can obtain a expansion of the joint characteristic function of $\sqrt{n}\,c_{\alpha g}$ ($\alpha=1,\ldots,p$; $\alpha\neq g$) and $\sqrt{n}(c_{gg}-1)$. We note that a similar result may be obtained, using an approach similar to that in Nagao [67].

We consider this result in terms of the latent vectors of R. The normalized latent vector corresponding to the gth latent root of R, say $\boldsymbol{f}_g$, is given by

$$\boldsymbol{f}_g = (f_{1g},\ldots,f_{pg})' = H\boldsymbol{c}_g.$$

The characteristic function of $\sqrt{n}(\boldsymbol{f}_g-\boldsymbol{h}_g)$ can be expressed as

$$E[\exp\{i\sum_{u=1}^{p}\sqrt{n}(f_{ug}-h_{ug})t_u\}]$$

$$= E[\exp\{i\sum_{\alpha\neq g}^{p}(\sum_{u=1}^{p}h_{u\alpha}t_u)\sqrt{n}\,c_{\alpha g} + i(\sum_{u=1}^{p}h_{ug}t_u)\sqrt{n}(c_{gg}-1)\}].$$

Hence, replacing t_α $(\alpha=1,\ldots,p)$ by $\sum_{u=1}^{p}h_{u\alpha}t_u$ in (4.8), (4.10)–(4.13) and combining these results, we obtain the following expression for the characteristic function of $\sqrt{n}(\boldsymbol{f}_g-\boldsymbol{h}_g)$:

$$(4.14)\quad \exp\Big(-\frac{1}{2}\boldsymbol{t}'H_g\Sigma_g H_g'\boldsymbol{t}\Big)\Big[1+\frac{1}{\sqrt{n}}\{i\sum_{u=1}^{p}d_u^{(1)}t_u + i^3\sum_{u=1}^{p}\sum_{v=1}^{p}\sum_{w=1}^{p}d_{uvw}^{(3)}t_ut_vt_w\} + O(n^{-1})\Big],$$

where $\boldsymbol{t}'=(t_1,\ldots,t_p)$, $H_g=(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_{g-1},\boldsymbol{h}_{g+1},\ldots,\boldsymbol{h}_p)$, $\Sigma_g=(\sigma_{\alpha\beta\cdot g})$ is given by (4.9) and the coefficients are

$$(4.15)\quad d_u^{(1)} = \frac{1}{2}\sum_{\alpha\neq g}\sum_{j}\sum_{k}\lambda_{\alpha g}\{-2\lambda_{\alpha g}a_{jk\cdot\alpha g}a_{jk\cdot gg} + \sum_{\beta\neq g}\lambda_{\beta g}a_{jk\cdot\alpha\beta}(a_{jk\cdot\beta g}+a_{kj\cdot\beta g}) - \rho_{jk}^3h_{j\alpha}h_{kg}\}h_{u\alpha} - \frac{1}{4}\sum_{\alpha\neq g}\lambda_{\alpha g}^2\sum_{j}\sum_{k}a_{jk\cdot\alpha g}(a_{jk\cdot\alpha g}+a_{kj\cdot\alpha g})h_{ug},$$

$$(4.16)\quad d_{uvw}^{(3)} = \frac{1}{4}\sum_{\alpha\neq g}\sum_{\beta\neq g}\sum_{\gamma\neq g}\lambda_{\alpha g}\lambda_{\beta g}\lambda_{\gamma g}\sum_{j}\sum_{k}\sum_{l}\sum_{m}\Big[-\lambda_{\alpha g}a_{jk\cdot\alpha g}a_{lm\cdot gg} + \sum_{\nu\neq g}\lambda_{\nu g}a_{jk\cdot\alpha\nu}a_{lm\cdot\nu g} - (\lambda_j\lambda_k\lambda_l\lambda_m)^{1/2}\Big\{\frac{1}{2}\sum_{s}\lambda_s b_{jks\alpha}b_{lmsg} - \delta_{j\alpha}b_{klmg} - \delta_{kg}b_{jlm\alpha} + \frac{3}{4}(\lambda_\alpha+\lambda_g)\sum_{s}h_{s\alpha}h_{sg}h_{sj}h_{sk}h_{sl}h_{sm}\Big\}\Big] \times (a_{jk\cdot\beta g}+a_{kj\cdot\beta g})(a_{lm\cdot\gamma g}+a_{ml\cdot\gamma g})h_{u\alpha}h_{v\beta}h_{w\gamma} - \frac{1}{8}\sum_{\alpha\neq g}\sum_{\beta\neq g}\sum_{\gamma\neq g}\lambda_{\alpha g}^2\lambda_{\beta g}\lambda_{\gamma g}\sum_{j}\sum_{k}\sum_{l}\sum_{m}a_{jk\cdot\alpha g}a_{lm\cdot\alpha g} \times (a_{jk\cdot\beta g}+a_{kj\cdot\beta g})(a_{lm\cdot\gamma g}+a_{ml\cdot\gamma g})h_{ug}h_{v\beta}h_{w\gamma} - \frac{\sqrt{2}}{12}\sum_{\alpha\neq g}\sum_{\beta\neq g}\sum_{\gamma\neq g}\lambda_{\alpha g}\lambda_{\beta g}\lambda_{\gamma g}\sum_{j}\sum_{k}\sum_{l}a_{jk\cdot\alpha g} \times (a_{kl\cdot\beta g}+a_{lk\cdot\beta g})(a_{jl\cdot\gamma g}+a_{lj\cdot\gamma g})h_{u\alpha}h_{v\beta}h_{w\gamma}.$$

Inverting this characteristic function, we have the following theorems.

THEOREM 4.1. *Let* $H=(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_p)$ *be an orthogonal matrix such that* $H'PH=\mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, *where* $\lambda_1\geqq\cdots\geqq\lambda_p>0$ *are the ordered latent roots of the population correlation matrix* P. *Let* $\boldsymbol{f}_g'=(f_{1g},\ldots,f_{pg})$ *be the normalized latent vector corresponding to the* g*th latent root of the sample correlation matrix* R, *where the sign of* $f_{\alpha g}$ *is taken to be equal to that of* $h_{\alpha g}$ $(\alpha=1,\ldots,p)$. *If* λ_g *is simple, then the limiting distribution of* $\sqrt{n}(\boldsymbol{f}_g-\boldsymbol{h}_g)$ *is normal with mean vector* $\mathbf{0}$ *and covariance matrix*

$$H_g\Sigma_g H_g' \qquad \text{(singular)}$$

where $H_g=(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_{g-1},\boldsymbol{h}_{g+1},\ldots,\boldsymbol{h}_p)$ *and* $\Sigma_g=(\sigma_{\alpha\beta\cdot g})$ $(\alpha,\beta\neq g)$ *is a* $(p-1)\times(p-1)$ *symmetric matrix with* $\sigma_{\alpha\beta\cdot g}$ *given by* (4.9).

THEOREM 4.2. *Let* $\tau^2=\sum_{\alpha\neq g}^{p}\sum_{\beta\neq g}^{p}h_{\mu\alpha}h_{\mu\beta}\sigma_{\alpha\beta\cdot g}$ *and assume that the* τ^2 *is not zero. Then, under the same notations and assumptions as in Theorem* 4.1, *an asymptotic expansion for the distribution of the* μ*th element of the normalized latent vector* $\boldsymbol{f}_g$ *of* R *is given by*

$$\Pr\{\sqrt{n}(f_{\mu g}-h_{\mu g})/\tau<x\}$$
$$=\Phi(x)-\frac{1}{\sqrt{n}}\{d_\mu^{(1)}\Phi^{(1)}(x)/\tau+d_\mu^{(3)}\Phi^{(3)}(x)/\tau^3\}+O(n^{-1}),$$

where $d_\mu^{(1)}$ *and* $d_\mu^{(3)}$ *are given by* (4.15) *and* (4.16) *with* $u=v=w$ $(=\mu)$, *respectively.*

Sugiura [91] has obtained asymptotic expansions for the distributions of the latent vectors of the Wishart and multivariate F matrices under two different normalizations, when the corresponding population roots are simple. The exact distributions of the latent vectors of some Wishart matrices have been studied by Khatri and Pillai [42] and Sugiyama [93, 94].

Consider now testing the null hypothesis

$$H_0:\boldsymbol{h}_g=\boldsymbol{h}_{g0} \tag{4.17}$$

that the normalized latent vector corresponding to the distinct latent root λ_g of P is equal to a specified vector $\boldsymbol{h}_{g0}$ such that $\boldsymbol{h}_{g0}'\boldsymbol{h}_{g0}=1$. From Theorem 4.1 it follows that

$$n(\boldsymbol{f}_g-\boldsymbol{h}_g)'H_g\Sigma_g^{-1}H_g'(\boldsymbol{f}_g-\boldsymbol{h}_g) \tag{4.18}$$

has a limiting chi-square distribution with $p-1$ degrees of freedom. Testing the null hypothesis H_0, we replace $\boldsymbol{h}_g$ in (4.18) by a specified vector $\boldsymbol{h}_{g0}$ and also replace the unknown parameters λ_α, h_{ij} $(j\neq g)$ and ρ_{ij} by their sample estimates.

Anderson [6] has considered the null hypothesis (4.17) concerning the vector γ_1, say, of coefficients of the first principal component extracted from the covariance matrix Σ, and has given the criterion

$$n\{d_1\gamma_1' S^{-1}\gamma_1 + (1/d_1)\gamma_1' S\gamma_1 - 2\}, \tag{4.19}$$

where d_1 is the largest latent root of S and $\gamma_1'\gamma_1 = 1$. An asymptotic expansion for the distribution of (4.19) has recently been derived by Hayakawa [34].

5. Tests for latent vectors of a correlation matrix

Consider a p-variate random vector $\boldsymbol{X}$ having a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and positive definite covariance matrix $\Sigma = (\sigma_{ij})$. The population correlation matrix P can be expressed as

$$P = \Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2},$$

where $\Sigma_0 = \mathrm{diag}(\sigma_1^2,\ldots,\sigma_p^2)$ with $\sigma_i^2 = \sigma_{ii}$. Let $\lambda_1,\ldots,\lambda_p$ be the latent roots of P and let $\boldsymbol{h}_1,\ldots,\boldsymbol{h}_p$ be the corresponding orthonormal latent vectors of P. Throughout this section we do not impose the restriction that the latent roots of P are ordered.

Given N independent observations $\boldsymbol{X}_\alpha$, $\alpha = 1,\ldots,N$, on $\boldsymbol{X}$, we wish to test the following hypotheses on the latent roots and vectors of P.

(5.1) $H_{(1)}$: A specified set of a $(<p)$ orthonormal vectors $\boldsymbol{h}_{10},\ldots,\boldsymbol{h}_{a0}$ are latent vectors of P, namely,

$$\boldsymbol{h}_\alpha = \boldsymbol{h}_{\alpha 0} \qquad (\alpha = 1,\ldots,a),$$

(5.2) $H_{(2)}$: A specified set of a $(<p)$ orthonormal vectors $\boldsymbol{h}_{10},\ldots,\boldsymbol{h}_{a0}$ are latent vectors of P and the corresponding latent roots are $\lambda_{10},\ldots,\lambda_{a0}$, namely,

$$\boldsymbol{h}_\alpha = \boldsymbol{h}_{\alpha 0} \quad \text{and} \quad \lambda_\alpha = \lambda_{\alpha 0} \qquad (\alpha = 1,\ldots,a).$$

We shall now derive test statistics for testing $H_{(1)}$ and $H_{(2)}$. The logarithm of the likelihood function after maximization with respect to $\boldsymbol{\mu}$ may be written as

$$\log L(\Sigma_0, P) = -\frac{1}{2}N(p\log 2\pi + \log|\Sigma_0| + \log|P| + \mathrm{tr}\, P^{-1}B_0RB_0), \tag{5.3}$$

where R is the sample correlation matrix and

$$B_0 = \mathrm{diag}(s_1/\sigma_1,\ldots,s_p/\sigma_p)$$

with $s_i^2 = s_{ii}/N$. Here $S = (s_{ij})$ is the matrix of the corrected sums of squares and products of the observations. It is easy to see that the maximum of $\log L$ when

all the parameters are unrestricted is

$$\log \max_{\Omega} L(\Sigma_0, P) = -\frac{1}{2}N(p \log 2\pi + \log |S_0| + \log |R| + p),$$

where Ω is the parameter space of (Σ_0, P) and $S_0 = \text{diag}\,(s_1^2, \ldots, s_p^2)$.

Let

$$H_{10} = (\boldsymbol{h}_{10}, \ldots, \boldsymbol{h}_{a0}), \quad \Lambda_a = \text{diag}\,(\lambda_1, \ldots, \lambda_a), \tag{5.4}$$

and let $H_2 = (\boldsymbol{h}_{a+1}, \ldots, \boldsymbol{h}_p)$ be any $p \times (p-a)$ matrix such that

$$H = [H_{10}, H_2] \tag{5.5}$$

is an orthogonal matrix. Under the null hypothesis $H_{(1)}$, it follows that

$$[H_{10}, H_2]' P [H_{10}, H_2] = \begin{pmatrix} \Lambda_a & 0 \\ 0 & P_{22}^* \end{pmatrix}$$

or equivalently

$$P = H_{10}\Lambda_a H_{10}' + H_2 P_{22}^* H_2',$$

where $P_{22}^* = H_2' P H_2$, and hence (5.3) can be rewritten as

$$\begin{aligned} \log L(\Sigma_0, \Lambda_a, P_{22}^*) = &-\frac{1}{2}N\{p \log 2\pi + \log |\Sigma_0| + \log |\Lambda_a| \\ &+ \log |P_{22}^*| + \text{tr}\, H_{10}\Lambda_a^{-1} H_{10}' B_0 R B_0 + \text{tr}\, H_2 P_{22}^{*-1} H_2' B_0 R B_0\}. \end{aligned} \tag{5.6}$$

The likelihood ratio criterion is given by

$$\max_{\omega_1} L(\Sigma_0, \Lambda_a, P_{22}^*) / \max_{\Omega} L(\Sigma_0, P), \tag{5.7}$$

where ω_1 is a subspace of Ω when $H_{(1)}$ is true. It is however difficult to obtain $\max_{\omega_1} L(\Sigma_0, \Lambda_a, P_{22}^*)$ in closed form and so an approximate test is proposed here.

Differentiating $\log L(\Sigma_0, \Lambda_a, P_{22}^*)$ with respect to the parameters σ_j $(j = 1, \ldots, p)$, λ_α $(\alpha = 1, \ldots, a)$, P_{22}^* and setting the results equal to zero gives

$$\sum_{i=1}^{p}\sum_{\alpha=1}^{a} \lambda_\alpha^{-1} h_{i\alpha}^{(10)} h_{j\alpha}^{(10)} r_{ij} b_i b_j + \sum_{i=1}^{p}\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a} h_{i\alpha}^{(2)} h_{j\beta}^{(2)} \rho^{*\alpha\beta} r_{ij} b_i b_j = 1,$$

$$\lambda_\alpha = \sum_{i=1}^{p}\sum_{j=1}^{p} h_{i\alpha}^{(10)} h_{j\alpha}^{(10)} r_{ij} b_i b_j, \tag{5.8}$$

$$P_{22}^* = H_2' B_0 R B_0 H_2,$$

where $b_j = s_j/\sigma_j$, $H_{10} = (h_{ij}^{(10)})$, $H_2 = (h_{ij}^{(2)})$ and $P_{22}^{*-1} = (\rho^{*\alpha\beta})$. If σ_j is estimated

by a consistent estimate $\hat{\sigma}_j$, then it follows from (5.8) that the estimates of λ_α and P_{22}^* are, respectively, given by

$$\hat{\lambda}_\alpha = \sum_{i=1}^{p}\sum_{j=1}^{p} h_{i\alpha}^{(10)} h_{j\alpha}^{(10)} r_{ij}\hat{b}_i\hat{b}_j,$$

(5.9)

$$\hat{P}_{22}^* = H_2'\hat{B}_0 R\hat{B}_0 H_2,$$

where $\hat{b}_j = s_j/\hat{\sigma}_j$ and $\hat{B}_0 = \mathrm{diag}\,(s_1/\hat{\sigma}_1,\ldots, s_p/\hat{\sigma}_p)$. Substituting (5.9) into (5.6), we obtain an approximate likelihood ratio criterion

$$N\log\{\prod_{\alpha=1}^{a}(H_{10}'\hat{B}_0 R\hat{B}_0 H_{10})_{\alpha\alpha}\,|\,H_2'\hat{B}_0 R\hat{B}_0 H_2\,|\,|R|^{-1}|\hat{B}_0|^{-2}\},$$

where $(B)_{\alpha\alpha}$ denotes the (α, α)th element of a matrix B. It may be seen that $\hat{b}_j = s_j/\hat{\sigma}_j$ converges to 1 in probability as N tends to infinity, namely, $\hat{B}_0 \to I$ in probability as $N\to\infty$. Therefore, we suggest the following statistic for testing the hypothesis $H_{(1)}$:

$$T_1 = N\log\{\prod_{\alpha=1}^{a}(H_{10}' R H_{10})_{\alpha\alpha}|H_2' R H_2|\,|R|^{-1}\}. \tag{5.10}$$

It may be noted that the statistic (5.10) replaced R with S/n gives the likelihood ratio statistic for testing the hypothesis $H_{(1)}$ on the latent vectors of the covariance matrix Σ, which was obtained by Gupta [31] and Mallows [60].

By an argument similar to that discussed above, we derive a statistic for testing the hypothesis $H_{(2)}$. Proceeding from the same notations as in (5.4) and (5.5), except that Λ_a is specified by $\Lambda_{a0} = \mathrm{diag}\,(\lambda_{10},\ldots, \lambda_{a0})$, we obtain

$$\log L_{\omega_2} = -\frac{1}{2}N\{p\log 2\pi + \log|\Sigma_0| + \log|\Lambda_{a0}| + \log|P_{22}^*|$$

$$+ \mathrm{tr}\, H_{10}\Lambda_{a0}^{-1}H_{10}'B_0 R B_0 + \mathrm{tr}\, H_2 P_{22}^{*-1}H_2' B_0 R B_0\},$$

where ω_2 is the subspace of Ω when $H_{(2)}$ is true. Differentiating $\log L_{\omega_2}$ with respect to σ_j $(j=1,\ldots, p)$ and P_{22}^* and setting the results equal to zero gives

$$\sum_{i=1}^{p}\sum_{\alpha=1}^{a}\lambda_{\alpha 0}^{-1}h_{i\alpha}^{(10)}h_{j\alpha}^{(10)}r_{ij}b_i b_j + \sum_{i=1}^{p}\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}h_{i\alpha}^{(2)}h_{j\beta}^{(2)}\rho^{*\alpha\beta}r_{ij}b_i b_j = 1,$$

$$P_{22}^* = H_2' B_0 R B_0 H_2.$$

If P_{22}^* is estimated by $\hat{P}_{22}^* = H_2'\hat{B}_0 R\hat{B}_0 H_2$ where $\hat{B}_0$ is defined in (5.9), the likelihood ratio criterion for testing $H_{(2)}$ may be approximated as

$$N[\log\{|H_2'\hat{B}_0 R\hat{B}_0 H_2|\,|\Lambda_{a0}|\,|R|^{-1}|\hat{B}_0|^{-2}\} + \mathrm{tr}\, H_{10}\Lambda_{a0}^{-1}H_{10}'\hat{B}_0 R\hat{B}_0 - a].$$

Since $\hat{B}_0 \to I$ in probability as $N\to\infty$, we propose the following statistic for testing $H_{(2)}$:

$$(5.11)\qquad T_2 = N[\log\{|H_2'RH_2|\,|\Lambda_{a0}|\,|R|^{-1}\} + \operatorname{tr}\Lambda_{a0}^{-1}H_{10}'RH_{10} - a].$$

We note that the statistic (5.11) replaced R with S/n gives the likelihood ratio statistic for testing the hypothesis $H_{(2)}$ on the latent roots and vectors of Σ, which was obtained by Gupta [31] and Mallows [60].

The problem of deriving the sampling distributions of T_1 and T_2 is intractable, because in general these statistics have not, even asymptotically, chi-square (χ^2) distributions. Reasonable approximations to the distributions of T_1 and T_2 might be given in the form of χ_d^2, where d is determined from the expectation of T_1 or T_2. Using Lemma 2.1 in Siotani [82] and neglecting the term of order $1/n$, we obtain

$$\begin{aligned}(5.12)\quad E[T_1] &= \frac{1}{2}p(p-2) - a - \frac{1}{2}(p-a)(p-a+1)\\
&\quad + \frac{1}{2}\sum_{\alpha=1}^{a}\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_\alpha^{-1}\rho_{ij}^3 h_{i\alpha}^{(10)}h_{j\alpha}^{(10)} - \sum_{\alpha=1}^{a}\sum_{i=1}^{p}\sum_{j=1}^{p}\rho_{ij}^2 h_{i\alpha}^{(10)2}h_{j\alpha}^{(10)2}\\
&\quad + 2\sum_{\alpha=1}^{a}\sum_{i=1}^{p}\lambda_\alpha h_{i\alpha}^{(10)4} + \frac{1}{2}\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\xi^{\alpha\beta}\sum_{i=1}^{p}\sum_{j=1}^{p}\rho_{ij}^3 h_{i\alpha}^{(2)}h_{j\beta}^{(2)}\\
&\quad + \frac{1}{2}\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\sum_{\gamma=1}^{p-a}\sum_{\delta=1}^{p-a}(\xi^{\alpha\gamma}\xi^{\beta\delta} + \xi^{\beta\gamma}\xi^{\alpha\delta})\\
&\quad \times \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}(2\rho_{ik}\rho_{jk}\rho_{kl} - \rho_{ik}^2\rho_{ij}\rho_{kl})h_{i\alpha}^{(2)}h_{j\beta}^{(2)}h_{k\gamma}^{(2)}h_{l\delta}^{(2)},\end{aligned}$$

$$\begin{aligned}(5.13)\quad E[T_2] &= \frac{1}{2}p(p-2) - \frac{1}{2}(p-a)(p-a+1)\\
&\quad + \frac{1}{2}\sum_{\alpha=1}^{a}\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_{\alpha 0}^{-1}\rho_{ij}^3 h_{i\alpha}^{(10)}h_{j\alpha}^{(10)} + \frac{1}{2}\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\xi^{\alpha\beta}\sum_{i=1}^{p}\sum_{j=1}^{p}\rho_{ij}^3 h_{i\alpha}^{(2)}h_{j\beta}^{(2)}\\
&\quad + \frac{1}{2}\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\sum_{\gamma=1}^{p-a}\sum_{\delta=1}^{p-a}(\xi^{\alpha\gamma}\xi^{\beta\delta} + \xi^{\beta\gamma}\xi^{\alpha\delta})\\
&\quad \times \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}(2\rho_{ik}\rho_{jk}\rho_{kl} - \rho_{ik}^2\rho_{ij}\rho_{kl})h_{i\alpha}^{(2)}h_{j\beta}^{(2)}h_{k\gamma}^{(2)}h_{l\delta}^{(2)},\end{aligned}$$

where $(H_2'PH_2)^{-1} = (\xi^{\alpha\beta})$.

In practice the unknown parameters included in (5.12) and (5.13) have to be replaced by their sample estimates. Nagao [66] has studied asymptotic null and nonnull distributions of a likelihood ratio criterion for testing the hypothesis $H_{(2)}$ on the latent roots and vectors of Σ. The exact likelihood ratio criterion for testing $H_{(2)}$ on the ordered latent roots and vectors of Σ has been derived by Fujikoshi [22].

Part II. Various Functions of the Sample Correlation Matrix

6. Generalized asymptotic expansions

The limiting joint distributions of various statistics expressed as functions of the sample correlation matrix R have been obtained by Olkin and Siotani [71]. In this section we shall derive an asymptotic expansion for the joint density of certain functions of R. To obtain asymptotic nonnull distributions of test statistics proposed in Section 5, an asymptotic expansion for the distribution of a function of R is also derived, up to and including the term of order $1/n$. The approach used here is based on that discussed in Konishi [48].

6.1. *Asymptotic expansion for the joint density of functions of the sample correlation matrix.* As in Part I, suppose a sample is drawn from a p-variate normal population $N_p(\boldsymbol{\mu}, \Sigma)$. Let $f^{(\alpha)}(R)=f^{(\alpha)}(r_{12}, r_{13},\ldots, r_{p-1,p})$ be an analytic function of the sample correlation matrix R in a neighborhood of $R=P$. Put

$$V=(v_{ij})=\sqrt{m}\left(\frac{1}{m}\Sigma_0^{-1/2}S\,\Sigma_0^{-1/2}-P\right)$$

with $\Sigma_0=\mathrm{diag}\,(\sigma_{11},\ldots,\sigma_{pp})$, where $m=n-2\Delta$ with the correction factor Δ and S is the usual matrix of corrected sums of squares and products in normal sample. The expansion of $f^{(\alpha)}(R)$, given in Konishi [48], is of the form

$$\begin{aligned}(6.1)\qquad f^{(\alpha)}(R) &= f^{(\alpha)}(P)+\frac{1}{\sqrt{m}}\frac{1}{2}\sum_{i\neq j}v_{ij}^{(1)}f_{ij}^{(\alpha)}\\ &\quad+\frac{1}{m}\left(\frac{1}{2}\sum_{i\neq j}v_{ij}^{(2)}f_{ij}^{(\alpha)}+\frac{1}{8}\sum_{i\neq j}\sum_{k\neq l}v_{ij}^{(1)}v_{kl}^{(1)}f_{ij\cdot kl}^{(\alpha)}\right)\\ &\quad+\frac{1}{m\sqrt{m}}\left(\frac{1}{2}\sum_{i\neq j}v_{ij}^{(3)}f_{ij}^{(\alpha)}+\frac{1}{4}\sum_{i\neq j}\sum_{k\neq l}v_{ij}^{(1)}v_{kl}^{(2)}f_{ij\cdot kl}^{(\alpha)}\right.\\ &\quad\left.+\frac{1}{48}\sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r}v_{ij}^{(1)}v_{kl}^{(1)}v_{qr}^{(1)}f_{ij\cdot kl\cdot qr}^{(\alpha)}\right)+O_p(m^{-2}),\end{aligned}$$

where $f_{ij\cdot kl\cdot qr}^{(\alpha)}=\dfrac{\partial^3 f^{(\alpha)}(R)}{\partial r_{qr}\partial r_{kl}\partial r_{ij}}\Big|_{R=P}$ and

$$v_{ij}^{(1)}=v_{ij}-\frac{1}{2}\rho_{ij}(v_{ii}+v_{jj}),$$

$$(6.2)\qquad v_{ij}^{(2)}=\frac{1}{4}\rho_{ij}v_{ii}v_{jj}-\frac{1}{2}(v_{ii}+v_{jj})v_{ij}+\frac{3}{8}\rho_{ij}(v_{ii}^2+v_{jj}^2),$$

$$v_{ij}^{(3)} = \frac{1}{4}v_{ii}v_{jj}v_{ij} - \frac{3}{16}\rho_{ij}(v_{ii}+v_{jj})v_{ii}v_{jj}$$

$$+ \frac{3}{8}(v_{ii}^2+v_{jj}^2)v_{ij} - \frac{5}{16}\rho_{ij}(v_{ii}^3+v_{jj}^3).$$

The summation $\sum_{i\neq j}$ stands for $\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}$ throughout this part. The $v_{ij}^{(\alpha)}$ in (6.2) are found to be the (i,j)th elements of $V^{(\alpha)}$ with $H=I$ in (2.4). The joint characteristic function of $\sqrt{m}\{f^{(\alpha)}(R)-f^{(\alpha)}(P)\}$, $\alpha=1,\ldots,a$, can be expressed as

$$E\Big[\operatorname{etr}(iAV)\Big\{1+\frac{i}{\sqrt{m}}\sum_{\alpha=1}^{a}t_\alpha\Big(\frac{1}{2}\sum_{i\neq j}v_{ij}^{(2)}f_{ij}^{(\alpha)}$$

$$+\frac{1}{8}\sum_{i\neq j}\sum_{k\neq l}v_{ij}^{(1)}v_{kl}^{(1)}f_{ij\cdot kl}^{(\alpha)}\Big)+O_p(m^{-1})\Big\}\Big],$$

where A is a $p\times p$ symmetric matrix with

$$a_{ii} = -\frac{1}{2}\sum_{\alpha=1}^{a}\sum_{\substack{j=1\\ j\neq i}}^{p}t_\alpha\rho_{ij}f_{ij}^{(\alpha)}, \quad a_{ij}=\frac{1}{2}\sum_{\alpha=1}^{a}t_\alpha f_{ij}^{(\alpha)} \qquad (i\neq j).$$

Hence, by an argument similar to that in Section 2, we obtain the following form for the joint characteristic function of $\sqrt{m}\{f^{(\alpha)}(R)-f^{(\alpha)}(P)\}$, $\alpha=1,\ldots,a$:

$$\exp\Big(-\frac{1}{2}\boldsymbol{t}'\Omega\boldsymbol{t}\Big)\Big\{1+\frac{1}{\sqrt{m}}\Big(i\sum_{\alpha=1}^{a}b_\alpha t_\alpha \tag{6.3}$$

$$+i^3\sum_{\alpha=1}^{a}\sum_{\beta=1}^{a}\sum_{\gamma=1}^{a}b_{\alpha\beta\gamma}t_\alpha t_\beta t_\gamma\Big)+O(m^{-1})\Big\},$$

where $\Omega=(\omega_{\alpha\beta})$ is an $a\times a$ symmetric matrix with

$$\omega_{\alpha\beta}=\frac{1}{2}\sum_{i\neq j}\sum_{k\neq l}(\rho_{jk}-\rho_{ij}\rho_{ik})(\rho_{il}-\rho_{ik}\rho_{kl})f_{ij}^{(\alpha)}f_{kl}^{(\beta)}$$

and the coefficients b_α, $b_{\alpha\beta\gamma}$ are

$$b_\alpha = -\frac{1}{4}\sum_{i\neq j}\rho_{ij}(1-\rho_{ij}^2)f_{ij}^{(\alpha)}+\frac{1}{4}\sum_{i\neq j}\sum_{k\neq l}\rho_{jk\cdot i}\rho_{il\cdot k}f_{ij\cdot kl}^{(\alpha)},$$

$$b_{\alpha\beta\gamma}=\frac{1}{8}\sum_{i\neq j}\{\rho_{ij}(3d_{ii\cdot\beta}+d_{jj\cdot\beta})-4d_{ij\cdot\beta}\}d_{ii\cdot\gamma}f_{ij}^{(\alpha)}$$

$$+\frac{1}{8}\sum_{i\neq j}\sum_{k\neq l}(d_{ij\cdot\beta}-\rho_{ij}d_{ii\cdot\beta})(d_{kl\cdot\gamma}-\rho_{kl}d_{kk\cdot\gamma})f_{ij\cdot kl}^{(\alpha)} \tag{6.4}$$

$$+\frac{1}{6}\sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r}\rho_{jk\cdot i}\rho_{ir\cdot q}\rho_{ql\cdot k}f_{ij}^{(\alpha)}f_{kl}^{(\beta)}f_{qr}^{(\gamma)}$$

with $\rho_{jk\cdot i}=\rho_{jk}-\rho_{ij}\rho_{ik}$ and $d_{ij\cdot\alpha}=\sum_{k\neq l}\rho_{ik}(\rho_{jl}-\rho_{jk}\rho_{kl})f_{kl}^{(\alpha)}$.

If Ω is nonsingular, then by inverting (6.3) we have the following theorem.

THEOREM 6.1. *Suppose $f^{(\alpha)}(R)$ is an analytic function of the sample correlation matrix R in a neighborhood of $R=P$. Let*

$$\boldsymbol{x}' = (\sqrt{m}\{f^{(1)}(R) - f^{(1)}(P)\},\ldots, \sqrt{m}\{f^{(a)}(R) - f^{(a)}(P)\}),$$

where $m=n-2\Delta$. Then an asymptotic expansion for the joint density function of $\boldsymbol{x}$ is given by

$$\phi(\boldsymbol{x};\Omega)\Big[1+\frac{1}{\sqrt{m}}\{\sum_{\alpha=1}^{a} b_\alpha H_\alpha(\boldsymbol{x};\Omega) + \sum_{\alpha=1}^{a}\sum_{\beta=1}^{a}\sum_{\gamma=1}^{a} b_{\alpha\beta\gamma}H_{\alpha\beta\gamma}(\boldsymbol{x};\Omega)\} + O(m^{-1})\Big], \tag{6.5}$$

where

$$\phi(\boldsymbol{x};\Omega) = \frac{1}{(2\pi)^{a/2}|\Omega|^{1/2}}\exp\Big(-\frac{1}{2}\boldsymbol{x}'\Omega^{-1}\boldsymbol{x}\Big)$$

and for $1\leqq i_\alpha\leqq a$

$$H_{i_1\cdots i_r}(\boldsymbol{x};\Omega) = \frac{(-1)^r}{\phi(\boldsymbol{x};\Omega)}\frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}}\phi(\boldsymbol{x};\Omega)$$

with $\Omega=(\omega_{\alpha\beta})$ given in (6.3), *and the coefficients b_α, $b_{\alpha\beta\gamma}$ are given by* (6.4).

As a simple application of this theorem, consider an asymptotic expansion for the joint density function of $R=(r_{ij})$. Taking

$$f^{(1)}(R) = r_{12},\ f^{(2)}(R) = r_{13},\ldots, f^{(p(p-1)/2)}(R) = r_{p-1,p}$$

in Theorem 6.1 and differentiating $f^{(\alpha)}(R)$ with respect to r_{ij} $(i<j)$, we obtain an asymptotic formula in the form of (6.5) with covariance matrix $\Omega=(\cdot)$ and coefficients $b_\cdot$, $b_{\cdots}$ given in the following: $\Omega=(\omega_{ij\cdot kl})$ is a symmetric matrix of order $p(p-1)/2$ with

$$\omega_{ij\cdot kl} = \frac{1}{2}(\rho_{jk\cdot i}\rho_{il\cdot k} + \rho_{jl\cdot i}\rho_{ik\cdot l} + \rho_{ik\cdot j}\rho_{jl\cdot k} + \rho_{il\cdot j}\rho_{jk\cdot l})$$

and

$$b_{ij} = -\frac{1}{2}\rho_{ij}(1-\rho_{ij}^2),$$

$$b_{ij\cdot kl\cdot qr} = \frac{1}{8}\{3\rho_{ij}(d_{ii\cdot kl}d_{ii\cdot qr} + d_{jj\cdot kl}d_{jj\cdot qr}) + 2\rho_{ij}d_{ii\cdot kl}d_{jj\cdot qr} - 4d_{ij\cdot kl}(d_{ii\cdot qr} + d_{jj\cdot qr})\}$$

$$+\frac{1}{6}\{\rho_{jk\cdot i}(\rho_{ir\cdot q}\rho_{lq\cdot k}+\rho_{iq\cdot r}\rho_{lr\cdot k})$$

$$+\rho_{jl\cdot i}(\rho_{ir\cdot q}\rho_{kq\cdot l}+\rho_{iq\cdot r}\rho_{kr\cdot l})+\rho_{ik\cdot j}(\rho_{jr\cdot q}\rho_{lq\cdot k}$$

$$+\rho_{jq\cdot r}\rho_{lr\cdot k})+\rho_{il\cdot j}(\rho_{jr\cdot q}\rho_{kq\cdot l}+\rho_{jq\cdot r}\rho_{kr\cdot l})\},$$

where $\rho_{jk\cdot i}=\rho_{jk}-\rho_{ij}\rho_{ik}$ and $d_{ij\cdot kl}=\rho_{ik}\rho_{jl\cdot k}+\rho_{il}\rho_{jk\cdot l}$.

The limiting distribution of $\sqrt{n}(r_{ij}-\rho_{ij})$ is normal with mean 0 and variance $\omega_{ij\cdot ij}=(1-\rho_{ij}^2)^2$, which follows immediately from the above result. Asymptotic joint distributions of correlation coefficients were given in Olkin and Siotani [71, p. 238].

6.2. *Asymptotic expansion for the distribution of a function of the sample correlation matrix.* To obtain asymptotic nonnull distributions of test statistics concerning the structure of the population correlation matrix, Konishi [48] has derived an asymptotic expansion for the distribution of an analytic function of R up to the terms of order $1/\sqrt{n}$. Proceeding from the same expansion as in (6.1), except that $f^{(\alpha)}(R)$ is replaced by an analytic function $f(R)$, we shall derive an asymptotic expansion for the distribution of $f(R)$ up to and including the term of order $1/n$. Rewrite (6.1) as

$$f(R)=f(P)+\frac{1}{\sqrt{m}}u_1+\frac{1}{m}u_2+\frac{1}{m\sqrt{m}}u_3+O_p(m^{-2})$$

for simplicity. Then the characteristic function of $\sqrt{m}\{f(R)-f(P)\}$ can be expressed as

$$E\Big[\operatorname{etr}(itAV)\Big\{1+\frac{(it)}{\sqrt{m}}u_2+\frac{1}{m}\Big((it)u_3+\frac{(it)^2}{2}u_2^2\Big)+O_p(m^{-3/2})\Big\}\Big], \tag{6.6}$$

where $A=(a_{ij})$ is a $p\times p$ symmetric matrix with

$$a_{ii}=-\frac{1}{2}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{p}\rho_{i\alpha}f_{i\alpha}\quad\text{and}\quad a_{ij}=\frac{1}{2}f_{ij}\qquad(i\neq j).$$

By Lemma 5.1 in Sugiura [87], it is straightforward to obtain each expectation in (6.6), but the calculation is tedious. Inverting the resulting characteristic function under the assumption that the variance of the limiting distribution of $\sqrt{m}\{f(R)-f(P)\}$ is not zero, we have the following theorem.

THEOREM 6.2. *Suppose $f(R)$ is an analytic function of the sample correlation matrix R in a neighborhood of $R=P$. Let $\Phi^{(j)}(x)$ be the jth derivatives of the standard normal distribution function $\Phi(x)$. Put $m=n-2\Delta$. Then an asymptotic expansion for the distribution of $f(R)$ is given by*

$$\Pr[\sqrt{m}\{f(R)-f(P)\}/\tau < x] \tag{6.7}$$

$$= \Phi(x) - \frac{1}{\sqrt{m}}\{a_1\Phi^{(1)}(x)/(2\tau) + a_3\Phi^{(3)}(x)/(2\tau)^3\}$$

$$+ \frac{1}{m}\sum_{j=1}^{3} b_{2j}\Phi^{(2j)}(x)/(2\tau)^{2j} + O(m^{-3/2}),$$

where

$$\tau^2 = \frac{1}{2}\sum_{i\neq j}\sum_{k\neq l}(\rho_{jk}-\rho_{ij}\rho_{ik})(\rho_{il}-\rho_{ik}\rho_{kl})f_{ij}f_{kl} \quad (>0) \tag{6.8}$$

and the coefficients are

$$a_1 = -\frac{1}{2}\sum_{i\neq j}\rho_{ij}(1-\rho_{ij}^2)f_{ij} + \frac{1}{2}\sum_{i\neq j}\sum_{k\neq l}\rho_{jk\cdot i}\rho_{il\cdot k}f_{ij\cdot kl},$$

$$a_3 = \sum_{i\neq j}\{\rho_{ij}(3d_{ii}+d_{jj}) - 4d_{ij}\}d_{ii}f_{ij}$$

$$+ \sum_{i\neq j}\sum_{k\neq l}(d_{ij}-\rho_{ij}d_{ii})(d_{kl}-\rho_{kl}d_{kk})f_{ij\cdot kl}$$

$$+ \frac{4}{3}\sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r}\rho_{ir\cdot q}\rho_{jk\cdot i}\rho_{lq\cdot k}f_{ij}f_{kl}f_{qr},$$

$$b_2 = \frac{1}{2}a_1^2 + 2\Delta\{2\sum_{i\neq j}(\rho_{ij}d_{ii}-d_{ij})f_{ij} + \sum_{i\neq j}\sum_{k\neq l}\rho_{jk\cdot i}\rho_{il\cdot k}f_{ij}f_{kl}\}$$

$$+ \sum_{i\neq j}(1-3\rho_{ij}^2)(\rho_{ij}d_{ii}-d_{ij})f_{ij}$$

$$+ \frac{1}{4}\sum_{i\neq j}\sum_{k\neq l}\rho_{ik\cdot j}\{\rho_{jl}(\rho_{ik}^2+3\rho_{jl}^2+12\rho_{jk}^2) - \rho_{jk}\rho_{kl}(\rho_{il}^2+6\rho_{jl}^2$$

$$+ 9\rho_{jk}^2)\}f_{ij}f_{kl} + \sum_{i\neq j}\sum_{k\neq l}\Big[\Big\{\frac{1}{2}\rho_{ij}\rho_{kl}(1-\rho_{kl}^2) + 3\rho_{ij}\rho_{ik}\rho_{il\cdot k}$$

$$- \rho_{jl\cdot k}(2\rho_{ik}-\rho_{ij}\rho_{jk})\Big\}d_{ii} - \Big\{\frac{1}{2}\rho_{kl}(1-\rho_{kl}^2) + 2\rho_{ik}\rho_{il\cdot k}\Big\}d_{ij}$$

$$+ 2(\rho_{jl\cdot i}-\rho_{ij}\rho_{il\cdot k})d_{ik}\Big]f_{ij\cdot kl}$$

$$+ \frac{1}{2}\sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r}[\rho_{kq}(\rho_{lr\cdot k}-\rho_{kl}\rho_{kr\cdot q})(d_{ij}-\rho_{ij}d_{ii})f_{ij\cdot kl\cdot qr}$$

$$+ \rho_{kq\cdot r}\{\rho_{iq}\rho_{kr}\rho_{kl}\rho_{jq\cdot i} - \rho_{ir}\rho_{lr}\rho_{jr\cdot i}$$

$$+ \rho_{lr\cdot k}(3\rho_{ij}\rho_{ir}^2 - \rho_{iq}\rho_{jq} - 2\rho_{ir}\rho_{jr})\}f_{ij\cdot kl}f_{qr}] \tag{6.9}$$

$$+ \frac{1}{4}\sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r}\sum_{s\neq t}\rho_{ir\cdot q}\rho_{jq\cdot i}\rho_{kt\cdot s}\rho_{sl\cdot k}f_{ij\cdot kl}f_{qr\cdot st},$$

$$
\begin{aligned}
b_4 = a_1 a_3 &+ \sum_{i\neq j} \{2(3d_{ii} + d_{jj})(2\rho_{ij}d^*_{ii} + d_{ij}d_{ii}) \\
&- 8(d_{ij}d^*_{ii} + d_{ii}d^*_{ij}) - \rho_{ij}(5d_{ii} + 3d_{jj})d^2_{ii}\} f_{ij} \\
&+ 2\sum_{i\neq j}\sum_{k\neq l} [\{\rho_{jl}\rho_{ik\cdot j} + \rho_{jk}\rho_{il\cdot j} - 3\rho_{ij}(\rho_{il}\rho_{ik\cdot l} + \rho_{ik}\rho_{il\cdot k}) \\
&+ \frac{1}{2}\rho_{ij}\rho_{kl}(\rho^2_{jl} + 3\rho^2_{ik})\} d_{ii}d_{kk} + 2\rho^2_{ik}d_{ij}d_{kl} - 2\{\rho_{ij}(\rho^2_{jk} + \rho^2_{ik}) \\
&- 2\rho_{ik}\rho_{jk\cdot i}\} d_{ii}d_{kl}] f_{ij}f_{kl} + \sum_{i\neq j}\sum_{k\neq l} (d_{ij} - \rho_{ij}d_{ii})\{d_{kk}(\rho_{kl}d_{ll} \\
&+ 3\rho_{kl}d_{kk} - 4d_{kl}) + 4(d^*_{kl} - \rho_{kl}d^*_{kk})\} f_{ij\cdot kl} \\
&+ \sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r} [2\rho_{ql\cdot k}(d_{ij} - \rho_{ij}d_{ii})\{\rho_{qr}\rho_{qk}(3d_{qq} + d_{rr}) \\
&- 2(\rho_{qk}d_{qr} + \rho_{kr}d_{rr})\} f_{ij\cdot kl}f_{qr} + \frac{1}{3}\{(d_{qr} - 3\rho_{qr}d_{qq})d_{ij}d_{kl} \\
&+ \rho_{ij}\rho_{kl}(3d_{qr} - \rho_{qr}d_{qq})d_{ii}d_{kk}\} f_{ij\cdot kl\cdot qr}] \\
&+ \sum_{i\neq j}\sum_{k\neq l}\sum_{q\neq r}\sum_{s\neq t} [2\rho_{it\cdot s}\rho_{jk\cdot i}\rho_{lq\cdot k}\rho_{rs\cdot q} f_{ij}f_{kl}f_{qr}f_{st} \\
&+ \{\rho_{ks}(\rho_{lt\cdot s} - \rho_{st}\rho_{sl\cdot k})(d_{ij}d_{qr} + \rho_{ij}\rho_{qr}d_{ii}d_{qq}) \\
&- \rho_{qr}\rho_{kt\cdot s}\rho_{sl\cdot k}d_{ij}d_{qq}\} f_{ij\cdot kl}f_{qr\cdot st}],
\end{aligned}
$$

$$
b_6 = \frac{1}{2}a_3^2,
$$

where $f_{ij}, f_{ij\cdot kl}, f_{ij\cdot kl\cdot qr}$ *denote the partial derivatives of* $f(R)$ *at* $R=P$, $\rho_{jk\cdot i}$ *is given in* (6.4) *and*

$$
d_{ij} = \sum_{\alpha\neq\beta} \rho_{i\alpha}(\rho_{j\beta} - \rho_{j\alpha}\rho_{\alpha\beta}) f_{\alpha\beta}, \quad d^*_{ij} = \sum_{\alpha\neq\beta} d_{i\alpha}(\rho_{j\beta} - \rho_{j\alpha}\rho_{\alpha\beta}) f_{\alpha\beta}.
$$

As an example of Theorem 6.2 consider an asymptotic expansion for the nonnull distribution of the likelihood ratio statistic $-\log|R|$ for testing the hypothesis H_0^*: $P=I$ discussed in Section 3.2. Put $f(R)=-\log|R|$ in Theorem 6.2. The partial derivatives of $f(R)$ at $R=P$ are

$$
\begin{aligned}
&f_{ij} = -2\rho^{ij},\ f_{ij\cdot kl} = 2(P^{-1}E_{kl}P^{-1})_{ij} = 2(\rho^{ik}\rho^{jl} + \rho^{il}\rho^{jk}), \\
&f_{ij\cdot kl\cdot qr} = -2(P^{-1}(E_{qr}P^{-1}E_{kl} + E_{kl}P^{-1}E_{qr})P^{-1})_{ij},
\end{aligned} \tag{6.10}
$$

where $P^{-1}=(\rho^{ij})$, $(A)_{ij}$ denotes the (i, j)th element of a matrix A, and E_{kl} is a $p\times p$ matrix with 1 in the (k, l)th and (l, k)th elements and 0 otherwise. From (6.8) and (6.10), it follows that

$$\tau^2 = \frac{1}{2}\sum_{i\neq j}\sum_{k\neq l}(\rho_{jk} - \rho_{ij}\rho_{ik})(\rho_{il} - \rho_{ik}\rho_{kl})(-2\rho^{ij})(-2\rho^{kl})$$

$$= 2\sum_i\sum_j\sum_k\sum_l(\rho_{jk}\rho^{ij} - \rho_{ij}\rho^{ij}\rho_{ik})(\rho_{il}\rho^{kl} - \rho_{ik}\rho_{kl}\rho^{kl})$$

$$= 2\sum_i\sum_k(\delta_{ik} - \rho_{ik})^2 = 2(\mathrm{tr}\,P^2 - p),$$

where the summation $\sum_i$ stands for $\sum_{i=1}^{p}$. Then an asymptotic expansion for the nonnull distribution of $-\log|R|$ is given by

$$\Pr\{\sqrt{m}(-\log|R| + \log|P|)/\tau < x\}$$

$$= \Phi(x) - \frac{1}{\sqrt{m}}\{a_1\Phi^{(1)}(x)/(2\tau) + a_3\Phi^{(3)}(x)/(2\tau)^3\} + O(m^{-1}),$$

where

$$a_1 = p(p-1),$$

$$a_3 = -16\sum_i\sum_j\sum_k\rho_{ij}^2\rho_{ik}^2 + 16\,\mathrm{tr}\,P^2 + \frac{32}{3}(\mathrm{tr}\,P^3 - p).$$

Similarly, the coefficients in the term of order $1/m$ are given by b_2, b_4 and b_6 in (6.9) with the partial derivatives (6.10) and

$$d_{ij} = 2(\sum_{\alpha=1}^{p}\rho_{i\alpha}\rho_{j\alpha} - \rho_{ij}), \quad d_{ij}^* = 2(\sum_{\alpha=1}^{p}d_{i\alpha}\rho_{j\alpha} - d_{ij}).$$

The limiting distribution of $|R|$ was obtained by Olkin and Siotani [71]. In the null case, namely, $P=I$, Nagarsenker [70] has obtained the exact distribution of $|R|$.

7. Asymptotic nonnull distributions of test statistics

7.1. *Test statistics concerning latent vectors of a correlation matrix.* The statistics (5.10) and (5.11) proposed in Section 5 are functions only of the sample correlation matrix R. Using Theorem 6.2 in the last section, we give asymptotic expansions for the nonnull distributions of these statistics. Put

$$f(R) = \frac{1}{N}T_1 = \sum_{\alpha=1}^{a}\log(H_{10}'RH_{10})_{\alpha\alpha} + \log|H_2'RH_2| - \log|R|$$

in Theorem 6.2. The partial derivatives of $f(R)$ with respect to r_{ij} $(i<j)$ at $R=P$ are given by

$$f_{ij} = 2\rho_{ij}^{(1)} - 2\rho^{ij},$$

$$f_{ij\cdot kl} = -4\sum_{\alpha=1}^{a} h_{i\alpha}^{(10)}h_{j\alpha}^{(10)}h_{k\alpha}^{(10)}h_{l\alpha}^{(10)}(H_{10}'PH_{10})_{\alpha\alpha}^{-2}$$

$$-2\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\sum_{\alpha'=1}^{p-a}\sum_{\beta'=1}^{p-a} h_{i\alpha}^{(2)}h_{j\beta}^{(2)}h_{k\alpha'}^{(2)}h_{l\beta'}^{(2)}(P_{22}^{*-1}E_{\alpha'\beta'}P_{22}^{*-1})_{\alpha\beta}$$

$$+2(P^{-1}E_{kl}P^{-1})_{ij},$$

$$(7.1)\quad f_{ij\cdot kl\cdot qr} = 16\sum_{\alpha=1}^{a} h_{i\alpha}^{(10)}h_{j\alpha}^{(10)}h_{k\alpha}^{(10)}h_{l\alpha}^{(10)}h_{q\alpha}^{(10)}h_{r\alpha}^{(10)}(H_{10}'PH_{10})_{\alpha\alpha}^{-3}$$

$$+2\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\sum_{\alpha'=1}^{p-a}\sum_{\beta'=1}^{p-a}\sum_{\alpha''=1}^{p-a}\sum_{\beta''=1}^{p-a} h_{i\alpha}^{(2)}h_{j\beta}^{(2)}h_{k\alpha'}^{(2)}h_{l\beta'}^{(2)}h_{q\alpha''}^{(2)}h_{r\beta''}^{(2)}$$

$$\times (P_{22}^{*-1}(E_{\alpha''\beta''}P_{22}^{*-1}E_{\alpha'\beta'} + E_{\alpha'\beta'}P_{22}^{*-1}E_{\alpha''\beta''})P_{22}^{*-1})_{\alpha\beta}$$

$$-2(P^{-1}(E_{qr}P^{-1}E_{kl} + E_{kl}P^{-1}E_{qr})P^{-1})_{ij},$$

where $H_{10} = (h_{ij}^{(10)})$, $H_2 = (h_{ij}^{(2)})$, $P_{22}^{*} = H_2'PH_2$ and

$$\rho_{ij}^{(1)} = (H_{10}\Lambda_a^{-1}H_{10}' + H_2P_{22}^{*-1}H_2')_{ij}$$

with $\Lambda_a = \mathrm{diag}\,[(H_{10}'PH_{10})_{11},\ldots,(H_{10}'PH_{10})_{aa}]$, and $E_{\alpha\beta}$ is defined in (6.10). Then we have

THEOREM 7.1. *An asymptotic expansion for the nonnull distribution of the statistic T_1 given by* (5.10) *for testing the hypothesis* (5.1) *is*

$$\Pr\Big[\sqrt{m}\{T_1/N - \sum_{\alpha=1}^{a}\log(H_{10}'PH_{10})_{\alpha\alpha} - \log|H_2'PH_2| + \log|P|\}/\tau_1 < x\Big]$$

$$= \Phi(x) - \frac{1}{\sqrt{m}}\{a_1\Phi^{(1)}(x)/(2\tau_1) + a_3\Phi^{(3)}(x)/(2\tau_1)^3\}$$

$$+\frac{1}{m}\sum_{j=1}^{3} b_{2j}\Phi^{(2j)}(x)/(2\tau_1)^{2j} + O(m^{-3/2}),$$

where

$$\tau_1^2 = 2\sum_{i\neq j}\sum_{k\neq l}\rho_{jk\cdot i}\rho_{il\cdot k}(\rho_{ij}^{(1)} - \rho^{ij})(\rho_{kl}^{(1)} - \rho^{kl})$$

and the coefficients are given by (6.9) *with the partial derivatives* (7.1) *and*

$$(7.2)\quad d_{ij} = 2\Big(\sum_{\alpha\neq\beta}\rho_{i\alpha}\rho_{j\beta\cdot\alpha}\rho_{\alpha\beta}^{(1)} + \sum_{\substack{\alpha=1\\ \alpha\neq j}}^{p}\rho_{i\alpha}\rho_{j\alpha}\Big),$$

$$d_{ij}^{*} = 2\Big(\sum_{\alpha\neq\beta} d_{i\alpha}\rho_{j\beta\cdot\alpha}\rho_{\alpha\beta}^{(1)} + \sum_{\substack{\alpha=1\\ \alpha\neq j}}^{p} d_{i\alpha}\rho_{j\alpha}\Big).$$

Similarly, taking

$$f(R) = T_2/N = \log|H_2'RH_2| + \log|\Lambda_{a0}| - \log|R| + \operatorname{tr}\Lambda_{a0}^{-1}H_{10}'RH_{10} - a$$

in Theorem 6.2 and differentiating with respect to r_{ij} $(i<j)$ at $R=P$, we have the following theorem.

THEOREM 7.2. *An asymptotic expansion for the nonnull distribution of the statistic T_2 given by* (5.11) *for testing the hypothesis* (5.2) *is*

$$\begin{aligned}\Pr[&\sqrt{m}\{T_2/N - \log|H_2'PH_2| - \log|\Lambda_{a0}| + \log|P| \\ &- \operatorname{tr}\Lambda_{a0}^{-1}H_{10}'PH_{10} + a\}/\tau_2 < x] \\ &= \Phi(x) - \frac{1}{\sqrt{m}}\{a_1\Phi^{(1)}(x)/(2\tau_2) + a_3\Phi^{(3)}(x)/(2\tau_2)^3\} \\ &\quad + \frac{1}{m}\sum_{j=1}^{3} b_{2j}\Phi^{(2j)}(x)/(2\tau_2)^{2j} + O(m^{-3/2}),\end{aligned}$$

where

$$\tau_2^2 = 2\sum_{i\neq j}\sum_{k\neq l}\rho_{jk\cdot i}\rho_{il\cdot k}(\rho_{ij}^{(2)} - \rho^{ij})(\rho_{kl}^{(2)} - \rho^{kl})$$

with $\rho_{ij}^{(2)} = (H_{10}\Lambda_{a0}^{-1}H_{10}' + H_2P_{22}^{*-1}H_2')_{ij}$, *and the coefficients* a_1, a_3, b_2, b_4, b_6 *are given by* (6.9) *with partial derivatives*

$$f_{ij} = 2(\rho_{ij}^{(2)} - \rho^{ij}),$$

$$\begin{aligned}f_{ij\cdot kl} = &-2\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\sum_{\alpha'=1}^{p-a}\sum_{\beta'=1}^{p-a} h_{i\alpha}^{(2)}h_{j\beta}^{(2)}h_{k\alpha'}^{(2)}h_{l\beta'}^{(2)}(P_{22}^{*-1}E_{\alpha'\beta'}P_{22}^{*-1})_{\alpha\beta} \\ &+ 2(P^{-1}E_{kl}P^{-1})_{ij},\end{aligned}$$

$$\begin{aligned}f_{ij\cdot kl\cdot qr} = &\,2\sum_{\alpha=1}^{p-a}\sum_{\beta=1}^{p-a}\sum_{\alpha'=1}^{p-a}\sum_{\beta'=1}^{p-a}\sum_{\alpha''=1}^{p-a}\sum_{\beta''=1}^{p-a} h_{i\alpha}^{(2)}h_{j\beta}^{(2)}h_{k\alpha'}^{(2)}h_{l\beta'}^{(2)}h_{q\alpha''}^{(2)}h_{r\beta''}^{(2)} \\ &\times (P_{22}^{*-1}(E_{\alpha''\beta''}P_{22}^{*-1}E_{\alpha'\beta'} + E_{\alpha'\beta'}P_{22}^{*-1}E_{\alpha''\beta''})P_{22}^{*-1})_{\alpha\beta} \\ &- 2(P^{-1}(E_{qr}P^{-1}E_{kl} + E_{kl}P^{-1}E_{qr})P^{-1})_{ij}\end{aligned}$$

and (7.2) *replaced* $\rho_{\alpha\beta}^{(1)}$ *by* $\rho_{\alpha\beta}^{(2)}$.

7.2. *Some other test statistics.* Given $N=n+1$ observations from a p-variate normal population $N_p(\boldsymbol{\mu}, \Sigma)$, Anderson [6] and Bartlett [7] have considered a test of the equality of the last $p-1$ latent roots of the population correlation matrix $P=(\rho_{ij})$ and proposed the test statistic

$$N[(p-1)\log\{\sum_{\alpha=2}^{p} l_\alpha/(p-1)\} - \sum_{\alpha=2}^{p}\log l_\alpha], \tag{7.3}$$

where $l_1 \geqq \cdots \geqq l_p > 0$ are the ordered latent roots of R. This hypothesis is precisely equivalent (see Anderson [6, p. 142]) to

$$H_3: \rho_{ij} = \rho \qquad \text{for all} \quad i < j,$$

where ρ is unspecified. It is difficult to obtain a likelihood ratio criterion for H_3 in closed form and so Aitkin, Nelson and Reinfurt [2] have proposed the approximate likelihood ratio criterion in the form

$$N[\log\{1 + (p-1)\bar{r}\}(1-\bar{r})^{p-1} - \log|R|], \tag{7.4}$$

where $\bar{r} = \left\{\frac{1}{2}p(p-1)\right\}^{-1} \sum_{i<j}^{p} r_{ij}$ for $R = (r_{ij})$. The asymptotic null distributions of (7.3), (7.4) and Lawley's test statistic [58] have been studied by Aitkin, Nelson and Reinfurt [2], Anderson [6], Bartlett [7] and Gleser [30]. The asymptotic nonnull distribution of (7.3) is included in Theorem 3.5 in Section 3.2, and that of (7.4) was given by Konishi [48].

Another interesting test concerning a correlation matrix is to test the null hypothesis

$$H_4: P = P_0 \qquad \text{for specified} \quad P_0.$$

Bartlett and Rajalakshman [8] and, in view of information theory, Kullback [55, 56] have proposed the statistic

$$N\{\log(|P_0|/|R|) - p + \operatorname{tr}(P_0^{-1}R)\}. \tag{7.5}$$

The asymptotic null distribution of (7.5) has been studied by Aitkin [1], Bartlett [7], Bartlett and Rajalakshman [8], Kullback [55, 56], and the asymptotic nonnull distribution by Konishi [48].

8. Further consideration of the use of asymptotic expansion formula

8.1. *Sample correlation coefficient.* In the case of $p=2$, the sample correlation matrix R and the population correlation matrix P are, respectively, reduced to

$$\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $r_{12} = r_{21} = r$ and $\rho_{12} = \rho_{21} = \rho$. The r is called the sample correlation coefficient, based on a sample of size $N = n+1$ drawn from a bivariate normal distribution with population correlation coefficient ρ. The asymptotic formula (6.7) in the bivariate case gives an asymptotic expansion for the distribution of a function of r, which was given by Konishi [48]. As special cases of the result, we can obtain asymptotic expansions for the distributions of various statistics

expressed as functions of r. The following results are due to Konishi [48].

$$\text{(8.1)}\quad \Pr\{\sqrt{m}(r-\rho)/(1-\rho^2) < x\}$$
$$= \Phi(x) + \frac{1}{\sqrt{m}}\rho\left(x^2 - \frac{1}{2}\right)\phi(x) + \frac{1}{m}\left\{\left(\Delta - \frac{3}{4} + \frac{1}{8}\rho^2\right)x + \frac{1}{4}(1+6\rho^2)x^3 - \frac{1}{2}\rho^2 x^5\right\}\phi(x) + O(m^{-3/2}),$$
$$\Pr\{\sqrt{m}(\sin^{-1} r - \sin^{-1}\rho)/(1-\rho^2)^{1/2} < x\}$$
$$= \Phi(x) + \frac{1}{\sqrt{m}}\frac{\rho}{2}(x^2-1)\phi(x) + \frac{1}{m}\left\{\left(\Delta - \frac{3}{4} + \frac{1}{8}\rho^2\right)x + \frac{1}{12}(1+5\rho^2)x^3 - \frac{1}{8}\rho^2 x^5\right\}\phi(x) + O(m^{-3/2}),$$

where $m = n - 2\Delta$ with a correction factor Δ depending on ρ, $\Phi(x)$ and $\phi(x)$ are the standard normal distribution function and its first derivative.

Another use of asymptotic formula for $f(r)$ may be found in Konishi [47]. Solving certain differential equation after deriving an asymptotic expansion for the distribution of $f(r)$, Konishi [47] has obtained a simple and accurate approximation to the distribution of r, which is of the form

$$\text{(8.2)}\quad \Pr\left\{\sqrt{m}\left(\frac{1}{2}\log\frac{1+r}{1-r} - \frac{1}{2}\log\frac{1+\rho}{1-\rho}\right) < x\right\}$$
$$= \Phi(x) - \frac{1}{2}\left(\frac{\rho}{\sqrt{m}} + \frac{x^3}{6m}\right)\phi(x) + O(m^{-3/2}),$$

where $m = n - 3/2 + \rho^2/4$.

It may be noted that $(1/2)\log\{(1+r)/(1-r)\}$ $(=z(r))$ is the well known Fisher's z-transformation [19]. Konishi [47] has made, using exact values as given by David [16], an overall comparison with previous approximations; a normalization approximation due to Ruben [78], a t-approximation due to Kraemer [49] and normal approximations for $z(r)$ due to Fisher [19], Gayen [28], Nabeya [65]. Further comparisons of these approximations will be given in Section 8.2.

8.2. *Numerical comparisons.* Various approximations to the distribution of r as stated in the last subsection are compared in Tables 8.1–8.3. Table 8.1 gives a comparison of the values of the probability integral $\Pr(r \leqq r_0)$ $(|r_0|<1)$ approximated by using (8.1) and (8.2). Tables 8.2 and 8.3 contain comparisons of the accuracies of various approximations. Exact values are taken from tables in David [16].

In the accompanying tables, N, R_a, Z_a, F_1, F_2, K and R are the notations standing for the following:

N: sample size $(=n+1)$,

R_a: the case that the values of $\Pr(r \leqq r_0)$ are approximated by (8.1) with $\Delta = 3/4-\rho^2/8$,

Z_a: the case that the values of $\Pr(r \leqq r_0)$ are approximated by (8.2),

F_1: the case that $z(r)=(1/2)\log\{(1+r)/(1-r)\}$ is approximated by a normal variate with mean $z(\rho)$ and variance $1/(N-3)$,

F_2: the case that $z(r)$ is approximated by a normal variate with mean $z(\rho)+\rho/2n+\rho(5+\rho^2)/8n^2$ and variance $\{1+(4-\rho^2)/2n+(22-6\rho^2-3\rho^4)/6n^2\}/n$,

K: the case that $(N-2)^{1/2}(r-\rho^*)/\{(1-r^2)(1-\rho^{*2})\}^{1/2}$ is approximated by a t-variate with $(N-2)$ degrees of freedom where ρ^* is the median of the distribution of r,

R: the case that the values of $\Pr(r \leqq r_0)$ are approximated by

Table 8.1. *Comparison of the values of* $\Pr(r \leqq r_0)$ *approximated by* (8.1) *and* (8.2) *for* $N=25$ *and* $\rho=0.9$

r_0		term of $O(1)$	term of $O(1/\sqrt{n})$	term of $O(1/n)$	total	exact
.80	R_a	.00608	.01880	.01292	.03779	.03129
	Z_a	.03753	−.00773	.00169	.03149	
.81	R_a	.01201	.02711	.00946	.04858	.04178
	Z_a	.05001	−.00974	.00168	.04195	
.82	R_a	.02242	.03550	.00313	.06105	.05574
	Z_a	.06645	−.01218	.00161	.05587	
.83	R_a	.03959	.04167	−.00442	.07685	.07423
	Z_a	.08794	−.01507	.00145	.07432	
.89	R_a	.40099	−.03192	−.00039	.36869	.36883
	Z_a	.40531	−.03661	.00002	.36872	
.90	R_a	.50000	−.03768	.00000	.46232	.46244
	Z_a	.50000	−.03768	.00000	.46232	
.91	R_a	.59901	−.03192	.00039	.56747	.56762
	Z_a	.60392	−.03639	−.00003	.56750	
.95	R_a	.89506	.03681	.01307	.94494	.94612
	Z_a	.95666	−.00868	−.00170	.94628	
.955	R_a	.91609	.04082	.01237	.96928	.96838
	Z_a	.97560	−.00541	−.00161	.96858	
.96	R_a	.93379	.04285	.01058	.98722	.98350
	Z_a	.98800	−.00295	−.00132	.98373	
.965	R_a	.94845	.04305	.00784	.99935	.99263
	Z_a	.99508	−.00135	−.00090	.99283	

$$\Pr(r \leqq r_0) = \Phi\left(\frac{\left\{r_0(1-r_0^2)^{-1}\left(N-\frac{5}{2}\right)\right\}^{1/2} - \left\{\rho(1-\rho^2)^{-1}\left(N-\frac{3}{2}\right)\right\}^{1/2}}{\left\{1+\frac{1}{2}r_0^2(1-r_0^2)^{-1}+\frac{1}{2}\rho^2(1-\rho^2)^{-1}\right\}^{1/2}} \right).$$

It may be seen from these tables that the asymptotic formula (8.2) provides high accuracy over the whole domain of r, even for relatively small N. Table 8.1 shows that the normal approximation based on the limiting term of (8.1) can remarkably be improved by the transformation $z(r)$ in the tail areas of the distribution curve. We can also see the efficacy of the terms of orders $1/\sqrt{n}$ and $1/n$ in the expansion formulae. The normal approximation with mean $z(\rho)$ and variance $1/(N-3)$, which is the most commonly used for the distribution of

Table 8.2. *Comparison of errors in approximating the values of* $\Pr(r \leqq r_0)$: *Error=(approximate value−exact value)*$\times 10^5$

r_0	F_1	F_2	K	R	Z_a	exact
			$N=11$, $\rho=0.5$			
−.10	597	−90	95	67	34	.02710
−.05	836	−40	116	74	25	.03666
.00	1120	26	138	80	12	.04893
.05	1447	109	160	83	−3	.06449
.10	1806	203	181	84	−20	.08400
.45	3516	286	139	12	−77	.39234
.50	3329	104	90	7	−79	.46671
.55	2994	−105	35	2	−85	.54751
.75	1057	−298	−100	−228	−81	.87402
.80	717	−70	−81	−346	−48	.93270
.85	454	124	−47	−400	−3	.97267
.90	213	141	−15	−313	31	.99335
			$N=11$, $\rho=0.9$			
.625	664	−84	218	340	100	.01165
.675	1165	−43	314	402	101	.02086
.725	1989	60	440	441	81	.03864
.775	3253	247	579	414	34	.07439
.80	4071	378	649	377	22	.10461
.89	6217	241	297	83	−42	.38127
.90	6043	92	167	79	−43	.43957
.91	5698	−80	22	77	−44	.50517
.96	1802	−183	−359	−495	18	.89182
.97	1040	84	−240	−669	64	.94987
.975	723	176	−166	−678	81	.97083
.98	450	200	−97	−606	85	.98571

Table 8.3. *Comparison of errors in approximating the values of* Pr $(r \leqq r_0)$: *Error=(approximate value−exact value)*$\times 10^5$

r_0	F_1	F_2	K	R	Z_a	exact
			$N=25,\ \rho=0.9$			
.80	857	−5	153	177	20	.03129
.81	1093	19	179	180	17	.04178
.82	1378	48	207	178	13	.05574
.83	1712	84	235	168	9	.07423
.84	2091	122	260	149	3	.09859
.89	3793	103	191	22	−11	.36883
.90	3756	6	113	18	−12	.46244
.91	3472	−104	22	16	−12	.56762
.95	803	−3	−100	−231	16	.94612
.955	540	43	−74	−236	20	.96838
.96	335	66	−48	−210	23	.98350
.97	85	42	−12	−104	14	.99733
			$N=50,\ \rho=0.9$			
.82	245	−20	46	72	9	.01285
.83	396	−13	66	83	8	.02177
.84	619	1	90	89	6	.03643
.85	929	26	121	88	5	.05998
.86	1323	56	150	75	2	.09681
.89	2539	65	165	11	−2	.33974
.90	2597	2	123	8	−2	.47403
.91	2312	−71	52	3	−3	.62459
.93	1037	−56	−36	−78	2	.88871
.94	467	10	−31	−108	6	.96114
.95	141	30	−12	−74	7	.99174
.96	22	10	−2	−23	3	.99920

$z(r)$, is not so accurate, though this approximation is much superior, in the tail areas, to that based on the limiting distribution of r.

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