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On Oriented G-Manifolds of Baas-Sullivan Type

Dedicated to Professor A. Komatu on his 70th birthday

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Introduction

N. A. Baas [1] has studied a bordism theory based on manifolds with a certain type of singularities, by reformulating a theory due to D. Sullivan.

The purpose of this paper is to study corresponding equivariant oriented bordism theories. By the same way as [1; § 2], we can define the notion of oriented $\mathscr{S}_n$-manifolds for each singularity class $\mathscr{S}_n=\{P_0=\mathrm{pt}, P_1,\dots, P_n\}$ of closed oriented manifolds, and those with G-actions for each finite group G while G acts trivially on P_i. Thus we obtain naturally a bordism group $\Omega(\mathscr{S}_n)_*^G(X, Y)$ based on oriented $\mathscr{S}_n$-manifolds with free G-actions for each pair (X, Y) of G-spaces. When $n=0$, $\Omega(\mathscr{S}_0)_*^G(-)$ coincides with the usual equivariant bordism group $\Omega_*^G(-)$, due to Conner-Floyd [3], based on (closed) oriented manifolds with free G-actions.

We study in § 1 (and § 5) some basic properties of oriented $\mathscr{S}_n$-manifolds and the above bordism group, and obtain an exact sequence in Theorem 1.16 which is similar to that in [1; Th. 3.2]. In case that $G=\boldsymbol{Z}_p$ for odd prime p, we can define in § 2 the Smith homomorphism, and extend some results on $\Omega_*^G(-)$, due to P. E. Conner [2] and C. M. Wu [8], to those on $\Omega(\mathscr{S}_n)_*^G(-)$ for each n. Furthermore, we obtain in § 3 a theorem on the Ω_*-module structure of $\Omega(\mathscr{S}_1)_*^{\boldsymbol{Z}_p}$ for an odd dimensional manifold P_1 of Dold type. Finally in § 4, we study oriented $\mathscr{S}_n$-manifolds with semi-free G-actions.

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§ 1. Oriented singular $(G, \mathscr{S}_n)$-manifolds

Throughout this paper, we will work in the category of compact *oriented* smooth manifolds, and we allow the manifolds to have general corners, (see [4] for manifolds with corners, whose coordinate neighborhoods are defined by using open subsets of $\{(x_1,\dots, x_m)\in \boldsymbol{R}^m \mid x_1\geqq 0,\dots, x_m\geqq 0\}$).

DEFINITION 1.1. Let an oriented manifold V and oriented submanifolds

$\partial_0 V, \partial_1 V, \dots, \partial_n V$ of the boundary ∂V (of codimension zero if $\partial_i V \neq \emptyset$) with induced orientations satisfy the following conditions:

$$\partial V = \partial_0 V \cup \partial_1 V \cup \cdots \cup \partial_n V$$

and $\partial_i V \cap \partial_j V$ is the intersection of the boundaries of $\partial_i V$ and $\partial_j V$ if $i \neq j$. Then we call

$$V = (V; \partial_0 V, \partial_1 V, \dots, \partial_n V)$$

a *decomposed* (*oriented*) *manifold of type n*.

$$-V = (-V; -\partial_0 V, -\partial_1 V, \dots, -\partial_n V)$$

with opposite orientations is a decomposed manifold.

Each $\partial_i V$ is again a decomposed manifold by defining

$$\partial_j(\partial_i V) = \begin{cases} \partial_i V \cap \partial_j V & \text{for } j \neq i, \\ \emptyset & \text{for } j = i. \end{cases}$$

For each closed oriented manifold P, the product manifolds $V \times P$ and $P \times V$ with product orientations are decomposed manifolds by defining

$$\partial_i(V \times P) = \partial_i V \times P, \quad \partial_i(P \times V) = (-1)^{\dim P} P \times \partial_i V.$$

We may regard each decomposed manifold V of type n as the one of type $n+1$ by defining $\partial_{n+1} V = \emptyset$.

By a *smooth map* $\varphi\colon V \to V'$ *of decomposed manifolds* of type n, we mean a smooth map $\varphi\colon V \to V'$ with $\varphi(\partial_i V) \subset \partial_i V'$. We denote simply the restriction by $\varphi_i = \varphi \mid \partial_i V\colon \partial_i V \to \partial_i V'$. If φ and each φ_i are diffeomorphisms, then we say that $\varphi\colon V \to V'$ is a *diffeomorphism of decomposed manifolds*.

Let us now fix a class of closed oriented manifolds

$$\mathscr{S} = \{P_0 = \mathrm{pt}, P_1, \dots, P_j, \dots\}, \quad p_j = \dim P_j,$$

where $P_0 = \mathrm{pt}$ is a one-point manifold, and put

$$\mathscr{S}_n = \{P_0 = \mathrm{pt}, P_1, \dots, P_n\}.$$

NOTATIONS. Let $I(n)$ be the set of all finite sequences of integers in $\{0,1,\dots, n\}$. For each $\omega = (i_1, \dots, i_k) \in I(n)$ and $j \in \{0,1,\dots,n\}$, set

$$(j,\omega) = (j, i_1, \dots, i_k) \in I(n), \quad (\omega, j) = (i_1, \dots, i_k, j) \in I(n);$$

$$|\omega| = k, \quad p(\omega) = \textstyle\sum_{l=1}^{k} p_{i_l}, \quad p(j \mid \omega) = \sum_{i_l < j} p_{i_l}.$$

DEFINITION 1.2. A decomposed manifold A of type n is called an (*oriented*) *$\mathscr{S}_n$-manifold* if it satisfies the following conditions (i) and (ii):

(i) With each $\omega \in I(n)$, there is associated a decomposed (oriented) manifold $A(\omega)$ of type n such that

a) $A(\emptyset)=A$, and $A(\omega)=\emptyset$ if $i_j=i_l$ for some $j \neq l$ $(\omega=(i_1,\dots,i_k))$; and

b) for each $\sigma \in S_\omega$ (the set of all permutations on ω), $A(\sigma\omega)=A(\omega)$ without considering orientations, and the identity map id: $A(\sigma\omega)\to A(\omega)$ is a diffeomorphism of decomposed manifolds of degree $\varepsilon(\sigma)$ (the sign of σ), i.e., $A(\sigma\omega)=\varepsilon(\sigma)A(\omega)$.

(ii) With each $\omega \in I(n)$ and $i \in \{0, 1,\dots, n\}$, there is associated a diffeomorphism of decomposed manifolds

$$\alpha(i,\omega)\colon \partial_i A(\omega) \approx A(i,\omega) \times P_i \qquad \text{of degree} \quad (-1)^{p_i p(i|\omega)}$$

if ω consists of distinct integers and $i \notin \omega$ ($\alpha(i,\omega)$ is called a *structure map*), and $\partial_i A(\omega)=\emptyset$ if $i \in \omega$, so that the diagrams

$$(1.3)\qquad \begin{array}{ccc} \partial_i A(\omega) & \xrightarrow{\alpha(i,\omega)} & A(i,\omega)\times P_i \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}\times\mathrm{id}} \\ \partial_i A(\sigma\omega) & \xrightarrow{\alpha(i,\sigma\omega)} & A(i,\sigma\omega)\times P_i, \end{array}$$

$$\begin{array}{ccccc} \partial_j\partial_i A(\omega) & \xrightarrow{\alpha(i,\omega)_j} & \partial_j A(i,\omega)\times P_i & \xrightarrow{\alpha(j,i,\omega)\times \mathrm{id}} & A(j,i,\omega)\times P_j\times P_i \\ \| & & & & \\ \partial_j A(\omega)\cap\partial_i A(\omega) & & & & \Big\downarrow{\scriptstyle \mathrm{id}\times T} \\ \| & & & & \\ \partial_i\partial_j A(\omega) & \xrightarrow{\alpha(j,\omega)_i} & \partial_i A(j,\omega)\times P_j & \xrightarrow{\alpha(i,j,\omega)\times \mathrm{id}} & A(i,j,\omega)\times P_i\times P_j \end{array}$$

of diffeomorphisms are commutative, where $\sigma \in S_\omega$ and T is the twisting map.

The *dimension* of $\mathscr{S}_n$-manifold $A=\{A(\omega), \alpha(i,\omega)\}$ is defined to be that of the ambient manifold $A=A(\emptyset)$.

We see easily the following lemma by definition:

LEMMA 1.4. *Let $A=\{A(\omega), \alpha(i,\omega)\}$ be an $\mathscr{S}_n$-manifold and Q be a closed oriented manifold. Then we have an $\mathscr{S}_n$-manifold $Q\times A=\{(Q\times A)(\omega), \alpha'(i,\omega)\}$ by setting*

$$(Q\times A)(\omega) = (-1)^{|\omega|\dim Q} Q \times A(\omega), \quad \alpha'(i,\omega) = \mathrm{id}\times\alpha(i,\omega),$$

where $\times$ is the product of decomposed manifolds. Furthermore, we have an $\mathscr{S}_n$-manifold $A\times Q=\{(A\times Q)(\omega), \alpha''(i,\omega)\}$ by setting

$$(A \times Q)(\omega) = (-1)^{p(\omega)\dim Q} A(\omega) \times Q, \quad \alpha''(i,\omega) = (\mathrm{id} \times T)\circ(\alpha(i,\omega) \times \mathrm{id}).$$

DEFINITION 1.5. Let $A=\{A(\omega), \alpha(i,\omega)\}$ and $B=\{B(\omega), \beta(i,\omega)\}$ be $\mathscr{S}_n$-manifolds. By an *$\mathscr{S}_n$-map f: $A\to B$*, we mean a system of smooth maps $f(\omega)$: $A(\omega)\to B(\omega)$ of decomposed manifolds such that the following diagram is commutative:

$$\begin{tikzcd}
\partial_i A(\omega) \arrow[r, "\alpha(i,\omega)"] \arrow[d, "f(\omega)_i"'] & A(i,\omega) \times P_i \arrow[d, "f(i,\omega)\times \mathrm{id}"] \\
\partial_i B(\omega) \arrow[r, "\beta(i,\omega)"] & B(i,\omega) \times P_i.
\end{tikzcd}$$

If each $f(\omega)$ is an orientation preserving diffeomorphism, then we say that *f: $A\to B$ is an $\mathscr{S}_n$-isomorphism.*

If $A(\omega)\subset B(\omega)$, $\partial_i A(\omega)\subset \partial_i B(\omega)$ and the inclusion map i: $A\subset B$, $i(\omega)$: $A(\omega)\subset B(\omega)$, is an $\mathscr{S}_n$-map, then we say that A is an *$\mathscr{S}_n$-submanifold of B of codimension* $\dim B-\dim A$.

Now, let G be a finite group, and let $(X, Y; \tau)$ be a pair of topological G-spaces, i.e., τ: $G\times(X, Y)\to(X, Y)$ be a G-action.

DEFINITION 1.6. If G acts on an $\mathscr{S}_n$-manifold A by $\mathscr{S}_n$-isomorphisms, i.e., if there is given a system φ: $G\times A\to A$ of G-actions $\varphi(\omega)$: $G\times A(\omega)\to A(\omega)$ such that each

$$\varphi(g, \ ) = \{\varphi(\omega)(g, \ )\}: A \longrightarrow A \qquad (g \in G)$$

is an $\mathscr{S}_n$-isomorphism, then we say that (A, φ) is a *$(G, \mathscr{S}_n)$-manifold.* (We say that it is *(semi-)free* if each G-action $\varphi(\omega)$ is (semi-)free.)

Furthermore, if there is given a system f: $A\to X$ of G-equivariant (continuous) maps $f(\omega)$: $A(\omega)\to X$ such that

$$f(\omega)(A(\omega)) \subset Y \qquad \text{for any} \quad \omega = (i_1,\dots,i_k)\ni 0$$

(in the special case $Y=\emptyset$, this means that $A(\omega)=\emptyset$ for $\omega\ni 0$) and the diagram

$$\begin{tikzcd}
\partial_i A(\omega) \arrow[rr, "\subset"] \arrow[d, "\alpha(i,\omega)"'] & & A(\omega) \arrow[r, "\mathrm{id}"] \arrow[d, "f(\omega)"] & A(\sigma\omega) \arrow[d, "f(\sigma\omega)"] \\
A(i,\omega)\times P_i \arrow[r, "\mathrm{pr}"'] & A(i,\omega) \arrow[r, "f(i,\omega)"'] & X \arrow[r, equal] & X
\end{tikzcd}$$

$(\sigma\in S_\omega)$ is commutative, then we say that (A, φ, f) is a *singular ((semi-)free) $(G, \mathscr{S}_n)$-manifold in $(X, Y; \tau)$.*

We see easily the following lemma by Lemma 1.4 and by definition:

LEMMA 1.7. *Let (A, φ, f) be a singular ((semi-)free) $(G, \mathscr{S}_n)$-manifold in $(X, Y; \tau)$ and Q be a closed oriented manifold. Then we have ((semi-)free) singular $(G, \mathscr{S}_n)$-manifolds*

$$(Q \times A, \mathrm{id} \times \varphi, f \circ \mathrm{pr}) \quad and \quad (A \times Q, \varphi \times \mathrm{id}, f \circ \mathrm{pr})$$

in $(X, Y; \tau)$, where $Q \times A$ and $A \times Q$ are the product $\mathscr{S}_n$-manifolds in Lemma 1.4 *and* $(\mathrm{id} \times \varphi)(\omega) = \mathrm{id} \times \varphi(\omega)$, $(\varphi \times \mathrm{id})(\omega) = \varphi(\omega) \times \mathrm{id}$, $(f \circ \mathrm{pr})(\omega) = f(\omega) \circ \mathrm{pr}$.

DEFINITION 1.8. For any singular $(G, \mathscr{S}_n)$-manifolds $A = (A, \varphi, f)$ and $A' = (A', \varphi', f')$ in $(X, Y; \tau)$, the *disjoint sum* $A + A'$, the *oppositely oriented manifold* $-A$ and $\delta_i A = (\delta_i A, \delta_i \varphi, \delta_i f)$ $(i \in \{0, 1, \dots, n\})$ are defined naturally by

$$A + A' = (A + A', \varphi + \varphi', f + f'), \quad -A = (-A, \varphi, f);$$

$$(\delta_i A)(\omega) = A(\omega, i), \quad (\delta_i \varphi)(\omega) = \varphi(\omega, i), \quad (\delta_i f)(\omega) = f(\omega, i).$$

Then $\delta_0 A$ is a singlular $(G, \mathscr{S}_n)$-manifold in $(Y; \tau) = (Y, \emptyset; \tau)$ by definition, and we say that A is *closed* if $\delta_0 A = \emptyset$, i.e., if $A(\omega) = \emptyset$ for $\omega \ni 0$. Further, $\delta_n A$ can be regarded naturally as a singular $(G, \mathscr{S}_{n-1})$-manifold in $(X, Y; \tau)$.

DEFINITION 1.9. We say that a singular ((semi-)free) $(G, \mathscr{S}_n)$-manifold (A, φ, f) in $(X, Y; \tau)$ *bords* if there exists a singular ((semi-)free) $(G, \mathscr{S}_n)$-manifold (B, Φ, F) in $(X, X; \tau)$ satisfying the following conditions (i) and (ii):

(i) (A, φ) is an *oriented $(G, \mathscr{S}_n)$-submanifold of $(\delta_0 B, \delta_0 \Phi)$ of codimension zero except 0-boundary*, i.e., for each $\omega \in I(n)$ with $\omega \not\ni 0$ and $i \neq 0$,

$$A(\omega) \subset (\delta_0 B)(\omega) = B(\omega, 0), \quad \partial_i A(\omega) \subset \partial_i(\delta_0 B)(\omega)$$

as oriented manifolds, and

$$\beta(i, \omega, 0) \mid \partial_i A(\omega) = \alpha(i, \omega), \quad \Phi(\omega, 0) \mid G \times A(\omega) = \varphi(\omega),$$

where α and β are the structure maps of A and B, respectively.

(ii) $F(\omega, 0) \mid A(\omega) = f(\omega), \quad F(\omega, 0)(B(\omega, 0) - \mathrm{Int}\, A(\omega)) \subset Y$

for each $\omega \in I(n)$ with $\omega \not\ni 0$, where Int means the interior in $B(\omega, 0)$.

(In the special case $Y = \emptyset$, these mean that $\delta_0(B, \Phi, F) = (A, \varphi, f)$.)

We say that $A = (A, \varphi, f)$ *is bordant to* $A' = (A', \varphi', f')$ if

$$A + (-A') = (A + (-A'), \varphi + \varphi', f + f')$$

bords. We write this as $A \sim A'$.

Then we have the following basic lemma:

LEMMA 1.10. *The relation $\sim$ of bordism between singular ((semi-)free) $(G, \mathscr{S}_n)$-manifolds in $(X, Y; \tau)$ is an equivalence relation.*

This lemma can be proved by the same way as the proof of Baas [1; Lemma 3.1.] for the unoriented case. Especially, the transitivity can be proved by using the following

LEMMA 1.11 (Pasting Lemma). *Let (A_k, φ_k) $(k=1, 2)$ and (B, φ) be $(G, \mathscr{S}_n)$- manifolds, and suppose that (B, φ) and $(-B, \varphi)$ are oriented $(G, \mathscr{S}_n)$-submanifolds of $(\delta_0 A_1, \delta_0\varphi_1)$ and $(\delta_0 A_2, \delta_0\varphi_2)$ of codimension zero except 0-boundary, respectively. Then, by identifying the disjoint sum A_1+A_2 along B, we obtain naturally the pasted $(G, \mathscr{S}_n)$-manifold*

$$(A, \varphi) = (A_1, \varphi_1) \cup_{(B,\varphi)} (A_2, \varphi_2)$$

of (A_1, φ_1) and (A_2, φ_2) along (B, φ).

In § 5, we will prove this lemma which may be clear intuitively.

By Lemma 1.10, we can define a bordism group as follows:

DEFINITION 1.12. We denote the bordism class of a singular free $(G, \mathscr{S}_n)$-manifold (A, φ, f) in $(X, Y; \tau)$ by $[A, \varphi, f]$, and set

$$\Omega(\mathscr{S}_n)^G_m(X, Y; \tau) = \{[A, \varphi, f] \mid \dim A = m\}$$

which is an abelian group by the disjoint sum. Thus, we have a graded abelian group

$$\Omega(\mathscr{S}_n)^G_*(X, Y; \tau) = \textstyle\sum_m \Omega(\mathscr{S}_n)^G_m(X, Y; \tau),$$

the *bordism group of singular free $(G, \mathscr{S}_n)$-manifolds in $(X, Y; \tau)$*. This has a natural left Ω_*-module structure given by the product $Q \times A$ in Lemma 1.7, where Ω_* is the oriented cobordism ring [7].

By replacing the term "free" by "semi-free" in the above definition, we may define the bordism group

$$\mathcal{O}(\mathscr{S}_n)^G_*(X, Y; \tau) = \textstyle\sum_m \mathcal{O}(\mathscr{S}_n)^G_m(X, Y; \tau).$$

Now, we have the following *Baas-Sullivan exact sequence* in our bordism theory. Let

$$(1.13) \qquad \beta\colon \Omega(\mathscr{S}_n)^G_*(X, Y; \tau) \longrightarrow \Omega(\mathscr{S}_n)^G_*(X, Y; \tau),$$

$$\beta[A, \varphi, f] = [A \times P_{n+1}, \varphi \times \mathrm{id}, f \circ \mathrm{pr}] \qquad \text{(cf. Lemma 1.7)},$$

be the homomorphism of degree $p_{n+1} = \dim P_{n+1}$ induced from the right multiplication by the closed manifold P_{n+1}. By regarding each singular $(G, \mathscr{S}_n)$-

manifold (A, φ, f) naturally as a singular $(G, \mathscr{S}_{n+1})$-manifold with $\partial_{n+1}A(\omega) = \emptyset$, we have the homomorphism

$$\gamma\colon \Omega(\mathscr{S}_n)_*^G(X, Y; \tau) \longrightarrow \Omega(\mathscr{S}_{n+1})_*^G(X, Y; \tau) \tag{1.14}$$

of degree zero. Finally, the Bockstein homomorphism

$$\delta\colon \Omega(\mathscr{S}_{n+1})_*^G(X, Y; \tau) \longrightarrow \Omega(\mathscr{S}_n)_*^G(X, Y; \tau) \tag{1.15}$$

of degree $-p_{n+1}-1$ is defined by

$$[A, \varphi, f] = [\delta_{n+1}A, \delta_{n+1}\varphi, \delta_{n+1}f] \qquad \text{(cf. Def. 1.8)}.$$

Then we can prove the following theorem:

THEOREM 1.16 (cf. Baas [1; Th. 3.2]). *We have the following Baas-Sullivan (BS) exact sequence*:

$$\cdots \longrightarrow \Omega(\mathscr{S}_n)_*^G(X, Y; \tau) \xrightarrow{\beta} \Omega(\mathscr{S}_n)_*^G(X, Y; \tau) \xrightarrow{\gamma} \Omega(\mathscr{S}_{n+1})_*^G(X, Y; \tau)$$
$$\xrightarrow{\delta} \Omega(\mathscr{S}_n)_*^G(X, Y; \tau) \longrightarrow \cdots.$$

Similarly we obtain the BS exact sequence for the bordism group $\mathcal{O}(\mathscr{S}_n)_*^G(X, Y; \tau)$ of singular semi-free $(G, \mathscr{S}_n)$-manifolds.

PROOF. This theorem can be proved by the same way as the proof of [1; Th. 3.2]. We only give the proof of $\operatorname{Im}\beta = \operatorname{Ker}\gamma$ in the absolute case $Y=\emptyset$.

For any singular $(G, \mathscr{S}_n)$-manifold (A, φ, f) in $(X; \tau)$, we obtain a singular $(G, \mathscr{S}_{n+1})$-manifold $(B, \bar{\varphi}, \bar{f})$ in $(X, X; \tau)$ such that

$$B(\omega) = (-1)^{|\omega|} I \times (A \times P_{n+1})(\omega), \quad B(n+1,\omega) = (-1)^{|\omega|+1} A(\omega);$$

$$\begin{cases} \partial_0 B(\omega) = (-1)^{|\omega|} 0 \times (A \times P_{n+1})(\omega), \\ \partial_{n+1} B(\omega) = (-1)^{|\omega|} 1 \times (A \times P_{n+1})(\omega), \\ \partial_i B(\omega) = (-1)^{|\omega|+1} I \times \partial_i (A \times P_{n+1})(\omega) & (1 \leqq i \leqq n), \\ \partial_i B(n+1,\omega) = (-1)^{|\omega|+1} \partial_i A(\omega) & (0 \leqq i \leqq n); \end{cases}$$

$$\bar{\varphi}(\omega) = \mathrm{id} \times (\varphi \times \mathrm{id})(\omega), \quad \bar{\varphi}(n+1,\omega) = \varphi(\omega);$$

$$\bar{f}(\omega) = (f \circ \mathrm{pr})(\omega), \quad \bar{f}(n+1,\omega) = f(\omega);$$

where $\omega \in I(n)$ and $I=[0,1]$ is the unit interval. Then

$$(\delta_0 B)(\omega) = B(\omega, 0) = (-1)^{|\omega|} \partial_0 B(\omega) = (A \times P_{n+1})(\omega),$$

$$\delta_0(B, \bar{\varphi}, \bar{f}) = (A \times P_{n+1}, \varphi \times \mathrm{id}, f \circ \mathrm{pr}).$$

Therefore, $\gamma\circ\beta=0$ by definition.

Conversely, assume that $[A, \varphi, f]\in\Omega(\mathscr{S}_n)^G_*(X; \tau)$ belongs to $\operatorname{Ker}\beta$. Then there exists a singular $(G, \mathscr{S}_{n+1})$-manifold (B, ψ, F) in $(X, X; \tau)$ such that

$$\delta_0(B, \psi, F) = (A, \varphi, f)$$

and hence

$$\partial_0 B(\omega) \cap \partial_{n+1}B(\omega) = \partial_{n+1}\partial_0 B(\omega) = \partial_{n+1}A(\omega) = \emptyset \qquad (\omega\in I(n)).$$

Therefore, (B, ψ, F) can be considered as a singular $(G, \mathscr{S}_n)$-manifold (B', ψ, F) in $(X, X; \tau)$ by setting

$$B'(\omega) = B(\omega), \quad B'(0,\omega) = \partial_0 B'(\omega) = \partial_0 B(\omega) + \partial_{n+1}B(\omega),$$

$$\partial_i B'(\omega) = \partial_i B(\omega), \quad \partial_i B'(0,\omega) = \partial_i\partial_0 B(\omega) + \partial_i\partial_{n+1}B(\omega)$$

for $\omega\in I(n)$ with $\omega\not\ni 0$ and $1\leqq i\leqq n$. Thus, we see easily that

$$(\delta_0 B')(\omega) = (-1)^{|\omega|}\partial_0 B'(\omega) = A(\omega) + (\delta_{n+1}B \times P_{n+1})(\omega),$$

which shows that $[A, \varphi, f] = -\beta[\delta_{n+1}(B, \psi, F)]\in\operatorname{Im}\beta$ as desired. q.e.d.

For any free $(G, \mathscr{S}_n)$-manifold (A, φ), let

$$\pi(\omega)\colon A(\omega) \longrightarrow \bar{A}(\omega) = A(\omega)/\varphi(\omega)$$

denote the orbit G-bundle. Then there exist a unique smooth structure and a unique orientation on $A(\omega)$ such that $\pi(\omega)$ is a local diffeomorphism and preserves orientation locally. Furthermore, $\bar{A}(\omega)$ is a decomposed manifold by defining $\partial_i\bar{A}(\omega)=\pi(\omega)(\partial_i A(\omega))$ and the structure map $\alpha(i,\omega)\colon \partial_i A(\omega)\to A(i,\omega)\times P_i$ induces a unique diffeomorphism

$$\bar{\alpha}(i,\omega)\colon \partial_i\bar{A}(\omega) \longrightarrow \bar{A}(i,\omega) \times P_i$$

so that $\bar{A}=\{\bar{A}(\omega), \bar{\alpha}(i,\omega)\}$ is an $\mathscr{S}_n$-manifold. We call $\bar{A}$ the *orbit* $\mathscr{S}_n$*-manifold of* (A, φ).

Then, by using the result of Conner-Floyd [3; (19.1)] in case $n=0$ and by the same way as its proof, we have the following

THEOREM 1.17 (cf. [3; (19.1)]). *The bordism group*

$$\Omega(\mathscr{S}_n)^G_* = \Omega(\mathscr{S}_n)^G_*(\mathrm{pt})$$

of free closed $(G, \mathscr{S}_n)$*-manifolds is isomorphic to the bordism group*

$$\Omega(\mathscr{S}_n)_*(BG) = \Omega(\mathscr{S}_n)^e_*(BG) \qquad (e\text{: the trivial group})$$

of singular $\mathcal{S}_n$-manifolds in the classifying space BG of G, i.e., there exists an isomorphism

$$L\colon \Omega(\mathcal{S}_n)^G_* \xrightarrow{\simeq} \Omega(\mathcal{S}_n)_*(BG), \quad L[A, \varphi] = [\bar{A}, \bar{f}],$$

of degree zero, where $\bar{A}=\{A(\omega)/\varphi(\omega)\}$ is the orbit $\mathcal{S}_n$-manifold of (A, φ) and each $\bar{f}(\omega)\colon \bar{A}(\omega)\to BG$ is a classifying map of the G-bundle $\pi(\omega)\colon A(\omega)\to\bar{A}(\omega)$.

PROOF. (1) Construction of L. For each $\omega\in I(n)$, we choose a classifying map $\bar{f}'(\omega)\colon \bar{A}(\omega)\to BG$ of the G-bundle $\pi(\omega)$ such that $\bar{f}'(\sigma\omega)=\bar{f}'(\omega)$ $(\sigma\in S_\omega)$. Then, by induction on $|\omega|$, we can construct a classifying map $\bar{f}(\omega)\colon \bar{A}(\omega)\to BG$ such that the diagrams

$$\begin{tikzcd} \partial_i\bar{A}(\omega) \arrow[r, "\bar{f}(\omega)_i"] \arrow[d, "\bar{\alpha}(i,\omega)"'] & BG \\ \bar{A}(i,\omega)\times P_i \arrow[r, "\mathrm{pr}"] & \bar{A}(i,\omega), \arrow[u, "\bar{f}(i,\omega)"'] \end{tikzcd} \qquad \begin{tikzcd} \bar{A}(\omega) \arrow[dr, "\bar{f}(\omega)"] \arrow[dd, "\mathrm{id}"'] & \\ & BG \\ \bar{A}(\sigma\omega) \arrow[ur, "\bar{f}(\sigma\omega)"'] & \end{tikzcd}$$

are commutative, as follows.

If $|\omega|\geqq\max\{|\omega'|\mid A(\omega')\neq\emptyset\}$ and $A(\omega)\neq\emptyset$, then $\partial A(\omega)=\emptyset$ and we can take $\bar{f}(\omega)=\bar{f}'(\omega)$.

Assume inductively that we have constructed $\bar{f}(\omega)$ as desired for each ω with $|\omega|\geqq r$. If $|\omega|=r-1$ and $\partial A(\omega)=\emptyset$, then we can take $\bar{f}(\omega)=\bar{f}'(\omega)$.

Consider the case $|\omega|=r-1$ and $\partial A(\omega)\neq\emptyset$, and denote by $\partial_i\bar{f}(\omega)$ the composition of

$$\partial_i\bar{A}(\omega) \xrightarrow{\bar{\alpha}(i,\omega)} \bar{A}(i,\omega)\times P_i \xrightarrow{\mathrm{pr}} \bar{A}(i,\omega) \xrightarrow{\bar{f}(i,\omega)} BG.$$

Then, by the inductive assumption and the pasting condition (ii) in Definition 1.2, it is not difficult to see that $\partial_i\bar{f}(\omega)=\partial_j\bar{f}(\omega)$ on $\partial_i\bar{A}(\omega)\cap\partial_j\bar{A}(\omega)$. Therefore, we obtain a map

$$\partial\bar{f}(\omega)\colon \partial\bar{A}(\omega)\longrightarrow BG, \quad \partial\bar{f}(\omega)\mid\partial_i\bar{A}(\omega)=\partial_i\bar{f}(\omega).$$

Since $\partial_i\bar{f}(\omega)$ is a classifying map of the G-bundle $\pi(\omega)_i\colon \partial_iA(\omega)\to\partial_i\bar{A}(\omega)$, $\partial\bar{f}(\omega)$ is that of $\pi(\omega)\mid\partial A(\omega)\colon \partial A(\omega)\to\partial\bar{A}(\omega)$. Thus, by the usual homotopy extension property, there exists a classifying map

$$\bar{f}(\omega)\colon \bar{A}(\omega)\longrightarrow BG, \quad \bar{f}(\omega)\mid\partial\bar{A}(\omega)=\partial\bar{f}(\omega),$$

of $\pi(\omega)$, which is homotopic to $\bar{f}'(\omega)$. Further, we may take as $\bar{f}(\sigma\omega)=\bar{f}(\omega)$ by the inductive assumption.

Therefore, the desired $\bar{f}(\omega)$ is constructed for each ω with $|\omega|=r-1$.

(2) L is isomorphic. It is easy to see that L is commutative with the BS exact sequences, i.e., the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega(\mathscr{S}_n)^G_* \arrow[r, "\beta"] \arrow[d, "L"] & \Omega(\mathscr{S}_n)^G_* \arrow[r, "\gamma"] \arrow[d, "L"] & \Omega(\mathscr{S}_{n+1})^G_* \arrow[r, "\delta"] \arrow[d, "L"] & \Omega(\mathscr{S}_n)^G_* \arrow[r] \arrow[d, "L"] & \cdots \\
\cdots \arrow[r] & \Omega(\mathscr{S}_n)_*(BG) \arrow[r, "\beta"] & \Omega(\mathscr{S}_n)_*(BG) \arrow[r, "\gamma"] & \Omega(\mathscr{S}_{n+1})_*(BG) \arrow[r, "\delta"] & \Omega(\mathscr{S}_n)_*(BG) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative. When $n=0$, L is isomorphic by [3; (19.1)]. Thus L is isomorphic for each n by the five lemma. q.e.d.

By the same way, we have the following

PROPOSITION 1.18 (cf. Wu [8; Prop. 2]). *There exists an isomorphism*

$$L\colon \Omega(\mathscr{S}_n)^{Z_p}_*(X;\tau) \xrightarrow{\simeq} \Omega(\mathscr{S}_n)_*(X \times S^\infty/\tau \times \rho) \qquad (p\colon \textit{odd prime})$$

of degree zero, where $\rho=\exp(2\pi i/p)$ *is the standard* $\boldsymbol{Z}_p$*-action on the infinite sphere* S^∞.

In conclusion of this section, we notice that the bordism groups

$$\Omega(\mathscr{S}_n)^{Z_p}_*(X, Y;\tau), \quad \mathcal{O}(\mathscr{S}_n)^{Z_p}_*(X, Y;\tau) \qquad (p\colon \text{odd prime})$$

are $\boldsymbol{Z}_p$-equivariant homology theories respectively, i.e., we can prove the following theorem by using the results of Wu [8; Prop. 1] in case $n=0$:

THEOREM 1.19 (cf. [1; Th. 3.3], [8; Prop. 1]). *Let p be an odd prime. Then the bordism group* $\Omega(\mathscr{S}_n)^{Z_p}_*(-)$ *forms a homology theory on the category of pairs of topological* $\boldsymbol{Z}_p$*-spaces and equivariant maps, where the induced homomorphism*

$$h_*\colon \Omega(\mathscr{S}_n)^{Z_p}_*(X, Y;\tau) \longrightarrow \Omega(\mathscr{S}_n)^{Z_p}_*(X', Y';\tau')$$

of an equivariant map $h\colon (X, Y;\tau)\to(X', Y';\tau')$ *is given naturally by* $h_*[A, \varphi, f]=[A, \varphi, h\circ f]$ *and the boundary homomorphism*

$$\delta_0\colon \Omega(\mathscr{S}_n)^{Z_p}_m(X, Y;\tau) \longrightarrow \Omega(\mathscr{S}_n)^{Z_p}_{m-1}(Y;\tau)$$

is given by $\delta_0[A, \varphi, f]=[\delta_0 A, \delta_0\varphi, \delta_0 f]$ (cf. Def. 1.8).
The same holds for $\mathcal{O}(\mathscr{S}_n)^{Z_p}_*(-)$.

PROOF. The homotopy, excision and exactness axioms can be proved by the entirely analogous proof to that of [1; Th. 3.3], and we only give the proof of the excision axiom.

Let U be an invariant open subset of Y with $\bar{U}\subset \operatorname{Int} Y$, and $i\colon (X-U, Y-U)$

$=(X', Y')\to(X, Y)$ be the inclusion. Then, by definition, we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega(\mathscr{S}_n)_*^{Z_p}(X, Y; \tau) \arrow[r, "\beta"] & \Omega(\mathscr{S}_n)_*^{Z_p}(X, Y; \tau) \arrow[r, "\gamma"] & \Omega(\mathscr{S}_{n+1})_*^{Z_p}(X, Y; \tau) \arrow[r, "\delta"] & \cdots \\
\cdots \arrow[r] & \Omega(\mathscr{S}_n)_*^{Z_p}(X', Y'; \tau) \arrow[r, "\beta"] \arrow[u, "i_*"] & \Omega(\mathscr{S}_n)_*^{Z_p}(X', Y'; \tau) \arrow[r, "\gamma"] \arrow[u, "i_*"] & \Omega(\mathscr{S}_{n+1})_*^{Z_p}(X', Y'; \tau) \arrow[r, "\delta"] \arrow[u, "i_*"] & \cdots
\end{tikzcd}$$

of the BS exact sequences. When $n=0$, i_* is isomorphic by [8; Prop. 1]. Thus i_* is isomorphic for any n by the five lemma. q. e. d.

§2. The Smith homomorphisms

In this section, we study free $\boldsymbol{Z}_p$-actions on $\mathscr{S}_n$-manifolds, where p is an odd prime.

DEFINITION 2.1. Let $A=(A, \varphi, f)$ be a singular free $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold in a $\boldsymbol{Z}_p$-space $(X; \tau)$ of dimension m. Then we say that an equivariant map

$$g = \{g(\omega)\}: (A, \varphi) \longrightarrow (S^{2k+1}, \rho), \quad m < 2k+1,$$

($\rho=\exp(2\pi i/p)$ is the standard $\boldsymbol{Z}_p$-action on the $(2k+1)$-sphere S^{2k+1}) is *t-regular on* S^{2k-1}, if the following conditions (i) and (ii) are satisfied:

(i) $B = \{B(\omega), \alpha(i, \omega) \mid B(\omega) \cap \partial_i A(\omega)\}, \quad B(\omega) = g(\omega)^{-1}(S^{2k-1})$,

is an invariant $\mathscr{S}_n$-manifold of A of condimension two.

(ii) The invariant normal bundle $N=\{N(\omega)\}$ of B in A is trivial, i.e., the one of $B(\omega)$ in $A(\omega)$ (cf. [4; Th. 1]) is trivial and is induced from the trivial normal bundle $S^{2k-1}\times \boldsymbol{R}^2$ of S^{2k-1} in S^{2k+1} by the map $g(\omega)\mid B(\omega)$.

Thus, $(B, \varphi\mid B, f\mid B)$ is a singular free $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold in $(X; \tau)$.

THEOREM 2.2 (cf. [3; (34.7)]). *Such a manifold $(B, \varphi\mid B, f\mid B)$ always exists, and the homomorphism (called the Smith homomorphism)*

$$\Delta: \Omega(\mathscr{S}_n)_m^{Z_p}(X; \tau) \longrightarrow \Omega(\mathscr{S}_n)_{m-2}^{Z_p}(X; \tau)$$

is well-defined by setting $\Delta[A, \varphi, f]=[B, \varphi\mid B, f\mid B]$. Further, Δ is commutative with the BS exact sequences.

PROOF. Let (A, φ, f) be a singular free $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold of dimension m in $(X; \tau)$, and choose an equivariant map $g'(\omega): (A(\omega), \varphi(\omega))\to(S^{2k+1}, \rho)$ for each ω such that $g'(\sigma\omega)=g'(\omega)$ $(\sigma\in S_\omega)$. Then there exists an equivariant map

$$g(\omega): (A(\omega), \varphi(\omega)) \longrightarrow (S^{2k+1}, \rho) \qquad \text{for each} \quad \omega,$$

satisfying the following conditions (2.3.1–3);

(2.3.1) The induced maps $\bar{g}(\omega)$, $\bar{g}'(\omega)$: $\bar{A}(\omega)\to S^{2k+1}/\rho$ are homotopic.

(2.3.2) $\bar{g}(\omega)$ is t-regular on S^{2k-1}/ρ.

(2.3.3) The following diagrams are commutative:

$$\begin{tikzcd} \partial_i A(\omega) \arrow[r, "g(\omega)_i"] \arrow[d, "\alpha(i,\omega)"'] & S^{2k+1} \\ A(i,\omega)\times P_i \arrow[r, "\mathrm{pr}"] & A(i,\omega), \arrow[u, "g(i,\omega)"'] \end{tikzcd} \qquad \begin{tikzcd} A(\omega) \arrow[dr, "g(\omega)"] \arrow[dd, "\mathrm{id}"'] & \\ & S^{2k+1} \\ A(\sigma\omega) \arrow[ur, "g(\sigma\omega)"'] & \end{tikzcd}$$

In fact, we can construct g inductively by the same way as the construction in the proof of the Theorem 1.17, by using a collar system $\Gamma(\bar{A})=\{\Gamma(\bar{A}(\omega))\}$ of $\bar{A}$ (cf. Th. 5.4), as follows.

If $|\omega|\geqq\max\{|\omega'|\,|\,A(\omega')\neq\emptyset\}$ and $A(\omega)\neq\emptyset$, then we obtain $g(\omega)$ satisfying (2.3.1–2) by using the usual t-regularity theorem [3; (10.4)] and the homotopy lifting property. Since $\partial A(\omega)=\emptyset$, (2.3.3) is trivial. We notice that $g(\omega)$ is also t-regular on S^{2k-1} by (2.3.2).

Assume inductively that $g(\omega)$ is constructed as desired for each ω with $|\omega|\geqq r$. If $|\omega|=r-1$ and $\partial A(\omega)=\emptyset$, then we can obtain $g(\omega)$ by the same way as above.

Consider the case $|\omega|=r-1$ and $\partial A(\omega)\neq\emptyset$. By using a collar $\Gamma(\bar{A}(\omega))=\{U_i(\bar{A}(\omega)),\ \lambda_i(\bar{A}(\omega))\}$ in $\Gamma(\bar{A})$, let

$$\bar{h}(\omega)\colon\ U=\cup_i U_i(\bar{A}(\omega))\longrightarrow S^{2k+1}/\rho$$

be a map whose restriction on $U_i(\bar{A}(\omega))$ is the composition of

$$U_i(\bar{A}(\omega))\xrightarrow[\approx]{\lambda_i(\bar{A}(\omega))}\partial_i\bar{A}(\omega)\times[0,1)\xrightarrow{\mathrm{pr}}\partial_i\bar{A}(\omega)$$

$$\xrightarrow[\approx]{\bar{\alpha}(i,\omega)}\bar{A}(i,\omega)\times P_i\xrightarrow{\mathrm{pr}}\bar{A}(i,\omega)\xrightarrow{\bar{g}(i,\omega)}S^{2k+1}/\rho.$$

Then, by the inductive hypothesis, we see that $\bar{h}(\omega)$ is t-regular on S^{2k-1}/ρ. Further, it is homotopic to $\bar{g}'(\omega)\,|\,U$. Therefore, by the t-regularity theorem and the homotopy lifting property, there exists an equivariant map $g(\omega)$: $(A(\omega),\varphi(\omega))\to(S^{2k+1},\rho)$ such that $\bar{g}(\omega)\,|\,U=\bar{h}(\omega)$ and it satisfies (2.3.1–3) as desired.

Thus we have a desired $\mathscr{S}_n$-manifold B. Further, we can prove that Δ is well-defined as usual by using a similar consideration to the above construction of g. We can also see easily that Δ is commutative with the BS exact sequences.

q.e.d.

We have the following exact sequence containing the Smith homomorphism Δ in the above theorem.

PROPOSITION 2.4 (cf. Wu [8; Th. 2]). *The sequence*

$$\cdots \longrightarrow \Omega(\mathscr{S}_n)_m^{Z_p}(S^1\times X;\ \rho\times\tau) \xrightarrow{\pi} \Omega(\mathscr{S}_n)_m^{Z_p}(X;\ \tau) \xrightarrow{\Delta}$$

$$\Omega(\mathscr{S}_n)_{m-2}^{Z_p}(X;\ \tau) \xrightarrow{\mathscr{P}} \Omega(\mathscr{S}_n)_{m-1}^{Z_p}(S^1\times X;\ \rho\times\tau) \longrightarrow \cdots$$

is exact, where the two homomorphisms π and $\mathscr{P}$ are defined by

$$\pi[A,\ \varphi, f] = [A,\ \varphi.\ \mathrm{pr}\circ f],$$

$$\mathscr{P}[A,\ \varphi, f] = [S^1 \times A,\ \rho\times\varphi,\ \mathrm{id}\times f].$$

Further, they are commutative with the BS exact sequences.

PROOF. We can prove the proposition by the analogous proof to that of [8; Th. 2] in case $n=0$, and we only give the proof of $\mathrm{Im}\,\mathscr{P}\supset\mathrm{Ker}\,\pi$ based on the t-regularity theorem and the similar inductive process to the construction in the proof of Theorem 2.2.

Assume $\pi[A,\ \varphi, f]=0$ for $[A,\ \varphi, f]\in\Omega(\mathscr{S}_n)_{m-1}^{Z_p}(S^1\times X;\ \rho\times\tau)$, i.e., assume that there is a singular $(\boldsymbol{Z}_p,\ \mathscr{S}_n)$-manifold $(W,\ \varphi', f')$ in $(X,\ X;\ \tau)$ with

$$\delta_0(W,\ \varphi', f') = (A,\ \varphi,\ \mathrm{pr}\circ f).$$

Then we can construct a smooth approximation

$$F_A = \{F_A(\omega)\}\colon (A,\ \varphi) \longrightarrow (S^1,\ \rho)$$

of $\mathrm{pr}_1\circ f=\{\mathrm{pr}_1\circ f(\omega)\}\colon (A,\ \varphi)\to(S^1,\ \rho)$ by the similar construction to that of $\bar{f}$ in the proof of Theorem 1.17, by using the smooth approximation theorem and the extension theorem [3; (10.1)].

Furthermore, we can extend F_A to an equivariant smooth map

$$F = \{F(\omega)\}\colon (W,\ \varphi') \longrightarrow (D^2,\ \rho)$$

by the same way, and it can be replaced by the one such that there exists a regular value $\bar{y}\in D^2/\rho-(S^1/\rho\cup\{0\})$ of $\bar{F}(\omega)$ for each ω, by the similar construction to that of g in the proof of Theorem 2.2. Thus, there exist p-regular values y, $\rho(y),\dots,\ \rho^{p-1}(y)\in D^2-(S^1\cup\{0\})$ of $F(\omega)$, and

$$N(\omega) = F(\omega)^{-1}(\{y,\ \rho(y),\dots,\ \rho^{p-1}(y)\}),\quad \partial_i N(\omega) = N(\omega)\cap\partial_i W(\omega),$$

with natural orientation form an $\mathscr{S}_n$-submanifold of W of codimension two. Since $F(\omega)(W(\omega,0))=F_A(A(\omega))\subset S^1$, we see that

$$\partial_0 N(\omega) = N(\omega)\cap\partial_0 W(\omega) = \emptyset,\qquad N(\omega)=\emptyset\quad\text{if}\quad \omega\ni 0;$$

and we obtain a singular free $(\boldsymbol{Z}_p,\ \mathscr{S}_n)$-manifold

$$(N, \varphi'_N, f'_N), \qquad \varphi'_N(\omega) = \varphi'(\omega) \mid N(\omega), \quad f'_N(\omega) = f'(\omega) \mid N(\omega),$$

in $(X; \tau)$.

Consider a closed invariant tubular neighborhood

$$(D^2 \times N, \rho \times \varphi'_N) = \{(D^2 \times N)(\omega) = D^2 \times N(\omega), \rho \times \varphi'_N(\omega)\}$$

of N in W [4; Th. 1], such that $D^2 \times N(\omega) \cap \partial_0 W(\omega) = \emptyset$ and

$$\partial_i(D^2 \times N)(\omega) = D^2 \times \partial_i N(\omega) \quad (i \neq 0), \qquad = S^1 \times N(\omega) \quad (i = 0).$$

Then, $\tilde{W}(\omega) = W(\omega) - \mathrm{Int}\,(D^2 \times N(\omega))$ satisfies the equality

$$\partial_0 \tilde{W}(\omega) = \partial_0 W(\omega) + (-S^1 \times N(\omega)),$$

which implies $(\delta_0 \tilde{W})(\omega) = A(\omega) + (-1)(S^1 \times N)(\omega)$. Thus

$$[A, \varphi, f] = [S^1 \times N, \rho \times \varphi'_N, \mathrm{id} \times f'_N] = \mathscr{P}[N, \varphi'_N, f'_N],$$

and $[A, \varphi, f] \in \mathrm{Im}\,\mathscr{P}$ as desired. q.e.d.

Now, we consider the reduced group.

LEMMA 2.5. *There holds the split exact sequence*

$$0 \longrightarrow \tilde{\Omega}(\mathscr{S}_n)_m^{Z_p}(X;\tau) \xrightarrow{\subset} \Omega(\mathscr{S}_n)_m^{Z_p}(X;\tau) \xrightarrow{\varepsilon_*} \Omega(\mathscr{S}_n)_m \longrightarrow 0,$$

where ε_* is the homomorphism sending (A, φ, f) to its orbit $\mathscr{S}_n$-manifold A/φ and $\tilde{\Omega}(\mathscr{S}_n)_m^{Z_p}(X; \tau) = \mathrm{Ker}\,\varepsilon_*$. Thus, the reduced group $\tilde{\Omega}(\mathscr{S}_n)_*^{Z_p}(X; \tau)$ has its own BS exact sequences.

PROOF. Fix a point $x \in X$. Then a right inverse homomorphism of ε_* is given by sending $[A] \in \Omega(\mathscr{S}_n)_m$ to $[\boldsymbol{Z}_p \times A, \sigma \times \mathrm{id}, f] \in \Omega(\mathscr{S}_n)_m^{Z_p}(X; \tau)$ where $\sigma(k,l) = k + l \bmod p$ and $f(\omega)(k \times A(\omega)) = \tau^k(x)$ for $0 \leqq k,l < p$. q.e.d.

It is easy to see that the Smith homomorphism Δ maps $\tilde{\Omega}(\mathscr{S}_n)_m^{Z_p}(X; \tau)$ to $\tilde{\Omega}(\mathscr{S}_n)_{m-2}^{Z_p}(X; \tau)$. Thus, Δ is commutative with the BS exact sequences of reduced groups.

The following theorem is well-known when $n=0$ by the results of Conner-Floyd [3; Ch. VII].

THEOREM 2.6. *Let $\mathscr{S}_n = \{P_0 = \mathrm{pt}, P_1, \ldots, P_n\}$.*

(1) *If each $p_i = \dim P_i$ is odd for $i \geqq 1$, then the Smith homomorphism*

$$\Delta\colon \tilde{\Omega}(\mathscr{S}_n)_{2k+1}^{Z_p} \longrightarrow \tilde{\Omega}(\mathscr{S}_n)_{2k-1}^{Z_p} \qquad (k \geqq 1)$$

is an epimorphism, and $\tilde{\Omega}(\mathscr{S}_n)_{2k}^{Z_p} = 0$ $(k \geqq 0)$.

(2) *In particular, if* $p_i \equiv 3 \pmod 4$ *for* $i \geqq 1$, *then*

$$\varDelta\colon \tilde{\Omega}(\mathscr{S}_n)^{Z_p}_{4k+3} \xrightarrow{\simeq} \tilde{\Omega}(\mathscr{S}_n)^{Z_p}_{4k+1} \qquad (k \geqq 0).$$

Proof. (1) Since $\tilde{\Omega}^{Z_p}_{2k}=0$ $(k \geqq 0)$ by [3; (34.3)], we see the latter half by the induction on n, by using the BS exact sequence. Hence, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\Omega}(\mathscr{S}_{n-1})^{Z_p}_{2k+1} & \xrightarrow{\gamma} & \tilde{\Omega}(\mathscr{S}_n)^{Z_p}_{2k+1} & \xrightarrow{\delta} & \tilde{\Omega}(\mathscr{S}_{n-1})^{Z_p}_{2k-p_n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varDelta} & & \downarrow{\scriptstyle \varDelta} & & \downarrow{\scriptstyle \varDelta} & & \\
0 & \longrightarrow & \tilde{\Omega}(\mathscr{S}_{n-1})^{Z_p}_{2k-1} & \xrightarrow{\gamma} & \tilde{\Omega}(\mathscr{S}_n)^{Z_p}_{2k-1} & \xrightarrow{\delta} & \tilde{\Omega}(\mathscr{S}_{n-1})^{Z_p}_{2k-p_n-2} & \longrightarrow & 0
\end{array}$$

of the BS exact sequences. Since $\varDelta$ is an epimorphism when $n=0$ by [3; (34.9)], the same holds for any n by induction.

(2) By the same way, we see (2) since $\varDelta$ is isomorphic when $n=0$ by [3; (36.4)]. q.e.d.

§3. $(G, \mathscr{S}_1)$-manifolds

In this section, we study the module structure of

$$\tilde{\Omega}(P)^{Z_p}_* = \tilde{\Omega}(\mathscr{S}_1)^{Z_p}_* \qquad (\mathscr{S}_1 = \{\mathrm{pt}, P\}, \quad p\text{: odd prime}).$$

Theorem 3.1. *Let P be a torsion element in Ω_* of odd dimension m. Then we have*

(1) $\tilde{\Omega}(P)^{Z_p}_{2k} = 0 \qquad (k \geqq 0)$,

(2) $\tilde{\Omega}(P)^{Z_p}_{2k+1} \simeq \tilde{\Omega}^{Z_p}_{2k-m} \oplus \tilde{\Omega}^{Z_p}_{2k+1} \qquad (k \geqq 0)$.

Here $\tilde{\Omega}^{Z_p}_* = \tilde{\Omega}(\mathrm{pt})^{Z_p}_*$ is the reduced bordism group of oriented closed manifolds with free $\boldsymbol{Z}_p$-actions.

Proof. (1) is shown in Theorem 2.6 (1). Hence we have the BS exact sequence

$$0 \longrightarrow \tilde{\Omega}^{Z_p}_{2k+1} \xrightarrow{\gamma} \tilde{\Omega}(P)^{Z_p}_{2k+1} \xrightarrow{\delta} \tilde{\Omega}^{Z_p}_{2k-m} \longrightarrow 0$$

Let p^u be the order of $\tilde{\Omega}^{Z_p}_{2k-m}$ [3; (34.2)], and let t and t' be the integers such that $2t+p^u t'=1$. Further, let W be an oriented manifold bounded by $2P$. Then, for each $[M, \varphi] \in \tilde{\Omega}^{Z_p}_{2k-m}$, we can associate a $(\boldsymbol{Z}_p, \mathscr{S}_1)$-manifold $(\mathscr{S}_1=\{\mathrm{pt}, P\})$ $(M \times W, \varphi \times \mathrm{id})$ with

$$\partial_0(M \times W) = \emptyset, \quad \partial_1(M \times W) = \partial(M \times W) = (-1)^{2k-m} 2M \times P,$$

$$(M \times W)(1) = (-1)^{2k-m}2M, \quad \alpha(1, \emptyset) = \mathrm{id}, \qquad \text{(cf. Lemma 1.7)}.$$

It is easy to see that a right inverse homomorphism is given by sending $[M, \varphi]$ to $(-1)^{2k-m}t[M \times W, \varphi \times \mathrm{id}] \in \tilde{\Omega}(P)^{Z_p}_{2k+1}$. q.e.d.

Further, we obtain some results on the Ω_*-module structure of $\tilde{\Omega}(P)^{Z_p}_*$.

THEOREM 3.2. *Let P be an odd dimensional closed oriented manifold of Dold type, i.e., the kernel of the homomorphism*

$$\beta\colon \Omega_* \longrightarrow \Omega_*, \quad \beta[M] = [M \times P],$$

is equal to $2\Omega_*$. Then we have a natural isomorphism

$$\theta\colon A_* = \tilde{\Omega}^{Z_p}_* \otimes_{\Omega_*} \Omega(P)_* \xrightarrow{\sim} \tilde{\Omega}(P)^{Z_p}_*$$

by sending $[M, \varphi] \otimes A$ to $[M \times A, \varphi \times \mathrm{id}]$ given in Lemma 1.7.

PROOF. By using the BS exact sequence and the assumption on the map β, we have the following short exact sequence of Ω_*-modules:

$$0 \longrightarrow \Omega_*/P\Omega_* \xrightarrow{\gamma} \Omega(P)_* \xrightarrow{\partial} 2\Omega_* \longrightarrow 0.$$

Now, we consider the following commutative diagram

$$\begin{array}{ccccccc}
\tilde{\Omega}^{Z_p}_* \otimes_{\Omega_*} (\Omega_*/P\Omega_*) & \xrightarrow{\mathrm{id}\otimes\gamma} & \tilde{\Omega}^{Z_p}_* \otimes_{\Omega_*} \Omega(P)_* & \xrightarrow{\mathrm{id}\otimes\partial} & \tilde{\Omega}^{Z_p}_* \otimes_{\Omega_*} 2\Omega_* & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_2} & & \\
0 \longrightarrow \tilde{\Omega}^{Z_p}_* & \xrightarrow{\gamma} & \tilde{\Omega}(P)^{Z_p}_* & \xrightarrow{\partial} & \tilde{\Omega}^{Z_p}_* & \longrightarrow & 0
\end{array}$$

with exact rows, where the vertical maps θ_i $(i=1, 2)$ are defined as follows:

$$\theta_1([M, \varphi] \otimes \{N\}) = [M \times N, \varphi \times \mathrm{id}],$$

$$\theta_2([M, \varphi] \otimes 2N) = [M \times 2N, \varphi \times \mathrm{id}].$$

We notice that the image $[M \times N, \varphi \times \mathrm{id}]$ of θ_1 is independent of the choice of N in a class $\{N\} \in \Omega_*/P\Omega_*$, since $\tilde{\Omega}^{Z_p}_*$ is p-torsion. Further, these maps θ_i $(i=1, 2)$ are isomorphic, because they have natural inverse ones respectively. Thus, the middle map θ is also isomorphic by the five lemma. q.e.d.

REMARK 3.3. In general, the two modules A_* and $\tilde{\Omega}(P)^{Z_p}_*$ are not necessarily isomorphic, as is seen from the following simple counterexample when $p=3$ and $P=\boldsymbol{CP}(2)$ (the complex projective plane).

Using of the BS exact sequence:

$$\cdots \longrightarrow \tilde{\Omega}^{Z_3}_1 \xrightarrow{\beta} \tilde{\Omega}^{Z_3}_5 \xrightarrow{\gamma} \tilde{\Omega}(\boldsymbol{CP}(2))^{Z_3}_5 \xrightarrow{\partial} \tilde{\Omega}^{Z_3}_0 = 0,$$

and $\beta[S^1, \rho]=\boldsymbol{CP}(2)[S^1, \rho]=3[S^5, \rho]$ [3; (46.3)], we see that $\tilde{\Omega}(\boldsymbol{CP}(2))_5^{Z_3}\simeq \boldsymbol{Z}_3$. On the other hand, $A_5=\{x\in A^* \mid \deg x=5\}=\tilde{\Omega}_5^{Z_3}\simeq \boldsymbol{Z}_{3^2}$ by the above relation between $[S^1, \rho]$ and $[S^5, \rho]$. Thus, $A^5 \not\simeq \tilde{\Omega}(\boldsymbol{CP}(2))_5^{Z_3}$ as abelian groups.

§4. $\boldsymbol{Z}_p$-actions for an odd prime p

In this section, we study $\mathscr{S}_n$-manifolds with $\boldsymbol{Z}_p$-actions for an odd prime p. We notice that every $\boldsymbol{Z}_p$-action for an odd prime p is always semi-free. Thus, the bordism group

$$\mathcal{O}(\mathscr{S}_n)_*^{Z_p}(X, Y; \tau) = \sum_m \mathcal{O}(\mathscr{S}_n)_m^{Z_p}(X, Y; \tau)$$

of singular $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifolds is defined in §1.

Now, we consider another bordism group.

Definition 4.1. Let $A=(A, \varphi, f)$ be an m-dimensional singular $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold in $(X, X; \tau)$ such that $\varphi(\omega)$ is free if $0\in\omega$, i.e., $\delta_0 A=(\delta_0 A, \delta_0\varphi, \delta_0 f)$ is an $(m-1)$-dimensional singular free $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold in $(X; \tau)$. We say that such two manifolds $A=(A, \varphi, f)$ and $A'=(A', \varphi', f')$ are *bordant*, if $A+(-A')$ bords by a singular $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold $W=(W, \Phi, F)$ in $(X, X; \tau)$ (cf. Def. 1.9) satisfying the following additional condition: For each ω, $\Phi(\omega)$ is free on

$$(\delta_0 W)(\omega) - \operatorname{Int}\{A(\omega) + (-A'(\omega))\}.$$

Then we see that this bordism relation is an equivalence relation by the same way as Lemma 1.10.

The set of bordism classes of such singular $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifolds is denoted by $\mathscr{M}(\mathscr{S}_n)_m^{Z_p}(X; \tau)$ which is an abelian group by the disjoint sum. Thus we have a bordism group

$$\mathscr{M}(\mathscr{S}_n)_*^{Z_p}(X; \tau) = \sum_m \mathscr{M}(\mathscr{S}_n)_m^{Z_p}(X: \tau)$$

which is naturally an Ω^*-module (cf. Def. 1.12). Further, the same BS exact sequence as Theorem 1.16 holds for this bordism groups.

The following proposition can be proved by the same way as the proof of Wu [8; Prop. 4] in case $n=0$:

Proposition 4.2. *The diagram*

$$\begin{array}{ccc}
\mathscr{M}(\mathscr{S}_n)_m^{Z_p}(X; \tau) & \xrightarrow{\delta_0} & \Omega(\mathscr{S}_n)_{m-1}^{Z_p}(X; \tau) \\
\downarrow{\scriptstyle \iota_*} & & \uparrow{\scriptstyle \Delta} \\
\mathscr{M}(\mathscr{S}_n)_{m+2}^{Z_p}(X; \tau) & \xrightarrow{\delta_0} & \Omega(\mathscr{S}_n)_{m+1}^{Z_p}(X; \tau)
\end{array}$$

is commutative, where Δ is the Smith homomorphism, δ_0 is a natural homomorphism of degree -1 given by $\delta_0[A, \varphi, f]=[\delta_0 A, \delta_0\varphi, \delta_0 f]$, and ι_* is defined by

$$\iota_*[A, \varphi, f] = [D^2 \times A, \rho \times \varphi, f\circ \mathrm{pr}], \quad (D^2 \times A)(\omega) = D^2 \times A(\omega),$$

$$\partial_i(D^2 \times A)(\omega) = \begin{cases} S^1 \times A(\omega) \cup D^2 \times (\delta_0 A)(\omega) & (i=0) \\ D^2 \times \partial_i A(\omega) & (i \neq 0), \end{cases}$$

(ρ is the standard $\boldsymbol{Z}_p$-action on the two-disk D^2).

Further, δ_0 and ι_* are commutative with the BS exact sequences.

DEFINITION 4.3. Let (A, φ) be a $(\boldsymbol{Z}_p, \mathscr{S}_n)$-manifold and let $F_{2k}(\omega)$ be the fixed point set of codimension $2k$ in $A(\omega)$. $F_{2k}(\omega)$ has a natural orientation induced by that of $\boldsymbol{Z}_p$-invariant neighborhood of $F_{2k}(\omega)$ [4; Th. 1]. Further, $F_{2k}=\{F_{2k}(\omega), \alpha_F(i,\omega)\}$ has a natural $\mathscr{S}_n$-structure by defining

$$\partial_i F_{2k}(\omega) = F_{2k}(\omega) \cap \partial_i A(\omega), \quad \alpha_F(i,\omega) = \alpha(i,\omega) \,|\, \partial_i F_{2k}(\omega).$$

In this sense, we call F_{2k} the *fixed point set* of codimension $2k$ in A.

For the next theorem, we consider a homomorphism

$$\nu\colon \mathcal{O}(\mathscr{S}_n)_*^{\boldsymbol{Z}_p}(X;\tau) \longrightarrow \mathscr{M}(\mathscr{S}_n)_*^{\boldsymbol{Z}_p}(X;\tau) \tag{4.4.1}$$

of degree zero given as follows. Let $[A, \varphi, f] \in \mathcal{O}(\mathscr{S})_*^{\boldsymbol{Z}_p}(X;\tau)$, and let F_{2k} be the fixed point set of (A, φ) as above. We then have a normal disk bundle $\nu_{2k}(\omega)\colon D^{2k}(\omega) \to A(\omega)$ (i.e., a closed $\boldsymbol{Z}_p$-invariant tubular neighborhood in $A(\omega)$ with natural orientation) for each ω. $D_{2k}(\omega)$ is a decomposed manifold by defining

$$\begin{cases} \partial_0 D_{2k}(\omega)\colon & \text{the associated sphere bundle of } D_{2k}(\omega), \\ \partial_i D_{2k}(\omega) = \nu_{2k}(\omega)^{-1}(\partial_i A(\omega)) & (i \neq 0). \end{cases}$$

Since we may suppose that the map $\alpha(i,\omega)$ is an isometry for the action $\varphi(\omega)$, the induced one $\alpha_D(i,\omega)=\alpha(i,\omega)\,|\,\partial_i D_{2k}(\omega)$ is a diffeomorphism

$$\alpha_D(i,\omega)\colon \partial_i D_{2k}(\omega) \approx D_{2k}(i,\omega) \times P_i, \quad \deg \alpha_D(i,\omega) = \deg \alpha(i,\omega).$$

Let $\varphi_{2k}(\omega)=\varphi(\omega)\,|\,\boldsymbol{Z}_p \times D_{2k}(\omega)$ and $f_{2k}(\omega)=f(\omega)\,|\,D_{2k}(\omega)$. Then the class $[D_{2k}, \varphi_{2k}, f_{2k}]$ is an element of $\mathscr{M}(\mathscr{S}_n)_*^{\boldsymbol{Z}_p}(X;\tau)$. Put

$$\nu[A, \varphi, f] = \sum_{0 \leq k \leq [\dim A/2]} [D_{2k}, \varphi_{2k}, f_{2k}]. \tag{4.4.2}$$

THEOREM 4.5. *The sequence*

$$\cdots \longrightarrow \Omega(\mathscr{S}_n)_m^{Z_p}(X;\tau) \xrightarrow{i_*} \mathcal{O}(\mathscr{S}_n)_m^{Z_p}(X;\tau)$$
$$\xrightarrow{\nu} \mathscr{M}(\mathscr{S}_n)_m^{Z_p}(X;\tau) \xrightarrow{\delta_0} \Omega(\mathscr{S}_n)_{m-1}^{Z_p}(X;\tau) \longrightarrow \cdots$$

is exact, where i_* forgets freeness and δ_0 is the homomorphism treated in Proposition 4.2. *Further, these homomorphisms are commutative with the corresponding BS exact sequences.*

The proof is entirely analogous to those given by Strong [6; Prop. 2] and Wu [8; Th. 1], so we omit the proof here.

In conclusion of this section, we study the module structure of the bordism group $\mathcal{O}(P)_*^{Z_p}=\mathcal{O}(\mathscr{S}_1)_*^{Z_p}$, $\mathscr{S}_1=\{\mathrm{pt}, P\}$, as is done for the bordism group $\tilde{\Omega}(P)_*^{Z_p}$ in § 3.

When $p=3$, we have the following

THEOREM 4.6. *Let $P=kH$ (the disjoint sum of k copies of H), where H is a free generator of Ω_* and k is an odd integer. Then the map*

$$\theta\colon A_* = \mathcal{O}_*^{Z_3} \otimes_{\Omega_*} \Omega(P)_* \longrightarrow \mathcal{O}(P)_*^{Z_3},\ \theta([M,\varphi]\otimes A) = [M\times A,\ \varphi\times \mathrm{id}],$$

is an isomorphism.

PROOF. Because $\mathcal{O}_*^{Z_3}$ is a free Ω_*-module [8; § 5] and any torsion in Ω_* has order two, we have

$$\mathcal{O}(P)_*^{Z_3} \simeq \mathcal{O}_*^{Z_3}/P\Omega_{*-m} \quad \text{and} \quad \Omega(P)_* \simeq \Omega_*/P\Omega_{*-m}$$

($m=\dim P$) by the BS exact sequences. Using these results, we obtain the result easily. q.e.d.

REMARK 4.7. If we consider the case when k is even in the above theorem, then the map θ is no longer isomorphic in general. For example, put $P=\boldsymbol{CP}(2)$. By the analogous method to Remark 3.3 and the explicit abelian group structure of $\mathcal{O}_*^{Z_3}$[9; § 3], we have

$$\mathcal{O}(P)_6^{Z_3} \simeq \boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}_k,$$
$$A_6 \simeq \boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}_k.$$

§ 5. Collar systems of $(G, \mathscr{S}_n)$-manifolds

In this section, we consider the notion of collar systems of $(G, \mathscr{S}_n)$-manifolds according to K. Jänich [5; § 3], and prove the pasting lemma (Lemma 1.11), which is used in the previous sections.

For the present, we do not consider the orientations of manifolds.

DEFINITION 5.1. Let A be a *decomposed G-manifold of type n*, i.e., A is a decomposed manifold of type n and G acts on A by isomorphisms of decomposed manifolds. A *G-collar*

$$\Gamma(A) = \{U_i(A),\ \lambda_i(A) \mid 0 \leqq i \leqq n\}$$

of A is defined to be a system of open invariant neighborhoods $U_i(A)$ of $\partial_i A$ in A and G-equivariant diffeomorphisms

$$\lambda_i = \lambda_i(A)\colon\ \partial_i A \times [0,1) \approx U_i(A) \qquad (G \text{ acts on } [0,1) \text{ trivially})$$

such that $\lambda_i \mid \partial_i A \times 0\colon \partial_i A \to U_i(A)$ is the inclusion and

$$\lambda_i(\partial_j\partial_i A \times [0,1)) = U_i(A) \cap \partial_j A,$$

$$\lambda_i(\lambda_j(x,s),t) = \lambda_j(\lambda_i(x,t),s), \qquad (i \neq j,\quad x \in \partial_j\partial_i A).$$

Furthermore, $\Gamma(A)$ induces a G-collar $\Gamma(\partial_k A)$ with $U_k(\partial_k A) = \emptyset$, $U_i(\partial_k A) = U_i(A) \cap \partial_k A$ and

$$\lambda_i(\partial_k A) = \lambda_i(A) \mid \partial_i\partial_k A \times [0,1)\colon\ \partial_i\partial_k A \times [0,1) \approx U_i(\partial_k A) \qquad (i \neq k).$$

K. Jänich has proved the following lemma by induction on the number of corners:

LEMMA 5.2 ([5; 3.4]). *Any decomposed G-manifold A has a G-collar* $\Gamma(A)$ *which induces the same G-collar* $\Gamma(\partial_i\partial_j A) = \Gamma(\partial_j\partial_i A)$ *of* $\partial_i\partial_j A = \partial_j\partial_i A$.

DEFINITION 5.3. Let (A, φ) be a $(G, \mathscr{S}_n)$-manifold. Then we say that

$$\Gamma(A) = \{\Gamma(A(\omega)) \mid \omega \in I(n)\}$$

is a *collar system* of (A, φ), if it satisfies the following conditions (i) and (ii):

(i) $\Gamma(A(\omega)) = \{U_i(A(\omega)),\ \lambda_i(A(\omega)) \mid 0 \leqq i \leqq n\} \qquad (\omega \in I(n))$

is a G-collar of the decomposed G-manifold $A(\omega)$ satisfying the condition of the above lemma, and $\Gamma(A(\omega)) = \Gamma(A(\sigma\omega))$ $(\sigma \in S_\omega)$.

(ii) The structure map $\alpha(i,\omega)\colon \partial_i A(\omega) \approx A(i,\omega) \times P_i$ maps $U_j(\partial_i A(\omega))$ onto $U_j(A(i,\omega)) \times P_i$ and the diagram

$$\begin{tikzcd}
\partial_j\partial_i A(\omega) \times [0,1) \arrow[r, "\alpha(i,\omega)_j \times \mathrm{id}"] \arrow[d, "\lambda_j(\partial_i A(\omega))"'] & \partial_j A(i,\omega) \times P_i \times [0,1) \arrow[r, "\mathrm{id} \times T"] & \partial_j A(i,\omega) \times [0,1) \times P_i \arrow[d, "\lambda_j(A(i,\omega)) \times \mathrm{id}"] \\
U_j(\partial_i A(\omega)) \arrow[rr, "\alpha(i,\omega) \mid U_j(\partial_i A(\omega))"] & & U_j(A(i,\omega)) \times P_i
\end{tikzcd}$$

of diffeomorphisms is commutative, where $\Gamma(\partial_i A(\omega)) = \{U_j(\partial_i A(\omega)), \lambda_j(\partial_i A(\omega))\}$ is the induced G-collar of $\Gamma(A(\omega))$ on $\partial_i A(\omega)$.

Then we have the following theorem by using the results of Jänich [5; 3.4]:

THEOREM 5.4. *Any $(G, \mathscr{S}_n)$-manifold (A, φ) has a G-collar system.*

PROOF. For each ω with $|\omega| \geqq \max\{|\omega'| \mid \partial A(\omega') \neq \emptyset\}$, we can choose $\Gamma(A(\omega))$ satisfying (i) by the above lemma. Then $\partial_i A(\omega) \cap \partial_j A(\omega) = \emptyset$ if $i \neq j$ and (ii) is trivial.

Assume inductively that we have constructed $\Gamma(A(\omega))$ as desired for each ω with $|\omega| \geqq r$. Take $\omega \in I(n)$ with $|\omega| = r-1$ and $\partial A(\omega) \neq \emptyset$. Then the inductive assumption implies that

$$\Gamma(\partial_i A(\omega)) = \{U_j(\partial_i A(\omega)), \lambda_j(\partial_i A(\omega)) \mid 0 \leqq j \leqq n\}$$

is a G-collar of $\partial_i A(\omega)$ for each i, where

$$U_j(\partial_i A(\omega)) = \alpha(i,\omega)^{-1}(U_j(A(i,\omega)) \times P_i)$$

and $\lambda_j(\partial_i A(\omega))$ is the diffeomorphism defined by the commutative diagram in (ii). Thus, by the collar germ extension lemma [5; 3.4], we have a collar $\Gamma(A(\omega))$ which induces the above G-collar $\Gamma(\partial_i A(\omega))$, and the construction of $\Gamma(A(\omega))$ is complete by induction on $|\omega|$. q.e.d.

Now, we prove the pasting lemma for $(G, \mathscr{S}_n)$-manifolds.

PROOF OF LEMMA 1.11. For $k=1, 2$, we choose a G-collar system $\Gamma(A_k)$ of (A_k, φ_k) by the above theorem. Then $\Gamma(A_k)$ induces an equivariant diffeomorphism

$$\lambda_0(A_k(\omega))\colon \partial_0 A_k(\omega) \times [0,1) \approx U_0(A_k(\omega)) \qquad (k = 1, 2). \tag{5.5.1}$$

By pasting the restrictions of $\lambda_0(A_k(\omega))$ on $B(\omega) \subset \partial_0 A_k(\omega)$, we obtain an equivariant homeomorphism

$$\lambda(\omega)\colon B(\omega) \times (-1,1) \approx V(\omega) \tag{5.5.2}$$

where $V(\omega) = U_0(A_1(\omega)) \cup U_0(A_2(\omega))$, such that

$$\lambda(\omega)(x,0) = x,$$

$$\lambda(\omega)(x,t) = \lambda_0(A_1(\omega))(x,t) \quad \text{and} \quad \lambda(\omega)(x,-t) = \lambda_0(A_2(\omega))(x,t)$$

for $x \in B(\omega)$ and $0 < t < 1$. Then we have the following commutative diagram of homeomorphisms and inclusions from the property of the collar system (cf. Def. 5.3):

$$\begin{array}{ccccc}
B(\omega)\times(-1,1) & \xrightarrow{\lambda(\omega)} & V(\omega) & \subset & A(\omega) \\
\cup & & \cup & & \cup \\
\partial_i B(\omega)\times(-1,1) & \xrightarrow{\lambda(\omega)_i} & V(\omega)\cap\partial_i A(\omega) & \subset & \partial_i A(\omega) \\
{\scriptstyle (\mathrm{id}\times T)\circ(\alpha(i,\omega)\times\mathrm{id})}\downarrow & & \downarrow & & {\scriptstyle \alpha(i,\omega)}\downarrow \\
B(i,\omega)\times(-1,1)\times P_i & \xrightarrow{\lambda(i,\omega)\times\mathrm{id}} & V(i,\omega)\times P_i & \subset & A(i,\omega)\times P_i
\end{array}$$

where $\alpha(i,\omega)$ is a homeomorphism induced by $\alpha_k(i,\omega)$ $(k=1, 2)$. We give the smooth structure of the pasted space $A(\omega)=A_1(\omega)\cup_{B(\omega)}A_2(\omega)$ such that $\lambda(\omega)$ is smooth. Since we have the above commutative diagram, such a process makes a well-defined smooth structure of the ambient manifold $A=A(\emptyset)$. q.e.d.

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