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Supplement to "Holomorphic curves in algebraic varieties"*)

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Let V be a complex projective algebraic smooth variety of dimension n and Σ a hypersurface of V. Using the same notation and terminologies as in [1], we recall Main Theorem in [1, section 3]:

Assume that there exists a system $\{\omega_i\}_{i=1}^{n+1}$ *of* $n+1$ *logarithmic* 1-*forms* $\omega_i \in H^0(V, \Omega_V^1(\log \Sigma))$ *such that* $\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1}$, $1 \leqq i \leqq n+1$, *are linearly independent over* $\boldsymbol{C}$. *Let* $f: \boldsymbol{C} \to V$ *be a holomorphic curve such that* $f(\boldsymbol{C}) \not\subset \Sigma$ *and* f *is non-degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$. *Then there is a positive constant* κ *such that*

$$\kappa T_f(r) < \bar{N}_f(r, \Sigma) + S_f(r). \tag{1}$$

The purpose of this supplementary note is to show the existence of a positive κ which is independent of f (cf. [1, Remark 1, p. 846]). In the present note we shall use the same notation and terminologies as in [1].

PROPOSITION (2). *Under the same assumptions as in the above theorem, there is a positive constant* κ *independent of* f *such that* (1) *holds.*

PROOF. We take a finite affine covering $\{W_\alpha\}_{\alpha=1}^{l}$ of $\{x \in V - \Sigma; \omega_1 \wedge \cdots \wedge \omega_n(x) \neq 0\}$. Then there is one W_α such that $f(\boldsymbol{C}) \cap W_\alpha \neq \phi$. In the sequel, we simply write W for the W_α and fix an embedding $W \subset \boldsymbol{C}^N$ with an affine coordinate system $\{T_1, \dots, T_N\}$. Set $f^*\omega_i = \zeta_i dz$. By the method of the proof of (1) in [1, section 3], it suffices to show the lemma:

LEMMA (3). *Let the notation be as above. Then there are only finitely many polynomials depending only on* $\{\omega_i\}_{i=1}^{n+1}$ *and the* T_k's

$$F(X_{ij}; Y_k) = P_0(X_{ij})Y_k^d + P_1(X_{ij})Y_k^{d-1} + \cdots + P_d(X_{ij}),$$

where $i=1, \dots, n+1$, $j=0, \dots, n-1$ *and* $k=1, \dots, N$, *such that for any* $f: \boldsymbol{C} \to V$ *non-degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$ *whose image meets* W, *there is one* $F(X_{ij}; Y_k)$ *satisfying that* $F(\zeta_i^{(j)}; f^*T_k) \equiv 0$ *and the leading coefficient* $P_0(\zeta_i^{(j)}) \not\equiv 0$.

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PROOF. The proof of this lemma is done along the same line as in Ochiai [2, Theorem A], by using the elimination theorem (cf., e.g., [3, Chap. XI]). Let $J_n(W)\to W$ be the holomorphic n-th jet bundle of germs of holomorphic mappings $g:(\boldsymbol{C};0)\to W$ from a neighborhood of the origin of $\boldsymbol{C}$ into W. Let $j_n(g)$ denote the n-th jet of $g:(\boldsymbol{C};0)\to W$ and set $g^*\omega_i=\xi_i dz$ for $g:(\boldsymbol{C};0)\to W$. Then we have the following canonical mapping I attached to $\{\omega_i\}_{i=1}^{n+1}$ ([2, section 2])

$$I: J_n(W) = W\times \boldsymbol{C}^{n^2} \longrightarrow \boldsymbol{C}^{n^2+n} \tag{4}$$

which is defined by

$$I((g(0),\ (\xi_i^{(j)}(0))_{1\leqq i\leqq n, 0\leqq j\leqq n-1}) = (\xi_i^{(j)}(0))_{1\leqq i\leqq n+1, 0\leqq j\leqq n-1} = (X_{ij})$$

for $j_n(g)=(g(0),\ (\xi_i^{(j)}(0))$, where (X_{ij}) is the natural coordinate system of $\boldsymbol{C}^{n^2+n}$. By [2, Lemma 2.4] the locus S where the differential dI is not regular is a proper subvariety of $W\times\boldsymbol{C}^{n^2}$ and $J_n(f)\,(f^{-1}(W))\not\subset S$, where $J_n(f):\boldsymbol{C}\to J_n(V)$ is the n-th prolongation of f. We may assume that the projection, $W\ni(T_1,\dots,T_N)\to(T_1,\dots,T_n)\in\boldsymbol{C}^n$, is a finite morphism. Let $\{A_\alpha\}_{\alpha=1}^t$ be a system of generators of the ideal $\{P\in\boldsymbol{C}[T_1,\dots,T_N];\ P=0 \text{ on } W\}$, where $\boldsymbol{C}[T_1,\dots,T_N]$ denotes the ring of polynomials in the T_i's with coefficients in $\boldsymbol{C}$. We may assume that the system $\{A_\alpha\}$ contains polynomials of the following form

$$T_k^d + A_{k1}(T_1,\dots,T_n)T_k^{d-1} + \cdots + A_{kd}(T_1,\dots,T_n)$$

for $k=n+1,\dots,N$. By the elimination theorem, it is sufficient to prove Lemma (3) for $k=1,2,\dots,n$, say for $k=1$. From (4) and $\{A_\alpha\}$ we obtain algebraic equations

$$\begin{cases} I_k(T_1,\dots,T_N,(X_{ij})_{1\leqq i\leqq n, 0\leqq j\leqq n-1}) - X_{n+1\,k} = 0, & 0\leqq k\leqq n-1, \\ A_\alpha(T_1,\dots,T_N) = 0, & 1\leqq\alpha\leqq t, \end{cases} \tag{5}$$

where we put $I=(I_{ij})$ and $I_k=I_{n+1\,k}$. Using the elimination theorem, we eliminate $T_{n+1},\dots,T_N$ in (5), and so we have

$$H_\alpha(T_1,\dots,T_n,(X_{ij})) = 0, \quad 1\leqq\alpha\leqq t'. \tag{6}$$

Put $H_\alpha(T_1,\dots,T_n,(X_{ij}))=H_{\alpha 0}(T_1,\dots,T_{n-1},(X_{ij}))T_n^{h_\alpha}+\cdots+H_{\alpha h_\alpha}(T_1,\dots,T_{n-1},(X_{ij}))$ for every α. Then there is some h'_α, $0\leqq h'_\alpha\leqq h_\alpha$, such that $T_k\circ f(z)$ and $X_{ij}\circ f(z)$ satisfy the equations

$$H_{\alpha 0}(T_1,\dots,T_{n-1},(X_{ij})) = \cdots = H_{\alpha h_\alpha - h'_\alpha+1}(T_1,\dots,T_{n-1},(X_{ij})) = 0 \tag{7}$$

for all $z\in f^{-1}(W)$ and that $H_{\alpha h_\alpha - h'_\alpha}(T_1\circ f(z),\dots,T_{n-1}\circ f(z),(X_{ij}\circ f(z))\not\equiv 0$. If all $h'_\alpha=0$, then $\dim_{J_n(f)(z)} I^{-1}(I(J_n(f)(z))>0$ for all $z\in f^{-1}(W)$. This is absurd, so that one $h'_\alpha > 0$. We eliminate T_n in the equations

$$(8) \qquad H_{\alpha h_\alpha - h'_\alpha}(T_1,\dots, T_{n-1}, (X_{ij}))T_n^{h'_\alpha} + \cdots + H_{\alpha h_\alpha}(T_1,\dots, T_{n-1}, (X_{ij})) = 0$$

for those α with $h'_\alpha > 0$, so that we get

$$(9) \qquad G'_\alpha(T_1,\dots, T_{n-1}, (X_{ij})) = 0, \qquad 1 \leqq \alpha \leqq t''.$$

We gather equations (7) and (9) and rewrite them as

$$(10) \qquad G_\alpha(T_1,\dots, T_{n-1}, (X_{ij})) = 0, \qquad 1 \leqq \alpha \leqq t'''.$$

We eliminate T_{n-1} in (10) in the same way as above. Continuing this process of elimination, we finally have

$$(11) \qquad F_\alpha(T_1, (X_{ij})) = F_{\alpha 0}(X_{ij})T_1^{d_\alpha} + \cdots + F_{\alpha d_\alpha}(X_{ij}) = 0, \qquad 1 \leqq \alpha \leqq t''''.$$

By the definition, $F_\alpha(T_1 \circ f(z), (X_{ij} \circ f(z)) \equiv 0$ for all α. Suppose that $F_{\alpha\nu}(X_{ij} \circ f(z)) \equiv 0$ for all α and ν. Take $z \in \boldsymbol{C}$ so that $f(z) \in W$, dI is regular at $J_n(f)(z)$, every leading coefficient in (8) does not vanish at $(T_1 \circ f(z),\dots, T_{n-1} \circ f(z), (X_{ij} \circ f(z)))$ and the same holds in each step of the above process of elimination. Then for any v in a neighborhood of z, there are $T_2(v),\dots, T_N(v)$ such that

$$I_k(T_1 \circ f(v), T_2(v),\dots, T_N(v), (X_{ij} \circ f(z))) - X_{n+1\,k} \circ f(z) = 0$$

for $k = 1,\dots, n$. It follows that the dimension of $I^{-1}(I(J_n(f)(z))$ at $J_n(f)(z)$ is positive. This is a contradiction. Thus one of $F_{\alpha\nu}(X_{ij} \circ f(z))$ does not vanish identically. By the construction we have a finite number of polynomials $F_\alpha(T_1, (X_{ij}))$ with the required property.

References

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