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Weakly ascendant subalgebras of Lie algebras

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Introduction

Maruo [4] introduced the notion of weak ideals generalizing that of subideals to study some kind of coalescence in Lie algebras. Recently Kawamoto [3] has considered N_k-pairs ($k \in \mathbf{N}$) and N_∞-pairs of subalgebras to study criteria for subideality and ascendancy in Lie algebras. For a subalgebra H of a Lie algebra L, the fact that (H, L) is an N_k-pair means that H is a k-step weak ideal of L. In this paper we shall introduce the notion of weakly ascendant subalgebras of a Lie algebra generalizing those of weak ideals and N_∞-pairs and investigate their properties.

The main results are as follows. If L is a hyperabelian Lie algebra of length λ and H is a μ-step weakly ascendant subalgebra of L, then H is a $\mu\lambda$-step ascendant subalgebra of L (Theorem 1). Therefore a subalgebra of a hyperabelian Lie algebra is weakly ascendant if and only if it is ascendant (Theorem 2). Every finitely generated, weakly ascendant subalgebra of a Lie algebra is at most of ω-step (Theorem 4). For a subset S of a generalized solvable Lie algebra L such that $\langle S \rangle$ is finite-dimensional and nilpotent, S is a left Engel subset of L if and only if $\langle S \rangle$ is weakly ascendant and if and only if $\langle S \rangle$ is ascendant (Theorem 5). For subalgebras $H \leq K_i$ ($i = 1, \cdots, n$) of a finite-dimensional Lie algebra, H is weakly ascendant of finite step in $\langle K_1, \cdots, K_n \rangle$ if and only if so is it in each K_i (Theorem 7).

1.

Throughout the paper, let L be a not necessarily finite-dimensional Lie algebra over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified, and let λ and μ be arbitrary ordinals.

We write $H \leq L$ when H is a subalgebra of L and $H \lhd L$ when H is an ideal of L.

A subalgebra H of L is a λ-step ascendant subalgebra of L, denoted by $H \lhd^\lambda L$, provided there is a series $(H_\alpha)_{\alpha \leq \lambda}$ of subalgebras of L such that

(a) $H_0 = H$ and $H_\lambda = L$,

(b) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \lambda$,

(c) $H_\beta = \bigcup_{\alpha < \beta} H_\alpha$ for any limit ordinal $\beta \leq \lambda$.

H is an ascendant subalgebra of L, denoted by H asc L, provided $H \lhd^{\lambda} L$ for some λ. Especially when $\lambda = n < \omega$, H is respectively an n-step subideal and a subideal of L, denoted by H si L.

We shall generalize these notions as follows. We say a subalgebra H of L to be a *λ-step weakly ascendant subalgebra* of L, provided there exists an ascending chain $(M_{\alpha})_{\alpha \leq \lambda}$ of subspaces of L such that

(a) $M_0 = H$ and $M_{\lambda} = L$,

(b) $[M_{\alpha+1}, H] \subseteq M_{\alpha}$ for any ordinal $\alpha < \lambda$,

(c) $M_{\beta} = \bigcup_{\alpha < \beta} M_{\alpha}$ for any limit ordinal $\beta \leq \lambda$.

We then write $H \leq^{\lambda} L$. We simply call such a chain $(M_{\alpha})_{\alpha \leq \lambda}$ a weakly ascending chain for H in L. We say a subalgebra H of L to be a *weakly ascendant subalgebra* of L provided $H \leq^{\lambda} L$ for some ordinal λ. We then write H wasc L. Especially when $\lambda < \omega$, we call H a weak subideal of L and write H wsi L.

We recall the definitions of some classes of Lie algebras. $\mathfrak{A}$ and $\mathrm{E}\mathfrak{A}$ denote respectively the classes of abelian and solvable Lie algebras over a field $\mathfrak{k}$. L belongs to $\acute{\mathrm{E}}\mathfrak{A}$ provided there is an ascending abelian series $(K_{\alpha})_{\alpha \leq \lambda}$ of L, that is, a series $(K_{\alpha})_{\alpha \leq \lambda}$ of subalgebras of L such that

(a) $K_0 = (0)$ and $K_{\lambda} = L$,

(b) $K_{\alpha} \lhd K_{\alpha+1}$ and $K_{\alpha+1}/K_{\alpha} \in \mathfrak{A}$ for any ordinal $\alpha < \lambda$,

(c) $K_{\beta} = \bigcup_{\alpha < \beta} K_{\alpha}$ for any limit ordinal $\beta \leq \lambda$.

L belongs to $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ provided there is an ascending abelian series $(K_{\alpha})_{\alpha \leq \lambda}$ of ideals of L. L is called hyperabelian if $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

For a subalgebra H of L, we say that L belongs to $\acute{\mathrm{E}}(H)\mathfrak{A}$ provided there is an ascending abelian series $(K_{\alpha})_{\alpha \leq \lambda}$ of H-invariant subalgebras of L. Obviously $\acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \acute{\mathrm{E}}(H)\mathfrak{A} \leq \acute{\mathrm{E}}\mathfrak{A}$.

When we emphasize the role of the ordinal λ in the definitions of $\acute{\mathrm{E}}\mathfrak{A}$, $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\acute{\mathrm{E}}(H)\mathfrak{A}$, we write $\acute{\mathrm{E}}_{\lambda}\mathfrak{A}$, $\acute{\mathrm{E}}_{\lambda}(\lhd)\mathfrak{A}$ and $\acute{\mathrm{E}}_{\lambda}(H)\mathfrak{A}$ respectively.

For subalgebras H, K of L, Kawamoto [3] has considered the following conditions: (H, K) is an N_n-pair ($n \in \mathbf{N}$) if $[K, {}_nH] \subseteq H$, and an N_{∞}-pair if for any $a \in K$ there is an $n = n(a) \in \mathbf{N}$ such that $[a, {}_nH] \subseteq H$. These conditions for (H, L) are special cases of weak ascendancy, as is seen in the following

LEMMA 1. *Let H be a subalgebra of a Lie algebra L.*

(a) *For $n \in \mathbf{N}$, $H \leq^{n} L$ if and only if (H, L) is an N_n-pair.*

(b) *$H \leq^{\omega} L$ if and only if (H, L) is an N_{∞}-pair.*

PROOF. (a) If (H, L) is an N_n-pair, put

$$M_i = [L, {}_{n-i}H] + H \qquad (0 \leq i \leq n).$$

Then $(M_i)_{i \leq n}$ is a weakly ascending chain for H in L and $H \leq^{n} L$. The converse is evident.

(b) If (H, L) is an N_∞-pair, put

$$M_i = \{a \in L \mid [a, {}_iH] \subseteq H\} \qquad (0 \leq i < \omega),$$
$$M_\omega = L.$$

Then $(M_\alpha)_{\alpha \leq \omega}$ is a weakly ascending chain for H in L and $H \leq^\omega L$. The converse is evident.

2.

We begin by showing some elementary properties of weakly ascendant subalgebras.

LEMMA 2. *Let L be a Lie algebra over $\mathfrak{k}$.*

(a) *If $H \leq^\lambda L$ and $K \leq L$, then $H \cap K \leq^\lambda K$.*

(b) *If $H \leq^\lambda L$ and $K \lhd L$, then $H + K \leq^\lambda L$.*

(c) *Let f be a homomorphism of L onto a Lie algebra $\bar{L}$. If $H \leq^\lambda L$, then $f(H) \leq^\lambda \bar{L}$. If $\bar{H} \leq^\lambda \bar{L}$, then $f^{-1}(\bar{H}) \leq^\lambda L$.*

PROOF. Assume that $H \leq^\lambda L$ and let $(M_\alpha)_{\alpha \leq \lambda}$ be a weakly ascending chain for H in L. Then

(a) $(M_\alpha \cap K)_{\alpha \leq \lambda}$ is a weakly ascending chain for $H \cap K$ in K.

(b) $(M_\alpha + K)_{\alpha \leq \lambda}$ is a weakly ascending chain for $H + K$ in L.

(c) $(f(M_\alpha))_{\alpha \leq \lambda}$ is a weakly ascending chain for $f(H)$ in $\bar{L}$.

If $(\bar{M}_\alpha)_{\alpha \leq \lambda}$ is a weakly ascending chain for $\bar{H}$ in $\bar{L}$, then $(f^{-1}(M_\alpha))_{\alpha \leq \lambda}$ is such a chain for $f^{-1}(\bar{H})$ in L.

We shall next show the following lemma, which generalizes [3, Lemma 3] as is seen by Lemma 1.

LEMMA 3. *Let L be a Lie algebra over $\mathfrak{k}$ such that $L = H + K$ with $H \leq L$, $K \lhd L$ and $K \in \mathfrak{A}$. Then $H \leq^\lambda L$ if and only if $H \lhd^\lambda L$.*

PROOF. Assume that $H \leq^\lambda L$ and let $(M_\alpha)_{\alpha \leq \lambda}$ be a weakly ascending chain for H in L. Then for any $\alpha \leq \lambda$ $[M_\alpha, H] \subseteq M_\alpha$ and

$$M_\alpha = M_\alpha \cap (H + K) = H + (M_\alpha \cap K).$$

If follows that for any $\alpha < \lambda$

$$\begin{aligned}[M_\alpha, M_{\alpha+1}] &= [H + (M_\alpha \cap K), H + (M_{\alpha+1} \cap K)] \\ &\subseteq H^2 + [H, M_{\alpha+1} \cap K] + [M_\alpha \cap K, H] + K^2 \\ &\subseteq H + (M_\alpha \cap K) \\ &= M_\alpha.\end{aligned}$$

Hence $M_\alpha \leq L$ and $M_\alpha \lhd M_{\alpha+1}$ for any $\alpha<\lambda$. Therefore $H \lhd^{\lambda} L$. The converse is evident.

By making use of Lemmas 2 and 3 we now show the following theorem, where $\mathrm{Core}_L(H)$ denotes the largest ideal of L contained in H.

THEOREM 1. *Let L be a Lie algebra over a field* $\mathfrak{k}$ *and let H be a subalgebra of L such that* $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}_\lambda(H/\mathrm{Core}_L(H))\mathfrak{A}$. *Assume that* $H \leq^{\mu} L$. *Then*

$$H \lhd^{\mu\lambda} L.$$

Especially, if λ *is not a limit ordinal (and even if* $H \leq^{\mu} H+L^2$),

$$H \lhd^{\mu(\lambda-1)+1} L.$$

PROOF. We may assume that $L \in \acute{\mathrm{E}}_\lambda(H)\mathfrak{A}$. This can be easily seen by using Lemma 2 (c).

If λ is not a limit ordinal, there exists an ascending abelian series $(K_\alpha)_{\alpha\leq\lambda}$ of H-invariant subalgebras of L such that $K_{\lambda-1}=L^2$. In fact, if $(L_\alpha)_{\alpha\leq\lambda}$ is an ascending abelian series of H-invariant subalgebras of L, put $K_\alpha=L_\alpha \cap L^2$ for $\alpha \leq \lambda-1$ and $K_\lambda=L_\lambda$. Then each K_α is H-invariant. Since $L/L_{\lambda-1} \in \mathfrak{A}$, it follows that $L^2 \subseteq L_{\lambda-1}$ and so $K_{\lambda-1}=L^2$. For any $\alpha<\lambda-1$

$$\begin{aligned} K_{\alpha+1}^2 &= (L_{\alpha+1} \cap L^2)^2 \\ &\subseteq L_{\alpha+1}^2 \cap L^2 \\ &\subseteq L_\alpha \cap L^2 = K_\alpha. \end{aligned}$$

Therefore $K_\alpha \lhd K_{\alpha+1}$ and $K_{\alpha+1}/K_\alpha \in \mathfrak{A}$. Thus we see that $(K_\alpha)_{\alpha\leq\lambda}$ is a desired series.

Now let $(K_\alpha)_{\alpha\leq\lambda}$ be an ascending abelian series of H-invariant subalgebras of L such that $K_{\lambda-1}=L^2$ if λ is a non-limit ordinal. Then for any $\alpha\leq\lambda$

$$K_\alpha \lhd H + K_\alpha \leq L.$$

Assume that λ is a non-limit (resp. limit) ordinal and

$$H \leq^{\mu} H + L^2 \qquad (\text{resp. } H \leq^{\mu} L).$$

For each $\alpha<\lambda-1$ (resp. $\alpha<\lambda$), put $\bar{K}_{\alpha+1}=K_{\alpha+1}/K_\alpha$ and $\bar{H}=(H+K_\alpha)/K_\alpha$. Then $\bar{K}_{\alpha+1} \lhd \bar{H}+\bar{K}_{\alpha+1}$ and $\bar{K}_{\alpha+1} \in \mathfrak{A}$. Since $H \leq^{\mu} H+K_{\alpha+1}$ by Lemma 2 (a), we have $\bar{H} \leq^{\mu} \bar{H}+\bar{K}_{\alpha+1}$ by Lemma 2 (c). Hence by using Lemma 3 we see that $\bar{H} \lhd^{\mu} \bar{H}+\bar{K}_{\alpha+1}$. It follows that

$$H + K_\alpha \lhd^{\mu} H + K_{\alpha+1}.$$

For a limit ordinal $\beta \leq \lambda$

$$H + K_{\beta} = H + (\bigcup_{\alpha<\beta} K_{\alpha})$$

$$= \bigcup_{\alpha<\beta} (H + K_{\alpha}).$$

Therefore

$$H \lhd^{\mu(\lambda-1)} H + K_{\lambda-1} \lhd L \qquad (\text{resp. } H \lhd^{\mu\lambda} L).$$

Observing that if $H \leq^{\mu} L$ then $H \leq^{\mu} H + L^2$ by Lemma 2 (a) and that $\mu(\lambda-1)+1 \leq \mu\lambda$, we finish the proof.

Corollary 1. *Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L such that $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}_{\lambda}(\lhd)\mathfrak{A}$ (especially, $L \in \acute{\mathrm{E}}_{\lambda}(\lhd)\mathfrak{A}$). If $H \leq^{\mu} L$, then $H \lhd^{\nu} L$ where*

$$\nu = \begin{cases} \mu(\lambda - 1) + 1 & \text{for a non-limit ordianl } \lambda \\ \mu\lambda & \text{for a limit ordinal } \lambda. \end{cases}$$

Proof. This is immediate from Theorem 1, since $\acute{\mathrm{E}}_{\lambda}(\lhd)\mathfrak{A} \leq \acute{\mathrm{E}}_{\lambda}(H/\mathrm{Core}_L(H))\mathfrak{A}$ and $\acute{\mathrm{E}}_{\lambda}(\lhd)\mathfrak{A}$ is Q-closed.

Owing to Lemma 1 we furthermore have the following two corollaries, which are [3, Theorems 4 and 12].

Corollary 2. *Let H be a subalgebra of a Lie algebra L and assume that $L/\mathrm{Core}_L(H) \in \mathfrak{A}^m$. If (H, L^2) is an N_n-pair, then $H \lhd^{n(m-1)+1} L$.*

Proof. If (H, L^2) is an N_n-pair, then $(H, H+L^2)$ is also an N_n-pair. By Lemma 1 (a) $H \leq^{n} H + L^2$. Therefore by Theorem 1 $H \lhd^{n(m-1)+1} L$.

Corollary 3. *Let H be a subalgebra of a Lie algebra L and assume that $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. If (H, L) is an N_{∞}-pair, then H asc L.*

Proof. If (H, L) is an N_{∞}-pair, then by Lemma 1 (b) $H \leq^{\omega} L$. Therefore by Theorem 1 H asc L.

It is shown by the examples in Section 5 that in Theorem 1 the assumption $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(H/\mathrm{Core}_L(H))\mathfrak{A}$ cannot be removed.

Theorem 2. *Let L be a Lie algebra over a field $\mathfrak{k}$ and let H be a subalgebra of L such that $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(H/\mathrm{Core}_L(H))\mathfrak{A}$. Then the following conditions are equivalent:*

(a) H wasc L.

(b) H asc L.

(c) *There exists an ordinal* λ *such that* $H \leq^{\lambda} \langle H, x \rangle$ *for any* $x \in L$.

(d) *There exists an ordinal* λ *such that* $H \lhd^{\lambda} \langle H, x \rangle$ *for any* $x \in L$.

PROOF. (a)⇒(b) follows from Theorem 1.

(b)⇒(d) and (d)⇒(c) are evident.

(c)⇒(a): Assume that $H \leq^{\lambda} \langle H, x \rangle$ for any $x \in L$. Then for each $x \in L$ there exists a weakly ascending chain $(M_{\alpha}(x))_{\alpha \leq \lambda}$ for H in $\langle H, x \rangle$. For each $\alpha \leq \lambda$ denote by M_{α} the subspace of L spanned by $\{M_{\alpha}(x) \mid x \in L\}$. Then it is immediate that $(M_{\alpha})_{\alpha \leq \lambda}$ is a weakly ascending chain for H in L and $H \leq^{\lambda} L$. This completes the proof.

COROLLARY. *Let* L *be a Lie algebra over a field* $\mathfrak{k}$ *and let* H *be a subalgebra of* L.

(a) *Let* $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ *(especially,* $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$*). Then* H wasc L *if and only if* H asc L.

(b) *Let* $L/\mathrm{Core}_L(H) \in \mathrm{E}\mathfrak{A}$ *(especially,* $L \in \mathrm{E}\mathfrak{A}$*). Then* H wsi L *if and only if* H si L.

PROOF. (a) is immediate from Theorem 2, since $\acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \acute{\mathrm{E}}(H/\mathrm{Core}_L(H))\mathfrak{A}$ and $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ is Q-closed. (b) follows from Theorem 1.

The statement (b) in the above corollary is contained in [3, Theorem 11], as is seen by Lemma 1 (a).

As another consequence of Theorem 1 we have the following

THEOREM 3. *Let* L *be a Lie algebra over a field* $\mathfrak{k}$. *Let* H *be a subalgebra of* L *such that* $L/\mathrm{Core}_L(H) \in \acute{\mathrm{E}}_{\lambda}\mathfrak{A}$ *and* $\langle a^H \rangle$ *is finitely generated for any* $a \in L$. *If* $H \leq^{\mu} L$, *then* $H \lhd^{\mu\lambda} L$.

PROOF. We may assume that $L \in \acute{\mathrm{E}}_{\lambda}\mathfrak{A}$. Let $(K_{\alpha})_{\alpha \leq \lambda}$ be an ascending abelian series of L. For any $\alpha \leq \lambda$, let L_{α} be the sum of all H-invariant subspaces of K_{α}. Then it is easy to see that each L_{α} is a unique maximal H-invariant subalgebra of K_{α} and $(L_{\alpha})_{\alpha \leq \lambda}$ is an ascending abelian series of H-invariant subalgebras of L ([3, Lemmas 15 and 16]). Therefore $L \in \acute{\mathrm{E}}_{\lambda}(H)\mathfrak{A}$. The assertion now follows from Theorem 1.

The following corollary is [3, Theorem 17].

COROLLARY. *Under the same hypothesis as in Theorem* 3, *if* (H, L) *is an* N_{∞}*-pair, then* H asc L.

PROOF. If (H, L) is an N_{∞}-pair, by Lemma 1 (b) $H \leq^{\omega} L$. Hence the statement follows from Theorem 3.

3.

To show further properties of weakly ascendant subalgebras, we need the following lemma generalizing [1, Lemma 1.2.3].

LEMMA 4. *Let L be a Lie algebra over $\mathfrak{k}$. Let H be a finitely generated, weakly ascendant subalgebra of L and let K be a finite-dimensional subspace of L. Then there exists an $n=n(K)\in\mathbf{N}$ such that $[K, {}_nH]\subseteq H$.*

PROOF. Let $(M_\alpha)_{\alpha\leq\lambda}$ be a weakly ascending chain for H in L and let N be a finite-dimensional subspace of H generating H. Take a basis $\{x_1,\cdots, x_s\}$ of N and a basis $\{a_1,\cdots, a_t\}$ of K. For each $n\in\mathbf{N}$, let μ_n be the first ordinal such that

$$\{[a_i, x_{j_1},\cdots, x_{j_n}]\mid 1\leq i\leq t,\ 1\leq j_k\leq s\}\subseteq M_{\mu_n}.$$

Then μ_n is not a limit ordinal. Since $[M_{\alpha+1}, N]\subseteq M_\alpha$ for any $\alpha<\lambda$, we have $\mu_{n+1}<\mu_n$ unless $\mu_n=0$. Since the ordinals $\leq\lambda$ are well-ordered, it follows that $\mu_n=0$ for some $n\in\mathbf{N}$. Hence $[K, {}_nN]\subseteq M_0=H$. By the Jacobi identity we conclude that $[K, {}_nH]\subseteq H$.

We remark that for any finitely generated, weakly ascendant subalgebra H of L, $H^\omega=\bigcap_{i=1}^{\infty} H^i$ and $H^{(\omega)}=\bigcap_{i=0}^{\infty} H^{(i)}$ are characteristic ideals of L. This can be shown by using Lemma 4, as in the proof of [4, Theorem 2.2].

THEOREM 4. *Let L be a Lie algebra over a field $\mathfrak{k}$. Then every finitely generated, weakly ascendant subalgebra of L is at most of ω-step.*

PROOF. By Lemma 4 we see that for any $a\in L$ there exists an $n=n(a)\in\mathbf{N}$ such that $[a, {}_nH]\subseteq H$. Hence (H, L) is an N_∞-pair. By Lemma 1 (b) it follows that $H\leq^\omega L$.

It is shown by the second example in Section 5 that in the above theorem the index ω is best possible.

We shall here consider an application of Theorem 4. The set of left Engel elements of L is denoted by $\mathfrak{e}(L)$. We define $\mathfrak{e}^*(L)$ to be the family of subsets S of L satisfying the following condition: For any $a\in L$ there exists an $n=n(a, S)\in\mathbf{N}$ such that $[a, {}_nS]=(0)$. We may call $S\in\mathfrak{e}^*(L)$ a left Engel subset of L. Now we have

LEMMA 5. *Let S be a subset of a Lie algebra L such that $\langle S\rangle$ is nilpotent. Then $S\in\mathfrak{e}^*(L)$ if and only if $\langle S\rangle\leq^\omega L$.*

PROOF. Put $H=\langle S\rangle$ and let H be nilpotent of class m. If $H\leq^\omega L$, then for any $a \in L$ there is an $n \in \mathbf{N}$ such that $[a, {}_nH] \subseteq H$. It follows that

$$[a, {}_{n+m}H] \subseteq H^{m+1} = (0).$$

Hence $H \in \mathfrak{e}^*(L)$ and therefore $S \in \mathfrak{e}^*(L)$. The converse is evident.

THEOREM 5. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to* É$\mathfrak{A}$. *For a subset S of L such that $\langle S \rangle$ is finite-dimensional and nilpotent, the following conditions are equivalent*:

(a) $S \in \mathfrak{e}^*(L)$.

(b) $\langle S \rangle$ wasc L.

(c) $\langle S \rangle \leq^{\omega} L$.

(d) $\langle S \rangle$ asc L.

PROOF. (b)⇒(c) follows from Theorem 4.

(c)⇒(d): Put $H = \langle S \rangle$ and assume that $H \leq^{\omega} L$. Then by Lemma 5 $H \in \mathfrak{e}^*(L)$. Hence for any $a \in L$ there is an $n \in \mathbf{N}$ such that $[a, {}_nH] = (0)$. It follows that

$$\langle a^H \rangle = \langle a, [a, H], \cdots, [a, {}_{n-1}H] \rangle$$

is finitely generated. Therefore by Theorem 3 H asc L.

(d)⇒(b) is evident.

Since (a)⇔(c) by Lemma 5, the proof is complete.

As an immediate consequence of Theorem 5 we have the following

COROLLARY. *Let L be a Lie algebra over a field $\mathfrak{k}$ belonging to* É$\mathfrak{A}$. *For any $x \in L$, the following conditions are equivalent*:

(a) $x \in \mathfrak{e}(L)$.

(b) $\langle x \rangle$ wasc L.

(c) $\langle x \rangle \leq^{\omega} L$.

(d) $\langle x \rangle$ asc L.

This corollary generalizes [1, Theorem 16.4.2 (a)], which states the equivalence of (a) and (d) only for a field $\mathfrak{k}$ of characteristic 0.

As a slight generalization of [1, Proposition 1.3.5] we show the following

THEOREM 6. *Let L be a Lie algebra over a field $\mathfrak{k}$. Then every perfect weakly ascendant subalgebra of L is an ideal of L.*

PROOF. Assume that $H \leq^{\lambda} L$ and $H = H^2$. If $(M_\alpha)_{\alpha \leq \lambda}$ is a weakly ascending chain for H in L, then we can show by transfinite induction that $[M_\alpha, H] \subseteq H$ for any $\alpha \leq \lambda$. Taking $\alpha = \lambda$, we see that $H \triangleleft L$.

4.

In this section we shall observe weakly ascendant subalgebras of step $\leq \omega$.

LEMMA 6. *Let* $H \leq K_\sigma$ $(\sigma \in I)$ *be subalgebras of a Lie algebra* L. *If* $H \leq^\omega K_\sigma$ *for any* $\sigma \in I$, *then* $H \leq^\omega \langle K_\sigma \mid \sigma \in I \rangle$.

PROOF. We may assume that $L = \langle K_\sigma \mid \sigma \in I \rangle$. If we put

$$N_\infty(H) = \{a \in L \mid [a, {}_nH] \subseteq H \text{ for some } n \in \mathbf{N}\},$$

it is easy to see that $N_\infty(H) \leq L$ ([3, Lemma 1 (a)]). If $H \leq^\omega K_\sigma$, then by Lemma 1 (b) (H, K_σ) is an N_∞-pair and so $K_\sigma \leq N_\infty(H)$. Hence $L = N_\infty(H)$. Therefore (H, L) is an N_∞-pair and by Lemma 1 (b) $H \leq^\omega L$.

THEOREM 7. *Let* L *be a finite-dimensional Lie algebra over a field* $\mathfrak{k}$. *Let* $H \leq K_i$ $(i = 1, \cdots, n)$ *be subalgebras of* L. *If* H wsi K_i *for any* i, *then* H wsi $\langle K_1, \cdots, K_n \rangle$.

PROOF. When L is finite-dimensional, $H \leq^\omega L$ is equivalent to H wsi L. Hence the statement follows from Lemma 6.

By Theorem 7 we see that for any subalgebra H of a finite-dimensional Lie algebra L there exists a unique maximal subalgebra of L which contains H as a weak subideal.

As a consequence of Theorem 7 we have the following result ([2, Theorem 6]).

COROLLARY. *Let* L *be a finite-dimensional solvable Lie algebra over* $\mathfrak{k}$. *Let* $H \leq K_i$ $(i = 1, \cdots, n)$ *be subalgebras of* L. *If* H si K_i *for any* i, *then* H si $\langle K_1, \cdots, K_n \rangle$.

PROOF. When L is solvable, H wsi L is equivalent to H si L by Theorem 1. Hence the statement follows from Theorem 7.

THEOREM 8. *Let* L *be a Lie algebra over a field* $\mathfrak{k}$ *and let* H *be a finite-codimensional subalgebra of* L. *Then* H wsi L *if and only if for any* $a \in L$ *and any* $x \in H$ *there exists an* $n = n(a, x) \in \mathbf{N}$ *such that* $[a, {}_nx] \in H$.

PROOF. Assume that the condition holds. For any $x \in H$, $\mathrm{ad}_L x$ induces a linear transformation $\rho(x)$ of the space L/H. By assumption each $\rho(x)$ is nil. Since the space L/H is finite-dimensional, $\rho(x)$ is nilpotent. Therefore the enveloping associative algebra of $\rho(H)$ is nilpotent. Hence there exists a $k \in \mathbf{N}$ such that $\rho(x_1) \cdots \rho(x_k) = 0$ for any $x_1, \cdots, x_k \in H$. This means that $[L, {}_kH] \subseteq H$. By Lemma 1 (a) $H \leq^k L$. The converse is evident.

5.

Let $S=\langle x, y, z\rangle$ be the 3-dimensional simple Lie algebra over a field of characteristic $\neq 2$ with multiplication

$$[x, z] = 2x, \quad [y, z] = -2y, \quad [x, y] = z. \tag{$*$}$$

Then it is known [4] that $\langle y\rangle \leq^2 S$, $\langle y\rangle$ is not a subideal of S and $S \notin \mathrm{E}\mathfrak{A} = \mathrm{E}(\langle y\rangle)\mathfrak{A}$.

Let V be the vector space over a field $\mathfrak{k}$ of characteristic 0 with basis $\{e_1, e_2, \cdots\}$ and let x, y, z be respectively the linear transformations of V defined by

$$\begin{aligned} x&: \ e_i \longrightarrow e_{i+1} && (i \geq 1), \\ y&: \ e_1 \longrightarrow 0, \quad e_i \longrightarrow i(i-1)e_{i-1} && (i \geq 2), \\ z&: \ e_i \longrightarrow 2ie_i && (i \geq 1). \end{aligned}$$

Then $S=\langle x, y, z\rangle$ is a simple Lie subalgebra of $\mathrm{End}_k V$ satisfying ($*$). Consider V as an abelian Lie algebra so that every element of S is a derivation of V. We construct the split extension

$$L = V \dotplus S$$

(cf. [5, Example F]). Then it is easy to see that $\langle y\rangle \leq^\omega L$, $\langle y\rangle$ is not a weak subideal of L, $\langle y\rangle$ is not an ascendant subalgebra of L, $L \notin \acute{\mathrm{E}}\mathfrak{A}$ and a priori $L \notin \acute{\mathrm{E}}(\langle y\rangle)\mathfrak{A}$.

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