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Extremal solutions of general nonlinear differential equations

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Recently, one of the present authors [4] has studied various forms of maximal and minimal asymptotic behavior of positive solutions of the nonlinear differential equations

$$y^{(n)} + f(t, y) = 0, \quad y^{(n)} - f(t, y) = 0.$$

This paper extends the results of [4] to much more general differential equations of the form

$$(1^+) \qquad L_n y + f(t, y) = 0$$

$$(1^-) \qquad L_n y - f(t, y) = 0$$

where $n \geq 2$ and

$$(2) \qquad L_n = \frac{1}{p_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \frac{\bullet}{p_0(t)}.$$

It also establishes criteria for the absence of various forms of asymptotic behavior among the eventually positive solutions of (1^+) and (1^-) and, in some cases, the complete absence of eventually positive solutions.

We always assume that:

$$(3) \qquad \begin{aligned} &\text{(a)} \quad p_i \in C([a, \infty), (0, \infty)), \quad 0 \leq i \leq n; \\ &\text{(b)} \quad f \in C([a, \infty) \times (0, \infty), (0, \infty)). \end{aligned}$$

We introduce the notation:

$$(4) \qquad L_0 y(t) = \frac{y(t)}{p_0(t)}, \quad L_i y(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1} y(t), \quad 1 \leq i \leq n.$$

The domain $\mathscr{D}(L_n)$ of L_n is defined to be the set of all functions y: $[T_y, \infty) \to R$ such that $L_i y(t)$, $0 \leq i \leq n$, are continuous on $[T_y, \infty)$. By a positive solution of (1^+) $[(1^-)]$ we mean a function $y \in \mathscr{D}(L_n)$ which satisfies (1^+) $[(1^-)]$ and is positive for all sufficiently large t.

1. We first consider the case where

$$\int_a^\infty p_i(t)dt = \infty \qquad \text{for} \quad 1 \leq i \leq n-1. \tag{5}$$

LEMMA 1. *Let $y \in \mathscr{D}(L_n)$ satisfy $y(t)>0$ and $L_n y(t)<0$ on $[t_0, \infty)$, $t_0 \geq a$. Then there exist an integer $k \in \{0, 1,\dots, n-1\}$, $k \not\equiv n (\mathrm{mod}\, 2)$, and a $t_1 > t_0$ such that*

$$\begin{aligned} &L_i y(t) > 0 \quad on \quad [t_1, \infty) \qquad for \quad 0 \leq i \leq k, \\ &(-1)^{i-k} L_i y(t) > 0 \quad on \quad [t_1, \infty) \qquad for \quad k+1 \leq i \leq n. \end{aligned} \tag{6}$$

LEMMA 2. *Let $y \in \mathscr{D}(L_n)$ satisfy $y(t)>0$ and $L_n y(t)>0$ on $[t_0, \infty)$. Then, either*

$$L_i y(t) > 0 \quad on \quad [t_1, \infty) \qquad for \quad 0 \leq i \leq n, \tag{7}$$

or there exists an integer $k \in \{0, 1,\dots, n-2\}$, $k \equiv n (\mathrm{mod}\, 2)$, such that (6) holds on $[t_1, \infty)$, where $t_1 > t_0$ is sufficiently large.

The set of all positive solutions of (1^+) $[(1^-)]$ is denoted by $\mathscr{P}$. The set of all positive solutions of (1^+) $[(1^-)]$ satisfying (6) for some $k \in \{0, 1,\dots, n-1\}$ $[k \in \{0, 1,\dots, n-2\}]$ is denoted by $\mathscr{P}_k$; the set of all positive solutions of (1^-) satisfying (7) is denoted by $\mathscr{P}_n$. From Lemmas 1 and 2 it follows that

$$\begin{aligned} &\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_3 \cup \cdots \cup \mathscr{P}_{n-1} && \text{for} \quad (1^+) \quad \text{with } n \text{ even}, \\ &\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_2 \cup \cdots \cup \mathscr{P}_{n-1} && \text{for} \quad (1^+) \quad \text{with } n \text{ odd}, \\ &\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_2 \cup \cdots \cup \mathscr{P}_n && \text{for} \quad (1^-) \quad \text{with } n \text{ even}, \\ &\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_3 \cup \cdots \cup \mathscr{P}_n && \text{for} \quad (1^-) \quad \text{with } n \text{ odd}. \end{aligned}$$

These observations lead us to consider (1^+) $[(1^-)]$ when n is odd [even]. It is now natural to refer to a positive solution y of (1^+) or (1^-) as minimal in case $y \in \mathscr{P}_0$, that is, $(-1)^i L_i y(t)$, $0 \leq i \leq n$, are eventually positive, and the existence of such minimal solutions follows readily from a theorem of Hartman and Wintner [1]. Here we need only impose growth conditions on $f(t, y)$ which assure that solutions of (1^+) and (1^-) can be continued to $t=\infty$ and consider the vector $\mathbf{x} = (L_0 y, -L_1 y, L_2 y,\dots, (-1)^{n-1} L_{n-1} y)$ which satisfies the first order system

$$\mathbf{x}' = -\mathbf{f}(t, \mathbf{x}). \tag{8}$$

Writing $\mathbf{v}>0$ in case all components of $\mathbf{v}$ are positive, we note that in (8) $\mathbf{f}(t, \mathbf{x})>0$ for $\mathbf{x}>0$. According to Hartman and Wintner [1] (see also Kreith

[3]) this assures the existence of a "monotone solution" $\mathbf{x}(t)$ of (8) satisfying $\mathbf{x}(t)>0$ in $[a, \infty)$.

We now consider (1^+) $[(1^-)]$ when n is even [odd]. It is clear that if $y \in \mathscr{P}_1$, then $L_0 y(t) = y(t)/p_0(t)$ is eventually increasing. Accordingly, for n even [odd] we refer to positive solutions $y(t)$ of (1^+) $[(1^-)]$ as minimal in case $L_0 y(t)$ are bounded and seek growth conditions on $f(t, y)$ which guarantee their existence.

The growth conditions on $f(t, y)$ will be formulated in terms of a continuous function $F(t, y)$ which is monotone increasing or decreasing in y and satisfies

$$f(t, y) \le F(t, y) \qquad \text{for} \quad (t, y) \in [a, \infty) \times (0, \infty). \tag{9}$$

The following notation will be employed. Let $i_k \in \{1, \dots, n-1\}$, $1 \le k \le n-1$, and $t, s \in [a, \infty)$. We define $I_0 = 1$ and

$$I_k(t, s; p_{i_k}, \dots, p_{i_1}) = \int_s^t p_{i_k}(r) I_{k-1}(r, s; p_{i_{k-1}}, \dots, p_{i_1}) dr, \quad 1 \le k \le n-1. \tag{10}$$

For simplicity we put for $0 \le i \le n-1$

$$\begin{aligned} &J_i(t, s) = I_i(t, s; p_1, \dots, p_i), && J_i(t) = J_i(t, a); \\ &K_i(t, s) = I_i(t, s; p_{n-1}, \dots, p_{n-i}), && K_i(t) = K_i(t, a). \end{aligned} \tag{11}$$

THEOREM 1. *Suppose that n is even [odd]. A sufficient condition for (1^+) $[(1^-)]$ to have a minimal positive solution is that*

$$\int^\infty K_{n-1}(t) p_n(t) F(t, c p_0(t)) dt < \infty \qquad \textit{for some} \quad c > 0. \tag{12}$$

SKETCH OF PROOF. Let $b = c/2$ or $b = 2c$ according to whether $F(t, y)$ is increasing or decreasing in y. Choose $T > a$ so large that

$$\int_T^\infty K_{n-1}(t) p_n(t) F(t, c p_0(t)) dt \le \frac{b}{8}.$$

Denote by $\mathscr{C}$ the locally convex space of all continuous functions y: $[T, \infty) \to R$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$. Consider the set

$$Y = \left\{ y \in \mathscr{C} : \frac{b}{2} p_0(t) \le y(t) \le 2b p_0(t),\ t \ge T \right\},$$

and define the operators $\Phi_\pm$: $Y \to \mathscr{C}$ by

$$\Phi_\pm y(t) = b p_0(t) \pm (-1)^{n-1} p_0(t) \int_t^\infty K_{n-1}(s, t) p_n(s) f(s, y(s)) ds.$$

It is not difficult to verify that $\Phi_\pm$ are continuous and map Y, which is a closed convex subset of $\mathscr{C}$, into compact subsets of Y. Therefore, by the Schauder-Tychonoff fixed-point theorem, Φ_+ $[\Phi_-]$ has a fixed point in Y, which provides a minimal positive solution of equation (1^+) $[(1^-)]$. For the details the reader is referred to Kitamura and Kusano [2].

Turning now to the concept of maximal solutions, we note that even in the linear case of $y^{(n)}-q(t)y=0$ we would not expect to be able to bound the growth of solutions $y\in\mathscr{P}_n$ if $q(t)>0$. Accordingly, we restrict our considerations of maximal solutions to equation (1^+). A positive solution y of (1^+) satisfies $L_n y(t)<0$ and therefore, by integrating this inequality n times, we see that $y(t)$ cannot grow faster than $p_0(t)J_{n-1}(t)$ as $t\to\infty$. Thus we define a positive solution $y\in\mathscr{P}_{n-1}$ to be maximal if it is asymptotic to $cp_0(t)J_{n-1}(t)$ for some $c>0$, i.e., if there exists a constant $c>0$ such that

$$\lim_{t\to\infty}[y(t)/p_0(t)J_{n-1}(t)] = c.$$

The basic result regarding the existence of maximal solutions is the following

THEOREM 2. *A sufficient condition for* (1^+) *to have a maximal positive solution is that*

$$\int^{\infty} p_n(t)F(t, cp_0(t)J_{n-1}(t))dt < \infty \qquad \textit{for some} \quad c > 0. \tag{13}$$

SKETCH OF PROOF. Let b be as in the proof of Theorem 1, choose $T>a$ so that

$$\int_T^{\infty} p_n(t)F(t, cp_0(t)J_{n-1}(t))dt \leq \frac{b}{8},$$

and define

$$Z = \left\{y\in\mathscr{C}\colon \frac{b}{2}p_0(t)J_{n-1}(t) \leq y(t) \leq 2bp_0(t)J_{n-1}(t),\ t\geq T\right\}.$$

Consider the operator $\Psi\colon Z\to\mathscr{C}$ defined by

$$\Psi y(t) = bp_0(t)J_{n-1}(t)$$
$$+ p_0(t)\int_T^t I_{n-2}(t, s; p_1,\dots, p_{n-2})p_{n-1}(s)\int_s^{\infty} p_n(r)f(r, y(r))drds.$$

Proceeding as in [2], it can be shown that Ψ is a continuous operator mapping Z into a compact subset of Z. It follows that Ψ has a fixed point in Z, which is the desired maximal positive solution of equation (1^+).

REMARK. In case $f(t, y)$ itself is monotone increasing or decreasing in y, then the hypotheses of Theorems 1 and 2 with F replaced by f are necessary as well as sufficient. See [2] for details.

It may happen that the nonexistence of maximal or minimal positive solutions implies the nonexistence of any other kinds of positive solutions. This is the case if, for example, $f(t, y)$ is itself decreasing in y as the following theorem shows.

THEOREM 3. *Let $f(t, y)$ be decreasing in y. Then equation (1^+) has no positive solution if and only if*

$$\int^{\infty} p_n(t)f(t, cp_0(t)J_{n-1}(t))dt = \infty \qquad \textit{for all} \quad c > 0. \tag{14}$$

PROOF. Suppose $y \in \mathscr{P}_k$ for some $k \in \{0, 1,\dots, n-1\}$. Then it follows that

$$\int^{\infty} K_{n-k-1}(t)p_n(t)f(t, y(t))dt < \infty. \tag{15}$$

(See [2].) If $y \in \mathscr{P}_0$, then $y(t) \leq c_1 p_0(t)$ eventually for some $c_1 > 0$, and this combined with (15) yields

$$\int^{\infty} K_{n-1}(t)p_n(t)f(t, c_1 p_0(t))dt < \infty.$$

If $y \in \mathscr{P}_k$ for some $k \in \{1,\dots, n-1\}$, then there are positive constants c_2 and c_3 such that

$$c_2 p_0(t)J_{k-1}(t) \leq y(t) \leq c_3 p_0(t)J_k(t) \tag{16}$$

for all sufficiently large t. Using (16) in (15), we obtain

$$\int^{\infty} K_{n-k-1}(t)p_n(t)f(t, c_3 p_0(t)J_k(t))dt < \infty.$$

In summary, if $\mathscr{P}_k \neq \phi$ for some $k \in \{0, 1,\dots, n-1\}$, then

$$\int^{\infty} K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_k(t))dt < \infty \qquad \text{for some} \quad c > 0,$$

or equivalently, $\mathscr{P}_k = \phi$ for some $k \in \{0, 1,\dots, n-1\}$ if

$$\int^{\infty} K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_k(t))dt = \infty \qquad \text{for all} \quad c > 0. \tag{17_k}$$

In view of (5) we see that

$$\lim_{t\to\infty} [J_{i+1}(t)/J_i(t)] = \infty \quad \text{and} \quad \lim_{t\to\infty} [K_{i+1}(t)/K_i(t)] = \infty.$$

Using these and the decreasing nature of f, we have

$$K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_k(t)) \leq K_{n-k+1}(t)p_n(t)f(t, cp_0(t)J_{k-2}(t))$$

provided t is sufficiently large, and so (17_k) implies (17_{k-2}). Now (14) is nothing else but (17_{n-1}), and from the above observation it follows that

$$\mathscr{P}_{n-1} = \mathscr{P}_{n-3} = \cdots = \mathscr{P}_1 = \phi \qquad \text{if} \quad n \quad \text{is even,}$$

$$\mathscr{P}_{n-1} = \mathscr{P}_{n-3} = \cdots = \mathscr{P}_0 = \phi \qquad \text{if} \quad n \quad \text{is odd.}$$

This completes the "if" part of the theorem. The "only if" part is contained in the Remark following Theorem 2.

Noting that (13) is sufficient for equation (1^-) to have a positive solution $y(t)$ such that $\lim_{t\to\infty} [y(t)/p_0(t)J_{n-1}(t)] = \text{const} > 0$, and that (15) also holds for solutions of (1^-) belonging to $\mathscr{P}_k$, $k \in \{0, 1,\dots, n-2\}$, we have the following theorem.

THEOREM 4. *Suppose $f(t, y)$ in (1^-) is decreasing in y. If (14) holds for all $c>0$, then $\mathscr{P} = \mathscr{P}_n$ for (1^-), and every positive solution of (1^-) grows faster than $p_0(t)J_{n-1}(t)$ as $t\to\infty$.*

Next we examine the case where $f(t, y)$ is nondecreasing in y. We say that equation (1^+) $[(1^-)]$ is superlinear or sublinear according as $f(t, y)$ satisfies

$$f(t, y)/y \geq f(t, z)/z \qquad \text{for} \quad y \geq z > 0,$$

or

$$f(t, y)/y \leq f(t, z)/z \qquad \text{for} \quad y \geq z > 0.$$

THEOREM 5. *Let (1^+) be sublinear. Suppose*

$$\liminf_{t\to\infty} \frac{J_{k-1}(t)K_{n-k-1}(t)}{J_{k+1}(t)K_{n-k-3}(t)} > 0 \tag{18}$$

for $k=1, 3,\dots, n-3$ if n is even, and for $k=2, 4,\dots, n-3$ if n is odd. Suppose in addition

$$\int^{\infty} p_n(t)f(t, cp_0(t)J_{n-2}(t))dt = \infty \qquad \textit{for all} \quad c > 0. \tag{19}$$

Then, $\mathscr{P} = \phi$ if n is even, and $\mathscr{P} = \mathscr{P}_0$ if n is odd.

THEOREM 6. *Let (1^+) be superlinear. Suppose*

$$\liminf_{t\to\infty} \frac{J_{k+1}(t)K_{n-k-3}(t)}{J_{k-1}(t)K_{n-k-1}(t)} > 0 \tag{20}$$

for $k=1, 3,\dots, n-3$ *if* n *is even and for* $k=2, 4,\dots, n-3$ *if* n *is odd. Suppose in addition*

$$(21) \qquad \int^{\infty} K_{n-2}(t)p_n(t)f(t, cp_0(t))dt = \infty \qquad \textit{for all} \quad c>0$$

if n *is even, and*

$$(22) \qquad \int^{\infty} K_{n-3}(t)p_n(t)f(t, cp_0(t)J_1(t))dt = \infty \qquad \textit{for all} \quad c>0$$

if n *is odd. Then,* $\mathscr{P}=\phi$ *if* n *is even, and* $\mathscr{P}=\mathscr{P}_0$ *if* n *is odd.*

Proof of Theorems 5 and 6. Let y be a positive solution of (1^+) such that $y \in \mathscr{P}_k$ for some $k \in \{1,\dots, n-1\}$. Then, using (15), (16) and the fact that $f(t, y)$ is nondecreasing, we have

$$\int^{\infty} K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_{k-1}(t))dt < \infty \qquad \text{for some} \quad c>0.$$

Consequently, $\mathscr{P}_k=\phi$ if

$$(23_k) \qquad \int^{\infty} K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_{k-1}(t))dt = \infty \qquad \text{for all} \quad c>0.$$

Let (1^+) be sublinear. Then, for any $c>0$ and for all sufficiently large t,

$$\begin{aligned} &K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_{k-1}(t)) \\ &\le K_{n-k+1}(t)p_n(t)f(t, cp_0(t)J_{k-3}(t)) \cdot \frac{J_{k-1}(t)K_{n-k-1}(t)}{J_{k-3}(t)K_{n-k+1}(t)}. \end{aligned}$$

It follows that, in the presence of (18), (23_k) implies (23_{k-2}). Now condition (19) means that (23_{n-1}) is valid, so that

$$\mathscr{P}_{n-1} = \mathscr{P}_{n-3} = \cdots = \mathscr{P}_1 = \phi \qquad \text{if} \quad n \quad \text{is even,}$$

$$\mathscr{P}_{n-1} = \mathscr{P}_{n-3} = \cdots = \mathscr{P}_2 = \phi \qquad \text{if} \quad n \quad \text{is odd.}$$

This completes the proof of Theorem 5.

Let (1^+) be superlinear. Then,

$$\begin{aligned} &K_{n-k-1}(t)p_n(t)f(t, cp_0(t)J_{k-1}(t)) \\ &\le K_{n-k-3}(t)p_n(t)f(t, cp_0(t)J_{k+1}(t)) \cdot \frac{J_{k-1}(t)K_{n-k-1}(t)}{J_{k+1}(t)K_{n-k-3}(t)} \end{aligned}$$

for any $c>0$ and for all large t, so that, under condition (20), (23_k) implies (23_{k+2}).

Noting that (21) and (22) agree with (23_1) and (23_2), respectively, we conclude that

$$\mathscr{P}_1 = \mathscr{P}_3 = \cdots = \mathscr{P}_{n-1} = \phi \qquad \text{if} \quad n \quad \text{is even,}$$

$$\mathscr{P}_2 = \mathscr{P}_4 = \cdots = \mathscr{P}_{n-1} = \phi \qquad \text{if} \quad n \quad \text{is odd,}$$

thereby completing the proof of Theorem 6.

If in particular $p_i(t)=1$, $0 \leq i \leq n$, then conditions (18) and (20) are clearly satisfied, and so Theorems 5 and 6 specialized to the equation

$$y^{(n)} + f(t, y) = 0 \tag{24}$$

yield the following result.

COROLLARY 1. (i) *Let* (24) *be sublinear. If*

$$\int^{\infty} f(t, ct^{n-2})dt = \infty \qquad \text{for all} \quad c > 0, \tag{25}$$

then $\mathscr{P} = \phi$ *if* n *is even, and* $\mathscr{P} = \mathscr{P}_0$ *if* n *is odd.*

(ii) *Let* (24) *be superlinear. Suppose*

$$\int^{\infty} t^{n-2} f(t, c)dt = \infty \qquad \text{for all} \quad c > 0 \tag{26}$$

if n *is even, and*

$$\int^{\infty} t^{n-3} f(t, ct)dt = \infty \qquad \text{for all} \quad c > 0 \tag{27}$$

if n *is odd. Then,* $\mathscr{P} = \phi$ *if* n *is even, and* $\mathscr{P} = \mathscr{P}_0$ *if* n *is odd.*

Likewise we are able to prove the following theorems.

THEOREM 7. *Let* (1^-) *be sublinear. Suppose that* (18) *holds for* $k=2, 4,\dots, n-4$ *if* n *is even and for* $k=1, 3,\dots, n-4$ *if* n *is odd. Suppose in addition that*

$$\int^{\infty} K_1(t)p_n(t)f(t, cp_0(t)J_{n-3}(t))dt = \infty \qquad \text{for all} \quad c > 0. \tag{28}$$

Then, it follows for (1^-) *that* $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_n$ *if* n *is even, and* $\mathscr{P} = \mathscr{P}_n$ *if* n *is odd.*

THEOREM 8. *Let* (1^-) *be superlinear. Suppose that* (20) *holds for* $k=2, 4,\dots, n-4$ *if* n *is even and for* $k=1, 3,\dots, n-4$ *if* n *is odd. Suppose in addition that* (22) *holds if* n *is even and that* (21) *holds if* n *is odd. Then it follows for* (1^-) *that* $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_n$ *if* n *is even and* $\mathscr{P} = \mathscr{P}_n$ *if* n *is odd.*

COROLLARY 2. *Let the equation*

$$y^{(n)} - f(t, y) = 0 \tag{29}$$

be sublinear. Suppose

$$\int^{\infty} tf(t, ct^{n-3})dt = \infty \qquad \text{for all} \quad c > 0. \tag{30}$$

Then, $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_n$ *if n is even and* $\mathscr{P} = \mathscr{P}_n$ *if n is odd.*

(ii) *Let* (29) *be superlinear. Suppose* (26) *or* (27) *holds according as n is odd or even. Then,* $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_n$ *if n is even and* $\mathscr{P} = \mathscr{P}_n$ *if n is odd.*

2. We now turn to the case where condition (5) is not satisfied. Recently, Trench [5] has shown that any differential operator L_n of the form (2) can be rewritten as

$$L_n = \frac{1}{\tilde{p}_n(t)} \frac{d}{dt} \frac{1}{\tilde{p}_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{\tilde{p}_1(t)} \frac{d}{dt} \frac{\cdot}{\tilde{p}_0(t)}, \tag{31}$$

where $\tilde{p}_i \in C([a, \infty), (0, \infty))$, $0 \leq i \leq n$, and

$$\int_a^{\infty} \tilde{p}_i(t)dt = \infty \qquad \text{for} \quad 1 \leq i \leq n-1, \tag{32}$$

and that the representation is unique in the sense that the $\tilde{p}_i(t)$, $0 \leq i \leq n$, are determined up to positive multiplicative constants with product 1. From this fact it follows that there exist principal systems for general L_n, one of which is

$$\{\tilde{p}_0(t), \tilde{p}_0(t)\tilde{J}_1(t), \ldots, \tilde{p}_0(t)\tilde{J}_{n-1}(t)\}, \tag{33}$$

where $\tilde{J}_i(t)$ are constructed from $\tilde{p}_i(t)$, $1 \leq i \leq n-1$, according to the rule (11). Here, by a principal system for L_n we mean a set of n positive solutions $\{Y_1(t), \ldots, Y_n(t)\}$ of the equation $L_n y = 0$ which satisfy

$$\lim_{t\to\infty} \frac{Y_i(t)}{Y_j(t)} = 0, \qquad 1 \leq i < j \leq n,$$

It is known that if $\{Y_1(t), \ldots, Y_n(t)\}$ and $\{\tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)\}$ are principal systems for L_n, then the limits

$$\lim_{t\to\infty} \frac{\tilde{Y}_i(t)}{Y_i(t)} > 0, \qquad 1 \leq i \leq n, \tag{34}$$

exist and are finite. (See, for example, Lemma 3 of [5].)

On the basis of the above observation we define minimal and maximal positive solutions of (1^+) and (1^-) with general L_n as follows. Given a principal system $\{Y_1(t), \ldots, Y_n(t)\}$ of $L_n y = 0$, for n even [odd], we say that a positive solution $y(t)$ of (1^+) $[(1^-)]$ is minimal if $y(t)$ is asymptotic to $cY_1(t)$ for some $c>0$, that is, $\lim_{t\to\infty}[y(t)/Y_1(t)]=c$. A positive solution $y(t)$ of (1^+) is called maximal if it is asymptotic to $cY_n(t)$ for some $c>0$: $\lim_{t\to\infty}[y(t)/Y_n(t)]=c$.

All the theorems proven in the preceding section can easily be transferred to the present situation.

THEOREM 1′. *Let $\{Y_1(t),\dots, Y_n(t)\}$ be a principal system for L_n and let $\{Z_1(t),\dots, Z_n(t)\}$ be a principal system for the operator L_n^* defined by*

$$L_n^* = \frac{1}{p_0(t)}\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\frac{\cdot}{p_n(t)}. \tag{35}$$

A sufficient condition for (1^+) and (1^-) to have a minimal solution is that

$$\int^\infty Z_n(t)F(t, cY_1(t))dt < \infty \qquad \textit{for some} \quad c > 0. \tag{36}$$

PROOF. We rewrite L_n in the form (31) satisfying (32). Then the operator L_n^* is represented as

$$L_n^* = \frac{1}{\tilde{p}_0(t)}\frac{d}{dt}\frac{1}{\tilde{p}_1(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{\tilde{p}_{n-1}(t)}\frac{d}{dt}\frac{\cdot}{\tilde{p}_n(t)}.$$

Let $\{\tilde{Y}_1(t),\dots, \tilde{Y}_n(t)\}$ stand for the principal system for L_n given by (33), and let $\{\tilde{Z}_1(t),\dots, \tilde{Z}_n(t)\}$ denote the set of functions

$$\{\tilde{p}_n(t),\ \tilde{p}_n(t)\tilde{K}_1(t),\dots,\ \tilde{p}_n(t)\tilde{K}_{n-1}(t)\}, \tag{37}$$

where $\tilde{K}_i(t)$ are defined exactly as the functions without tilde. The set of functions (37) is a principal system for L_n^*.

Theorem 1 states that a minimal positive solution of (1^+) $[(1^-)]$ exists if

$$\int^\infty \tilde{K}_{n-1}(t)\tilde{p}_n(t)F(t, c\tilde{p}_0(t))dt < \infty \qquad \text{for some} \quad c > 0,$$

or if

$$\int^\infty \tilde{Z}_n(t)F(t, c\tilde{Y}_1(t))dt < \infty \qquad \text{for some} \quad c > 0. \tag{38}$$

Since the limits (34) and $\lim_{t\to\infty}[\tilde{Z}_i(t)/Z_i(t)]>0$, $1\le i\le n$, exist and are finite, (38) is equivalent to (36).

THEOREM 2′. *Equation (1^+) has a maximal positive solution if*

$$\int^\infty Z_1(t)F(t, cY_n(t))dt < \infty \qquad \textit{for some} \quad c > 0. \tag{39}$$

THEOREM 3′. *Let $f(t, y)$ be decreasing in y. Equation* (1^+) *has no positive solutions if and only if*

$$(40) \qquad \int^{\infty} Z_1(t) f(t, cY_n(t))dt = \infty \qquad \text{for all} \quad c > 0.$$

THEOREM 4′. *Let $f(t, y)$ be decreasing in y and suppose that* (40) *holds. Then any positive solution of* (1^-) *grows faster than $Y_n(t)$ as $t \to \infty$.*

THEOREM 5′. *Let* (1^+) *be sublinear. Suppose*

$$(41) \qquad \liminf_{t \to \infty} \frac{Y_{k-1}(t) Z_{n-k-1}(t)}{Y_{k+1}(t) Z_{n-k-3}(t)} > 0$$

for $k = 1, 3, \ldots, n-3$ if n is even and for $k = 2, 4, \ldots, n-3$ if n is odd. Suppose in addition

$$(42) \qquad \int^{\infty} Z_1(t) f(t, cY_{n-1}(t))dt = \infty \qquad \text{for all} \quad c > 0.$$

Then, (1^+) *has no positive solution if n is even, and every positive solution $y(t)$ of* (1^+) *is such that $\lim_{t \to \infty} [y(t)/Y_1(t)]$ exists and is finite if n is odd.*

Similarly, Theorems 6′, 7′ and 8′ could be derived from Theorems 6, 7 and 8, respectively.

EXAMPLE. Consider the operator L_n defined by (2) in which

$$\int_a^{\infty} p_i(t)dt < \infty \qquad \text{for} \quad 1 \le i \le n-1.$$

Define the functions $j_i(t)$ and $k_i(t)$ as follows:

$$\begin{cases} j_0(t) = 1, \\ j_i(t) = \int_t^{\infty} p_i(s) j_{i-1}(s)ds, \quad 1 \le i \le n-1, \end{cases}$$

$$\begin{cases} k_0(t) = 1, \\ k_i(t) = \int_t^{\infty} p_{n-i}(s) k_{i-1}(s)ds, \quad 1 \le i \le n-1. \end{cases}$$

Then, as principal systems for L_n and L_n^* we can take

$$\{p_0(t) j_{n-1}(t),\ p_0(t) j_{n-2}(t), \ldots, p_0(t)\}$$

and

$$\{p_n(t) k_{n-1}(t),\ p_n(t) k_{n-2}(t), \ldots, p_n(t)\},$$

respectively.

References

[1] P. Hartman and A. Wintner, On monotone solutions of systems of nonlinear differential equations, Amer. J. Math. **76** (1954), 860–866.
[2] Y. Kitamura and T. Kusano, Nonlinear oscillation of higher-order functional differential equations with deviating arguments, J. Math. Anal. Appl. (to appear).
[3] K. Kreith, Nonlinear differential equations with monotone solutions, Pacific J. Math. (to appear).
[4] K. Kreith, Extremal solutions for a class of nonlinear differential equations (to appear).
[5] W. F. Trench, Canonical forms and principal systems for general disconjugate equations, Trans. Amer. Math. Soc. **189** (1974), 319–327.

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