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Global transforms and Noetherian pairs

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Throughout this note R will be a commutative Noetherian ring with identity having total quotient ring $T(R)$. The *global transform* of R is the overring $R^g = \{x \in T(R) \mid (R: x)$ contains a finite product of maximal ideals of $R\}$. Thus $x \in R^g$ if and only if $x \in R$ or $R/(R: x)$ is zero-dimensional. Following Wadsworth [5], we say that a pair of rings (R, S) with $R \subseteq S$ is a Noetherian pair if every ring between R and S is Noetherian. Matijevic [4] showed that (1) if A is a ring with $R \subseteq A \subseteq R^g$, then A/xA is a finitely generated R-module for every nonzero-divisor x of R, and (2) if R is reduced, then (R, R^g) is a Noetherian pair. These results may be thought of as a generalization of the Krull-Akizuki Theorem. Further properties of R^g were investigated by Fujita and Itoh [3].

The purpose of this note is to determine conditions on R (more general than reducedness) so that (R, R^g) is a Noetherian pair. We also define two other "transforms" of R and relate them to R^g and to some results of Davis [2] and Wadsworth [5].

At several places in this paper we use the following facts: If A is a ring between R and $T(R)$ (here R is Noetherian), then the mapping $P \to PT(R) \cap A$ is a bijection from the set of prime (primary) ideals of R contained in $Z(R)$ onto the set of prime (primary) ideals of A contained in $Z(A)$. Moreover, if $0 = Q_1 \cap \cdots \cap Q_t$, where Q_i is P_i-primary, is an irredundant primary decomposition of 0 in R, then $0 = Q'_1 \cap \cdots \cap Q'_t$ is an irredundant primary decomposition of 0 in A where $Q'_1 = Q_i T(R) \cap A$ is $P_i T(R) \cap A$-primary. Here $Z(B)$ denotes the zero-divisors of a module B. We also use $G(I)$ to denote the grade of an ideal I (i.e., $G(I)$ is the length of a maximal R-sequence in I).

We first relate Matijevic's result that (R, R^g) is a Noetherian pair (when R is reduced) to a result of Wadsworth. For a Noetherian ring R we define $\tilde{R} = \{x \in T(R) \mid (R: x)$ is contained in no maximal ideals M of R with rank $M \geq 2\}$ and $\hat{R} = \{x \in T(R) \mid (R: x)$ is contained in no maximal ideals M of R with $G(M) \geq 2\}$. It is easily verified that for $x \in T(R)$, $x \in \tilde{R}$ $(x \in \hat{R})$ if and only if $x/1 \in R_M$ for every maximal ideal M of R with rank $M > 1$ $(G(M) > 1)$. Thus $\tilde{R}$ and $\hat{R}$ are overrings of R. If R is an integral domain, then $\tilde{R} = \cap \{R_M \mid M$ a maximal ideal with rank $M > 1\}$ and $\hat{R} = \cap \{R_M \mid M$ a maximal ideal with $G(M) > 1\}$. (If $\tilde{R}$ $(\hat{R})$ has no maximal ideals M with rank $M > 1$ $(G(M) > 1)$, then $\tilde{R}$ $(\hat{R})$ is defined to be the total quotient ring $T(R)$ of R.) For integral domains, $\tilde{R}$ was defined by

Wadsworth and he proved that $(R, \tilde{R})$ is a Noetherian pair [5, Theorem 8].

PROPOSITION 1. *Let R be a Noetherian ring. Then $R \subseteq \tilde{R} \subseteq R^g \subseteq \hat{R} \subseteq T(R)$. Thus if every maximal ideal of R of grade 1 has rank one, then $\tilde{R} = R^g = \hat{R}$. Conversely, if $\tilde{R} = R^g$, then every maximal ideal of R of grade one has rank one and hence $\tilde{R} = R^g = \hat{R}$.*

PROOF. Suppose that $x \in \tilde{R}$. If $x \in R$, then $x \in R^g$. If $x \notin R$, then $(R: x)$ is contained in no maximal ideals of rank greater than one. But since $(R: x)$ is a regular ideal of R, any prime containing it has rank at least one. Thus $R/(R: x)$ is zero-dimensional. Thus $\tilde{R} \subseteq R^g$. Next suppose that $x \in R^g - R$. Let M be a maximal ideal of R containing $(R: x)$. Then M is a minimal prime ideal over $(R: x)$ and hence M_M is the minimal prime ideal of R_M containing $(R: x)_M = (R_M: x/1)$. Hence there exists a positive integer n with $M_M^n \subseteq (R_M: x/1)$. Thus $G(M_M^n) = 1$ and hence $1 = G(M_M^n) = G(M_M) = G(M)$. Of the remaining two statements, the first is obvious. Suppose that $\tilde{R} = R^g$. Let M be a maximal ideal of R of grade one. Let a be any regular element of M. Since a is a maximal R-sequence in M, there exists an element $b \in R$–(a) such that $bM \subseteq (a)$. Hence $(b/a)M \subseteq R$ so that $b/a \in R^g = \tilde{R}$. But clearly $M = (R: b/a)$. Thus since $b/a \in \tilde{R}$, we must have rank $M = 1$.

Thus Wadsworth's result that $(R, \tilde{R})$ is a Noetherian pair (when R is a domain) follows from the result of Matijevic. We note that the inclusions in Proposition 1 may be proper. For example, if (R, M) is a local domain with $G(M) = 1$ but rank $M = 2$, then $R = \tilde{R} \subsetneq R^g \subsetneq \hat{R} = K$, the quotient field of R. Thus even if R is a domain, $(R, \hat{R})$ need *not* be a Noetherian pair. We remark that if R is an integral domain and if the integral closure $\bar{R}$ of R has no maximal ideals of rank one, then $R^g \subseteq \bar{R}$ and hence R^g is integral over R ([5, Corollary 11].)

Matijevic has shown that (R, R^g) is a Noetherian pair if R is reduced. Thus the same holds for $(R, \tilde{R})$ while $(R, \hat{R})$ need not be a Noetherian pair even if R is a domain. We next determine conditions more general than reducedness under which $(R, \tilde{R})$ and (R, R^g) will be Noetherian pairs.

LEMMA 1. *Let R be a Noetherian ring and A a ring with $R \subseteq A \subseteq R^g$. If Q is a (minimal) prime ideal of A, then A/Q is Noetherian. Thus A is Noetherian if and only if every minimal prime ideal of A is finitely generated.*

PROOF. If Q is a minimal prime ideal of A, then $Q \cap R$ is a minimal prime ideal of R. Then $R/Q \cap R \subseteq A/Q \subseteq (R/Q \cap R)^g$. Hence by Matijevic's result, A/Q is Noetherian. The second statement follows from the theorem of Cohen stating that a ring is Noetherian if every prime ideal is finitely generated.

LEMMA 2. *Let R be a Noetherian ring and A a ring between R and R^g. If M is a maximal ideal of R with $G(M)\neq 1$, then $R_M=A_M=(R^g)_M$.*

PROOF. By [3, Proposition 1 (b)] we have $R_M\subseteq A_M\subseteq(R^g)_M\subseteq(R_M)^g$. By [3, Proposition 2] we have $(R_M)^g=R_M$. Hence $R_M=A_M=(R^g)_M=(R_M)^g$.

THEOREM 1. *Let R be a Noetherian ring with the property that for every maximal ideal M of R with $G(M)=1$, R_M is reduced. Then (R, R^g) is a Noetherian pair.*

PROOF. Let A be a ring between R and R^g. Let $0=Q_1\cap\cdots\cap Q_t$ be an irredundant primary decomposition of 0 in A where Q_i is P_i-primary. It suffices to show that each ring A/Q_i is Noetherian. If Q_i is a minimal prime ideal of A, this follows from Lemma 1. Thus we may assume that Q_i is *not* a minimal prime ideal. Now $Q_i\cap R$ is $P_i\cap R$-primary and is the $P_i\cap R$-primary component of the irredundant primary decomposition $(Q_1\cap R)\cap\cdots\cap(Q_t\cap R)$ of 0 in R. Now $P_i\cap R$ cannot be contained in a maximal ideal M of R of grade one. For then R_M is reduced and hence $Q_i\cap R=P_i\cap R$ is a minimal prime ideal of R. From this it follows that $Q_i=P_i$ is a minimal prime ideal of A, contradicting our assumption to the contrary. Thus if $I\supseteq Q_i$ is an ideal of A, $I\cap R$ cannot be contained in any maximal ideal of grade one. We show that $I=(I\cap R)A$ from which it follows that I is finitely generated and hence that A/Q_i is Noetherian. If M is a maximal ideal of R with $G(M)=1$, then $(I\cap R)_M=R_M$ and hence $(I\cap R)A_M=A_M=I_M$. If M is a maximal ideal of R *not* of grade one, then $A_M=R_M$ by Lemma 2 and hence again $(I\cap R)A_M=I_M$.

Davis [2, Theorem 1] has proved the following result: for a Noetherian ring R, $(R, T(R))$ is a Noetherian pair if and only if for every regular maximal ideal M of R, R_M is one-dimensional and reduced. In Theorem 2 we prove that $(R, \tilde{R})$ is a Noetherian pair if and only if for every regular maximal ideal M of R, R_M is reduced. The "if" part of Theorem 2 along with Lemma 3 gives a new proof of half of Davis's result. The "only if" part of Theorem 2 however uses [2, Theorem 1].

THEOREM 2. *Let R be a Noetherian ring. Then $(R, \tilde{R})$ is a Noetherian pair if and only if for every regular maximal ideal of R with* rank $M=1$, *R_M is reduced.*

PROOF. The "if" part of Theorem 2 is almost identical to the proof of Theorem 1. If Q is a minimal prime ideal of a ring A between R and $\tilde{R}$, then A/Q is Noetherian by Lemma 1 since $A\subseteq\tilde{R}\subseteq R^g$. To finish this implication we only need show that the "rank version" of Lemma 2 is also true: if A is a ring between R and $\tilde{R}$ and M is a maximal ideal of R with rank $M\neq 1$, then $R_M=A_M$

$=\tilde{R}_M$. But this follows from the easily proved "rank versions" of Proposition 1(b) and Proposition 2 of [3].

Conversely, suppose that $(R, \tilde{R})$ is a Noetherian pair and let M be a regular maximal ideal of R of rank one. Let $a \in M$ be regular. Then we have an irredundant primary decomposition $(a)=Q \cap Q_1 \cap \cdots \cap Q_t$ for (a) where Q is M-primary (M is a minimal prime ideal of (a)) and Q_i is P_i-primary. Now Q and $Q_1 \cap \cdots \cap Q_t$ are comaximal so $(a)=Q(Q_1 \cap \cdots \cap Q_t)$. Hence for a large n, we have $M^n(Q_1 \cap \cdots \cap Q_t) \subseteq (a)$. Now $Q_1 \cap \cdots \cap Q_t \not\subseteq M$ so there exists $b \in Q_1 \cap \cdots \cap Q_t - M$ and hence $bM^n \subseteq (a)$. Now $M^n \subseteq (R\colon b/a)$ and $\operatorname{rank} M=1$ so $b/a \in \tilde{R}$. Since $b \notin M$, the image α of $1/a$ is in $\tilde{R}_M$. But then $R_M[\alpha] \subseteq \tilde{R}_M \subseteq T(R_M)$ and $R_M[\alpha]$ has dimension zero so that $R_M[\alpha]=T(R_M)$. Thus $\tilde{R}_M=T(R_M)$. Thus $(R_M, T(R_M))=(R_M, \tilde{R}_M)$ is a Noetherian pair. By [2, Theorem 1], R_M is reduced.

LEMMA 3. *Let R be a Noetherian ring. Then the following statements are equivalent*:

(1) $\tilde{R}=T(R)$,

(2) $R^g=T(R)$,

(3) *every regular maximal ideal of R has rank one.*

Furthermore, $\hat{R}=T(R)$ if and only if every regular maximal ideal of R has grade one.

PROOF. It is obvious that (1)$\Rightarrow$(2) and (3)$\Rightarrow$(1). Assume that $R^g=T(R)$ and let M be a regular maximal ideal of R. Let $r \in M$ be regular. Hence $r^{-1} \in T(R)=R^g$, so there exist maximal ideals $M_1,\dots, M_s$ of R so that $r^{-1}M_1 \cdots M_s \subseteq R$, i.e., $M_1 \cdots M_s \subseteq (r)$. Shrink M to a prime ideal P minimal over (r), so that $(r) \subseteq P \subseteq M$. Then $\operatorname{rank} P=1$. But $M_1 \cdots M_s \subseteq (r) \subseteq P$ implies, say, $M_1 \subseteq P$. But then $M_1 \subseteq P \subseteq M$ so $M=M_1=P$ has rank one. The proof of the last statement is similar.

An obvious sufficient condition for $(R, \hat{R})$ to be a Noetherian pair is that for every maximal ideal M of R with $G(M)=1$, R_M is reduced and has dimension one. We have been unable to determine necessary conditions for (R, R^g) and $(R, \hat{R})$ to Noetherian pairs. Originally, the following two conjectures seemed reasonable: (1) (R, R^g) is a Noetherian pair if and only if for every maximal ideal M of grade one, R_M is reduced, and (2) $(R, \hat{R})$ is a Noetherian pair if and only if for every maximal ideal M of grade one, R_M is reduced and one-dimensional. Professor K. Fujita has sent me counterexamples to these two conjectures and has kindly allowed me to include them.

Appendix

The following lemma and two examples are due to Professor K. Fujita. The author is grateful to Professor K. Fujita for allowing him to include them in this paper.

LEMMA. *Let A and B be Noetherian rings, and let $f\colon A\to B$ be a flat and integral homomorphism. If $G(M)\geq 1$ for each maximal ideal M of A, then $B^g\cong A^g\otimes_A B$.*

PROOF. By [3, Proposition 5], it follows that $A^g\otimes_A B=(\varinjlim \mathrm{Hom}_A(\mathfrak{A}, A))\otimes_A B=\varinjlim (\mathrm{Hom}_A(\mathfrak{A}, A)\otimes_A B)$, the direct limit running over all ideals $\mathfrak{A}$ of A with $\dim(A/\mathfrak{A})=0$. Since B is flat over A, this limit is isomorphic to $\varinjlim \mathrm{Hom}_B(\mathfrak{A}B, B)$. Since B is integral over A, $\dim(B/\mathfrak{A}B)=0$ and hence $\varinjlim \mathrm{Hom}_B(\mathfrak{A}B, B)=\bigcup_{\dim(B/\beta)=0}\beta^{-1}=B^g$. Therefore $A^g\otimes_A B\cong B^g$.

EXAMPLE 1. Let k be a field and X, Y, Z be indeterminates. Set $A=k[X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3]$, $M=(X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3)A$, $B=A[Z]/(Z^2)=A[z]$, and $N=(M, z)B$. Now $B\cong A\oplus A$, so B is flat and integral over A. Since $A^g=k[X, Y]$, by the Lemma we have $B^g\cong A^g\otimes_A B=k[X, Y]\otimes_A(A[Z]/(Z^2))\cong k[X, Y][Z]/(Z^2)=k[X, Y, z]$. Therefore B^g is finite over B, so that (B, B^g) is a Noetherian pair. However, $G(N)=1$ but B_N is not reduced because $\sqrt{(XYB\colon X^2Y)}=N$. This example defeats the first conjecture.

EXAMPLE 2. Let k be a field and X, Y, Z be indeterminates. Set $C=k[X, Y, 1/X][Z]/(Z^2)=k[X, Y, 1/X, z]$, $B=k[X, Y, z/X]$, and $A=k[X, Y, z, Yz/X]$. Then $A\subset B\subset C$. Since B is an equicodimensional Cohen-Macaulay ring with $\dim(B)=2$, $B^g=B$. Set $M=(X, Y, z, Yz/X)A$. Since $M=(XA\colon z)$, $G(M)=1$. Let a be any element of $\hat{A}$. If $(A\colon a)=A$, then $a\in A\subseteq A^g$. Suppose that $(A\colon a)\subsetneq A$. If $\mathrm{rank}(A\colon a)=1$, then there exists a maximal ideal N of A such that $(A\colon a)\subseteq N$ and with $G(N)=2$ because A is a Hilbert ring. This contradiction shows that $\mathrm{rank}(A\colon a)=2$. Hence $a\in A^g$. Thus $\hat{A}=A^g$. Since $(A\colon z/X)=M$, $z/X\in\hat{A}$, and hence $B\subseteq\hat{A}$. Since B is integral over A, $A^g\subseteq B^g$ ([3, Proposition 3]). Thus $\hat{A}=B$. Then $(A, \hat{A})$ is a Noetherian pair because $\hat{A}$ is finite over A. However, A_M is not reduced and $\dim(A_M)=2$. This is a strong counter-example to the second conjecture.

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