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A finite-difference method on a Riemannian manifold

Hisao MIZUMOTO

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Introduction

The aim of the present paper is to extend the results in the paper "A finite-difference method on a Riemann surface" [10] to the higher dimensional case.

In Chapter I, we establish orthogonal decomposition theorems concerning difference forms on an n-dimensional polyhedron ($2 \leqq n < \infty$) which give an analogue to de Rham-Kodaira's theory on a Riemannian manifold (cf. Kodaira [8] and de Rham [14]). Here our definition of a polyhedron differs from the ordinary one based on a triangulation and is based on a polyangulation of an n-dimensional manifold (see §**1.1**). A p-difference (p-th order difference form; $0 \leqq p \leqq n$) on a polyhedron is defined as a function on a p-chain which takes a complex value at each oriented p-simplex (see §**2.1**). In order to set the definition of a conjugate difference form which answers our purpose, we introduce the concepts of a conjugate polyhedron and of a complex polyhedron (see §**1.3**). A theory of harmonic difference forms on the complex polyhedron which is analogous to the theory of differential forms on a Riemannian manifold, is then established (cf. Mizumoto [10] and [11] in the 2- and 3-dimensional cases). Eckmann [6] treated a boundary value problem of a harmonic difference form on a polyhedron (Komplex). Our method, which makes an effective use of a conjugate difference on a conjugate polyhedron, is different from his.

In Chapter II, we shall concern ourselves with the problem of approximating a harmonic p-th order differential form on a Riemannian manifold by harmonic p-th order difference forms. We define a sequence $\{\boldsymbol{K}_i\}_{i=0}^{\infty}$ of normal subdivisions of a normal complex polyhedron $\boldsymbol{K}_0$ (see §**1.6**) and a Riemannian manifold M based on $\boldsymbol{K}_0$ (see §**1.7**). Then we shall discuss the norm convergence of smooth extensions of harmonic difference forms on $\boldsymbol{K}_i$, $i=0, 1, 2,\dots,$ to a harmonic differential on M (see Theorems 5.1, 5.2, 5.3 and 5.4, and cf. §**5.2** for the definition of smooth extension). In our present method, the harmonicity of the limit differential form of smooth extensions of harmonic difference forms and that of their conjugate difference forms are simultaneously shown. Our method is based on the fact that the smooth extensions of a harmonic difference form and its conjugate difference form are closed differential forms, so that their limit differential forms in the Hilbert space of differential forms are a pair of closed and conjugate closed ones, and thus a pair of harmonic and conjugate harmonic ones.

The method of orthogonal projection of difference forms and differential forms is also effectively used.

Dodziuk [5] obtained a finite-difference approximation theorem of somewhat different type. It seems that his method is closely related rather to the finite element method than to our method.

Chapter I. Theory of difference forms on a polyhedron

§1. Topological foundations

1. Polyangulation. Let E^n be the n-dimensional euclidean space ($n \geqq 2$). By a *euclidean* 0-*simplex* e^0 we mean a point on E^n. A *euclidean p-simplex* e^p ($1 \leqq p \leqq n$) is inductively defined as a bounded closed simply-connected domain on a p-dimensional plane or on a p-dimensional spherical surface in E^n (a bounded closed simply-connected domain in E^n itself if $p=n$), surrounded by a finite number of euclidean $(p-1)$-simplices $e_1^{p-1}, \dots, e_\nu^{p-1}$ ($\nu \geqq 2$), where we assume that if $e_i^{p-1} \cap e_j^{p-1} \neq \emptyset$ ($i \neq j$) then $e_i^{p-1} \cap e_j^{p-1}$ consists of common simplices of both boundaries of e_i^{p-1} and e_j^{p-1}. Each r-simplex e^r ($0 \leqq r \leqq p$) which is a composing element of the p-simplex e^p, is called an *r-face* of e^p and is called a *proper* r-face when $r<p$. A 0-face e^0 is also called a *vertex* of the p-simplex e^p. Then p-simplex e^p is its own unique p-face.

Let M be an n-dimensional orientable manifold ($n \geqq 2$).[1] By a *p-simplex* s^p ($0 \leqq p \leqq n$) on M we mean a pair of a euclidean p-simplex e^p and a one-to-one bicontinuous mapping ϕ of e^p into M. We shall write $s^p = [e^p, \phi]$ ($0 \leqq p \leqq n$). The image of e^p under ϕ is called the *carrier* of s^p, and is denoted by $|s^p|$; that is, $\phi(e^p) = |s^p|$. We will use the same terminologies "r-face" and "vertex" for those of each p-simplex s^p on the manifold M. We say that a point x of M *belongs to* s^p when $x \in |s^p|$ ($0 \leqq p \leqq n$).

A collection K of simplices on M is called a *polyangulation* of M or a *polyhedron*[2] if it satisfies the following conditions:

(i) Each point x on M belongs to at least one simplex in K;

(ii) Each face s^r ($0 \leqq r \leqq p$) of a simplex s^p of K is an element of K;

(iii) If $s^p, s^r \in K$ and $s^p \cap s^r \neq \emptyset$, then $s^p \cap s^r$ is a finite collection of simplices each of which is a common face of both s^p and s^r;

(iv) No 0-simplex is a vertex of an infinite number of simplices of K.

It is known that any differentiable manifold M is polyangulable (triangulable) (cf. Munkres [12]). A manifold M on which a polyangulation is defined, is called a *polyangulated manifold*. If for each n-simplex $s^n = [e^n, \phi]$ of a polyhedron

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[1] Throughout the present paper, the dimension n of a manifold M will be fixed.

[2] Throughout the present paper, the terminology "polyhedron" will be used in this sense.

K the euclidean n-simplex e^n is a cube, then K is called a *cubic polyhedron*. If M is closed (open resp.), then K is said to be *closed* (*open* resp.).

Let Ω be a compact bordered closed subdomain of M whose boundary consists of $(n-1)$-faces of a polyangulation K. Then the polyhedron L defined as the collection of p-simplices $(p=0,\dots, n)$ of K having their carrier on Ω is called a *compact bordered polyhedron*. By $|L|$ we denote the carrier of L: $|L|=\Omega$. The collection ∂L of p-simplices $(p=0,\dots, n-1)$ of L having their carrier on $\partial\Omega$ is called the *boundary* of L.

Let L and L_1 be two polyhedra. If every n-simplex of L is an n-simplex of L_1, then L is called a *subpolyhedron* of L_1 and L_1 is said to *contain* L.

2. Homology. On a polyhedron K we can define a homology in the same manner as in the case of a triangulated polyhedron. An orientation of each p-simplex $(0\leqq p\leqq n)$ can be easily defined. An oriented p-simplex is denoted by the same notation s^p as a p-simplex.

For a fixed dimension p $(0\leqq p\leqq n)$ a free Abelian group $C_p(K)$ is defined by the following conditions:

(i) All oriented p-simplices are generators of $C_p(K)$;

(ii) Each element c^p of $C_p(K)$ can be represented in the form of finite sum

$$c^p = \sum_i a_i s_i^p,$$

where the coefficients a_i are integers. Each element of $C_p(K)$ is called a *p-chain*.

The *boundary* ∂ of a p-simplex s^p $(1\leqq p\leqq n)$ is defined by

$$\partial s^p = \sum_{i=1}^{\nu} s_i^{p-1} \qquad (\nu = 2 \text{ if } p = 1;\ \nu \geqq 2 \text{ if } 2 \leqq p \leqq n), \tag{1.1}$$

where $s_1^{p-1},\dots, s_\nu^{p-1}$ are $(p-1)$-faces of s^p with the orientation induced by the orientation of s^p. If a $(p-1)$-simplex s^{p-1} is a $(p-1)$-face of s^p with the orientation induced by the orientation of s^p, then we write $s^{p-1}\subset\partial s^p$. The boundary of a p-chain $c^p=\sum_i a_i s_i^p$ $(1\leqq p\leqq n)$ is defined by

$$\partial c^p = \sum_i a_i \partial s_i^p.$$

A p-chain whose boundary is zero, is called a *cycle*. We assume that every 0-chain is a cycle. Since we can easily see that

$$\partial\partial s^p = 0 \tag{1.2}$$

for each p-simplex s^p $(2\leqq p\leqq n)$, we have

$$\partial\partial c^p = \sum_i a_i \partial\partial s_i^p = 0 \tag{1.3}$$

for each p-chain $c^p=\sum_i a_i s_i^p$.

Provided any confusion does not occur, for the present case of polyhedron we shall use the same usual terminologies of homology.

3. Complex polyhedron. Let K and K^* be two open or closed polyangulations of a common manifold M. The polyhedron K^* (K resp.) is called a *conjugate polyhedron* of K (K^* resp.), if they satisfy the following conditions:

(i) For each p $(0\leqq p\leqq n)$ and for each p-simplex s^p of K there exists one and only one q-simplex s^q of K^* $(p+q=n)$[1] such that the intersection $|s^p|\cap|s^q|$ is only one point which is an interior point of both $|s^p|$ and $|s^q|$, and the p-simplex s^p is disjoint from the other q-simplex of K^* than the q-simplex s^q; the simplex s^q (s^p resp.) is said to be *conjugate* to the simplex s^p (s^q resp.) and it is denoted by $*s^p$ ($*s^q$ resp.).

(ii) For a p-simplex s^p of K and an r-simplex s^r of K^*, if $|s^p|\cap|s^r|\neq\emptyset$ then the conjugate simplex $*s^p$ ($*s^r$ resp.) is a q-face ($(n-r)$-face resp.) of s^r (s^p resp.) and thus it follows that $p+r\geqq n$.

We shall introduce an orientation to the conjugate simplex $*s^p$ of an oriented p-simplex s^p $(0\leqq p\leqq n)$ in such a way that

$$s^p\times *s^p=1, \tag{1.4}$$

where the symbol $s^p\times *s^p$ expresses the intersection number of s^p and $*s^p$ (cf. p. 411 of [2] for the definition).

Lemma 1.1.

(i) $**s^p=*(*s^p)=(-1)^{pq}s^p$;

(ii) $s^{p-1}\subset\partial s^p$ *if and only if* $*s^p\subset\partial(-1)^p*s^{p-1}$ $(1\leqq p\leqq n)$.

Proof. (i) This follows immediately from the relation

$$s^p\times *s^p=(-1)^{pq}*s^p\times s^p$$

(cf. pp. 412–413 of [2]).

(ii) If follows from the definition (ii) of the conjugate polyhedron that

$$|s^{p-1}|\subset|\partial s^p| \quad \text{if and only if} \quad |*s^p|\subset|\partial *s^{p-1}|.$$

Hence $s^{p-1}\subset\partial s^p$ if and only if $*s^p\subset\partial(-1)^r*s^{p-1}$ for some r. Then the equation

$$s^p\times\partial(-1)^r*s^{p-1}=(-1)^p\partial s^p\times(-1)^r*s^{p-1}$$

(cf. Satz II of p. 413 of [2]) implies that

$$s^p\times *s^p=(-1)^p s^{p-1}\times(-1)^r*s^{p-1}.$$

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1) Throughout the present paper, the pair p and q will always express the non-negative integers with $p+q=n$ for the dimension n of M.

Then by (1.4) we have $(-1)^{p+r}=1$. Hence we may take $r=p$.

The pair of K and K^* is called a *complex polyangulation* of M or a *complex polyhedron*, and it is denoted by $\boldsymbol{K}=\langle K, K^*\rangle$. A manifold M on which a complex polyangulation is defined is called a *complex-polyangulated manifold.* If M is open or closed, then $\boldsymbol{K}=\langle K, K^*\rangle$ is said to be *open* or *closed* respectively. If both polyhedra K and K^* are cubic, then $\boldsymbol{K}$ is said to be *cubic.*

By a *p-chain* $(0\leqq p\leqq n)$ of a complex polyhedron $\boldsymbol{K}$, we mean a formal sum $\gamma=c_1+c_2$ of a p-chain c_1 of K and a p-chain c_2 of K^*. The *boundary* $\partial\gamma$ is defined by $\partial\gamma=\partial c_1+\partial c_2$. Each p-chain $\gamma=c_1+c_2$ with $\partial c_1=0$ and $\partial c_2=0$ is called a *cycle.* A p-cycle $\gamma=c_1+c_2$ is said to be *homologous to zero* and we write $\gamma\sim 0$, if both c_1 and c_2 are homologous to zero.

4. Compact bordered complex polyhedron. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be an open or closed complex polyhedron. Let L be a compact bordered subpolyhedron of K. Let L^{*s} and L^{*b} be the collections of p-simplices $(p=0,\dots, n)$ of K^* having their carrier on

$$\bigcup_{s^0\in L-\partial L} |{*s^0}| \text{ and } \bigcup_{s^0\in L} |{*s^0}| \quad \text{respectively.}$$

Let us suppose that $|L^{*s}|$ is not vacuous and is connected. Then the polyhedra L^{*s} and L^{*b} are the maximal and minimal compact bordered subpolyhedra of K^* respectively under the condition $|L^{*s}|\subset|L|\subset|L^{*b}|$.

Now we shall define a new compact bordered polyhedron L^* such that $L^{*s}\subset L^*$ and $|L^*|=|L|$. For each p-simplex s^p of ∂L $(0\leqq p\leqq n-1)$ the conjugate *half q-simplex* $\tilde{*}s^p$ of s^p is defined by the conditions:

(i) $|\tilde{*}s^p|=|{*s^p}|\cap|L|$;

(ii) $\tilde{*}s^p$ has the orientation induced by that of ${*s^p}$.

By L^* we denote the polyhedron defined as the collection of all p-simplices $(p=0,\dots, n)$ which are p-faces of n-simplices of L^{*s} and conjugate half n-simplices of 0-simplices of ∂L. The polyhedron L^* is called a *conjugate polyhedron* of L and the pair $\boldsymbol{L}=\langle L, L^*\rangle$ is called a *compact bordered complex polyhedron.* If the original $\boldsymbol{K}$ is cubic, then $\boldsymbol{L}$ is said to be *cubic.* The carrier $|\boldsymbol{L}|$ of $\boldsymbol{L}$ is defined by $|\boldsymbol{L}|=|L|=|L^*|$.

Let $\boldsymbol{L}=\langle L, L^*\rangle$ and $\boldsymbol{L}_1=\langle L_1, L_1^*\rangle$ be two complex polyhedra. If L is a subpolyhedron of L_1, then $\boldsymbol{L}$ is called a *complex subpolyhedron* of $\boldsymbol{L}_1$.

We can see that $\partial\boldsymbol{L}=\langle\partial L, \partial L^*\rangle$ defines a finite collection of $(n-1)$-dimensional closed complex polyhedra. $\partial\boldsymbol{L}$ is called the *boundary* of $\boldsymbol{L}$.

Let s^p $(0\leqq p\leqq n-1)$ be an arbitrary p-simplex of the boundary $\partial\boldsymbol{L}=\langle\partial L, \partial L^*\rangle$. Since $\partial\boldsymbol{L}$ is a finite collection of $(n-1)$-dimensional complex polyhedra, we can consider a conjugate $(q-1)$-simplex of s^p on $\partial\boldsymbol{L}$, which is denoted by ${*s^p}(\partial\boldsymbol{L})$.

LEMMA 1.2. *For each p-simplex* s^p $(0 \leqq p \leqq n-1)$ *of* ∂L *the* $(q-1)$*-simplex* $*s^p(\partial \boldsymbol{L})$ *is the unique* $(q-1)$*-face of the q-simplex* $(-1)^p \tilde{*} s^p$ *contained in* ∂L^*:

$$*s^p(\partial \boldsymbol{L}) \in \partial(-1)^p \tilde{*} s^p \qquad \textit{for each } s^p \in \partial L. \tag{1.5}$$

PROOF. The inclusion relation

$$|*s^p(\partial \boldsymbol{L})| \subset |\partial \tilde{*} s^p| \qquad (s^p \in \partial L)$$

follows from the relations $|\tilde{*}s^p| = |*s^p| \cap |L|$ and $|*s^p(\partial \boldsymbol{L})| = |*s^p| \cap |\partial L|$. Let s^{p+1} be a $(p+1)$-simplex such that $s^p \subset \partial s^{p+1}$ and $|s^{p+1}| \cap |L - \partial L| \neq \emptyset$. Then, by (ii) of Lemma 1.1 we have an inclusion relation

$$*s^{p+1} \subset \partial(-1)^{p+1} \tilde{*} s^p. \tag{1.6}$$

We can easily verify the relation

$$s^{p+1} \times *s^p(\partial \boldsymbol{L}) = s^{p+1} \times *s^{p+1}.$$

Then by (1.6) we have the inclusion relation

$$(-1)*s^p(\partial \boldsymbol{L}) \subset \partial(-1)^{p+1} \tilde{*} s^p$$

when we note the position of two $(q-1)$-simplices $*s^{p+1}$ and $*s^p(\partial \boldsymbol{L})$.

A p-simplex or a p-chain $(0 \leqq p \leqq n)$ is said to be *in the interior of* $\boldsymbol{L} = \langle L, L^* \rangle$, if its carrier is in the interior of $|\boldsymbol{L}|$.

5. Subdivision of a polyhedron. Here we shall make some agreement. We shall denote subsets of $N = \{1, \ldots, n\}$ by I_r, J_s, $L_t, \ldots$, etc. The subscripts r, s and t of I_r, J_s and L_t respectively show numbers of elements of the subsets. By the small letters $i_1, \ldots, i_r$; $j_1, \ldots, j_s$; $l_1, \ldots, l_t$ with subscripts we denote elements of the subsets I_r, J_s and L_t respectively, i.e. $I_r = \{i_1, \ldots, i_r\}$, $J_s = \{j_1, \ldots, j_s\}$ and $L_t = \{l_1, \ldots, l_t\}$. Here we shall agree that $i_1 < \cdots < i_r$, $j_1 < \cdots < j_s$ and $l_1 < \cdots < l_t$. If a family $\{L_s, M_t, \ldots, N_u\}$ of subsets of I_r is a decomposition of I_r, i.e. $I_r = L_s \cup M_t \cup \cdots \cup N_u$ and L_s, $M_t, \ldots, N_u$ are mutually disjoint, then we write $I_r = L_s + M_t + \cdots + N_u$. For each $I_p \subset N$, the complement of I_p in N is denoted by J_q: $N = I_p + J_q$. If a p-simplex is denoted by $e^p_{I_r L_s M_t \cdots}$ whose meaning is subsequently defined, then the subscripts I_r, L_s, $M_t, \ldots$ will show mutually disjoint subsets of N. We should note that $I_r L_s M_t \cdots$ does not mean a product set of I_r, L_s, $M_t, \ldots$.

Let $\boldsymbol{K} = \langle K, K^* \rangle$ be a cubic complex polyhedron and let $s^n = [e^n, \phi]$ be an n-simplex of K. We may assume that the euclidean n-simplex e^n is the unit cube

$$e^n_N = \{0 \leqq x_i \leqq 1 \quad (i \in N)\}. \tag{1.7}$$

We denote each p-face $(0 \leqq p \leqq n)$ of e^n_N by

(1.8) $$e^p_{I_pL_r} = \{0 \leqq x_i \leqq 1 \ (i \in I_p),\ x_i = 1 \ (i \in L_r),\ x_i = 0 \ (i \in J_q - L_r)\} \qquad (0 \leqq r \leqq q).$$

We write

$$s^p_{I_pL_r} = [e^p_{I_pL_r}, \phi].$$

We agree that the p-simplex $s^p_{I_pL_r}$ has the orientation induced by the orientation of the p-dimensional space $O\text{-}x_{i_1}\cdots x_{i_p}$.

The euclidean n-simplex e^n_N is divided into 2^n cubes

$$\natural e^n_{\bar{I}_r\tilde{I}_{n-r}} = \left\{0 \leqq x_i \leqq \frac{1}{2} \ (i \in \bar{I}_r),\ \frac{1}{2} \leqq x_i \leqq 1 \ (i \in \tilde{I}_{n-r})\right\} \quad (0 \leqq r \leqq n)$$

by the n hyperplanes $\{x_i = 1/2\}$ $(i = 1,\dots, n)$. Then a *subdivision* of the n-simplex s^n into 2^n new simplices

$$\natural s^n_{\bar{I}_r\tilde{I}_{n-r}} = [\natural e^n_{\bar{I}_r\tilde{I}_{n-r}}, \phi]$$

is defined. Further the subdivision of the n-simplex s^n induces subdivision of each p-face of s^n $(1 \leqq p \leqq n-1)$. We denote each p-simplex $(0 \leqq p \leqq n)$ of the subdivision of the euclidean n-simplex e^n_N by

$$\begin{aligned}\natural e^p_{\bar{I}_r\tilde{I}_{p-r}L_sM_t} = \Big\{&0 \leqq x_i \leqq \frac{1}{2} \ (i \in \bar{I}_r),\ \frac{1}{2} \leqq x_i \leqq 1 \ (i \in \tilde{I}_{p-r}),\\ &x_i = 1 \ (i \in L_s),\ x_i = \frac{1}{2} \ (i \in M_t),\ x_i = 0 \ (\text{for other } i \text{ of } N)\Big\}\\ &(0 \leqq r \leqq p,\ 0 \leqq s \leqq q,\ 0 \leqq t \leqq q - s),\end{aligned}$$

and we write

$$\natural s^p_{\bar{I}_r\tilde{I}_{p-r}L_sM_t} = [\natural e^p_{\bar{I}_r\tilde{I}_{p-r}L_sM_t}, \phi].$$

We agree that the p-simplex $\natural s^p_{\bar{I}_r\tilde{I}_{p-r}L_sM_t}$ has the orientation induced by the orientation of the p-dimensional space $O\text{-}x_{i_1}\cdots x_{i_p}$, where $I_p = \bar{I}_r + \tilde{I}_{p-r} = \{i_1,\dots, i_p\}$. We carry out this procedure for all n-simplices of K so that if a p-simplex s^p $(1 \leqq p \leqq n-1)$ is a common p-face of two n-simplices s^n and σ^n then the subdivision of s^n and σ^n induces a common subdivision of the p-face s^p, if necessary, by a suitable choice of each mapping ϕ. Then we have a new cubic polyhedron K_1 which is called the *subdivision* of the polyhedron K. Since the complex polyhedron $\boldsymbol{K} = \langle K, K^*\rangle$ is cubic, the conjugate polyhedron K_1^* of K_1 is also cubic and thus so is the complex polyhedron $\boldsymbol{K}_1 = \langle K_1, K_1^*\rangle$. The complex polyhedron $\boldsymbol{K}_1 = \langle K_1, K_1^*\rangle$ is called the *subdivision* of $\boldsymbol{K}$, where we should note that K_1^* is not a subdivision of K^*.

Let $s^n=[e^n, \phi]$ be an arbitrary n-simplex in the interior of the conjugate polyhedron K^* and let us assume that the euclidean n-simplex e^n is the unit cube e^n_N of (1.7). We denote e^n_N and its 3^n-1 adjacent euclidean n-simplices of e^n_N by

$$e^n_{\bar I_r \tilde I_t \hat I_{n-r-t}} = \{-1 \leqq x_i \leqq 0 \ (i \in \bar I_r),\ 1 \leqq x_i \leqq 2 \ (i \in \tilde I_t),$$

$$0 \leqq x_i \leqq 1 \ (i \in \hat I_{n-r-t})\} \qquad (0 \leqq r \leqq n,\ 0 \leqq t \leqq n-r).$$

Especially $e^n_{\bar I_0 \tilde I_0 \hat I_n}=e^n_N$. Then each euclidean p-simplex

$$e^p_{\bar I_r \tilde I_t \hat I_{p-r-t} L_s} = \{-1 \leqq x_i \leqq 0 \ (i \in \bar I_r),\ 1 \leqq x_i \leqq 2 \ (i \in \tilde I_t),$$

$$0 \leqq x_i \leqq 1 \ (i \in \hat I_{p-r-t}),\ x_i = 1 \ (i \in L_s),\ x_i = 0 \ (\text{for other } i \text{ of } N)\}$$

$$(0 \leqq r \leqq p,\ 0 \leqq t \leqq p-r,\ 0 \leqq s \leqq q)$$

is a p-face of one of these 3^n n-simplices. We may assume that the mapping ϕ of $s^n=[e^n_N, \phi]$ can be extended to a one-to-one bicontinuous mapping of the above 3^n euclidean n-simplices into the basic manifold M, and the 3^n n-simplices

$$s^n_{\bar I_r \tilde I_t \hat I_{n-r-t}} = [e^n_{\bar I_r \tilde I_t \hat I_{n-r-t}}, \phi] \quad (0 \leqq r \leqq n,\ 0 \leqq t \leqq n-r)$$

are the collection of s^n and its 3^n-1 adjacent n-simplices of the conjugate polyhedron K^*. Then each p-simplex

$$s^p_{\bar I_r \tilde I_t \hat I_{p-r-t} L_s} = [e^p_{\bar I_r \tilde I_t \hat I_{p-r-t} L_s}, \phi]$$

is a p-face of one of these 3^n n-simplices of K^*.

We define 3^n new euclidean n-simplices

$$\natural e^n_{\bar I_r \tilde I_t \hat I_{n-r-t}} = \left\{-\frac{1}{4} \leqq x_i \leqq \frac{1}{4} \ (i \in \bar I_r),\ \frac{3}{4} \leqq x_i \leqq \frac{5}{4} \ (i \in \tilde I_t),\right.$$

$$\left.\frac{1}{4} \leqq x_i \leqq \frac{3}{4} \ (i \in \hat I_{n-r-t})\right\} \qquad (0 \leqq r \leqq n,\ 0 \leqq t \leqq n-r)$$

and euclidean p-simplices

$$\natural e^p_{\bar I_r \tilde I_t \hat I_{p-r-t} L_s} = \left\{-\frac{1}{4} \leqq x_i \leqq \frac{1}{4} \ (i \in \bar I_r),\ \frac{3}{4} \leqq x_i \leqq \frac{5}{4} \ (i \in \tilde I_t),\right.$$

$$\left.\frac{1}{4} \leqq x_i \leqq \frac{3}{4} \ (i \in \hat I_{p-r-t}),\ x_i = \frac{3}{4} \ (i \in L_s),\ x_i = \frac{1}{4} \ (\text{for other } i \text{ of } N)\right\}$$

$$(0 \leqq r \leqq p,\ 0 \leqq t \leqq p-r,\ 0 \leqq s \leqq q).$$

Then we may assume that each

$$\natural s^n_{\bar{I}_r\tilde{I}_t\hat{I}_{n-r-t}} = [\natural e^n_{\bar{I}_r\tilde{I}_t\hat{I}_{n-r-t}}, \phi]$$

is an n-simplex of the conjugate polyhedron K_1^* of the subdivision K_1. Each p-simplex

$$\natural s^p_{\bar{I}_r\tilde{I}_t\hat{I}_{p-r-t}L_s} = [\natural e^p_{\bar{I}_r\tilde{I}_t\hat{I}_{p-r-t}L_s}, \phi]$$

is a p-face of one of these 3^n n-simplices. We agree that the p-simplex $\natural s^p_{\bar{I}_r\tilde{I}_t\hat{I}_{p-r-t}L_s}$ has the orientation induced by the orientation of the p-dimensional space O-$x_{i_1}\cdots x_{i_p}$, where $I_p=\bar{I}_r+\tilde{I}_t+\hat{I}_{p-r-t}=\{i_1,\dots, i_p\}$.

6. Normal coordinates. Let K be a cubic polyhedron and $s^n=[e^n, \phi]$ be an arbitrary n-simplex of K. We can choose the mapping ϕ so that e^n is a unit cube. Then there exists an affine transformation ψ of the unit cube e^n_N of (1.7) onto e^n: $e^n=\psi(e^n_N)$. To each point P of e^n we can assign the coordinates of the point $\psi^{-1}(P)$ of e^n_N. These coordinates are called the *normal coordinates* of (the point P of) e^n. Let ψ_1 be another affine transformation of e^n_N onto e^n. Then both normal coordinates assigned by ψ and ψ_1 are said to be *equivalent* to each other. We find that a point P on each p-face $\psi(e^p_{I_pL_r})$ of e^n has a normal coordinates $(x_1,\dots, x_n)$ such that $x_i=1$ $(i\in L_r)$ and $x_i=0$ $(i\in J_q-L_r)$. The essential class $(x_{i_1},\dots, x_{i_p})$ is called the *normal coordinates* of the point P on the p-face $\psi(e^p_{I_pL_r})$ (induced by the normal coordinates of e^n).

Let $s^n=[e^n, \phi]$ be an n-simplex of K with normal coordinates assigned on e^n. Then we can assign normal coordinates to each point x of s^n by giving the normal coordinates of $\phi^{-1}(x)\in e^n$ to the point x.

The set $\sigma^p_{I_p}$ $(I_p\subset N,\ 0\leqq p\leqq n-1)$ of points of s^n having normal coordinates $(x_1,\dots, x_n)$ with $x_i=1/2$ $(i\in J_q)$ is called a *median p-face* of s^n.

The following lemma can be proved by the method analogous to that in the case of normal (barycentric) coordinates of triangulation.

LEMMA 1.3. *For the collection $\{s^n=[e^n, \phi]\}$ of n-simplices of a cubic polyhedron K, a set of mappings ϕ can be so chosen that for each common p-face s^p $(1\leqq p\leqq n-1)$ of two n-simplices s^n and σ^n, the normal coordinates of s^p induced by s^n and σ^n are equivalent.*

A set of normal coordinates chosen in this way is called *normal coordinates* of K. A cubic polyhedron K to which such normal coordinates are assigned, is said to be *normal.*

Let $\boldsymbol{K}=\langle K, K^*\rangle$ be a cubic complex polyhedron such that K is normal. If for each common p-face $s^p\in K$ $(1\leqq p\leqq n-1)$ of 2^q n-simplices s^n_i $(i=1,\dots, 2^q)$, the carrier $|{*s^p}|$ lies on the union of some median q-faces of s^n_i $(i=1,\dots,2^q)$, then K^* and $\boldsymbol{K}$ are called a *normal conjugate polyhedron* of K and *a normal complex polyhedron* respectively. Here, if K is compact bordered, then it is moreover required that for each p-simplex $s^p \in \partial K$ $(1 \leqq p \leqq n-1)$ the carrier $|\tilde{*}s^p|$ lies on the union of some median q-faces.

A subdivision K_1 of a normal polyhedron K is called a *normal subdivision* of K, if for each p-simplex $s^p \in K_1$ $(0 \leqq p \leqq n-1)$, the carrier $|s^p|$ lies on the carrier of some p-simplex of K or lies on some median p-face of an n-simplex of K. We may assume that the normal subdivision K_1 is normal polyhedron which has the normal coordinates induced by that of K. Let K_1^* be the normal conjugate polyhedron of the normal subdivision K_1. Then the complex polyhedron $\boldsymbol{K}_1 = \langle K_1, K_1^* \rangle$ is called a *normal subdivision* of $\boldsymbol{K}$.

7. A Riemannian manifold based on a normal polyhedron. Let M be a manifold on which a normal complex polyhedron $\boldsymbol{K} = \langle K, K^* \rangle$ is defined. Then we can make M into a Riemannian manifold by the following procedure:

(i) With the notation in **6**, we can map each n-simplex $s^n = [e^n, \phi] \in K$ onto the unit cube e_N^n of (1.7) by the mapping $(\phi \circ \psi)^{-1}$. By these mappings, local coordinates in a neighborhood of each point in the interior of each n-simplex of K are defined.

(ii) If a point x lies on a p-face s^p of an n-simplex s_1^n $(0 \leqq p \leqq n-1)$ but does not lie on any $(p-1)$-face, then there exist just 2^q n-simplices s_i^n $(i = 1, \dots, 2^q)$ whose common p-face is the simplex s^p. Then we can map the union $\bigcup_{i=1}^{2^q} s_i^n$ onto a union $\bigcup_{i=1}^{2^q} e_i^n$ of 2^q unit cubes e_i^n $(i = 1, \dots, 2^q)$ on E^n which have a common p-face e^p in such a way that s^p is mapped onto e^p and the normal coordinates of s_i^n $(i = 1, \dots, 2^q)$ are preserved. The point x is mapped into a point $P \in e^p$. By this mapping, local coordinates in a neighborhood of x are defined. The restriction of these local coordinates to each s_i^n is an affine transformation of the local coordinates defined in (i).

(iii) The transformation between local coordinates defined in (i) or (ii) is a rotation or a parallel transformation of E^n and thus the length is invariant under the transformation. Hence, by making use of these local coordinates we can introduce a positive definite metric in M and can make M into a Riemannian manifold.

The Riemannian manifold M constructed by the above procedure (i), (ii) and (iii) is called a *Riemannian manifold based on a normal complex polyhedron* $\boldsymbol{K}$.

§2. Difference forms on a polyhedron

1. Difference calculus. Let $\boldsymbol{K} = \langle K, K^* \rangle$ be an open, closed or compact bordered complex polyhedron.

By a *p-difference* (*p-th order difference form*) φ^p on $\boldsymbol{K}$ $(0 \leqq p \leqq n)$, we mean a complex valued function φ^p on the collection of oriented and oppositely oriented

p-simplices of $\boldsymbol{K}$ such that φ^p has a value $\varphi^p(s^p)$ for each oriented p-simplex s^p and $\varphi^p(-s^p)=-\varphi^p(s^p)$.

In the case where $\boldsymbol{K}$ is compact bordered, for every 0-simplex s^0 of ∂K^*, let s^{n-1} be the $(n-1)$-simplex of ∂K with $s^0=*s^{n-1}(\partial\boldsymbol{K})$ and let s_1^0 be the 0-simplex in the exterior of $\boldsymbol{K}$ with $\partial(-1)^n*s^{n-1}=s_2^0-s_1^0$. Then for every 0-difference φ^0 on $\boldsymbol{K}$ we define $\varphi^0(s_1^0)$ by a relation

$$\varphi^0(s_1^0) = 2\,\varphi^0(s^0) - \varphi^0(s_2^0)\,. \tag{2.1}$$

For every p-difference $\varphi^p(1\leqq p\leqq n)$ on $\boldsymbol{K}$ and for every conjugate p-simplex $*s^q$ of $s^q\in\partial K$ we define $\varphi^p(*s^q)$ by a relation

$$\varphi^p(*s^q) = 2\,\varphi^p(\tilde{*}s^q)\,. \tag{2.2}$$

We define $c_1\varphi^p+c_2\psi^p$ by

$$(c_1\varphi^p + c_2\psi^p)(s^p) = c_1\varphi^p(s^p) + c_2\psi^p(s^p) \quad (0\leqq p \leqq n),$$

where φ^p and ψ^p are p-differences on $\boldsymbol{K}$, and c_1 and c_2 are complex constants.

The *exterior product* $\varphi^p\psi^q$ of a p-difference φ^p and a q-difference ψ^q $(0\leqq p\leqq n-1)$ is defined as an n-difference given by

$$\varphi^p\psi^q(s^n) = \sum_{s^p\ni *s^n}\frac{1}{v(s^p)}\,\varphi^p(s^p)\psi^q(*s^p) \tag{2.3}$$

for each n-simplex $s^n\in\boldsymbol{K}$, where if $s^n=\tilde{*}s^0$ then $*s^n$ is replaced by $\tilde{*}^{-1}s^n$, if $s^p=\tilde{*}s^q$ or $s^p\in\partial K$ then $*s^p$ is replaced by $(-1)^{pq}\tilde{*}^{-1}s^p$ or $\tilde{*}s^p$ respectively, and $v(s^p)$ is the number of 0-simplices $s^0\in\boldsymbol{K}-\partial K^*$ with $s^0\in s^p$. If $p=0$, then (2.3) is reduced to

$$\varphi^0\psi^n(s^n) = \varphi^0(*s^n)\psi^n(s^n)\,. \tag{2.4}$$

The *complex conjugate* $\overline{\varphi^p}$ of a p-difference φ^p $(0\leqq p\leqq n)$ is defined by $\overline{\varphi^p}(s^p)=\overline{\varphi^p(s^p)}$.

The *difference* of a p-difference φ^p $(0\leqq p\leqq n-1)$ is defined as a $(p+1)$-difference $\Delta\varphi^p$ given by

$$\Delta\varphi^p(s^{p+1}) = \sum_{s^p\subset\partial s^{p+1}}\varphi^p(s^p)$$

for each $(p+1)$-simplex $s^{p+1}\in\boldsymbol{K}$. If $\Delta\varphi^p=0$, then φ^p is said to be *closed.* We assume that every n-difference is closed. If for a p-difference φ^p $(1\leqq p\leqq n)$ there exists a $(p-1)$-difference ψ^{p-1} such that $\varphi^p=\Delta\psi^{p-1}$, then φ^p is said to be *exact.* We assume that no 0-difference is exact. We have

$$\Delta\Delta\varphi^p = \Delta(\Delta\varphi^p) = 0$$

for each p-difference φ^p $(0 \leqq p \leqq n-2)$. Hence, if a p-difference φ^p is exact, then it is closed.

We agree that every p-difference φ^p $(1 \leqq p \leqq n-1)$ on a compact bordered complex polyhedron $\boldsymbol{K} = \langle K, K^* \rangle$ satisfies the conditions

$$\Delta\varphi^p(\tilde{*}s^{q-1}) = 0 \tag{2.5}$$

and

$$\Delta\varphi^p(*s^{q-1}) = 0 \tag{2.6}$$

for each $(q-1)$-simplex $s^{q-1} \in \partial K$. These assumptions do not mean any essential restriction.

2. Summation of differences. We define the *sum* of a p-difference over a p-chain $(0 \leqq p \leqq n)$. Let $c^p = \sum_i a_i s_i^p$ be a p-chain of a complex polyhedron $\boldsymbol{K}$. The sum of a p-difference φ^p over the p-chain c^p is defined by

$$\mathscr{S}_{c^p} \varphi^p = \sum_i a_i \varphi^p(s_i^p).$$

The basic duality between a chain and a difference

$$\mathscr{S}_{c^p} \Delta\varphi^{p-1} = \mathscr{S}_{\partial c^p} \varphi^{p-1} \qquad (p = 1, \dots, n) \tag{2.7}$$

is obvious, where c^p is a p-chain and φ^{p-1} is a $(p-1)$-difference.

The following two criteria are also obvious:

A p-difference φ^p $(0 \leqq p \leqq n-1)$ is closed if and only if $\mathscr{S}_{c^p} \varphi^p = 0$ for every cycle c^p that is homologous to 0;

A p-difference φ^p $(0 \leqq p \leqq n)$ is exact if and only if $\mathscr{S}_{c^p} \varphi^p = 0$ for every cycle c^p.

If a p-difference φ^p $(0 \leqq p \leqq n-1)$ is closed, then the *period of* φ^p *along a* p-*cycle* c^p is defined by $\mathscr{S}_{c^p} \varphi^p$, which depends only on the homology class of c^p. From the basic duality (2.7), de Rham's duality theorem between the homology group of p-chains and the cohomology group of p-differences is derived.

Now we shall define the *sum* of an n-difference over a complex polyhedron $\boldsymbol{K} = \langle K, K^* \rangle$. First, let us assume that $\boldsymbol{K}$ is compact bordered or closed. When by the common notation $\boldsymbol{K}$ we denote the n-chain defined as a sum of oriented n-simplices contained in $\boldsymbol{K}$, the sum of an n-difference φ^n over $\boldsymbol{K}$

$$\mathscr{S}_{\boldsymbol{K}} \varphi^n$$

is defined as the sum of φ^n over the n-chain $\boldsymbol{K}$. If $\boldsymbol{K}$ is open, then we can set

$$\int_{\boldsymbol{K}} \varphi^n = \lim_{c^n \to \boldsymbol{K}} \int_{c^n} \varphi^n \tag{2.8}$$

provided that the limit exists, where c^n is an n-chain of $\boldsymbol{K}$ which approximates $\boldsymbol{K}$.

3. Conjugate differences. Let φ^p $(0 \leqq p \leqq n)$ be a p-difference on a complex polyhedron $\boldsymbol{K}$. Then the *conjugate difference* $*\varphi^p$ of φ^p is defined as a q-difference satisfying the condition

$$*\varphi^p(*s^p) = \varphi^p(s^p) \qquad (0 \leqq p \leqq n)$$

for each p-simplex $s^p \in K \cup \{*s^q | \, s^q \in K\}$. Then by (i) of Lemma 1.1 we can see that

$$**\varphi^p = (-1)^{pq}\varphi^p \tag{2.9}$$

and

$$\varphi^p \psi^q = (-1)^{pq} *\psi^q * \varphi^p. \tag{2.10}$$

By (2.9), the inverse operator $*^{-1}$ of the operator $*$ for a p-difference is given by

$$*^{-1} = (-1)^{pq} *. \tag{2.11}$$

A p-difference φ^p $(0 \leqq p \leqq n)$ is said to be *harmonic* if φ^p and $*\varphi^p$ are both closed. By (2.9) and the definition, φ^p and $*\varphi^p$ are simultaneously harmonic.

We introduce the operator

$$\delta = (-1)^p *^{-1} \varDelta * \tag{2.12}$$

for a p-difference. By (2.12) and (2.11), the operator δ for a p-difference has the expression

$$\delta = (-1)^{n(p+1)+1} * \varDelta *. \tag{2.13}$$

By (ii) of Lemma 1.1 it follows that

$$\begin{aligned} \delta\varphi^p(s^{p-1}) &= (-1)^p *^{-1} \varDelta * \varphi^p(s^{p-1}) = \varDelta * \varphi^p((-1)^p * s^{p-1}) \\ &= \sum_{*s^p \subset \partial(-1)^p * s^{p-1}} *\varphi^p(*s^p) = \sum_{s^{p-1} \subset \partial s^p} \varphi^p(s^p). \end{aligned}$$

Hence we see that the operator δ has the simple meaning

$$\delta\varphi^p(s^{p-1}) = \sum_{s^{p-1} \subset \partial s^p} \varphi^p(s^p) \tag{2.14}$$

for each $(p-1)$-simplex s^{p-1} in $\boldsymbol{K}$. By the definition of the operator δ, a p-difference φ^p $(1 \leqq p \leqq n-1)$ is harmonic if and only if

$$\Delta\varphi^p = \delta\varphi^p = 0.$$

§3. The Hilbert space of differences

1. The inner product. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be an open, closed or compact bordered complex polyhedron. Let φ^p and ψ^p $(0\leqq p\leqq n)$ be two p-differences on $\boldsymbol{K}$. We define the *inner product* $(\varphi^p, \psi^p) = (\varphi^p, \psi^p)_{\boldsymbol{K}}$ of φ^p and ψ^p by

$$(\varphi^p, \psi^p)_{\boldsymbol{K}} = \mathfrak{S}_{\boldsymbol{K}}\varphi^p * \overline{\psi^p} \qquad (0 \leqq p \leqq n).$$

Then we can easily verify that

$$\begin{aligned}(\varphi^p, \psi^p)_{\boldsymbol{K}} = &\sum_{s^p\in K-\partial K} \varphi^p(s^p)\overline{\psi^p}(s^p) + \frac{1}{2}\sum_{s^p\in\partial K} \varphi^p(s^p)\overline{\psi^p}(s^p) \\ &+ \sum_{s^q\in K-\partial K} \varphi^p(*s^q)\overline{\psi^p}(*s^q) + \frac{1}{2}\sum_{s^q\in\partial K} \varphi^p(*s^q)\overline{\psi^p}(*s^q),\end{aligned} \tag{3.1}$$

where we agree that the sum with respect to an empty set vanishes. By (3.1) we find that the relations

$$(*\varphi^p, *\psi^p) = (\varphi^p, \psi^p)$$

and

$$(\varphi^p, \psi^p) = (\overline{\psi^p, \varphi^p})$$

hold.

The *norm* $\|\varphi^p\| = \|\varphi^p\|_{\boldsymbol{K}}$ of a p-difference φ^p is defined by

$$\|\varphi^p\|_{\boldsymbol{K}} = (\varphi^p, \varphi^p)_{\boldsymbol{K}}^{1/2} \qquad (0 \leqq p \leqq n).$$

By $\Gamma=\Gamma(\boldsymbol{K})$ we denote the Hilbert space of all p-differences φ^p on $\boldsymbol{K}$ with a finite norm $\|\varphi^p\| < +\infty$ for a fixed p $(0\leqq p\leqq n)$. Furthermore, we define the closed subspaces of Γ as follows:

$$\Gamma_c = \{\varphi^p \mid \varphi^p \text{ is closed}, \quad \varphi^p\in\Gamma\},$$

$$\Gamma_e = \{\varphi^p \mid \varphi^p \text{ is exact}, \quad \varphi^p\in\Gamma\},$$

$$\Gamma_h = \{\varphi^p \mid \varphi^p \text{ is harmonic}, \quad \varphi^p\in\Gamma\},$$

$$\Gamma_c^* = \{\varphi^p \mid *\varphi^p \text{ is closed}, \quad \varphi^p\in\Gamma\},$$

$$\Gamma_e^* = \{\varphi^p \mid *\varphi^p \text{ is exact}, \quad \varphi^p\in\Gamma\},$$

$$\Gamma_h^* = \{\varphi^p \mid *\varphi^p \text{ is harmonic}, \quad \varphi^p\in\Gamma\}.$$

Then it is obvious that $\Gamma_e \subset \Gamma_c$, $\Gamma_h = \Gamma_c \cap \Gamma_c^*$ and $\Gamma_h^* = \Gamma_h$.

2. Fundamental theorem.

THEOREM 3.1. *If a complex polyhedron* **K** *is compact bordered or closed, then the summation formula*

$$(3.2) \qquad (\Delta\varphi^{p-1}, \psi^p)_{\boldsymbol{K}} = \int_{\partial\boldsymbol{K}} \varphi^{p-1} * \overline{\psi^p} + (\varphi^{p-1}, \delta\psi^p)_{\boldsymbol{K}} \qquad (1 \leqq p \leqq n)$$

holds, where if **K** *is closed then the first term of the right hand side vanishes.*

PROOF. The case of $2 \leqq p \leqq n-1$: By (3.1) we have

$$(3.3) \quad \begin{aligned} (\Delta\varphi, \psi)_{\boldsymbol{K}} = {} & \sum_{s^p \in K - \partial K} \Delta\varphi(s^p)\overline{\psi}(s^p) + \frac{1}{2} \sum_{s^p \in \partial K} \Delta\varphi(s^p)\overline{\psi}(s^p) \\ & + \sum_{s^q \in K - \partial K} \Delta\varphi(*s^q)\overline{\psi}(*s^q) + \frac{1}{2} \sum_{s^q \in \partial K} \Delta\varphi(*s^q)\overline{\psi}(*s^q), \end{aligned}$$

where $\varphi = \varphi^{p-1}$ and $\psi = \psi^p$.1) By (2.6) the last term of the right hand side of (3.3) vanishes and further by (ii) of Lemma 1.1, (2.5), (2.14) and Lemma 1.2 we see that

$$\begin{aligned} & \sum_{s^p \in K - \partial K} \Delta\varphi(s^p)\overline{\psi}(s^p) + \frac{1}{2} \sum_{s^p \in \partial K} \Delta\varphi(s^p)\overline{\psi}(s^p) \\ = {} & \sum_{s^{p-1} \in K - \partial K} \varphi(s^{p-1}) \sum_{s^{p-1} \subset \partial s^p} \overline{\psi}(s^p) \\ & \quad + \sum_{s^{p-1} \in \partial K} \varphi(s^{p-1}) \sum_{s^{p-1} \subset \partial s^p, s^p \in K - \partial K} \overline{\psi}(s^p) \\ & \qquad + \sum_{s^{p-1} \in \partial K} \varphi(s^{p-1}) \sum_{s^{p-1} \subset \partial s^p, s^p \in \partial K} \frac{1}{2} \overline{\psi}(s^p) \\ = {} & \sum_{s^{p-1} \in K - \partial K} \varphi(s^{p-1}) \delta\overline{\psi}(s^{p-1}) \\ & \quad + \sum_{s^{p-1} \in \partial K} \varphi(s^{p-1}) \Bigg\{ \sum_{*s^p \subset \partial(-1)^p * s^{p-1}, s^p \in K - \partial K} *\overline{\psi}(*s^p) \\ & \qquad + \sum_{*s^p \subset \partial(-1)^p * s^{p-1}, s^p \in \partial K} \frac{1}{2} *\overline{\psi}(*s^p) \Bigg\} \\ = {} & \sum_{s^{p-1} \in K - \partial K} \varphi(s^{p-1}) \delta\overline{\psi}(s^{p-1}) + \sum_{s^{p-1} \in \partial K} \varphi(s^{p-1}) * \overline{\psi}(*s^{p-1}(\partial\boldsymbol{K})). \end{aligned}$$

Similarly, we see that

---

1) Throughout the present paper, the upperscripts of φ^{p-1} and ψ^p are omitted in the obvious case.

$$
\begin{aligned}
(\varphi, \delta\psi)_{\boldsymbol{K}} = & \sum_{s^{p-1}\in K-\partial K} \varphi(s^{p-1})\delta\bar{\psi}(s^{p-1}) + \frac{1}{2} \sum_{s^{p-1}\in \partial K} \varphi(s^{p-1})\delta\bar{\psi}(s^{p-1}) \\
& + \sum_{s^{q+1}\in K-\partial K} \varphi(*s^{q+1})\delta\bar{\psi}(*s^{q+1}) + \frac{1}{2} \sum_{s^{q+1}\in \partial K} \varphi(*s^{q+1})\delta\bar{\psi}(*s^{q+1}) \\
= & \sum_{s^{p-1}\in K-\partial K} \varphi(s^{p-1})\delta\bar{\psi}(s^{p-1}) + \sum_{s^{q}\in K-\partial K} \Delta\varphi(*s^{q})\bar{\psi}(*s^{q}) \\
& - (-1)^{(p-1)q} \sum_{s^{q}\in \partial K} \varphi(*s^{q}(\partial\boldsymbol{K}))*\bar{\psi}(s^{q}).
\end{aligned}
$$

Hence we find that

$$
\begin{aligned}
& (\Delta\varphi, \psi)_{\boldsymbol{K}} - (\varphi, \delta\psi)_{\boldsymbol{K}} \\
& \quad = \sum_{s^{p-1}\in \partial \boldsymbol{K}} \varphi(s^{p-1})*\bar{\psi}(*s^{p-1}(\partial\boldsymbol{K})) = \int_{\partial\boldsymbol{K}} \varphi*\bar{\psi}.
\end{aligned}
$$

The case of $p=1$: By a method similar to that in the case of $2\leqq p\leqq n-1$, we can derive the relations

$$
\begin{aligned}
(\Delta\varphi, \psi)_{\boldsymbol{K}} = & \sum_{s^{0}\in K-\partial K} \varphi(s^{0})\delta\bar{\psi}(s^{0}) + \sum_{s^{0}\in \partial K} \varphi(s^{0})*\bar{\psi}(*s^{0}(\partial\boldsymbol{K})) \\
& + \sum_{s^{n-1}\in K-\partial K} \Delta\varphi(*s^{n-1})\bar{\psi}(*s^{n-1}) + \frac{1}{2} \sum_{s^{n-1}\in \partial K} \Delta\varphi(*s^{n-1})\bar{\psi}(*s^{n-1})
\end{aligned}
$$

and

$$
\begin{aligned}
(\varphi, \delta\psi)_{\boldsymbol{K}} = & \sum_{s^{0}\in K-\partial K} \varphi(s^{0})\delta\bar{\psi}(s^{0}) + \sum_{s^{n}\in K} \varphi(*s^{n})\delta\bar{\psi}(*s^{n}) \\
= & \sum_{s^{0}\in K-\partial K} \varphi(s^{0})\delta\bar{\psi}(s^{0}) + \sum_{s^{n-1}\in K-\partial K} \Delta\varphi(*s^{n-1})\bar{\psi}(*s^{n-1}) \\
& + (-1)^{n} \sum_{s^{n-1}\in \partial K} \bar{\psi}(*s^{n-1}) \sum_{*s^{n}\in \partial(-1)^{n}*s^{n-1},\, s^{n}\in K} \varphi(*s^{n}).
\end{aligned}
$$

Hence, by (ii) of Lemma 1.1 and (2.1) we can see that

$$
\begin{aligned}
& (\Delta\varphi, \psi)_{\boldsymbol{K}} - (\varphi, \delta\psi)_{\boldsymbol{K}} \\
& = \sum_{s^{0}\in \partial K} \varphi(s^{0})*\bar{\psi}(*s^{0}(\partial\boldsymbol{K})) \\
& \quad - \sum_{s^{n-1}\in \partial K} *\bar{\psi}(s^{n-1}) \left\{\frac{1}{2} \Delta\varphi((-1)^{n}*s^{n-1}) - \varphi(*s^{n})\right\} \\
& = \sum_{s^{0}\in \partial K} \varphi(s^{0})*\bar{\psi}(*s^{0}(\partial\boldsymbol{K})) + \sum_{s^{n-1}\in \partial K} \varphi(*s^{n-1}(\partial\boldsymbol{K}))*\bar{\psi}(s^{n-1}) \\
& = \int_{\partial\boldsymbol{K}} \varphi*\bar{\psi},
\end{aligned}
$$

where $s^{n-1}\subset\partial s^n\in K$.

The case of $p=n$ can be easily reduced to the case of $p=1$ when we put $\varphi^0=*\psi^n$ and $\psi^1=*\varphi^{n-1}$.

3. Orthogonal projection of a compact polyhedron. In **3** and **4**, we shall briefly state the method of orthogonal projection of the Hilbert space of differences which is analogous to de Rham-Kodaira's orthogonal decomposition theorem for differential forms on a Riemannian manifold.

THEOREM 3.2. *Let* $\boldsymbol{K}$ *be a closed complex polyhedron and let* Γ *be the Hilbert space of p-differences* $(0\leqq p\leqq n)$ *on* $\boldsymbol{K}$. *Then the orthogonal decompositions*:

$$\Gamma=\Gamma_c\dotplus\Gamma_e^*=\Gamma_c^*\dotplus\Gamma_e, \tag{3.4}$$

$$\Gamma=\Gamma_h\dotplus\Gamma_e\dotplus\Gamma_e^*, \tag{3.5}$$

$$\Gamma_c=\Gamma_h\dotplus\Gamma_e \quad \textit{and} \quad \Gamma_c^*=\Gamma_h\dotplus\Gamma_e^* \tag{3.6}$$

hold.

PROOF. By Theorem 3.1 we see that

$$(\psi^p, *\Delta\varphi^{q-1})=(-1)^q(\Delta\psi^p, *\varphi^{q-1}) \qquad (0\leqq p\leqq n-1).$$

Hence $\Delta\psi^p=0$ implies that $(\psi^p, *\Delta\varphi^{q-1})=0$ and thus ψ^p is orthogonal to every element of Γ_e^*.

Conversely, if

$$(\Delta\psi^p, *\varphi^{q-1})=0$$

holds for every $(q-1)$-difference φ^{q-1} on $\boldsymbol{K}$, then we can easily verify that $\Delta\psi^p=0$ on $\boldsymbol{K}$. Hence on a closed complex polyhedron $\boldsymbol{K}$, Γ_c is the orthogonal complement of Γ_e^*. Then by the general theory we have the decomposition $\Gamma=\Gamma_c\dotplus\Gamma_e^*$. In the case of $p=n$, by the definition in §**2.1** we find that $\Gamma_c=\Gamma$ and $\Gamma_e^*=\emptyset$ and thus we have $\Gamma=\Gamma_c\dotplus\Gamma_e^*$.

The decomposition $\Gamma=\Gamma_c^*\dotplus\Gamma_e$ immediately follows from the decomposition $\Gamma=\Gamma_c\dotplus\Gamma_e^*$ of the space Γ of q-differences. The decomposition (3.4) implies (3.5) and (3.6).

Let $\boldsymbol{L}=\langle L, L^*\rangle$ be a compact bordered complex polyhedron. A p-difference φ^p $(0\leqq p\leqq n-1)$ on $\boldsymbol{L}$ is said to *vanish on the boundary* $\partial\boldsymbol{L}=\langle\partial L, \partial L^*\rangle$ if $\varphi^p(s^p)=0$ for every p-simplex s^p of $\partial\boldsymbol{L}$. A closed p-difference φ^p $(0\leqq p\leqq n-1)$ is said to belong to the subspace Γ_{c0} of Γ_c if φ^p vanishes on $\partial\boldsymbol{L}$. Similarly, an exact p-difference $\varphi^p=\Delta\psi^{p-1}$ $(1\leqq p\leqq n)$ is said to belong to the subspace Γ_{e0} of

Γ_e if ψ^{p-1} vanishes on $\partial \boldsymbol{L}$. In the case of $p=n$, we interpret Γ_{c0} as $\Gamma_{c0}=\Gamma_c=\Gamma$. In the case of $p=0$, we interpret Γ_{e0} as $\Gamma_{e0}=\Gamma_e=\emptyset$. The subspaces Γ^*_{c0} and Γ^*_{e0} are defined by $\Gamma^*_{c0}=\{\varphi^p|\ *\varphi^p \in \Gamma_{c0}\}$ and $\Gamma^*_{e0}=\{\varphi^p|\ *\varphi^p \in \Gamma_{e0}\}$, where Γ_{c0} and Γ_{e0} are the ones of q-differences.

By Theorem 3.1 we obtain the formula

$$(3.7)\qquad (\psi^p, *\Delta\varphi^{q-1}) = \int_{\partial \boldsymbol{L}} \overline{\varphi^{q-1}}\psi^p + (-1)^q(\Delta\psi^p, *\varphi^{q-1}) \qquad (0 \leqq p \leqq n-1).$$

By making use of (3.7) and an argument similar to that in the proof of Theorem 3.2 we obtain the following theorem.

Theorem 3.3. *Let* $\boldsymbol{L}$ *be a compact bordered complex polyhedron and let* Γ *be the Hilbert space of p-differences* $(0 \leqq p \leqq n)$ *on* $\boldsymbol{L}$. *Then the orthogonal decompositions*:

$$\Gamma = \Gamma_{c0} \dotplus \Gamma^*_e = \Gamma^*_{c0} \dotplus \Gamma_e,$$

$$\Gamma = \Gamma_c \dotplus \Gamma^*_{e0} = \Gamma^*_c \dotplus \Gamma_{e0},$$

$$\Gamma = \Gamma_h \dotplus \Gamma_{e0} \dotplus \Gamma^*_{e0},$$

$$\Gamma_c = \Gamma_h \dotplus \Gamma_{e0},$$

$$\Gamma_{c0} = \Gamma_{h0} \dotplus \Gamma_{e0},$$

$$\Gamma_e = \Gamma_{he} \dotplus \Gamma_{e0},$$

$$\Gamma_h = \Gamma_{he} \dotplus \Gamma^*_{h0} = \Gamma_{h0} \dotplus \Gamma^*_{he}$$

hold, where $\Gamma_{he}=\Gamma_h \cap \Gamma_e$, $\Gamma_{h0}=\Gamma_h \cap \Gamma_{c0}$ *and* $\Gamma^*_{h0}=\Gamma_h \cap \Gamma^*_{c0}$.

4. Orthogonal projection on an open polyhedron. Let us suppose that $\boldsymbol{K}$ is an open or closed complex polyhedron. A p-difference φ^p $(0 \leqq p \leqq n)$ on $\boldsymbol{K}$ is said to have *compact support* if $\varphi^p(s^p)=0$ for every p-simplex $s^p \in \boldsymbol{K}$ except for a finite number of p-simplices of $\boldsymbol{K}$.

For $1 \leqq p \leqq n$, let Γ'_{e0} be the subclass of Γ_e which is defined as a collection of the p-difference φ^p such that $\varphi^p=\Delta\psi^{p-1}$ for some $(p-1)$-difference ψ^{p-1} with compact support. We define the subspace Γ_{e0} of Γ as the closure of Γ'_{e0} in Γ $(1 \leqq p \leqq n)$. In the case of $p=0$, we interpret Γ_{e0} as $\Gamma_e=\emptyset$. The subspace Γ^*_{e0} is defined by $\Gamma^*_{e0}=\{\varphi^p|\ *\varphi^p \in \Gamma_{e0}\}$, where Γ_{e0} consists of q-differences. From the definition it follows that $\Gamma_{e0}=\Gamma_e$ and $\Gamma^*_{e0}=\Gamma^*_e$ $(0 \leqq p \leqq n)$ for a closed complex polyhedron $\boldsymbol{K}$. For $0 \leqq p \leqq n-1$, let Γ'_{c0} be the subclass of Γ_c which is defined as a collection of the closed p-difference φ^p with compact support. In the case of $p=n$, we interpret Γ'_{c0} as $\Gamma_c=\Gamma$. The subclass Γ'^*_{c0} is defined by $\Gamma'^*_{c0}=\{\varphi^p|\ *\varphi^p \in \Gamma'_{c0}\}$, where Γ'_{c0} consists of p-differences. The subspaces Γ_{c0} and Γ^*_{c0}

are defined as the orthogonal complements of Γ_e^* and Γ_e respectively:

$$\Gamma = \Gamma_{c0} + \Gamma_e^* \quad \text{and} \quad \Gamma = \Gamma_{c0}^* + \Gamma_e.$$

Then we have $\Gamma'_{c0} \subset \Gamma_{c0} \subset \Gamma_c$ and $\Gamma'^*_{c0} \subset \Gamma^*_{c0} \subset \Gamma^*_c$.

By making use of (3.7) we can prove the following theorem.

THEOREM 3.4. *Let* $\boldsymbol{K}$ *be a closed or open complex polyhedron and let* Γ *be the Hilbert space of p-differences* $(0 \leqq p \leqq n)$ *on* $\boldsymbol{K}$. *Then the orthogonal decompositions of the same type as that in Theorem* 3.3 *hold.*

Chapter II. Difference forms and differential forms

§4. The convergence of differences with respect to subdivisions

1. Natural extension of difference. Let $\boldsymbol{K} = \langle K, K^* \rangle$ be a cubic complex polyhedron and $\boldsymbol{K}_1 = \langle K_1, K_1^* \rangle$ be a subdivision of $\boldsymbol{K}$. Let φ^p $(0 \leqq p \leqq n)$ be a closed p-difference on $\boldsymbol{K}$. Then we shall define the *natural extension* $\natural\varphi^p$ of φ^p to the subdivision $\boldsymbol{K}_1$ as a p-difference on $\boldsymbol{K}_1$ as follows.

We shall use the notation in **§1.5**. First, we assume that the difference φ^p vanishes on K^*. Then the natural extension φ^p is a p-difference on $\boldsymbol{K}_1$ which is defined by

$$\begin{aligned}(4.1) \qquad & \natural\varphi^p(\natural s^p_{I_r I_{p-r} L_s M_t}) \\ & = \frac{1}{2^{p+t}} \sum_{\nu=0}^{t} \sum_{N_\nu \subset M_t} \varphi^p(s^p_{I_p(L_s \cup N_\nu)}) \qquad (I_p = \bar{I}_r + \tilde{I}_{p-r})\end{aligned}$$

for each p-simplex $s^p \in K_1$ and which vanishes on K_1^*.

Secondly, we assume that the difference φ^p vanishes on K. If $\boldsymbol{K}$ is closed or open, then the natural extension $\natural\varphi^p$ is a p-difference on $\boldsymbol{K}_1$ which is defined by

$$\begin{aligned}(4.2) \qquad & \natural\varphi^p(\natural s^p_{I_r \tilde{I}_t \hat{I}_{p-r-t} L_s}) \\ & = \frac{1}{2^{p+r+t}} \sum_{\mu=0}^{r} \sum_{\nu=0}^{t} \sum_{K_\mu \subset I_r} \sum_{\tilde{K}_\nu \subset \tilde{I}_t} \\ & \qquad \sum_{u=0}^{q} \frac{3^{q-u}}{4^q} \sum_{i=\max(0,u-q+s)}^{\min(u,s)} \sum_{M_{s-i} \subset L_s} \sum_{N_{u-i} \subset J_q - L_s} \\ & \qquad\qquad \varphi^p(s^p_{K_\mu \tilde{K}_\nu \hat{K}_{p-\mu-\nu}(M_{s-i} \cup N_{u-i})}) \\ & \qquad\qquad (\bar{I}_r + \tilde{I}_t + \hat{I}_{p-r-t} = \bar{K}_\mu + \tilde{K}_\nu + \hat{K}_{p-\mu-\nu})\end{aligned}$$

for each p-simplex $s^p \in K_1^*$ and which vanishes on K_1. If $\boldsymbol{K}$ is compact bordered, then we may assume that the difference φ^p is defined on K^{*b} (cf. §**1.4** for the notation). Then the natural extension $\natural\varphi^p$ is a p-difference on K_1^{*b} which is defined by (4.2) for each p-simplex $s^p \in K_1^{*b}$.

For a generic φ^p on $\boldsymbol{K}$, the natural extension $\natural\varphi^p$ is defined as the sum $\natural\varphi^p = \natural\varphi^p_K + \natural\varphi^p_{K^*}$, where $\natural\varphi^p_K$ and $\natural\varphi^p_{K^*}$ are the natural extensions of the restrictions of φ^p to K and K^* respectively.

In what follows we write simply $\varphi_{I_p L_r}$ and $\varphi_{\bar{I}_r \tilde{I}_t \hat{I}_{p-r-t} L_s}$ for $\varphi^p(s^p_{I_p L_r})$ and $\varphi^p(s^p_{\bar{I}_r \tilde{I}_t \hat{I}_{p-r-t} L_s})$ respectively.

LEMMA 4.1. *$\natural\varphi^p$ is closed on $\boldsymbol{K}_1$.*

PROOF. If $s^{p+1} \in K_1$, then we may assume that $s^{p+1} = \natural s^{p+1}_{I_{p+1}\bar{I}_0 L_0 M_t}$ by a choice of a suitable normal coordinates. When we set $I_{p+1} = \bar{I}_{p+1} = N_p + L_1$ and

$$\operatorname{sgn}(L_1;\, N_p) = \operatorname{sgn}\begin{pmatrix} i_1 \cdots i_{p+1} \\ l_1\, n_1 \cdots n_p \end{pmatrix},$$

we obtain that

$$\begin{aligned}
&\varDelta\natural\varphi^p(\natural s^{p+1}_{I_{p+1}\bar{I}_0 L_0 M_t}) \\
&= \sum_{\bar{N}_p + L_1 = \bar{I}_{p+1}} \operatorname{sgn}(L_1;\, N_p)\, \{\natural\varphi^p(\natural s^p_{\bar{N}_p \bar{I}_0 L_0 (M_t \cup L_1)}) - \natural\varphi^p(\natural s^p_{\bar{N}_p \bar{I}_0 L_0 M_t})\} \\
&= \frac{1}{2^{p+t+1}} \sum_{N_p + L_1 = I_{p+1}} \operatorname{sgn}(L_1;\, N_p) \Bigl(\sum_{\nu=0}^{t+1} \sum_{K_\nu \subset M_t \cup L_1} \varphi_{N_p K_\nu} - 2 \sum_{\nu=0}^{t} \sum_{K_\nu \subset M_t} \varphi_{N_p K_\nu} \Bigr) \\
&= \frac{1}{2^{p+t+1}} \sum_{\nu=0}^{t} \sum_{K_\nu \subset M_t} \sum_{N_p + L_1 = I_{p+1}} \operatorname{sgn}(L_1; N_p)\, (\varphi_{N_p (K_\nu \cup L_1)} - \varphi_{N_p K_\nu}) \\
&= \frac{1}{2^{p+t+1}} \sum_{\nu=0}^{t} \sum_{K_\nu \subset M_t} \varDelta\varphi(s^{p+1}_{I_{p+1} K_\nu}) = 0.
\end{aligned}$$

If $s^{p+1} \in K_1^*$, then we may assume that $s^{p+1} = \natural s^{p+1}_{\bar{I}_r \tilde{I}_0 \hat{I}_{p+1-r} L_0}$ by a choice of a suitable normal coordinates. When we set $I_{p+1} = \bar{I}_r + \tilde{I}_0 + \hat{I}_{p+1-r} = M_p + L_1$ and $M_p = \bar{K}_\mu + \tilde{K}_0 + \hat{K}_{p-\mu}$, and we introduce the following new notation:

$$\begin{aligned}
(4.3) \qquad & e^p_{\bar{K}_\mu \tilde{K}_0 \hat{K}_{p-\mu} N_s L_1} \\
&= \{-1 \leqq x_i \leqq 0 \ (i \in \bar{K}_\mu),\ 0 \leqq x_i \leqq 1 \ (i \in \hat{K}_{p-\mu}), \\
&\qquad x_i = 1 \ (i \in N_s),\ x_i = -1 \ (i \in L_1),\ x_i = 0 \ (i \in J_{p-1} - N_s)\}, \\
& \natural e^p_{\bar{K}_\mu \tilde{K}_0 \hat{K}_{p-\mu} N_s L_1} \\
&= \Bigl\{ -\frac{1}{4} \leqq x_i \leqq \frac{1}{4} \ (i \in \bar{K}_\mu),\ \frac{1}{4} \leqq x_i \leqq \frac{3}{4} \ (i \in \hat{K}_{p-\mu}),
\end{aligned}$$

$$x_i = \frac{3}{4}\ (i \in N_s),\ \ x_i = -\frac{1}{4}\ (i \in L_1),\ \ x_i = \frac{1}{4}\ (i \in J_{q-1} - N_s)\Big\},$$

$$s^p_{K_\mu \breve{K}_0 \hat{K}_{p-\mu} N_s L_1} = [e^p_{K_\mu \breve{K}_0 \hat{K}_{p-\mu} N_s L_1},\ \phi]$$

and

$$\natural s^p_{K_\mu \breve{K}_0 \hat{K}_{p-\mu} N_s L_1} = [\natural e^p_{K_\mu \breve{K}_0 \hat{K}_{p-\mu} N_s L_1},\ \phi],$$

we obtain that

$$\Delta \natural \varphi^p(\natural s^{p+1}_{I_r \breve{I}_0 \hat{I}_{p+1-r} L_0})$$

$$= \sum_{\bar{M}_{r-1}+L_1=I_r} \operatorname{sgn}(L_1;\, M_p)\ \{\natural\varphi^p(\natural s^p_{\bar{M}_{r-1}\breve{I}_0\hat{I}_{p+1-r}L_0}) - \natural\varphi^p(\natural s^p_{\bar{M}_{r-1}\breve{I}_0\hat{I}_{p+1-r}N_0 L_1})\}$$

$$+ \sum_{\hat{M}_{p-r}+L_1=\hat{I}_{p+1-r}} \operatorname{sgn}(L_1;\, M_p)\ \{\natural\varphi^p(\natural s^p_{I_r\breve{I}_0\hat{M}_{p-r}L_1}) - \natural\varphi^p(\natural s^p_{I_r\breve{I}_0\hat{M}_{p-r}L_0})\}$$

$$= \frac{1}{2^{p+r-1}} \Big\{ \sum_{\bar{M}_{r-1}+L_1=I_r} \operatorname{sgn}(L_1;\, M_p) \sum_{\mu=0}^{r-1} \sum_{K_\mu \subset \bar{M}_{r-1}} \sum_{u=0}^{q} \frac{3^{q-u}}{4^q}\,\cdot$$

$$\cdot\Big(\sum_{N_u \subset J_{q-1} \cup L_1} \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_u} - \sum_{N_u \subset J_{q-1}} \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_u}$$

$$- \sum_{N_{u-1} \subset J_{q-1}} \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_{u-1} L_1} \Big)$$

$$+ \frac{1}{2} \sum_{\hat{M}_{p-r}+L_1=\hat{I}_{p+1-r}} \operatorname{sgn}(L_1;\, M_p) \sum_{\mu=0}^{r} \sum_{K_\mu \subset I_r} \sum_{u=0}^{q} \frac{3^{q-u}}{4^q}\,\cdot$$

$$\cdot\Big(\sum_{N_u \subset J_{q-1}} \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu}(N_u \cup L_1)} + \sum_{N_{u-1} \subset J_{q-1}} \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_{u-1}}$$

$$- \sum_{N_u \subset J_{q-1} \cup L_1} \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_u} \Big) \Big\}$$

$$= \frac{3^q}{2^{p+2q+r-1}} \sum_{u=1}^{q} \frac{1}{3^u} \Big[\sum_{\bar{M}_{r-1}+L_1=I_r} \operatorname{sgn}(L_1;\, M_p) \sum_{\mu=0}^{r-1} \sum_{K_\mu \subset \bar{M}_{r-1}} \sum_{N_{u-1} \subset J_{q-1}}$$

$$(\varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu}(N_{u-1} \cup L_1)} - \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_{u-1} L_1})$$

$$+ \frac{1}{2} \sum_{\hat{M}_{p-r}+L_1=\hat{I}_{p+1-r}} \operatorname{sgn}(L_1;\, M_p) \sum_{\mu=0}^{r} \sum_{K_\mu \subset I_r} \Big\{ \sum_{N_u \subset J_{q-1}} (\varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu}(N_u \cup L_1)}$$

$$- \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_u}) + \sum_{N_{u-1} \subset J_{q-1}} (\varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_{u-1}} - \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu}(N_{u-1} \cup L_1)}) \Big\} \Big]$$

$$= \frac{3^q}{2^{p+2q+r-1}} \sum_{u=0}^{q} \frac{1}{3^u} \sum_{N_{u-1} \subset J_{q-1}} \sum_{\mu=0}^{r} \sum_{K_\mu \subset I_r}$$

$$\Big\{ \sum_{\bar{K}_{\mu-1}+L_1=\bar{K}_\mu} \operatorname{sgn}(L_1;\, M_p)\ (\varphi_{\bar{K}_{\mu-1} \breve{K}_0 \hat{K}_{p+1-\mu} N_{u-1}} - \varphi_{\bar{K}_{\mu-1} \breve{K}_0 \hat{K}_{p+1-\mu} N_{u-1} L_1})$$

$$+ \sum_{\hat{K}_{p-\mu}+L_1=\hat{K}_{p+1-\mu}} \operatorname{sgn}(L_1;\, M_p)\ (\varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu}(N_{u-1} \cup L_1)} - \varphi_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p-\mu} N_{u-1}}) \Big\}$$

$$= \frac{3^q}{2^{p+2q+r-1}} \sum_{u=0}^{q} \frac{1}{3^u} \sum_{N_{u-1} \subset J_{q-1}} \sum_{\mu=0}^{r} \sum_{K_\mu \subset I_r} \Delta\varphi(s^{p+1}_{\bar{K}_\mu \breve{K}_0 \hat{K}_{p+1-\mu} N_{u-1}}) = 0.$$

Henceforth, we need to somewhat modify the definition of the inner product of p-differences. We define *length* of a 1-simplex s^1 of a cubic complex polyhedron $\boldsymbol{K}$ by giving a common positive number h to each $s^1 \in K \cup \{*s^{n-1} \mid s^{n-1} \in K\}$. If each $s^1 \in K \cup \{*s^{n-1} \mid s^{n-1} \in K\}$ has length h, then $\boldsymbol{K}$ is said to have *side length h*. We agree that if $\boldsymbol{K}$ has side length h, then the subdivision $\boldsymbol{K}_1$ of $\boldsymbol{K}$ has side length $h/2$. If $\boldsymbol{K}$ has side length h, then the inner product $(\varphi^p, \psi^p)_{\boldsymbol{K}}$ of φ^p and ψ^p is defined by

$$(\varphi^p, \psi^p)_{\boldsymbol{K}} = h^{n-2p}\left\{\sum_{s^p \in K-\partial K} \varphi^p(s^p)\overline{\psi^p}(s^p) + \frac{1}{2}\sum_{s^p \in \partial K} \varphi^p(s^p)\overline{\psi^p}(s^p)\right.$$
$$\left. + \sum_{s^q \in K-\partial K} \varphi^p(*s^q)\overline{\psi^p}(*s^q) + \frac{1}{2}\sum_{s^q \in \partial K} \varphi^p(*s^q)\overline{\psi^p}(*s^q)\right\}$$

(compare with (3.1)), where the sum with respect to an empty set vanishes. We agree that each $\boldsymbol{K}$ has side length 1 except in the case where $\boldsymbol{K}$ is taken as a subdivision of some complex polyhedron.

LEMMA 4.2. *If $\boldsymbol{K}$ is closed or open, or $\boldsymbol{K}$ is compact bordered and φ^p vanishes on K^*, then the following inequality holds*:

$$\|\natural\varphi^p\|_{\boldsymbol{K}_1} \leqq \|\varphi^p\|_{\boldsymbol{K}}.$$

PROOF. First, we assume that the difference φ^p vanishes on K^*. For an arbitrary n-simplex $s^n \in K$, its contribution $\|\varphi\|^2_{s^n}$ to the square norm $\|\varphi\|^2_{\boldsymbol{K}}$ is equal to

$$\|\varphi\|^2_{s^n} \equiv \frac{1}{2^q}\sum_{I_p+J_q=N}\sum_{s=0}^{q}\sum_{K_s\subset J_q}|\varphi_{I_pK_s}|^2. \tag{4.4}$$

Let c^n be the n-chain of K_1 which is the sum of the subdivision of the n-simplex s^n. Then the n-chain c^n is a sum of 2^n n-simplex of K_1 with $|c^n| = |s^n|$. By the definition (4.1) we see that the contribution $\|\natural\varphi\|^2_{c^n}$ of c^n to the square norm $\|\natural\varphi\|^2_{\boldsymbol{K}_1}$ is equal to

$$\|\natural\varphi\|^2_{c^n} \equiv \left(\frac{1}{2}\right)^{n-2p}\frac{1}{2^q}\sum_{I_p+J_q=N}\sum_{t=0}^{q}2^t\sum_{s=0}^{q-t}\sum_{r=0}^{p}\sum_{K_s+M_t\subset J_q}\sum_{I_p=I_r+\bar{I}_{p-r}}$$
$$|\natural\varphi^p(\natural s^p_{I_r\bar{I}_{p-r}K_sM_t})|^2$$
$$= \frac{1}{2^{2q}}\sum_{I_p+J_q=N}\sum_{t=0}^{q}\frac{1}{2^t}\sum_{s=0}^{q-t}\sum_{K_s+M_t\subset J_q}\left|\sum_{\nu=0}^{t}\sum_{N_\nu\subset M_t}\varphi_{I_p(K_s\cup N_\nu)}\right|^2$$
$$= \frac{1}{2^{2q}}\sum_{I_p+J_q=N}\left\{\sum_{\tau=0}^{q}\binom{q}{\tau}\frac{1}{2^\tau}\sum_{s=0}^{q}\sum_{K_s\subset J_q}|\varphi_{I_pK_s}|^2\right.$$
$$+ \sum_{\nu=1}^{q}\sum_{\tau=0}^{q-\nu}\binom{q-\nu}{\tau}\frac{1}{2^{\nu+\tau}}\sum_{s=0}^{q-\nu}\sum_{J_q=K_s+L_{q-s}}\sum_{\mu=0}^{[\nu/2]}\sum_{M_\mu+N_{\nu-\mu}\subset L_{q-s}}$$
$$\left.(\varphi_{I_p(K_s\cup M_\mu)}\bar{\varphi}_{I_p(K_s\cup N_{\nu-\mu})} + \bar{\varphi}_{I_p(K_s\cup M_\mu)}\varphi_{I_p(K_s\cup N_{\nu-\mu})})\right\}.$$

Hence we find that

$$\|\varphi\|^2_{s^n}-\|\natural\varphi\|^2_{c^n}$$

$$=\frac{3^q}{2^{3q}}\sum_{I_p+J_q=N}\sum_{\nu=1}^{q}\frac{1}{3^\nu}\sum_{s=0}^{q-\nu}\sum_{J_q=K_s+L_{q-s}}\sum_{\mu=0}^{[\nu/2]}\sum_{M_\mu+N_{\nu-\mu}\subset L_{q-s}}$$

$$|\varphi_{I_p(K_s\cup M_\mu)}-\varphi_{I_p(K_s\cup N_{\nu-\mu})}|^2\geqq 0,$$

and thus

$$\|\varphi\|^2_{\boldsymbol{K}}-\|\natural\varphi\|^2_{\boldsymbol{K}_1}$$

$$=\frac{3^q}{2^{3q}}\sum_{s^n\in K}\sum_{I_p+J_q=N}\sum_{\nu=1}^{q}\frac{1}{3^\nu}\sum_{s=0}^{q-\nu}\sum_{J_q=K_s+L_{q-s}}\sum_{\mu=0}^{[\nu/2]}\sum_{M_\mu+N_{\nu-\mu}\subset L_{q-s}}$$

$$|\varphi_{I_p(K_s\cup M_\mu)}-\varphi_{I_p(K_s\cup N_{\nu-\mu})}|^2\geqq 0.$$

Secondly, we assume that the difference φ^p vanishes on K. For an arbitrary n-simplex $s^n\in K^*$, the contribution $\|\varphi\|^2_{s^n}$ of the n-simplex s^n to the square norm $\|\varphi\|^2_{\boldsymbol{K}}$ is written in the same form as (4.4). By the definition (4.2) we can see that the contribution $\|\natural\varphi\|^2_{\boldsymbol{K}_1\cap|s^n|}$ of the portion of K_1^* restricted to support $|s^n|$ to the square norm $\|\natural\varphi\|^2_{\boldsymbol{K}_1}$ is equal to

$$\|\natural\varphi\|^2_{\boldsymbol{K}_1\cap|s^n|}$$

$$\equiv\left(\frac{1}{2}\right)^{n-2p}\sum_{I_p+J_q=N}\sum_{r=0}^{p}\sum_{t=0}^{p-r}\frac{1}{2^{r+t}}\sum_{I_p=I_r+\tilde I_t+\hat I_{p-r-t}}\sum_{s=0}^{q}\sum_{L_s\subset J_q}$$

$$|\natural\varphi^p(\natural s^p_{I_r\tilde I_t\hat I_{p-r-t}L_s})|^2$$

$$=\frac{1}{2^{4n+q}}\sum_{I_p+J_q=N}\sum_{r=0}^{p}\sum_{t=0}^{p-r}2^{3(p-r-t)}\sum_{I_p=I_r+\tilde I_t+\hat I_{p-r-t}}\sum_{s=0}^{q}\sum_{L_s\subset J_q}$$

$$\left|\sum_{\mu=0}^{r}\sum_{\nu=0}^{t}\sum_{\bar K_\mu\subset I_r}\sum_{\tilde K_\nu\subset\tilde I_t}\sum_{u=0}^{q}3^{q-u}\sum_{i=\max(0,u+s-q)}^{\min(u,s)}\sum_{M_{s-i}\subset L_s}\sum_{N_{u-i}\subset J_q-L_s}\right.$$

$$\left.\varphi_{\bar K_\mu\tilde K_\nu\hat K_{p-\mu-\nu}(M_{s-i}\cup N_{u-i})}\right|^2$$

$$=\frac{1}{2^{4n+q}}\sum_{I_p+J_q=N}\sum_{t=0}^{q}\sum_{j=0}^{p-t}\binom{p-t}{j}2^{3(p-t)-2j}\sum_{r=0}^{t}\sum_{I_p=I_r+\tilde I_{t-r}+\hat I_{p-t}}\sum_{\bar K_\rho\cup\bar L_\sigma=I_r}$$

$$\sum_{\tilde K_\kappa\cup\tilde L_\tau=\tilde I_{t-r}}\sum_{u=0}^{q}\sum_{k=0}^{q-u}\binom{q-u}{k}2^u3^{2q-u-2k}\sum_{s=0}^{q-u}\sum_{L_s\subset J_q}\sum_{\mu=0}^{[u/2]}\sum_{M_\mu+N_{u-\mu}\subset J_q-L_s}$$

$$(\varphi_{\bar K_\rho\tilde K_\kappa\hat K_{p-\rho-\kappa}(L_s\cup M_\mu)}\,\bar\varphi_{\bar L_\sigma\tilde L_\tau\hat L_{p-\sigma-\tau}(L_s\cup N_{u-\mu})}$$

$$+\bar\varphi_{\bar K_\rho\tilde K_\kappa\hat K_{p-\rho-\kappa}(L_s\cup M_\mu)}\,\varphi_{\bar L_\sigma\tilde L_\tau\hat L_{p-\sigma-\tau}(L_s\cup N_{u-\mu})})$$

$$(I_p = \bar{K}_\mu + \tilde{K}_\nu + \hat{K}_{p-\mu-\nu} = \bar{K}_\rho + \tilde{K}_\kappa + \hat{K}_{p-\rho-\kappa} = \bar{L}_\sigma + \tilde{L}_\tau + \hat{L}_{p-\sigma-\tau}),$$

where if $u=0$, $\bar{K}_\rho=\bar{L}_\sigma=\bar{I}_r$ and $\tilde{K}_\kappa=\tilde{L}_\tau=\tilde{I}_{t-r}$, then the term with respect to φ in parentheses is replaced by a term

$$|\varphi_{\bar{I}_r\tilde{I}_{t-r}\hat{I}_{p-t}L_s}|^2.$$

Hence we find that

$$\left(\frac{3}{4}\right)^p \|\varphi\|^2_{s^n} + \sum_{t=1}^{q}\left\{1\Big/\left(\binom{p}{t}2^t\right)\right\}\left\{\binom{p}{t}3^{p-t}\Big/4^p\right\}\cdot$$

$$\cdot\frac{1}{2^q}\sum_{I_p+J_q=N}\sum_{r=0}^{t}\sum_{I_p=\bar{I}_r+\tilde{I}_{t-r}+\hat{I}_{p-t}}\sum_{s=0}^{q}\sum_{L_s\subset J_q}|\varphi_{\bar{I}_r\tilde{I}_{t-r}\hat{I}_{p-t}L_s}|^2$$

$$-\|\natural\varphi\|^2_{\boldsymbol{K}_1\cap|s^n|}$$

$$=\frac{5^n}{2^{3n+q}}\sum_{I_p+J_q=N}\sum_{t=0}^{p}\frac{1}{10^t}\sum_{r=0}^{t}\sum_{I_p=\bar{I}_r+\tilde{I}_{t-r}+\hat{I}_{p-t}}$$

$$\sum_{\bar{K}_\rho\cup\bar{L}_\sigma=\bar{I}_r}\sum_{\tilde{K}_\kappa\cup\tilde{L}_\tau=\tilde{I}_{t-r}}\sum_{u=0}^{q}\left(\frac{3}{5}\right)^u\sum_{s=0}^{q-u}\sum_{L_s\subset J_q}\sum_{\mu=0}^{[u/2]}\sum_{M_\mu+N_{u-\mu}\subset J_q-L_s}$$

$$|\varphi_{\bar{K}_\rho\tilde{K}_\kappa\hat{K}_{p-\rho-\kappa}(L_s\cup M_\mu)}-\varphi_{\bar{L}_\sigma\tilde{L}_\tau\hat{L}_{p-\sigma-\tau}(L_s\cup N_{u-\mu})}|^2 \geqq 0.$$

Therefore, we obtain that

$$\|\varphi\|^2_{\boldsymbol{K}} - \|\natural\varphi\|^2_{\boldsymbol{K}_1}$$

$$=\frac{5^n}{2^{3n+q}}\sum_{s^n\in K^*}\sum_{I_p+J_q=N}\sum_{t=0}^{p}\frac{1}{10^t}\sum_{r=0}^{t}\sum_{I_p=\bar{I}_r+\tilde{I}_{t-r}+\hat{I}_{p-t}}$$

$$\sum_{\bar{K}_\rho\cup\bar{L}_\sigma=\bar{I}_r}\sum_{\tilde{K}_\kappa\cup\tilde{L}_\tau=\tilde{I}_{t-r}}\sum_{u=0}^{q}\left(\frac{3}{5}\right)^u\sum_{s=0}^{q-u}\sum_{L_s\subset J_q}\sum_{\mu=0}^{[u/2]}\sum_{M_\mu+N_{u-\mu}\subset J_q-L_s}$$

$$|\varphi_{\bar{K}_\rho\tilde{K}_\kappa\hat{K}_{p-\rho-\kappa}(L_s\cup M_\mu)}-\varphi_{\bar{L}_\sigma\tilde{L}_\tau\hat{L}_{p-\sigma-\tau}(L_s\cup N_{u-\mu})}|^2 \geqq 0.$$

By Lemmas 4.1 and 4.2 we know that if $\varphi^p\in\Gamma_c(\boldsymbol{K})$ then $\natural\varphi^p\in\Gamma_c(\boldsymbol{K}_1)$.

Let $\{\boldsymbol{K}_i=\langle K_i, K_i^*\rangle\}_{i=0}^{\infty}$ be a sequence of complex polyhedra such that $\boldsymbol{K}_0$ is cubic and $\boldsymbol{K}_i$ is a subdivision of $\boldsymbol{K}_{i-1}$ $(i=1, 2,\dots)$. Let φ^p be a p-difference of $\Gamma_c(\boldsymbol{K}_0)$ and $\natural^i\varphi^p$ $(i=1, 2,\dots)$ be the natural extension of $\natural^{i-1}\varphi^p$ to $\boldsymbol{K}_i$ where $\natural^0\varphi^p=\varphi^p$. The p-difference $\natural^i\varphi^p$ on $\boldsymbol{K}_i$ $(i=1, 2,\dots)$ is called the *natural extension* of a p-difference φ^p to $\boldsymbol{K}_i$.

2. Norm convergence with respect to subdivision. With the notation in **1**, let $\varphi^{p,i}=\varphi^i$ be an element of the Hilbert space $\Gamma_c(\boldsymbol{K}_i)$ of closed p-differences on $\boldsymbol{K}_i$ $(i=0, 1,\dots)$.

LEMMA 4.3. *Suppose that $\boldsymbol{K}_0$ is closed or open, or $\boldsymbol{K}_0$ is compact bordered and φ^i vanishes on K_i^* for each i. If the orthogonality*

$$(\varphi^j - \natural^{j-i}\varphi^i, \varphi^i)_{\boldsymbol{K}_j} = 0 \tag{4.5}$$

holds for every i, j $(j>i)$, *then the sequence* $\{\varphi^i\}_{i=0}^{\infty}$ *has the following properties*:

(i) $\|\varphi^i\|_{K_i}$ *is monotone decreasing with* i;

(ii) $\lim_{i\to\infty}\|\varphi^i\|_{K_i} = \lim_{i,j\to\infty}\|\natural^{j-i}\varphi^i\|_{K_j}$;

(iii) $\lim_{i,j\to\infty}\|\varphi^j - \natural^{j-i}\varphi^i\|_{K_j} = 0.$

PROOF. The orthogonality (4.5) and Lemma 4.2 imply that

$$\|\varphi^j - \natural^{j-i}\varphi^i\|_{K_j}^2 = \|\natural^{j-i}\varphi^i\|_{K_j}^2 - \|\varphi^j\|_{K_j}^2 \leqq \|\varphi^i\|_{K_i}^2 - \|\varphi^j\|_{K_j}^2.$$

Hence we have (i), (ii) and (iii).

REMARK. For instance, if $\varphi^i \in \Gamma_h(K_i)$ and $\varphi^i - \natural^i\varphi^0 \in \Gamma_{e0}(K_i)$ $(i=0, 1,\dots)$, then the assumption (4.5) of Lemma 4.3 is satisfied.

§5. The difference approximation of a differential on the Riemannian manifold based on a normal comlex polyhedron

1. Hilbert space of differentials. Let M be an open or closed orientable analytic Riemannian manifold with a positive-definite metric $ds^2 = \sum_{i,j} g_{ij}\,dx_i dx_j$, where $g_{ij} = g_{ij}(x_1,\dots, x_n)$ are assumed to be real analytic functions of $x_1,\dots, x_n$. For two p-differentials (p-th order differential forms) ω and τ on M $(0 \leqq p \leqq n)$, the *inner product* (ω, τ) of ω and τ is defined by

$$(\omega, \tau) = (\omega, \tau)_M = \int_M \omega * \bar{\tau},$$

where by $\omega * \bar{\tau}$ we denote the exterior product of the differential ω and the conjugate $*\bar{\tau}$ of $\bar{\tau}$. Let $\Gamma = \Gamma(M)$ be the Hilbert space consisting of all measurable p-differentials ω on M with finite norm $\|\omega\| = (\omega, \omega)^{1/2} < +\infty$.[1] We define the subclasses $\Gamma_c^1 = \Gamma_c^1(M)$ and $\Gamma_c^{1*} = \Gamma_c^{1*}(M)$ of Γ by

$$\Gamma_c^1 = \{\omega \mid d\omega = 0, \quad \omega \in \Gamma \cap C^1\},$$

$$\Gamma_c^{1*} = \{\omega \mid d{*}\omega = 0, \quad \omega \in \Gamma \cap C^1\}.$$

In the case of $p=n$ ($p=0$ resp.), we interpret Γ_c^1 (Γ_c^{1*} resp.) as $\Gamma_c^1 = \Gamma \cap C^1$ ($\Gamma_c^{1*} = \Gamma \cap C^1$ resp.).

Let $\Gamma_{e0}^1 = \Gamma_{e0}^1(M)$ ($\Gamma_{e0}^{1*} = \Gamma_{e0}^{1*}(M)$ resp.) be the subclass of Γ consisting of all

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[1] We shall use the common notation Γ with some subscript for both spaces of p-differences and p-differentials with finite norm. If any confusion may occur, then we shall indicate the polyhedron K and the Riemannian manifold M like $\Gamma(K)$ and $\Gamma(M)$ respectively.

p-differentials ω such that $\omega = d\tau$ ($\omega = *d\tau$ resp.) for some $(p-1)$-differential τ ($(q-1)$-differential τ resp.) of class C^2 with compact support. In the case of $p=0$ ($p=n$ resp.), we interpret Γ_{e0}^1 (Γ_{e0}^{1*} resp.) as $\emptyset$. The subspaces $\Gamma_{e0} = \Gamma_{e0}(M)$ and $\Gamma_{e0}^* = \Gamma_{e0}^*(M)$ of Γ are defined as the closures in Γ of Γ_{e0}^1 and Γ_{e0}^{1*} respectively. We define the subspaces $\Gamma_c = \Gamma_c(M)$ and $\Gamma_c^* = \Gamma_c^*(M)$ of Γ as the orthogonal complements of Γ_{e0}^* and Γ_{e0} respectively. By $\Gamma_h = \Gamma_h(M)$ we denote the subspace of Γ formed by harmonic p-differentials. Then it is known (cf. Kodaira [8]) that $\Gamma_c^1 \subset \Gamma_c$, $\Gamma_c^{1*} \subset \Gamma_c^*$ and $\Gamma_h = \Gamma_c \cap \Gamma_c^*$, and the orthogonal decompositions:

$$\Gamma = \Gamma_h \dotplus \Gamma_{e0} \dotplus \Gamma_{e0}^* \quad \text{(de Rham-Kodaira's decomposition)},$$

$$\Gamma_c = \Gamma_h \dotplus \Gamma_{e0},$$

$$\Gamma_c^* = \Gamma_h \dotplus \Gamma_{e0}^*$$

hold.

Let Ω be a compact bordered subdomain of the Riemannian manifold M. Then the domain Ω is itself a Riemannian manifold. Hence the above orthogonal decompositions can be applied to any such domain Ω.

2. Smooth extension of a difference. Let $\boldsymbol{K} = \langle K, K^* \rangle$ be an open, closed or compact bordered normal complex polyhedron and M be the Riemannian manifold based on the normal complex polyhedron $\boldsymbol{K}$. Let φ^p be a p-difference on $\boldsymbol{K}$ ($0 \leqq p \leqq n$). For each cubic n-simplex $s^n = [e^n, \phi]$ of $\boldsymbol{K}$ we can choose the n-simplex e^n and the mapping ϕ so that the normal coordinates of s^n are preserved and e^n is the unit cube

$$e_N^n = \{0 \leqq x_i \leqq 1 \ (i = 1, \dots, n)\}$$

of the n-dimensional euclidean space E^n. We can adopt the coordinate system $x_1, \dots, x_n$ as a local coordinate system on $|s^n| \subset M$. With the notation in §**1.5** we define the *smooth extension* $\sharp\varphi^p$ of φ^p to the support $|s^n|$ by the p-differential $\sharp\varphi^p$ on $|s^n|$ satisfying the condition

$$\sharp\varphi^p = \sum_{I_p \subset N} \omega_{I_p} \, dx_{i_1} \cdots dx_{i_p}, \tag{5.1}$$

$$\omega_{I_p} = \sum_{s=0}^{q} \sum_{J_q = K_s + L_{q-s}} \varphi_{I_p K_s} x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s}} \tag{5.2}$$

on the local coordinate neighborhood ($|s^n|$; $x_1, \dots, x_n$), where $x'_i = 1 - x_i$.

First, let us assume that φ^p vanishes on K^*. Then we define the smooth extension $\sharp\varphi^p$ of the difference φ^p to the Riemannian manifold M by the p-differential on M which is the smooth extension $\sharp\varphi^p$ of φ^p to $|s^n|$ for each $s^n \in K$. Here the coefficients ω_{I_p} of (5.1) are generally discontinuous at a point of the carrier $|s^r|$ of r-simplex $s^r \in K$ ($0 \leqq r \leqq p$). Then we define the coefficients of

(5.1) on the carrier $|s^r|$ by

$$\omega_{I_p}(x) = \frac{1}{\nu(s^r)} \sum_{s^r \in \partial s^n} \lim_{\xi \to x,\, \xi \in |s^n|^\circ} \omega_{I_p}(\xi)$$

for a fixed system of local coordinates about a point $x \in |s^r|$, where $\nu(s^r)$ is the number of n-simplices s^n such that s^r is a face of s^n, and by $|s^n|^\circ$ we denote the interior of $|s^n|$.

Secondly, let us assume that φ^p vanishes on K. If $\boldsymbol{K}$ is open or closed, then we can similarly define the smooth extension $\sharp\varphi^p$ of φ^p to the Riemannian manifold M. If $\boldsymbol{K}$ is compact bordered, then we define the smooth extension $\sharp\varphi^p$ of φ^p to M by the p-differential on M which is the smooth extension $\sharp\varphi^p$ of φ^p to $|s^n|$ for $s^n \in K^{*s}$ and vanishes on $M - |K^{*s}|$.

Now, let us assume that φ^p is a generic p-difference. Let φ_K^p and $\varphi_{K^*}^p$ be the restrictions of φ^p to K and K^* respectively, and let $\sharp\varphi_K^p$ and $\sharp\varphi_{K^*}^p$ be the smooth extensions of φ_K^p and $\varphi_{K^*}^p$ to M respectively. By the p-differential $\sharp\varphi^p = \sharp\varphi_K^p + \sharp\varphi_{K^*}^p$ on M we define the *smooth extension* of the p-difference φ^p to the Riemannian manifold M.

LEMMA 5.1. *If φ^p is closed on $\boldsymbol{K}$ and $\sharp\varphi^p \in \Gamma(M)$, then $\sharp\varphi^p \in \Gamma_c(M)$. Here, if $\boldsymbol{K}$ is compact bordered and if the support of φ^p contains some simplices of K^*, then $\sharp\varphi^p \in \Gamma_c(M)$ is replaced by $\sharp\varphi^p \in \Gamma_c(\Omega)$ $(\Omega = |K^{*s}|)$.*

PROOF. We may assume that $1 \leqq p \leqq n-1$, and φ^p is the restriction of a generic φ^p to K or K^*. Let Q be an arbitrary cubic n-chain (see **4**) of K or K^*. It is sufficient to prove that $\sharp\varphi^p \in \Gamma_e(U)$ $(U = |Q|)$. Since φ^p is closed on $\boldsymbol{K}$, φ^p is exact on Q and thus there exists a $(p-1)$-difference ψ^{p-1} such that $\Delta\psi^{p-1} = \varphi^p$.

First, we shall prove that

$$d\sharp\psi^{p-1} = \sharp\Delta\psi^{p-1}$$

on U. By (5.1) and (5.2), we can write

$$\sharp\psi^{p-1} = \sum_{I_{p-1} \subset N} (\sharp\psi)_{I_{p-1}} dx_{i_1} \cdots dx_{i_{p-1}},$$

$$(\sharp\psi)_{I_{p-1}} = \sum_{s=0}^{q+1} \sum_{J_{q+1} = K_s + L_{q-s+1}} \psi_{I_{p-1} K_s} x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s+1}}$$

for each $s^n \in Q$. Then, by the definition of $d\sharp\psi^{p-1}$ we have

$$d\sharp\psi^{p-1} = \sum_{I_p \subset N} (d\sharp\psi)_{I_p} dx_{i_1} \cdots dx_{i_p},$$

where

$$(d\sharp\psi)_{I_p} = \sum_{I_p = M_1 + N_{p-1}} \operatorname{sgn}(M_1;\, N_{p-1}) \frac{\partial(\sharp\psi)_{N_{p-1}}}{\partial x_{m_1}}.$$

Since

$$\frac{\partial(\sharp\psi)_{N_{p-1}}}{\partial x_{m_1}} = \sum_{s=0}^{q} \sum_{J_q=K_s+L_{q-s}} (\psi_{N_{p-1}(K_s \cup M_1)} - \psi_{N_{p-1}K_s}) x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s}},$$

we obtain

$$\begin{aligned}(d\sharp\psi)_{I_p} &= \sum_{s=0}^{q} \sum_{J_q=K_s+L_{q-s}} \sum_{I_p=M_1+N_{p-1}} \operatorname{sgn}(M_1;\, N_{p-1})(\psi_{N_{p-1}(K_s \cup M_1)} - \psi_{N_{p-1}K_s}) \cdot \\ &\qquad\qquad \cdot x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s}} \\ &= \sum_{s=0}^{q} \sum_{J_q=K_s+L_{q-s}} \Delta\psi(s_{I_pK_s}) x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s}},\end{aligned}$$

which implies that $d\sharp\psi^{p-1} = \sharp\Delta\psi^{p-1} = \sharp\varphi^p$ on U.

Secondly, we can easily construct a $(p-1)$-differential ω^{p-1} of C^2 such that

$$\|d\omega^{p-1} - d\sharp\psi^{p-1}\|_U < \varepsilon \qquad \text{for any } \varepsilon > 0.$$

Hence we have $\sharp\varphi^p \in \Gamma_e(U)$.

3. The relation between φ^p and $\sharp\varphi^p$.

LEMMA 5.2. *Let* $\boldsymbol{K}$ *be an open, closed or compact bordered normal complex polyhedron, and* M *be the Riemannian manifold based on* $\boldsymbol{K}$. *Let* φ^p *and* ψ^p *be p-differences of* $\Gamma_c(\boldsymbol{K})$. *Then the following inequalities hold*:

$$\|\sharp\varphi^p\|_M^2 \leqq \|\varphi^p\|_{\boldsymbol{K}}^2 \leqq 3^q \|\sharp\varphi^p\|_M^2, \tag{5.3}$$

$$\begin{aligned}&|(\varphi,^p\, \psi^p)_{\boldsymbol{K}} - (\sharp\varphi^p,\, \sharp\psi^p)_M| \\ &\qquad \leqq \{(\|\varphi^p\|_{\boldsymbol{K}}^2 - \|\sharp\varphi^p\|_M^2)(\|\psi^p\|_{\boldsymbol{K}}^2 - \|\sharp\psi^p\|_M^2)\}^{1/2},\end{aligned} \tag{5.4}$$

where if $\boldsymbol{K}$ *is compact bordered then* $\|\varphi^p\|_{\boldsymbol{K}}^2$ *of* (5.3) *is replaced*

$$\|\varphi^p\|_{K+K^{*s}}^2 \equiv \sum_{s^p \in K+K^{*s}-(\partial K+\partial K^{*s})} |\varphi^p(s^p)|^2 + \frac{1}{2} \sum_{s^p \in \partial K+\partial K^{*s}} |\varphi^p(s^p)|^2.$$

PROOF. By the definitions (5.1) and (5.2), for each n-simplex $s^n \in \boldsymbol{K}$ and for the smooth extension $\sharp\varphi^p$ of the restriction of φ^p to s^n, we can see that

$$\begin{aligned}\|\sharp\varphi\|_{|s^n|}^2 &= \sum_{I_p \subset N} \int_0^1 \cdots \int_0^1 |\omega_{I_p}|^2 dx_1 \cdots dx_n \\ &= \sum_{I_p+J_q=N} \int_0^1 \cdots \int_0^1 \left| \sum_{s=0}^{q} \sum_{J_q=K_s+L_{q-s}} \varphi_{I_pK_s} x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s}} \right|^2 dx_1 \cdots dx_n \\ &= \frac{1}{3^q} \sum_{N=I_p+J_q} \left\{ \sum_{s=0}^{q} \sum_{K_s \subset J_q} |\varphi_{I_pK_s}|^2 \right.\end{aligned}$$

$$+\sum_{t=1}^{q}\frac{1}{2^t}\sum_{s=0}^{q-t}\sum_{J_q=K_s+L_{q-s}}\sum_{\mu=0}^{[t/2]}\sum_{M_\mu+N_{t-\mu}\subset L_{q-s}}$$

$$\left.(\varphi_{I_p(K_s\cup M_\mu)}\bar{\varphi}_{I_p(K_s\cup N_{t-\mu})}+\bar{\varphi}_{I_p(K_s\cup M_\mu)}\varphi_{I_p(K_s\cup N_{t-\mu})})\right\}.$$

Hence by (4.4) we have

$$\|\varphi\|^2_{s^n}-\|\#\varphi\|^2_{|s^n|} \tag{5.5}$$

$$=\frac{1}{3^q}\sum_{I_p+J_q=N}\sum_{t=1}^{q}\frac{1}{2^t}\sum_{s=0}^{q-t}\sum_{J_q=K_s+L_{q-s}}\sum_{\mu=0}^{[t/2]}\sum_{M_\mu+N_{t-\mu}\subset L_{q-s}}$$

$$|\varphi_{I_p(K_s\cup M_\mu)}-\varphi_{I_p(K_s\cup N_{t-\mu})}|^2\geqq 0,$$

which implies the first inequality of (5.3). By analogous calculation and Schwarz's inequality we obtain the inequality (5.4).

The second inequality of (5.3) follows from the equalities

$$3^q\|\#\varphi^p\|^2_{|s^n|}-\|\varphi^p\|^2_{s^n}$$

$$=\frac{1}{2^q}\sum_{N=I_p+J_q}\sum_{\tau=1}^{q}\left\{\binom{q}{\tau}\sum_{s=0}^{q}\sum_{K_s\subset J_q}|\varphi_{I_pK_s}|^2\right.$$

$$+\sum_{t=1}^{\tau}\binom{q-t}{\tau-t}\sum_{s=0}^{q-t}\sum_{J_q=K_s+L_{q-s}}\sum_{\mu=0}^{[t/2]}\sum_{M_\mu+N_{t-\mu}\subset L_{q-s}}$$

$$\left.(\varphi_{I_p(K_s\cup M_\mu)}\bar{\varphi}_{I_p(K_s\cup N_{t-\mu})}+\bar{\varphi}_{I_p(K_s\cup M_\mu)}\varphi_{I_p(K_s\cup N_{t-\mu})})\right\}$$

$$=\frac{1}{2^q}\sum_{N=I_p+J_q}\sum_{\tau=1}^{q}\sum_{s=0}^{q-\tau}\sum_{J_q=K_s+L_{q-s}}\sum_{N_\tau\subset L_{q-s}}\left|\sum_{\mu=0}^{\tau}\sum_{M_\mu\subset N_\tau}\varphi_{I_p(K_s\cup M_\mu)}\right|^2.$$

By Lemma 5.2, $\varphi^p\in\Gamma_c(\boldsymbol{K})$ if and only if $\#\varphi^p\in\Gamma_c(M)$. Here, if $\boldsymbol{K}$ is compact bordered and if the support of φ^p contains some simplices of K^*, then $\#\varphi^p\in\Gamma_c(M)$ is replaced by $\#\varphi^p\in\Gamma_c(\Omega)$ ($\Omega=|K^{*s}|$).

4. Courant-Friedrichs-Lewy's Lemma. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be a cubic complex polyhedron. An n-chain Q of K is called a *cube* of K, if there exists a one-to-one bicontinuous mapping ϕ of a euclidean cube $Q^e=\{l\leqq x_i\leqq m\ (i=1,\dots,n)\}$ (l, m: integrers; $l+1<m$) onto Q and if each n-simplex $s^n\in Q$ is the image of a euclidean n-simplex $e^n=\{\mu_i\leqq x_i\leqq\mu_i+1\ (l\leqq\mu_i<m,\ \mu_i\text{: an integer};\ i=1,\dots,n)\}$ by the mapping ϕ. Let Q^* be the conjugate polyhedron of Q. We may assume that each n-simplex $s^n\in Q^{*s}$ is the image of a euclidean n-simplex $e^n=\{\mu_i-1/2\leqq x_i\leqq\mu_i+1/2\ (l<\mu_i<m;\ i=1,\dots,n)\}$ by the mapping ϕ. For the n-simplex $s^n\in Q$ corresponding to $e^n=\{\mu_i\leqq x_i\leqq\mu_i+1;\ 1\leqq i\leqq n\}$, we write

$$e^p_{I_p}=\{\mu_i\leqq x_i\leqq\mu_i+1\ (i\in I_p),\ x_i=\mu_i\ (i\in J_q)\}$$

and

$$s^{p}_{I_p} = [e^{p}_{I_p}, \phi] \qquad (0 \leqq p \leqq n),$$

which is a p-face of $s^n \in Q$. For the n-simplex $s^n \in Q^{*s}$, we write

$$e^{p}_{I_p} = \left\{\mu_i - \frac{1}{2} \leqq x_i \leqq \mu_i + \frac{1}{2}\ (i \in I_p),\ x_i = \mu_i - \frac{1}{2}\ (i \in J_q)\right\}$$

and

$$s^{p}_{I_p} = [e^{p}_{I_p}, \phi] \qquad (0 \leqq p \leqq n),$$

which is a p-face of $s^n \in Q^{*s}$.

Let $\{\boldsymbol{Q}_j = \langle Q_j, Q_j^*\rangle\}_{j=0}^{\nu+1}$ $(\nu \geqq 1)$ be an increasing sequence of complex subpolyhedra of $\boldsymbol{K}$ such that each Q_j is a cube and Q_j $(j=1,\dots,\nu+1)$ is the minimum cube under the condition $|Q_{j-1}| \subset |Q_j|^\circ$. Let φ^p $(0 \leqq p \leqq n)$ be a p-difference on $\boldsymbol{Q}_{\nu+1}$. From φ^p we can define a 0-difference u_{I_p} $(I_p \subset N)$ by setting

$$u_{I_p}(s^0_{I_0}) = \varphi^p(s^p_{I_p}) \tag{5.6}$$

for the vertex $s^0_{I_0}$ and the p-face $s^p_{I_p}$ of each n-simplex s^n of $Q_{\nu+1} + Q^{*s}_{\nu+1}$.

LEMMA 5.3. *If φ^p is harmonic on $\boldsymbol{Q}_{\nu+1}$, then the* 1*-difference Δu_{I_p} is harmonic on $\boldsymbol{Q}_\nu$.*

PROOF. It is sufficient to verify that $\delta \Delta u_{I_p} = 0$. We use the notation in §**1.5** and the notation (4.3). If we note the equality

$$\begin{aligned}&\operatorname{sgn}(K_1; M_{p-1}) \operatorname{sgn}(L_1; M_{p-1})\\ &\quad = -\operatorname{sgn}(L_1; M_{p-1} \cup K_1) \operatorname{sgn}(K_1; M_{p-1} \cup L_1) \qquad (M_{p-1} + K_1 + L_1 \subset N),\end{aligned}$$

then we obtain

$$\begin{aligned}&\delta \Delta u_{I_p}(s^0_{I_0 L_0})\\
&= \sum_{I_p = M_{p-1}+K_1} \{(\varphi_{I_0 \check{I}_0 \hat{I}_p L_0} - \varphi_{I_0 \tilde{K}_1 \hat{M}_{p-1} L_0}) + (\varphi_{I_0 \check{I}_0 \hat{I}_p L_0} - \varphi_{\tilde{K}_1 \check{I}_0 \hat{M}_{p-1} L_0})\}\\
&\quad + \sum_{L_1 \subset J_q} \{(\varphi_{I_0 \check{I}_0 \hat{I}_p L_0} - \varphi_{I_0 \check{I}_0 \hat{I}_p L_1}) + (\varphi_{I_0 \check{I}_0 \hat{I}_p L_0} - \varphi_{I_0 \check{I}_0 \hat{I}_p L_0 L_1})\}\\
&= \sum_{I_p = M_{p-1}+K_1} \operatorname{sgn}(K_1; M_{p-1}) \Bigg[\Bigg\{\operatorname{sgn}(K_1; M_{p-1})(\varphi_{I_0 \check{I}_0 \hat{I}_p L_0} - \varphi_{I_0 \tilde{K}_1 \hat{M}_{p-1} L_0})\\
&\quad + \sum_{L_1 \subset J_q} \operatorname{sgn}(L_1; M_{p-1})(\varphi_{L_1 \check{I}_0 \hat{M}_{p-1} K_1} - \varphi_{I_0 \check{I}_0 (M_{p-1} \cup L_1)^\wedge K_1})\Bigg\}\\
&\quad - \Bigg\{\operatorname{sgn}(K_1; M_{p-1})(\varphi_{\tilde{K}_1 \check{I}_0 \hat{M}_{p-1} L_0} - \varphi_{I_0 \check{I}_0 \hat{I}_p L_0})\\
&\quad + \sum_{L_1 \subset J_q} \operatorname{sgn}(L_1; M_{p-1})(\varphi_{L_1 \check{I}_0 \hat{M}_{p-1} L_0} - \varphi_{I_0 \check{I}_0 (M_{p-1} \cup L_1)^\wedge L_0})\Bigg\}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&-\sum_{L_1\subset J_q}\operatorname{sgn}(L_1;I_p)\Big[\Big\{\operatorname{sgn}(L_1;I_p)(\varphi_{I_0\bar I_0\hat I_pL_1}-\varphi_{I_0\bar I_0\hat I_pL_0})\\
&+\sum_{I_p=M_{p-1}+K_1}\operatorname{sgn}(K_1;M_{p-1}\cup L_1)(\varphi_{I_0\bar I_0(M_{p-1}\cup L_1)^\frown K_1}-\varphi_{I_0\bar I_0(M_{p-1}\cup L_1)^\frown L_0})\Big\}\\
&-\Big\{\operatorname{sgn}(L_1;I_p)(\varphi_{I_0\bar I_0\hat I_pL_0}-\varphi_{I_0\bar I_0\hat I_pL_0L_1})\\
&+\sum_{I_p=M_{p-1}+K_1}\operatorname{sgn}(K_1;M_{p-1}\cup K_1)(\varphi_{L_1\bar I_0\hat M_{p-1}K_1}-\varphi_{\bar L_1\bar I_0\hat M_{p-1}L_0})\Big\}\Big]\\
=&\sum_{I_p=M_{p-1}+K_1}\operatorname{sgn}(K_1;M_{p-1})\{\delta\varphi(s^{p-1}_{M_{p-1}K_1})-\delta\varphi(s^{p-1}_{M_{p-1}L_0})\}\\
&-\sum_{L_1\subset J_q}\operatorname{sgn}(L_1;I_p)\{\Delta\varphi(s^{p+1}_{I_0\bar I_0(I_p\cup L_1)^\frown L_0})-\Delta\varphi(s^{p+1}_{\bar L_1\bar I_0\hat I_pL_0})\},
\end{aligned}
$$

where $N_r=\bar N_r=\tilde N_r=\hat N_r$ for an arbitrary subset N_r of N. Since φ^p is harmonic, the last side vanishes.

LEMMA 5.4. (cf. pp. 49–51 of [4] and p. 315 of [10].)

$$
v^2\sum_{I_p\subset N}\|\Delta u_{I_p}\|^2_{Q_0}\leqq\|\varphi^p\|^2_{Q_{v+1}}. \tag{5.7}
$$

PROOF. First, we assume that φ^p vanishes on Q_{v+1}. Then, by the formula (3.2) we have

$$
\begin{aligned}
\|\Delta u_{I_p}\|^2_{Q_0}\leqq\|\Delta u_{I_p}\|^2_{Q_j}&=\mathfrak{S}_{\partial Q_j}u_{I_p}*\overline{\Delta u_{I_p}}\\
&\leqq\frac{1}{2}\Big(\mathfrak{S}_{A_{j+1}}|u_{I_p}|^2-\mathfrak{S}_{A_j}|u_{I_p}|^2\Big)\\
&\qquad\qquad (j=0,\dots,v-1),
\end{aligned}\tag{5.8}
$$

where $A_j\,(j=0,\dots,v-1)$ is the 0-chain defined as the sum of all 0-simplices of ∂Q_j^{*s}. When we add the inequalities (5.8) for j and $I_p\subset N$, we have

$$
\begin{aligned}
k\sum_{I_p\subset N}\|\Delta u_{I_p}\|^2_{Q_0}&\leqq\frac{1}{2}\Big(\mathfrak{S}_{A_k}\sum_{I_p\subset N}|u_{I_p}|^2-\mathfrak{S}_{A_0}\sum_{I_p\subset N}|u_{I_p}|^2\Big)\\
&\leqq\frac{1}{2}\mathfrak{S}_{A_k}\sum_{I_p\subset N}|u_{I_p}|^2\qquad (k=1,\dots,v).
\end{aligned}
$$

Furthermore, when we add the last inequalities for k, we have

$$
\begin{aligned}
v(v+1)\sum_{I_p\subset N}\|\Delta u_{I_p}\|^2_{Q_0}&\leqq\sum_{k=1}^{v}\mathfrak{S}_{A_k}\sum_{I_p\subset N}|u_{I_p}|^2\\
&\leqq\sum_{Q^{*s}_{v+1}-\partial Q^{*s}_{v+1}}|\varphi^p|^2.
\end{aligned}\tag{5.9}
$$

Secondly, let us assume that φ^p vanishes on Q^*_{v+1}. When we take Q_0 and

$Q_{\nu+1}$ for Q_0^{*s} and $Q_{\nu+1}^{*s}$ of (5.9) respectively, we have

$$(5.10) \qquad \nu(\nu+1) \sum_{I_p \subset N} \|\Delta u_{I_p}\|_{Q_0}^2 \leqq \sum_{Q_{\nu+1}-\partial Q_{\nu+1}} |\varphi^p|^2.$$

The inequalities (5.9) and (5.10) imply the present lemma.

LEMMA 5.5. *Let $\{\boldsymbol{K}_i = \langle K_i, K_i^* \rangle\}_{i=0}^{\infty}$ be a sequence of open or closed cubic complex polyhedra such that $\boldsymbol{K}_i$ is a subdivision of $\boldsymbol{K}_{i-1}$ $(i=1, 2,\ldots)$. Let φ^i $(i=0, 1,\ldots)$ be a p-difference of $\Gamma_h(\boldsymbol{K}_i)$ such that $\|\varphi^i\|_{\boldsymbol{K}_i}$ is bounded with respect to i. Then, the limit relations:*

$$(5.11) \quad E_{\boldsymbol{K}_i}(\varphi^i) \equiv \sum_{s^n \in \boldsymbol{K}_i} \sum_{I_p + J_q = N} \sum_{s=1}^{q} \sum_{L_s \subset J_q} |\varphi^i_{I_p L_s} - \varphi^i_{I_p(L_s - L_1)}|^2 \to 0 \qquad (i \to \infty)$$

and

$$(5.12) \quad E_{\boldsymbol{K}_i}(*\varphi^i) \equiv \sum_{s^n \in \boldsymbol{K}_i} \sum_{I_p + J_q = N} \sum_{r=1}^{p} \sum_{L_r \subset I_p} |(*\varphi^i)_{J_q L_r} - (*\varphi^i)_{J_q(L_r - L_1)}|^2 \to 0 \qquad (i \to \infty)$$

hold.

PROOF. We fix an arbitrary n-simplex $s^n \in K_1$. We can always find an increasing sequence $Q_0^3,\ldots, Q_4^3$ of concentric cubes of K_3 such that $|Q_0^3| = |s^n|$.

Let Q_j^i $(j=0,\ldots, 4;\ i=4, 5,\ldots)$ be the subdivision of Q_j^{i-1} which is a cube of K_i, and let $\boldsymbol{Q}_j^i = \langle Q_j^i, Q_j^{i*} \rangle$. Then, by Lemma 5.4, we have

$$(3 \cdot 2^{i-3} - 1)^2 \sum_{I_p \subset N} \|\Delta u^i_{I_p}\|^2_{\boldsymbol{Q}_1^i} \leqq \|\varphi^i\|^2_{\boldsymbol{Q}_4^i} \qquad (i = 3, 4,\ldots),$$

where $u^i_{I_p}$ is the 0-difference defined by (5.6) for the present φ^i. On the other hand, we can easily verify that

$$E_{Q_0^i + Q_0^{i*b}}(\varphi^i) + E_{Q_0^i + Q_0^{i*b}}(*\varphi^i) \leqq 2^n \sum_{I_p \subset N} \|\Delta u^i_{I_p}\|^2_{\boldsymbol{Q}_1^i} \qquad (i = 3, 4,\ldots).$$

Hence we have

$$E_{Q_0^i + Q_0^{i*b}}(\varphi^i) + E_{Q_0^i + Q_0^{i*b}}(*\varphi^i) \leqq \frac{2^n}{(3 \cdot 2^{i-3} - 1)^2} \|\varphi^i\|^2_{\boldsymbol{Q}_4^i} \qquad (i = 3, 4, \ldots).$$

Adding the last inequalities for all simplices $s^n \in K_1$, we obtain

$$E_{\boldsymbol{K}_i}(\varphi^i) + E_{\boldsymbol{K}_i}(*\varphi^i) \leqq \frac{6^n}{(3 \cdot 2^{i-3} - 1)^2} \|\varphi^i\|^2_{\boldsymbol{K}_i} \qquad (i = 3, 4,\ldots),$$

which implies the limit relations (5.11) and (5.12) because of the assumption of the lemma.

In the case where $\boldsymbol{K}_i$ is compact bordered, Lemma 5.5 is reduced to the following somewhat minor result.

LEMMA 5.6. *Let* $\{\boldsymbol{K}_i=\langle K_i, K_i^*\rangle\}_{i=0}^{\infty}$ *be a sequence of compact bordered cubic complex polyhedra such that* $\boldsymbol{K}_i$ *is a subdivision of* $\boldsymbol{K}_{i-1}$ $(i=1, 2,\dots)$. *Let* $\boldsymbol{L}_0=\langle L_0, L_0^*\rangle$ *be an arbitrary complex subpolyhedron of* $\boldsymbol{K}_0$ *with* $|L_0|\subset |K_0|^\circ$, *and let* $\{\boldsymbol{L}_i=\langle L_i, L_i^*\rangle\}_{i=0}^{\infty}$ *be a sequence of complex polyhedra such that* $\boldsymbol{L}_i\subset\boldsymbol{K}_i$ *and* $|\boldsymbol{L}_i|=|\boldsymbol{L}_0|$ $(i=1, 2,\dots)$. *Let* φ^i $(i=1, 2,\dots)$ *be a p-difference of* $\Gamma_h(\boldsymbol{K}_i)$ *such that* $\|\varphi^i\|_{\boldsymbol{K}_i}$ *is bounded with respect to i. Then, the limit relations*

$$E_{L_i+L_i^{*b}}(\varphi^i)\to 0 \qquad (i\to\infty) \tag{5.13}$$

and

$$E_{L_i+L_i^{*b}}(*\varphi^i)\to 0 \qquad (i\to\infty) \tag{5.14}$$

hold.

5. The estimation of $\|\sharp*\varphi-*\sharp\varphi\|$. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be an open, closed or compact bordered normal complex polyhedron and M be the Riemannian manifold based on $\boldsymbol{K}$. Let φ^p be an element of the Hilbert space $\Gamma_h(\boldsymbol{K})$ of harmonic p-differences $(0\leqq p\leqq n)$.

Let $s^n=[e^n, \phi]$ and $\sigma^n=[\varepsilon^n, \psi]$ be a pair of n-simplices such that $s^n\in K$ ($s^n\in K^*$ resp.), $\sigma^n\in K^*$ ($\sigma^n\in K$ resp.) and $|s^n|\cap|\sigma^n|\neq\emptyset$, where if $\boldsymbol{K}$ is compact bordered, then we interpret K^* as $K^*=K^{*s}$. We can choose the n-simplices e^n and ε^n, and the mapping ϕ and ψ so that the normal coordinates of s^n and σ^n are preserved, and e^n and ε^n are the unit cubes

$$e^n = \{0\leqq x_i\leqq 1\ (i=1,\dots, n)\}$$

and

$$\varepsilon^n = \left\{-\frac{1}{2}\leqq x_i\leqq\frac{1}{2}\ (i=1,\dots, n)\right\}$$

on the euclidean space E^n. We can adopt the coordinate system $x_1,\dots, x_n$ as a local coordinate system on $|s^n|\cap|\sigma^n|\subset M$. By the definitions (5.1) and (5.2), the smooth extension of the restriction of the p-difference $\varphi=\varphi^p$ to the n-simplex s^n is denoted by

$$\sharp\varphi = \sum_{I_p\subset N}\omega_{I_p}dx_{i_1}\cdots dx_{i_p},$$

$$\omega_{I_p} = \sum_{s=0}^{q}\sum_{J_q=K_s+L_{q-s}}\varphi_{I_pK_s}x_{k_1}\cdots x_{k_s}x'_{l_1}\cdots x'_{l_{q-s}}$$

on the local coordinate neighborhood $(|s^n|;\, x_1,\dots,\, x_n)$, where $x'_i = 1 - x_i$. The conjugate differential $*\sharp\varphi$ of the smooth extension $\sharp\varphi$ has the expression

$$*\sharp\varphi = \sum_{I_p+J_q=N} \operatorname{sgn}(I_p;\, J_q)\, \omega_{I_p} dx_{j_1}\cdots dx_{j_q}.$$

By the coordinate transformation

$$\chi:\quad \xi_i = \frac{1}{2} - x_i \quad (i = 1,\dots,\, n),$$

the unit cube ε^n is transformed to the unit cube

$$\tilde{\varepsilon}^n = \{0 \leqq \xi_i \leqq 1 \quad (i = 1,\dots,\, n)\}.$$

Then each q-face of the euclidean n-simplex $\tilde{\varepsilon}^n$ and each q-face of the n-simplex σ^n can be written in the forms

$$\tilde{\varepsilon}^q_{J_qK_r} = \{0 \leqq \xi_i \leqq 1\ (i \in J_q),\ \xi_i = 1\ (i \in K_r),\ \xi_i = 0\ (i \in I_p - K_r)\}$$
$$(0 \leqq r \leqq p)$$

and

$$\sigma^q_{J_qK_r} = [\tilde{\varepsilon}^q_{J_qK_r},\, \psi\circ\chi^{-1}]$$

respectively. We agree that the q-simplex $\sigma^q_{J_qK_r}$ has the orientation induced by the orientation of the q-dimensional space $O\text{-}x_{j_1}\cdots x_{j_q}$. For the restriction of the conjugate difference $*\varphi$ to the n-simplex σ^n which has a value at each q-face $\sigma^q_{J_qK_r}$ of σ^n, we introduce the notation

$$(*\varphi)_{J_qK_r} = *\varphi(\sigma^q_{J_qK_r}).$$

If we note that

$$\sigma^q_{J_qK_0} = \operatorname{sgn}(I_p;\, J_q)\, *s^p_{I_pL_0},$$

then we find that

$$(*\varphi)_{J_qK_0} = \operatorname{sgn}(I_p;\, J_q)\, \varphi_{I_pL_0}. \tag{5.15}$$

When we introduce a coordinate system

$$\xi'_i = 1 - \xi_i = x_i + \frac{1}{2} \qquad (i = 1,\dots,\, n)$$

on the local coordinate neighborhood $(|\sigma_n|;\, \xi'_1,\dots,\, \xi'_n)$, the smooth extension $\sharp*\varphi$ of the restriction of the difference $*\varphi$ to σ^n can be written in the form

$$\sharp*\varphi = \sum_{J_q\subset N} \omega_{J_q} d\xi'_{j_1}\cdots d\xi'_{j_q}$$

$$= \sum_{J_q \subset N} \omega_{J_q} dx_{j_1} \cdots dx_{j_q},$$

$$\omega_{J_q} = \sum_{r=0}^{p} \sum_{I_p = K_r + L_{p-r}} (*\varphi)_{J_q K_r} \xi_{k_1} \cdots \xi_{k_r} \xi'_{l_1} \cdots \xi'_{l_{p-r}}.$$

On setting

$$\tau_{J_q} = \operatorname{sgn}(I_p;\, J_q)\, \omega_{I_p} - \omega_{J_q},$$

we can write

$$*\sharp\varphi - \sharp*\varphi = \sum_{J_q \subset N} \tau_{J_q} dx_{j_1} \cdots dx_{j_q}$$

and

$$\text{(5.16)} \qquad \|*\sharp\varphi - \sharp*\varphi\|^2_{|s^n| \cap |\sigma^n|} = \int_0^{1/2} \cdots \int_0^{1/2} \Bigl(\sum_{J_q \subset N} |\tau_{J_q}|^2 \Bigr) dx_1 \cdots dx_n.$$

We shall estimate the integral (5. 16). First we note that ω_{I_p} can be written in the form

$$\begin{aligned}
\omega_{I_p} &= \sum_{s=0}^{q} \sum_{J_q = K_s + L_{q-s}} \varphi_{I_p K_s} x_{k_1} \cdots x_{k_s} x'_{l_1} \cdots x'_{l_{q-s}} \\
&= \sum_{s=0}^{q} \sum_{J_q = K_s + L_{q-s}} \varphi_{I_p K_s} x_{k_1} \cdots x_{k_s} \Bigl(1 + \sum_{\nu=1}^{q-s} (-1)^\nu \sum_{M_\nu \subset L_{q-s}} x_{m_1} \cdots x_{m_\nu} \Bigr) \\
&= \sum_{s=0}^{q} (-1)^s \sum_{K_s \subset J_q} x_{k_1} \cdots x_{k_s} \sum_{\nu=0}^{s} (-1)^\nu \sum_{M_\nu \subset K_s} \varphi_{I_p M_\nu} \\
&= \varphi_{I_p K_0} + \sum_{s=1}^{q} (-1)^s \sum_{K_s \subset J_q} x_{k_1} \cdots x_{k_s} \sum_{\nu=0}^{s-1} (-1)^\nu \sum_{M_\nu \subset K_s - K_1} (\varphi_{I_p M_\nu} - \varphi_{I_p (K_1 \cup M_\nu)}).
\end{aligned}$$

Similarly, we can write

$$\begin{aligned}
\omega_{J_q} &= \sum_{r=0}^{p} \sum_{I_p = K_r + L_{p-r}} (*\varphi)_{J_q K_r} \xi_{k_1} \cdots \xi_{k_r} \xi'_{l_1} \cdots \xi'_{l_{p-r}} \\
&= (*\varphi)_{J_q K_0} + \sum_{r=1}^{p} (-1)^r \sum_{K_r \subset I_p} \xi_{k_1} \cdots \xi_{k_r} \sum_{\nu=0}^{r-1} (-1)^\nu \sum_{M_\nu \subset K_r - K_1} \\
&\qquad \{(*\varphi)_{J_q M_\nu} - (*\varphi)_{J_q (K_1 \cup M_\nu)}\}.
\end{aligned}$$

Hence, by (5.15) we find that

$$\begin{aligned}
\tau_{J_q} &= \operatorname{sgn}(I_p;\, J_q)\, \omega_{I_p} - \omega_{J_q} \\
&= \operatorname{sgn}(I_p;\, J_q) \sum_{s=1}^{q} (-1)^s \sum_{K_s \subset J_q} x_{k_1} \cdots x_{k_s} \sum_{\nu=0}^{s-1} (-1)^\nu \sum_{M_\nu \subset K_s - K_1} (\varphi_{I_p M_\nu} - \varphi_{I_p (K_1 \cup M_\nu)}) \\
&\quad - \sum_{r=1}^{p} (-1)^r \sum_{K_r \subset I_p} \xi_{k_1} \cdots \xi_{k_r} \sum_{\nu=0}^{r-1} (-1)^\nu \sum_{M_\nu \subset K_r - K_1} \{(*\varphi)_{J_q M_\nu} - (*\varphi)_{J_q (K_1 \cup M_\nu)}\}.
\end{aligned}$$

For simplicity, we set

$$\Phi_{I_pK_s} = \sum_{\nu=0}^{s-1}(-1)^\nu \sum_{M_\nu \subset K_s - K_1} (\varphi_{I_pM_\nu} - \varphi_{I_p(K_1 \cup M_\nu)})$$

and

$$\Phi^*_{J_qK_r} = \sum_{\nu=0}^{r-1}(-1)^\nu \sum_{M_\nu \subset K_r - K_1} \{(*\varphi)_{J_qM_\nu} - (*\varphi)_{J_q(K_1 \cup M_\nu)}\}.$$

We sum the square norms (5.16) for all n-simplices $s^n \in K+K^*$, and for all pairs s^n and σ^n with $|s^n| \cap |\sigma^n| \neq \emptyset$. Then in the case where $\boldsymbol{K}$ is open or closed, we obtain

$$(5.17) \qquad \|*\sharp\varphi - \sharp*\varphi\|^2_M$$

$$= 2^n \sum_{s^n \in \boldsymbol{K}} \sum_{I_p+J_q=N} \sum_{s=1}^{q} \sum_{\sigma=1}^{q} (-1)^{s+\sigma} \sum_{K_s \subset J_q} \sum_{L_\sigma \subset J_q} \frac{1}{3^t \cdot 2^{n+2(s+\sigma-t)}} \cdot$$

$$\cdot \{\Phi_{I_pK_s}\bar{\Phi}_{I_pL_\sigma} + \bar{\Phi}_{I_pK_s}\Phi_{I_pL_\sigma}\}$$

$$+ 2^n \sum_{\sigma^n \in \boldsymbol{K}} \sum_{I_p+J_q=N} \sum_{r=1}^{p} \sum_{\rho=1}^{p} (-1)^{r+\rho} \sum_{K_r \subset I_p} \sum_{L_\rho \subset I_p} \frac{1}{3^u \cdot 2^{n+2(r+\rho-u)}} \cdot$$

$$\cdot \{\Phi^*_{J_qK_r}\bar{\Phi}^*_{J_qL_\rho} + \bar{\Phi}^*_{J_qK_r}\Phi^*_{J_qL_\rho}\}$$

$$\sum_{s^n, \sigma^n \in \boldsymbol{K}, |s^n|^\circ \cap |\sigma^n|^\circ \neq \emptyset} \sum_{I_p+J_q=N} \operatorname{sgn}(I_p; J_q) \sum_{s=1}^{q} \sum_{r=1}^{p} (-1)^{s+r}$$

$$\sum_{K_s \subset J_q} \sum_{L_r \subset I_p} \frac{1}{2^{n+2(s+r)}} \{\Phi_{I_pK_s}\bar{\Phi}^*_{J_qL_r} + \bar{\Phi}_{I_pK_s}\Phi^*_{J_qL_r}\},$$

where t and u are the numbers of elements of $K_s \cap L_\sigma$ and $K_r \cap L_\rho$ respectively, and if $t=s=\sigma$ ($u=r=\rho$ resp.) then $\{\Phi_{I_pK_s}\bar{\Phi}_{I_pL_\sigma} + \bar{\Phi}_{I_pK_s}\Phi_{I_pL_\sigma}\}$ $(\{\Phi^*_{J_qK_r}\bar{\Phi}^*_{J_qL_\rho} + \bar{\Phi}^*_{J_qK_r}\Phi^*_{J_qL_\rho}\}$ resp.) is replaced by $|\Phi_{I_pK_s}|^2$ ($|\Phi^*_{J_qK_r}|^2$ resp.). In the case where $\boldsymbol{K}$ is compact bordered, we also obtain an equation analogous to (5.17).

LEMMA 5.7. *If* $\boldsymbol{K}$ *is open or closed, then the inequality*

$$\|*\sharp\varphi - \sharp*\varphi\|^2_M \leqq A[E_{\boldsymbol{K}}(\varphi) + E_{\boldsymbol{K}}(*\varphi) + \{E_{\boldsymbol{K}}(\varphi)\}^{1/2}\{E_{\boldsymbol{K}}(*\varphi)\}^{1/2}]$$

holds, and if $\boldsymbol{K}$ *is compact bordered, then the inequality*

$$\|*\sharp\varphi - \sharp*\varphi\|^2_\Omega \leqq A[E_{K+K^{*s}}(\varphi) + E_{K+K^{*s}}(*\varphi)$$

$$+ \{E_{K+K^{*s}}(\varphi)\}^{1/2}\{E_{K+K^{*s}}(*\varphi)\}^{1/2}]$$

holds, where $\Omega = |K^{*s}|$, $E_{\boldsymbol{K}}(\varphi)$ *and* $E_{\boldsymbol{K}}(*\varphi)$, *etc. are the quantities defined in Lemma 5.5, and* A *is a constant depending only on the dimension* n.

PROOF. We note that generic terms appearing in the right hand side of (5.17) have the following three types:

$$(\varphi_{I_pM_\nu} - \varphi_{I_p(K_1\cup M_\nu)})\overline{(\varphi_{I_pN_\mu} - \varphi_{I_p(L_1\cup N_\mu)})},$$

$$\{(*\varphi)_{J_qM_\nu} - (*\varphi)_{J_q(K_1\cup M_\nu)}\}\,\overline{\{(*\varphi)_{J_qN_\mu} - (*\varphi)_{J_q(L_1\cup N_\mu)}\}}$$

and

$$(\varphi_{I_pM_\nu} - \varphi_{I_p(K_1\cup M_\nu)})\overline{\{(*\varphi)_{J_qN_\mu} - (*\varphi)_{J_q(L_1\cup N_\mu)}\}}$$

except their coefficients. Then by Schwarz's inequality we have the present lemma.

As a consequence of Lemmas 5.5 and 5.7 we obtain:

COROLLARY 5.1. *Let* $\{\boldsymbol{K}_i\}_{i=0}^{\infty}$ *be a sequence of open or closed normal complex polyhedra such that* $\boldsymbol{K}_i$ *is a subdivision of* $\boldsymbol{K}_{i-1}$ $(i=1, 2,\dots)$, *and let* M *be the Riemannian manifold based on* $\boldsymbol{K}_0$. *Let* $\varphi^{p,i}=\varphi^i$ $(i=0, 1,\dots)$ *be a* p-*difference of* $\Gamma_h(\boldsymbol{K}_i)$ *such that* $\|\varphi^i\|_{\boldsymbol{K}_i}$ *is bounded with respect to* i. *Then the following limit relation holds*:

$$\|*\sharp\varphi^i - \sharp*\varphi^i\|_M^2 \to 0 \qquad (i\to\infty).$$

As a consequence of Lemmas 5.6 and 5.7 we obtain:

COROLLARY 5.2. *Let* $\{\boldsymbol{K}_i\}_{i=0}^{\infty}$ *be a sequence of compact bordered normal complex polyhedra such that* $\boldsymbol{K}_i$ *is a subdivision of* $\boldsymbol{K}_{i-1}$ $(i=1, 2,\dots)$, *and let* M *be a compact bordered Riemannian manifold based on* $\boldsymbol{K}_0$. *Let* $\varphi^{p,i}=\varphi^i$ $(i=0, 1,\dots)$ *be a* p-*difference of* $\Gamma_h(\boldsymbol{K}_i)$ *such that* $\|\varphi^i\|_{\boldsymbol{K}_i}$ *is bounded with respect to* i. *Then the limit relation*

$$\|*\sharp\varphi^i - \sharp*\varphi^i\|_\Omega^2 \to 0 \qquad (i\to\infty)$$

holds for an arbitrary closed subregion Ω *of* M°.

6. Fundamental theorem.

THEOREM 5.1. *Let* $\{\boldsymbol{K}_i=\langle K_i, K_i^*\rangle\}_{i=0}^{\infty}$ *be a sequence of open, closed or compact bordered normal complex polyhedra such that* $\boldsymbol{K}_i$ *is the normal subdivision of* $\boldsymbol{K}_{i-1}$ $(i=1, 2,\dots)$. *Let* M *be a Riemannian manifold based on the normal complex polyhedron* $\boldsymbol{K}_0$. *Let* φ^i $(i=0, 1,\dots)$ *be a* p-*difference of* $\Gamma_h(\boldsymbol{K}_i)$. *If* $\{\varphi^i\}_{i=0}^{\infty}$ *forms a Cauchy sequence, i.e.*

$$\lim_{i,j\to\infty} \|\natural^{j-i}\varphi^i - \varphi^j\|_{\boldsymbol{K}_j} = 0 \qquad (j \geqq i) \tag{5.18}$$

holds, then the sequence $\{\sharp\varphi^i\}_{i=0}^{\infty}$ *of smooth extensions strongly converges to a harmonic* p-*differential* $\omega\in\Gamma_h(M)$, *i.e.*

$$\lim_{i\to\infty} \|\sharp\varphi^i - \omega\|_M = 0. \tag{5.19}$$

Furthermore, if $\boldsymbol{K}_i$ *are open or closed then the limit relations*

$$\lim_{i\to\infty} \|\varphi^i\|_{\boldsymbol{K}_i} = \lim_{i\to\infty} \|\sharp\varphi^i\|_M = \|\omega\|_M \tag{5.20}$$

and

$$\lim_{i\to\infty} \|\sharp{*}\varphi^i - {*}\omega\|_M = 0 \tag{5.21}$$

hold, and if $\boldsymbol{K}_i$ *are compact bordered then the limit relations*

$$\lim_{i\to\infty} \|\varphi^i\|_{K_i+K_i^{*s}} = \lim_{i\to\infty} \|\sharp\varphi^i\|_M = \|\omega\|_M \tag{5.22}$$

and

$$\lim_{i\to\infty} \|\sharp{*}\varphi^i - {*}\omega\|_\Omega = 0 \tag{5.23}$$

hold for an arbitrary closed subregion Ω *of* M°, *where*

$$\|\varphi^i\|^2_{K_i+K_i^{*s}} \equiv \sum_{s^p\in K_i+K_i^{*s}-(\partial K_i+\partial K_i^{*s})} |\varphi^i(s^p)|^2 + \frac{1}{2}\sum_{s^p\in\partial K_i+\partial K_i^{*s}} |\varphi^i(s^p)|^2.$$

Proof. First, let us assume that $\boldsymbol{K}_i$ $(i=0, 1,\ldots)$ are open or closed. Lemma 5.2 and the limit relation (5.18) imply that

$$\lim_{i,j\to\infty} \|\sharp\varphi^j - \sharp\natural^{j-i}\varphi^i\|_M = 0. \tag{5.24}$$

Here we note that each coefficient of the differentials $\sharp\natural^{j-i}\varphi^i$ uniformly converges as $j\to\infty$ on each compact subregion Ω of M for a fixed i. In the inequality

$$\begin{aligned} \|\sharp\varphi^j - \sharp\varphi^k\|_M & \\ \leqq \|\sharp\varphi^j - \sharp\natural^{j-i}\varphi^i\|_M &+ \|\sharp\varphi^k - \sharp\natural^{k-i}\varphi^i\|_M \\ &+ \|\sharp\natural^{j-i}\varphi^i - \sharp\natural^{k-i}\varphi^i\|_M \qquad (k>j>i), \end{aligned} \tag{5.25}$$

if for any $\varepsilon>0$ we choose a compact subpolyhedron $\boldsymbol{L}_i$ of $\boldsymbol{K}_i$ so that

$$\|\sharp\natural^{j-i}\varphi^i - \sharp\natural^{k-i}\varphi^i\|_{M-\Omega} \leqq 2\,\|\varphi^i\|_{\boldsymbol{K}_i-\boldsymbol{L}_i} < \frac{\varepsilon}{2} \qquad (\Omega = |\boldsymbol{L}_i|),$$

then the inequality

$$\|\sharp\natural^{j-i}\varphi^i - \sharp\natural^{k-i}\varphi^i\|_M < \varepsilon \tag{5.26}$$

holds for sufficiently large j, k and for a fixed i. From (5.25), (5.24) and (5.26), if follows that

$$\lim_{j,k\to\infty} \|\sharp\varphi^j - \sharp\varphi^k\|_M = 0. \tag{5.27}$$

The limit relation (5.27) and Lemma 5.1 assure that there exists a p-differential $\omega \in \Gamma_c(M)$ satisfying (5.19) and the second equality of (5.20).

By (5.5) we can see that there exists a constant C not depending on i such that the inequalities

$$\|\varphi^i\|^2_{\boldsymbol{K}_i} - \|\sharp\varphi^i\|^2_M \leqq C\, E_{\boldsymbol{K}_i}(\varphi^i) \qquad (i = 0, 1,\dots)$$

hold. Then, by Lemma 5.5, we obtain the first equality of (5.20).

By Corollary 5.1 and (5.19) we have (5.21) which implies $*\omega \in \Gamma_c(M)$. Hence $\omega \in \Gamma_h(M)$.

Secondly, let us assume that $\boldsymbol{K}_i$ $(i=0, 1,\dots)$ are compact bordered. If we use Lemma 5.6 and Corollary 5.2 in place of Lemma 5.5 and Corollary 5.1 respectively, then the same proof with some modification holds also for this case.

7. The method of orthogonal projection.

THEOREM 5.2. *Let $\{\boldsymbol{K}_i = \langle K_i, K_i^*\rangle\}_{i=0}^{\infty}$ be a sequence of open, closed or compact bordered normal complex polyhedra such that $\boldsymbol{K}_i$ is the normal subdivision of $\boldsymbol{K}_{i-1}$ $(i=1, 2,\dots)$. Let M be a Riemannian manifold based on the normal complex polyhedron $\boldsymbol{K}_0$. Let χ be an arbitrary p-difference of $\Gamma_c(\boldsymbol{K}_0)$. Here, if $\boldsymbol{K}_0$ is compact bordered, then χ is assumed to vanish on K_0^*. Let φ^i $(i=0, 1,\dots)$ be the projection of the natural extension $\natural^i\chi$ on $\Gamma_h(\boldsymbol{K}_i)$. Then, we obtain the same conclusion as in Theorem 5.1. Furthermore, the monotone convergence of norms*

$$\|\varphi^i\|_{\boldsymbol{K}_i} \searrow \|\omega\|_M \qquad (i \to \infty) \tag{5.28}$$

holds. If χ vanishes on K_0^*, then the inequalities

$$\|\varphi^i\|_{\boldsymbol{K}_i} \geqq \|\sharp\varphi^i\|_M \geqq \|\omega\|_M \tag{5.29}$$

hold for every i, the limit differential ω is the projection of the smooth extension $\sharp\chi$ on $\Gamma_h(M)$, and hence $\sharp\chi - \omega \in \Gamma_{e0}(M)$.

PROOF. The assumption of the theorem implies that

$$\natural^i\chi = \varphi^i + \psi^i, \quad \psi^i \in \Gamma_{e0}(\boldsymbol{K}_i) \qquad (i = 0, 1,\dots). \tag{5.30}$$

Hence we find that

$$\varphi^i - \natural^i\varphi^0 = \natural^i\psi^0 - \psi^i \in \Gamma_{e0}(\boldsymbol{K}_i) \qquad (i = 0, 1,\dots). \tag{5.31}$$

Therefore, by Lemma 4.3 the assumption (5.18) of Theorem 5.1 is satisfied, and thus the same conclusion as in Theorem 5.1 holds. The monotone convergency (5.28) follows from Lemma 4.3, and (5.20) or (5.22).

The first inequality of (5.29) follows from (5.3). Let us assume that χ vanishes on K_0^*. Then we can verify that $\sharp\chi = \sharp\natural^i\chi$. In fact, noting

$$\begin{aligned}&\int_{|s^p|} \sharp\chi \\ &= \sum_{\sigma=0}^{q} \sum_{J_q = K_\sigma + N_{q-\sigma}} \chi_{I_p K_\sigma} \int_{|e^p|} x_{k_1} \cdots x_{k_\sigma} x'_{n_1} \cdots x'_{n_{q-\sigma}} dx_{i_1} \cdots dx_{i_p} \\ &= \frac{1}{2^{p+t}} \sum_{\nu=0}^{t} \sum_{N_\nu \subset M_t} \chi(s^p_{I_p(L_s \cup N_\nu)}) = \natural\chi(s^p)\end{aligned}$$

for each p-simplex $s^p \in K_1$, we easily see that $\sharp\chi = \sharp\natural\chi$, where $s^p = \natural s^p_{\bar{I}_r \tilde{I}_{p-r} L_s M_t}$, $e^p = \natural e^p_{\bar{I}_r \tilde{I}_{p-r} L_s M_t}$ and $\bar{I}_r + \tilde{I}_{p-r} = I_p$ with the notation in §**1.5**. The assumption and (5.31) imply that $\sharp\varphi^i - \sharp\varphi^0 = \sharp\varphi^i - \sharp\natural^i\varphi^0 \in \Gamma_{e0}(M)$. Hence, by (5.19) we see that $\omega - \sharp\varphi^i \in \Gamma_{e0}(M)$ for every i. Because of $\omega \in \Gamma_h(M)$, ω is the projection of $\sharp\varphi^i$ on $\Gamma_h(M)$ for every i. Thus we have the second inequality of (5.29). Furthermore, by (5.30), ω is the projection of $\sharp\chi = \sharp\natural^i\chi$ on $\Gamma_h(M)$.

8. Difference approximation of a differential. Let $\boldsymbol{K} = \langle K, K^* \rangle$ be an open, closed or compact bordered normal complex polyhedron, and M be the Riemannian manifold based on $\boldsymbol{K}$. Let Θ be a closed p-differential on M, of class C^1 $(1 \leqq p \leqq n)$. For an arbitrary n-simplex $s^n = [e^n_N, \phi] \in \boldsymbol{K}$, we choose a local coordinate neighborhood $(|s^n|; x_1, \ldots, x_n)$ so that

$$(5.32) \qquad s^n\colon \ e^n_N = \{0 \leqq x_i \leqq 1 \quad (i = 1, \ldots, n)\}.$$

The p-differential Θ has a local representation

$$\Theta = \sum_{I_p \subset N} \Theta_{I_p} dx_{i_1} \cdots dx_{i_p}$$

on the local coordinate neighborhood $(|s^n|; x_1, \ldots, x_n)$, where each coefficient Θ_{I_p} is a complex valued function on the unit cube e^n_N. By a *difference approximation* ψ of Θ on the normal complex polyhedron $\boldsymbol{K}$, we mean the closed p-difference on $\boldsymbol{K}$ defined by

$$\psi(s^p_{I_p L_r}) = \int_{|e^p|} \Theta \qquad (e^p = e^p_{I_p L_r})$$

for each p-face $s^p_{I_p L_r}$ of s^n, where the notation $s^p_{I_p L_r}$ and $e^p_{I_p L_r}$ follows the definition in §**1.5**. Here, if $\boldsymbol{K}$ is compact bordered, then by the similar method the difference approximation ψ is also defined for each p-face of each half n-simplex of K^*.

THEOREM 5.3. *Let* $\{\boldsymbol{K}_i = \langle K_i, K_i^* \rangle\}_{i=0}^{\infty}$ *be a sequence of open, closed or compact bordered normal complex polyhedra such that* $\boldsymbol{K}_i$ *is the normal subdivision of* $\boldsymbol{K}_{i-1}$ *for each* i, *and let* M *be the Riemannian manifold based on* $\boldsymbol{K}_0$. *Let* Θ *be a closed* p-*differential on* M, *of class* C^1 $(1 \leqq p \leqq n)$, *and let* ψ^i

($i=0, 1,\dots$) *be the difference approximation of* Θ *on* $\boldsymbol{K}_i$. *Here, in the case where* $\boldsymbol{K}_i$ *are open, we assume that for a compact bordered subpolyhedron* $\boldsymbol{L}_i$ *approximating* $\boldsymbol{K}_i$ *the limit relation*

$$(5.33) \qquad \lim_{\boldsymbol{L}_i\to\boldsymbol{K}_i} \|\psi^i\|_{\boldsymbol{K}_i-\boldsymbol{L}_i} = 0$$

holds uniformly with respect to i. Then, $\psi^i \in \Gamma_c(\boldsymbol{K}_i)$ ($i=0, 1,\dots$) *and* $\Theta \in \Gamma_c(M)$, *and for the sequence* $\{\varphi^i\}_{i=0}^{\infty}$ *of the harmonic component* φ^i *of* ψ^i *the same conclusion as in Theorem* 5.1 *is obtained. Furthermore, when we denote the restrictions of* φ^i *to* K_i *and* K_i^* *by* $\varphi_{K_i}^i$ *and* $\varphi_{K_i^*}^i$ *respectively, the sequences* $\{\sharp\varphi_{K_i}^i\}_{i=0}^{\infty}$ *and* $\{\sharp\varphi_{K_i^*}^i\}_{i=0}^{\infty}$ *strongly converge to the common limit p-differential* $\omega/2$ *which is the harmonic component of* Θ.

PROOF. We note that the coefficients of $\sharp\psi_{K_i}^i$, $\sharp\psi_{K_i^*}^i$, $\sharp\natural^{j-i}\psi_{K_i}^i$ and $\sharp\natural^{j-i}\psi_{K_i^*}^i$ ($j>i$) uniformly converge to the corresponding common coefficients of Θ as i, $j\to\infty$ on each compact subregion Ω of M. By this fact, the assumption (5.33) and Lemma 5.2, we can easily verify that $\psi^i \in \Gamma_c(\boldsymbol{K}_i)$ for every i, the limit relations

$$(5.34) \qquad \lim_{i\to\infty} \|\sharp\psi^i - 2\Theta\|_M = 0$$

and

$$(5.35) \qquad \lim_{i,j\to\infty} \|\sharp\natural^{j-i}\psi^i - 2\Theta\|_M = 0$$

hold, and $\Theta \in \Gamma_c(M)$.

The limit relations (5.34) and (5.35) imply that

$$(5.36) \qquad \lim_{i,j\to\infty} \|\sharp\psi^j - \sharp\natural^{j-i}\psi^i\|_M = 0 \qquad (j>i).$$

Let φ^{ij} ($j>i$) be the harmonic component of $\natural^{j-i}\psi^i$ on $\boldsymbol{K}_j$. Then, by Lemma 5.2 the limit relation (5.36) implies that

$$(5.37) \qquad \lim_{i,j\to\infty} \|\sharp\varphi^j - \sharp\varphi^{ij}\|_M = 0 \qquad (j > i).$$

By making use of the limit relation (5.37), we can prove the present theorem by method analogous to that in Theorem 5.1. The remaining parts are obvious.

9. Difference approximation on a compact bordered region.

THEOREM 5.4. *Let* $\{\boldsymbol{K}_i = \langle K_i, K_i^*\rangle\}_{i=0}^{\infty}$ *be a sequence of an open or closed normal complex polyhedra such that* $\boldsymbol{K}_i$ *is the normal subdivision of* $\boldsymbol{K}_{i-1}$ *for each i, and let M be the Riemannian manifold based on* $\boldsymbol{K}_0$. *Let* Ω *be an arbitrary compact bordered subregion of the Riemannian manifold M. Let* $\{\boldsymbol{L}_i$

$=\langle L_i, L_i^*\rangle\}_{i=0}^{\infty}$ be a sequence of compact bordered normal complex polyhedra such that $|\boldsymbol{L}_{i-1}|\subset|\boldsymbol{L}_i|^{\circ}$, $|\boldsymbol{L}_i|\to\Omega^{\circ}$ $(i\to\infty)$ and $\boldsymbol{L}_i\subset\boldsymbol{K}_i$ for each i.

Let Θ be a closed p-differential of $\Gamma_c^1(\Omega)$ $(1\leqq p\leqq n)$, let ψ^i $(i=0, 1,\dots)$ be the difference approximation of Θ on $\boldsymbol{L}_i$, and let $\{\varphi^i\}_{i=0}^{\infty}$ be the sequence of the harmonic components of ψ^i on $\boldsymbol{L}_i$. Then, the sequence $\{\sharp\varphi^i\}_{i=0}^{\infty}$ of smooth extensions strongly converges to a harmonic p-differential $\omega\in\Gamma_h(\Omega)$ which is the harmonic component of 2Θ on Ω, i.e. the limit relation

$$\lim_{i\to\infty}\|\sharp\varphi^i-\omega\|_{\Omega'}=0$$

holds for each compact subregion Ω' of Ω°, and the following limit relations hold:

$$\lim_{i\to\infty}\|\varphi^i\|_{L_{ki}+L_{ki}^{*s}}=\lim_{i\to\infty}\|\sharp\varphi^i\|_{|\boldsymbol{L}_k|}=\|\omega\|_{|\boldsymbol{L}_k|}$$

for a fixed number k, where $\boldsymbol{L}_{ki}=\langle L_{ki}, L_{ki}^*\rangle$ $(i>k)$ is the subpolyhedron of $\boldsymbol{L}_i$ with $|\boldsymbol{L}_{ki}|=|\boldsymbol{L}_k|$. Furthermore, the sequences $\{\sharp\varphi_{L_i}^i\}_{i=0}^{\infty}$ and $\{\sharp\varphi_{L_i^*}^i\}_{i=0}^{\infty}$ strongly converge to the common limit p-differential $\omega/2$ which is the harmonic component of Θ.

PROOF. We can easily verify that the limit relations

$$\lim_{i\to\infty}\|\sharp\psi^i-2\Theta\|_{\Omega'}=0$$

and

$$\lim_{i,j\to\infty}\|\sharp\natural^{j-i}\psi^i-2\Theta\|_{\Omega'}=0\qquad(j>i)$$

hold for each compact subregion Ω' of Ω°. Thus we have

$$\lim_{i,j\to\infty}\|\sharp\natural^{j-i}\psi^i-\sharp\psi^j\|_{\Omega'}=0. \tag{5.38}$$

Let φ^{ij} be the harmonic component of $\natural^{j-i}\psi^i$ on $\boldsymbol{L}_{ij}$. Then, by Lemma 5.2 the limit relation (5.38) implies that

$$\lim_{i,j\to\infty}\|\sharp\varphi^{ij}-\sharp\varphi^j\|_{\Omega'}=0\qquad(j>i).$$

By making use of Theorems 5.1 and 5.2, the remaining parts are proved. The detailed argument is omitted.

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Department of Mathematics,
Faculty of Integrated Arts and Sciences,
Hiroshima University