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Necessary and sufficient condition for eventual decay of oscillations in general functional equations with delays

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1. Introduction

Our main purpose in this paper is to study the equation

$$(r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = f(t) \tag{1}$$

and present a necessary and sufficient condition so that all oscillatory solutions of equation (1) converge to zero asymptotically.

In [2, 3], this author showed that subject to

$$\int^{\infty} t^{n-2} |a(t)| dt < \infty \tag{2}$$

$$\int^{\infty} t^{n-2} |f(t)| dt < \infty \tag{3}$$

and boundedness of $(t^{n-k})/r(t)$, $0 \leq k < 1$ for $t \in [T, \infty)$, $T > 0$ all oscillatory solutions approach zero as $t \to \infty$. There are examples given in [2] to show that condition on $r(t)$ cannot be weakened. This restriction on $r(t)$ eliminates a very important class of equations of type (1) that requires $\int^{\infty} 1/r(t)dt = \infty$. We find a set of conditions in Theorem 3.2 which essentially ensure that all oscillatory solutions of (1) eventually vanish while retaining $\int^{\infty} 1/r(t)dt = \infty$. We, then, use this theorem to find a necessary and sufficient condition to accomplish the stated goal of this work in section 4.

2. Definition and assumptions

Unless otherwise stated, following assumptions apply throughout this work:

(i) $g(t)$, $r(t)$, $a(t)$, $f(t)$ and $h(t)$ are $R \to R$ and continuous, R being the real line;

(ii) $r(t) > 0$, $r'(t) \geq 0$ for $t \geq t_0$ where $t_0 > 0$ will be assumed fixed;

(iii) $th(t) > 0$, $t \neq 0$ and there exists an $m > 0$ such that $\dfrac{h(t)}{t} \leq m$ for $t \neq 0$;

(iv) $\int_{t_0}^{\infty} 1/r(t)dt = \infty$.

(v) $0 < g(t) \leq t$ and $g(t) \to \infty$ as $t \to \infty$.

We call a function $Q(t) \in C[t_0, \infty)$ oscillatory if $Q(t)$ has arbitrarily large zeros in $[t_0, \infty)$; otherwise $Q(t)$ is nonoscillatory.

Conditions (i), (ii), (iii) and (v) guarantee that all solutions of (1) can be continuously extended on all of $[t_0, \infty)$. In fact following the proof of our Theorem 3.1 in [4] with minor changes we obtain the following theorem with regard to the extendability of the solutions of equation (1).

THEOREM 2.1. *The continuity of $a(t)$, $r(t)$, $f(t)$, $h(t)$ and $g(t)$ for $t \geq t_0$ is sufficient to allow any solution of equation* (1) *to be continued indefinitely to the right of T for some $T \geq t_0$.*

From now on, the term "solution" only applies to a solution of an equation in the spirit of Theorem 2.1.

3. Main result

THEOREM 3.1. *Suppose*

$$\int^{\infty} t^{n-1}|a(t)|dt < \infty \tag{4}$$

and

$$\int^{\infty} |f(t)|dt < \infty. \tag{5}$$

Let $y(t)$ be a solution of equation (1). *Then $y(g(t)) = O(t^{n-1})$.*

PROOF. Let S_0 be large enough positive number so that for $t \geq S_0$, $g(t) \geq t_0$. Integrating equation (1) $(n-1)$ times between t_0 and t where we choose $t \geq S_0$ we get

$$\begin{aligned} r(t)y'(t) &= C_1 + C_2(t - t_0) + \cdots + C_{n-1}(t - t_0)^{n-2} \\ &\quad - \int_{t_0}^{t} \frac{(t-x)^{n-2}}{(n-2)!} a(x)h(y(g(x)))\,dx + \int_{t_0}^{t} \frac{(t-x)^{n-2}}{(n-2)!} f(x)dx \end{aligned} \tag{6}$$

where

$$C_i = \frac{(r(t_0)y'(t_0))^{(i-1)}}{(i-1)!},\ i = 1, 2, \ldots, n-1.$$

Dividing by $r(t)$ and integrating between t_0 and $g(t)$ for $t \geq S_0$ we have

$$y(g(t)) = y(g(t_0)) + C_1 \int_{t_0}^{g(t)} \frac{1}{r(s)}\,ds + \cdots + C_{n-1} \int_{t_0}^{g(t)} \frac{(s-t_0)^{n-2}}{r(s)}\,ds$$

$$- \int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^{s} \frac{(s-x)^{n-2} a(x) h(y(g(x))) dx ds}{(n-2)!}$$
$$+ \int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^{s} \frac{(s-x)^{n-2} f(x) dx ds}{(n-2)!}.$$

Since $r'(t) \geq 0$, $t \geq g(t) \geq t_0$ for $t \geq S_0$ we have from above

(7)
$$|y(g(t))| \leq |C_0| + |C_1| \frac{1}{r(t_0)} (t - t_0) + \cdots + \frac{|C_{n-1}|(t-t_0)^{n-1}}{r(t_0)(n-1)}$$
$$+ \frac{m}{r(t_0)} \int_{t_0}^{t} \frac{(t-s)^{n-1} |a(s)| |y(g(s))| ds}{(n-1)!}$$
$$+ \frac{1}{r(t_0)} \int_{t_0}^{t} \frac{(t-s)^{n-1} |f(s)| ds}{(n-1)!},$$

where we have set $y(t_0) = C_0$. Dividing (7) by t^{n-1} we get

(8)
$$\frac{|y(g(t))|}{t^{n-1}} \leq K_0 + K_1 \int_{t_0}^{t} s^{n-1} |a(s)| \frac{|y(g(s))|}{s^{n-1}} ds$$

where

$$K_0 \geq |C_0| + \frac{|C_1|}{r(t_0)} + \cdots + \frac{|C_{n-1}|}{(n-1)\, r(t_0)} + \frac{1}{r(t_0)} \int_{t_0}^{\infty} |f(s)| ds$$

and

$$K_1 \geq \frac{m}{r(t_0)}.$$

The conclusion of the theorem now follows by Gronwall's inequality. The proof is complete.

The following lemma is given in a remark in [3]

LEMMA 3.1. *If $t_1 < t_2 < t_3 < \cdots < t_n$, then*

$$\left| \int_{t_1}^{t_2} \int_{s_3}^{t_3} \cdots \int_{s_n}^{t_n} a_0(x) dx ds_n ds_{n-1} \cdots ds_3 \right|$$
$$\leq \int_{t_1}^{\infty} \int_{s_3}^{\infty} \cdots \int_{s_n}^{\infty} |a_0(x)| dx ds_n \cdots ds_3$$

for any function $a_0(x) \in C[t_1, \infty)$.

THEOREM 3.2. *Assume that*

(9)
$$\int^{\infty} t^{2n-2} |a(t)| dt < \infty$$

and

$$\int^{\infty} t^{n-1}|f(t)|dt<\infty. \tag{10}$$

Let $y(t)$ be an oscillatory solution of equation (1). *Then $y(t)\to 0$ as $t\to\infty$.*

PROOF. Suppose to the contrary that

$$\limsup_{t\to\infty} |y(t)|>2d>0 \tag{11}$$

for some number d. By Theorem 3.1 there are numbers $C>0$ and $T\geq t_0$ such that $g(t)>t_0$ and

$$\frac{y(g(t))}{t^{n-1}}\leq C \tag{12}$$

for $t\geq T$. Let $T_1>T$ be such that

$$\frac{mC}{(n-1)!\,r(t)}\int_{T_1}^{\infty} s^{2n-2}|a(s)|ds<\frac{d}{4} \tag{13}$$

and

$$\frac{1}{(n-1)!\,r(t)}\int_{T_1}^{\infty} s^{n-1}|f(s)|ds<\frac{d}{4} \tag{14}$$

for all $t\geq T_1$. (13) and (14) are possible in view of (9), (10) and the fact that $r'(t)\geq 0$. Since (11) holds, let t_1 and t_2, $t_1<t_2$ be consecutive zeros of $y(t)$ such that $t_1>T_1$ and

$$M=\operatorname{Max}|y(t)|>d \tag{15}$$

for $t\in[t_1, t_2]$. Let $T_0\in[t_1, t_2]$ be such that $M=|y(T_0)|$. Since $(r(t)y'(t))^{(i-1)}$, $i=1, 2,\dots, n-1$, are oscillatory, choose $p_1<p_2<p_3<\cdots<p_{n-1}$ as zeros of $(r(t)y'(t))$, $(r(t)y'(t))',\dots, (r(t)y'(t))^{(n-2)}$ respectively such that $p_1>t_2$.

On repeated integration from equation (1) we get

$$\begin{aligned}(16)\quad &\pm y'(t)+\frac{1}{r(t)}\int_t^{p_1}\int_{s_2}^{p_2}\int_{s_3}^{p_3}\cdots\int_{s_{n-1}}^{p_{n-1}} a(x)h(y(g(x)))dx\,ds_{n-1}\cdots ds_2\\ &=\frac{1}{r(t)}\int_t^{p_1}\int_{s_2}^{p_2}\int_{s_3}^{p_3}\cdots\int_{s_{n-1}}^{p_{n-1}} f(x)dx\,ds_{n-1}\cdots ds_2\end{aligned}$$

Integrating (16) between t_1 and T_0 we get

$$\begin{aligned}(17)\quad &\pm M+\int_{t_1}^{T_0}\frac{1}{r(t)}\int_t^{p_1}\int_{s_2}^{p_2}\cdots\int_{s_{n-1}}^{p_{n-1}} a(x)h(y(g(x)))dx\,ds_{n-1}\cdots ds_2dt\\ &=\int_{t_1}^{T_0}\frac{1}{r(t)}\int_t^{p_1}\int_{s_2}^{p_2}\cdots\int_{s_{n-1}}^{p_{n-1}} f(x)dx\,ds_{n-1}\cdots ds_2dt.\end{aligned}$$

By Lemma 3.1, (17) yields

$$(18)\qquad M \le \frac{1}{r(t_1)} \int_{t_1}^{\infty} \int_{t}^{\infty} \int_{s_2}^{\infty} \cdots \int_{s_{n-1}}^{\infty} |a(x)||h(y(g(x)))|\,dx\,ds_{n-2} \cdots ds_2\,dt$$
$$+ \frac{1}{r(t_1)} \int_{t_1}^{\infty} \int_{t}^{\infty} \int_{s_2}^{\infty} \cdots \int_{s_{n-1}}^{\infty} |f(x)|\,dx\,ds_{n-2} \cdots ds_2\,dt.$$

(18) gives

$$(19)\qquad M \le \frac{m}{r(t_1)} \int_{t_1}^{\infty} \frac{(x-t_1)^{n-1}}{(n-1)!} |a(x)y(g(x))|\,dx.$$
$$+ \frac{1}{r(t_1)} \int_{t_1}^{\infty} \frac{(x-t_1)^{n-1}}{(n-1)!} |f(x)|\,dx.$$

From (12) and (19) and the fact that $M > d$ we have

$$(20)\qquad d \le \frac{mC}{r(t_1)} \int_{t_1}^{\infty} \frac{x^{2n-2}}{(n-1)!} |a(x)|\,dx$$
$$+ \frac{1}{r(t_1)} \int_{t_1}^{\infty} \frac{x^{n-1}}{(n-1)!} |f(x)|\,dx$$

(13), (14) and (20) yield a contradiction. The proof is now complete.

REMARK. The requirement that $r'(t) \ge 0$ can be improved. The same proof with hardly any change gives us the following

COROLLARY 3.1. *Suppose* (i), (iii), (iv), (v) *hold. Let* $r(t) \ge \alpha > 0$,

$$(21)\qquad \int^{\infty} 1/r(s) \int_{s}^{\infty} x^{2n-3} |a(x)|\,dx\,ds < \infty$$

and

$$(22)\qquad \int^{\infty} 1/r(s) \int_{s}^{\infty} x^{n-2} |f(x)|\,dx\,ds < \infty.$$

Then oscillatory solutions of equation (1) *approach zero as* $t \to \infty$.

The following example shows that given the conditions on $r(t)$ and $f(t)$, the condition on $a(t)$ cannot be violated.

EXAMPLE 3.1. The equation

$$(23)\qquad y''(t) + \frac{5}{4t^2} y(t) = \frac{13}{4} \cdot \frac{1}{t^3}$$

has the oscillatory solution $\sqrt{t}\sin(\ln t) + \frac{1}{t}$. Only the condition on $a(t)$ is violated.

EXAMPLE 3.2. The equation

$$y^{(4)}(t) + e^{-t}y(\sqrt{t}) = -4e^{-t}\sin t + e^{-t-\sqrt{t}}\sin(\sqrt{t}) \tag{24}$$

for $t>0$ satisfies the conditions and conclusion of Theorem 3.2. It has $y(t)=e^{-t}\sin t$ as an oscillatory solution converging to zero as $t\to\infty$.

4. Necessary and sufficient condition

THEOREM 4.1. *Suppose* $a(t)>0$ *and* $\int^{\infty} t^{2n-2}a(t)dt<\infty$ *for* $t\geq t_0$. *Further suppose that* $f(t)/(t^{n-1}a(t))$ *approaches a limit as* $t\to\infty$. *Then a necessary and sufficient condition for all oscillatory solutions of equation* (1) *to approach zero asymptotically is that*

$$\lim_{t\to\infty}\left(\frac{f(t)}{t^{n-1}a(t)}\right) = 0. \tag{25}$$

PROOF. (Sufficiency). Suppose (25) holds. Then $t^{n-1}|f(t)|<t^{2n-2}a(t)$ for sufficiently large t. Since $\int^{\infty} t^{2n-2}a(t)dt<\infty$, we have $\int^{\infty} t^{n-1}|f(t)|dt<\infty$ and the conclusion follows by theorem 3.2.

(Necessity). Let $y(t)$ be an oscillatory solution of equation (1) approaching zero as $t\to\infty$. Suppose that

$$\lim_{t\to\infty}\frac{|f(t)|}{t^{n-1}a(t)} \geq \beta > 0.$$

Dividing equation (1) by $(t^{n-1}a(t))$ and taking the limit as $t\to\infty$ we find that $(r(t)y'(t))^{(n-1)}$ has one sign to the right of sufficiently large t. This forces $y'(t)$ to eventually assume a constant sign which contradicts the fact that $y(t)$ is oscillatory. The proof is, now, complete.

EXAMPLE 4.1. Consider the equation

$$y''(t) + e^{-t}y(\sqrt{t}) = -2e^{-t}\cos t + e^{-t-\sqrt{t}}\sin(\sqrt{t}) \tag{26}$$

for $t>0$. Since condijtions of Theorem 4.1. hold, all oscillatory solutions of (25) approach zero. $y(t)=e^{-t}\sin t$ is one such solution.

REMARK. Our next theorem is recapitulation of Theorem 3.2 in terms of the ratio $f(t)/a(t)$ for $a(t)>0$.

THEOREM 4.2. *Suppose* $\int^{\infty} t^{2n-2}a(t)dt<\infty$ *and* $(f(t)/t^{n-1}a(t))$ *is bounded. Then all oscillatory solutions of equation* (1) *tend to zero as* $t\to\infty$.

PROOF. As was observed in the sufficiency part of Theorem 4.1. we have $\int^{\infty} t^{2n-2}a(t)dt<\infty$ and $\int^{\infty} t^{n-1}|f(t)|dt<\infty$. Theorem 3.2 applies and the proof is complete.

EXAMPLE 4.2. The equation

$$y''(t) + te^{-t}y(t) = e^{-t}\sin t^2 - 4te^{-t}\cos t^2 + 2e^{-t}\cos t^2 - 4t^2e^{-t}\sin t^2 + te^{-2t}\sin t^2 \tag{27}$$

has $y(t)=e^{-t}\sin t^2$ as an oscillatory solution. We notice that $(f(t)/ta(t))$ is bounded. We also notice that this ratio does not approach a limit as $t\to\infty$. Thus equation (27) satisfies conditions and conclusion of Theorem 4.2 but not of Theorem 4.1.

THEOREM 4.3. *Suppose for* $t\geq t_0$, $a(t)>0$, *the ratio* $(f(t)/t^{n-1}a(t))$ *is bounded, and* $g'(t)\geq 0$. *Further suppose n is even and*

$$y^{(n)}(t) + t^{n-1}a(t)h(y(g(t))) = 0 \tag{28}$$

has a bounded nonoscillatory solution. Then all oscillatory solutions of equation (1) *approach zero as* $t\to\infty$.

PROOF. By a well known result (see this author [5]) we must have

$$\int^{\infty} t^{2n-2}a(t)dt<\infty.$$

Since all the conditions of Theorem 4.2 now hold, the proof follows.

THEOREM 4.4. *Suppose* $a(t)>0$ *for* $t\geq t_0$ *and* $\liminf_{t\to\infty}\frac{|f(t)|}{a(t)}>0$. *Let* $y(t)$ *be an oscillatory solution of equation* (1). *Then* $\limsup_{t\to\infty}|y(t)|>0$.

PROOF. Suppose to the contrary that $y(t)\to 0$ as $t\to\infty$. Dividing equation (1) by $a(t)$ and taking limit we get $(r(t)y'(t))^{(n-1)}$ has same sign eventually. But this forces $y(t)$ to be nonoscillatory. This contradiction completes the proof of this theorem.

EXAMPLE 4.3. Consider the equation

$$y''(t) + y(t-2\pi) = \frac{1}{2} \tag{29}$$

$$\liminf_{t\to\infty}\frac{|f(t)|}{a(t)} = \frac{1}{2}.$$

all oscillatory solutions of (29) satisfy the conclusion of Theorem 4.4. In fact $y(t)=\frac{1}{2}+\sin t$ is one such solution.

REMARK. Notice that none of the conditions of previous theorems are needed here.

THEOREM 4.5. *Suppose* $a(t)>0$ *for* $t\geq t_0$, $\int^{\infty} t^{2n-2}a(t)dt<\infty$, $(f(t)/t^{n-1}a(t))$ *is bounded and* $\liminf_{t\to\infty}\frac{|f(t)|}{t^{n-1}a(t)}>0$. *Then all solutions of equation* (1) *are nonoscillatory.*

PROOF. Suppose to the contrary that a solution $y(t)$ of equation (1) is oscillatory. Since conditions of Theorem 4.2 hold $y(t)\to 0$ as $t\to\infty$. We also observe that

$$\liminf_{t\to\infty}\frac{|f(t)|}{a(t)}\geq\liminf_{t\to\infty}\frac{|f(t)|}{t^{n-1}a(t)}>0.$$

By Theorem 4.4, the conclusion is obvious.

EXAMPLE 4.4. All solutions of equation

$$y''(t)+\frac{1}{t^9}y(t)=\frac{1}{t^8},\ t>0 \tag{30}$$

are nonoscillatory. In fact $y(t)=t$ is one such solution. Here all conditions of Theorem 4.5 hold.

5. Discussion

Kusano and Onose [1, Theo.5] showed that all oscillatory solutions of $(r(t)y'(t))'+a(t)h(y(g(t)))=f(t)$ approach zero asymptotically subject to the conditions:

$$\int^{\infty} t|a(t)|dt<\infty \quad\text{and}\quad \int^{\infty} t|f(t)|dt<\infty.$$

This seems to suggest that the condition $\int^{\infty} t^{2n-2}|a(t)|dt<\infty$ could possibly be weakened to $\int^{\infty} t^{n-1}|a(t)|dt<\infty$. However we have not been able to prove or disprove it. From proof of Theorem 3.2 in inequality (19), it follows that this later conclusion is true for bounded solutions of (1). We state this as a theorem.

THEOREM 5.1. *Suppose* $\int^{\infty} t^{n-1}|a(t)|dt<\infty$ *and* $\int^{\infty} t^{n-1}|f(t)|dt<\infty$ *then bounded oscillatory solutions of equation* 1 *approach zero at* $t\to\infty$.

REMARK. This theorem extends our Theorem 3.1 in [6].

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