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Continuity of contractions in a functional Banach space

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In the Dirichlet space theory, contractions on the real line play an important role in connection with potential theoretic properties. A. Ancona [1] proved that contractions are continuous in Dirichlet space. Our aim in this note is to prove that the contractions considered in [3] are continuous in a certain functional Banach space.

Let X be a locally compact space and ξ be a positive (Radon) measure on X. For measurable functions u and v on X, we define

$$u \vee v = \max\{u, v\}, \quad u \wedge v = \min\{u, v\},$$

$$u^+ = u \vee 0 \quad \text{and} \quad u^- = -(u \wedge 0).$$

Let $\mathscr{X} = \mathscr{X}(X; \xi)$ be a real reflexive Banach space whose elements are measurable functions on X. We denote by $\|u\|$ the norm of $u \in \mathscr{X}$, by $\mathscr{X}^*$ the dual space of $\mathscr{X}$, and by $\langle u^*, u\rangle$ the value of $u^* \in \mathscr{X}^*$ at $u \in \mathscr{X}$.

Throughout this note, let Φ be a strictly convex function on $\mathscr{X}$ such that

(i) $\Phi(u) \geqq 0$ for all $u \in \mathscr{X}$ and $\Phi(u) = 0$ if and only if $u = 0$;

(ii) if $\{u_n\} \subset \mathscr{X}$ and $\lim_{n\to\infty} \Phi(u_n) = 0$, then $u_n \to 0$ in $\mathscr{X}$;

(iii) Φ is bounded on each bounded subset of $\mathscr{X}$; and

(iv) Φ is differentiable in the sense of Gâteaux, i.e., there is an operator $G: \mathscr{X} \to \mathscr{X}^*$ such that for any $u, v \in \mathscr{X}$,

$$\langle Gu, v\rangle = \lim_{t \downarrow 0} \frac{\Phi(u+tv) - \Phi(u)}{t}.$$

The operator G is called the gradient of Φ and denoted by $\nabla\Phi$.

We shall use the following elementary properties of Φ and $\nabla\Phi$ without proof:

(Φ_1) Let $u \in \mathscr{X}$ and $u^* \in \mathscr{X}^*$. Then $u^* = \nabla\Phi(u)$ if and only if

$$\langle u^*, v - u\rangle \leqq \Phi(v) - \Phi(u) \qquad \text{for any} \quad v \in \mathscr{X}.$$

(Φ_2) $\nabla\Phi$ is bounded, i.e., it maps bounded sets in $\mathscr{X}$ to bounded sets in $\mathscr{X}^*$.

For a non-negative measurable function g on X, we define an operator T_g^+ by

$$T_g^+ u = u^+ \wedge g \qquad \text{for} \quad u \in \mathscr{X}.$$

The operator $T^+=T_g^+$ with $g\equiv\infty$ will be called the positive contraction. We shall say that T_g^+ operates in $\mathscr{X}$ (with respect to Φ) if $T_g^+u\in\mathscr{X}$ and

$$\Phi(u+T_g^+(v-u))+\Phi(v-T_g^+(v-u))\leqq\Phi(u)+\Phi(v)$$

for any $u, v\in\mathscr{X}$. If T_g^+ operates in $\mathscr{X}$, then it is continuous at $0\in\mathscr{X}$. From this it follows that if the positive contraction T^+ operates in $\mathscr{X}$ and $u_n\to u$ in $\mathscr{X}$, then $u\wedge u_n\in\mathscr{X}$ and $u\wedge u_n\to u$ in $\mathscr{X}$.

Hereafter we assume that $\mathscr{X}$ is a functional space, i.e., the following axiom is satisfied (cf. [2]):

AXIOM a. For any compact set $K\subset X$, there exists a positive constant M such that

$$\int_K |u|d\xi\leqq M\|u\| \qquad \text{for all} \quad u\in\mathscr{X}.$$

LEMMA 1. *If T_g^+ operates in $\mathscr{X}$ and $u_n\to u$ in $\mathscr{X}$, then $T_g^+u_n\to T_g^+u$ weakly in $\mathscr{X}$.*

PROOF. It is easy to see that T_g^+ is a bounded operator in $\mathscr{X}$, so that $\{T_g^+u_n\}$ is bounded. By using Axiom a, we see, in the same way as [2; Lemma 2.1], that $T_g^+u_n\to T_g^+u$ weakly in $\mathscr{X}$.

In the same way as [2; Proposition 2.1], we have the next lemma.

LEMMA 2. *The contraction T_g^+ operates in $\mathscr{X}$ if and only if $T_g^+u\in\mathscr{X}$ and*

$$\langle\nabla\Phi(u+T_g^+v)-\nabla\Phi(u), v-T_g^+v\rangle\geqq 0$$

for any $u, v\in\mathscr{X}$.

LEMMA 3. *Let $\{u_n\}\subset\mathscr{X}$ be a sequence converging to $u\in\mathscr{X}$ and set $v_n=u\wedge u_n$. If T^+ operates in $\mathscr{X}$, then $T^+v_n\to T^+u$ in $\mathscr{X}$.*

PROOF. By (Φ_1) we have

$$\begin{aligned}\Phi(u^+-v_n^+)&\leqq\langle\nabla\Phi(u^+-v_n^+), u^+-v_n^+\rangle\\&=\langle\nabla\Phi(u^+-v_n^+), u-v_n\rangle+\langle\nabla\Phi(u^+-v_n^+), u^--v_n^-\rangle.\end{aligned}$$

Since $v_n\to u$ in $\mathscr{X}$ and $\{\nabla\Phi(u^+-v_n^+)\}$ is bounded in $\mathscr{X}^*$ by virtue of (Φ_2), the first term tends to zero as $n\to\infty$. Since $v_n^+\wedge(v_n^--u^-)=0, (u^+-v_n^+)+w^+=u^+$ and $w-w^+=u^--v_n^-$, where $w=v_n+u^-$. Hence by Lemma 2, we obtain

$$\limsup_{n\to\infty}\langle\nabla\Phi(u^+-v_n^+), u^--v_n^-\rangle\leqq\limsup_{n\to\infty}\langle\nabla\Phi(u^+), u^--v_n^-\rangle=0.$$

It follows that $\limsup_{n\to\infty} \Phi(u^+ - v_n^+) \leqq 0$, which implies that $v_n^+ \to u^+$ in $\mathscr{X}$ on account of (ii).

COROLLARY. *If T^+ operates in $\mathscr{X}$, then T^+ is continuous.*

PROOF. Let $\{u_n\}$ be a sequence in $\mathscr{X}$ which converges to $u \in \mathscr{X}$. Then, by the above lemma we find that

$$(u \vee u_n)^+ = ((-u) \wedge (-u_n))^+ - (-u) \wedge (-u_n) \longrightarrow (-u)^+ - (-u) = u^+$$

in $\mathscr{X}$. Hence we have again by Lemma 3 that

$$u_n^+ = (u \vee u_n)^+ + (u \wedge u_n)^+ - u^+ \longrightarrow u^+ \quad \text{in} \quad \mathscr{X},$$

which means that T^+ is continuous.

Now we are ready to prove our main result.

THEOREM. *If T^+ and T_g^+ operate in $\mathscr{X}$, then T_g^+ is continuous.*

PROOF. Suppose $u_n \to u$ in $\mathscr{X}$ and $u_n \wedge 0 = 0$ for each n. Set $w_n = u \vee u_n$. Then $w_n \to u$ in $\mathscr{X}$, and hence $(w_n - u) \wedge g \to 0$ in $\mathscr{X}$ by the continuity of T_g^+ at 0. Using (Φ_1), we have

$$\begin{aligned}\Phi(u \wedge g - w_n \wedge g) &\leqq \langle \nabla\Phi(u \wedge g - w_n \wedge g), u \wedge g - w_n \wedge g\rangle \\ &= \langle \nabla\Phi(u \wedge g - w_n \wedge g), (u \wedge g + (w_n - u) \wedge g) - w_n \wedge g\rangle \\ &\quad - \langle \nabla\Phi(u \wedge g - w_n \wedge g), (w_n - u) \wedge g\rangle .\end{aligned}$$

Since $T_g^+(u \wedge g + (w_n - u) \wedge g) = w_n \wedge g = T_g^+ w_n$, Lemma 2 yields

$$\begin{aligned}&\limsup_{n\to\infty} \langle \nabla\Phi(u \wedge g - w_n \wedge g), (u \wedge g + (w_n - u) \wedge g) - w_n \wedge g\rangle \\ &\leqq \limsup_{n\to\infty} \langle \nabla\Phi(u \wedge g), u \wedge g + (w_n - u) \wedge g - w_n \wedge g\rangle = 0\end{aligned}$$

with the aid of Lemma 1. Hence $\limsup_{n\to\infty} \Phi(u \wedge g - w_n \wedge g) \leqq 0$, which implies that $w_n \wedge g \to u \wedge g$ in $\mathscr{X}$ by (ii). If we write

$$u_n \wedge g = w_n \wedge g + (u \wedge g) \wedge u_n - u \wedge g,$$

then we see that $u_n \wedge g \to u \wedge g$ in $\mathscr{X}$ by using the fact that $v \wedge u_n \to v \wedge u$ in $\mathscr{X}$ for $v \in \mathscr{X}$ because T^+ is continuous. Thus our theorem is proved.

References

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