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Scalar curvatures of left invariant metrics on some Lie groups

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In [1], J. Milnor gave many facts concerning curvatures of left invariant metrics on Lie groups. About scalar curvatures, he showed that

(1) if a Lie group G is solvable, then every left invariant metric on G is either flat, or else has strictly negative scalar curvature, and

(2) every left invariant metric on $SL(2, \mathbf{R})$ has strictly negative scalar curvature.

And he conjectured that if the universal covering group of a Lie group G is homeomorphic to Euclidean space then the conclusion of (1) holds. In this note we shall show that this conjecture is affirmative, that is, we have the following

Theorem. *Let G be a Lie group such that the universal covering group of G is homeomorphic to Euclidean space. Then every left invariant metric on G is either flat or else has strictly negative scalar curvature.*

Let G be a Lie group with a left invariant metric $\langle\ ,\ \rangle$ and H a closed normal subgroup. In this note, we always consider the left invariant metrics $\langle\ ,\ \rangle_H$ on H and $\langle\ ,\ \rangle_{G/H}$ on G/H obtained from the metric of G naturally, so that the natural embedding from H into G is an isometry and the natural projection π from G to G/H is a submersion. We denote the sectional curvatures of G, G/H and H by κ, κ_* and $\bar{\kappa}$, and the scalar curvatures by $\rho(G)$, $\rho(G/H)$ and $\rho(H)$ respectively.

Lemma 1. *Let G be a Lie group whose universal covering group is homeomorphic to Euclidean space and $\mathfrak{g}$ its Lie algebra. If G is not solvable, then $\mathfrak{g}=\mathfrak{s}_1+\mathfrak{g}_0$ (direct sum) where $\mathfrak{s}_1$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ such that the connected simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_0$ is homeomorphic to Euclidean space.*

Proof. Let $\mathfrak{g}=\mathfrak{s}+\mathfrak{r}$ (direct sum) be a Levi decomposition, where $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{r}$ is the radical of $\mathfrak{g}$. By the assumption, $\mathfrak{s}\neq 0$ and the connected simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{s}$ is homeomorphic to Euclidean space. Because of the fact that a connected simply connected simple Lie group homeomorphic to Euclidean space is locally isomorphic to $SL(2, \mathbf{R})$, we have $\mathfrak{s}=\mathfrak{s}_1+\cdots+\mathfrak{s}_l$ where $\mathfrak{s}_i$ $(i=1,\ldots, l)$ are ideals of $\mathfrak{s}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. We put $\mathfrak{g}_0=\mathfrak{s}_2+\cdots+\mathfrak{s}_l+\mathfrak{r}$. Then $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ and the corresponding connected simply connected Lie group is homeomorphic to Euclidean space. Q.E.D.

LEMMA 2. *Let G be a Lie group with a left invariant metric and H a closed normal subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Let $\{e_1,\ldots, e_n\}$ be an orthonormal base of $\mathfrak{g}$ such that e_α $(\alpha=r+1,\ldots, n)$ are in $\mathfrak{h}$ where r is the dimension of G/H. Then for $1\leqq s, t\leqq r$,*

$$\kappa(e_s, e_t) = \kappa_*(\pi_*(e_s), \pi_*(e_t)) - \frac{3}{4}\|[e_s, e_t]_{\mathfrak{h}}\|^2,$$

where $[e_s, e_t]_{\mathfrak{h}}$ is the $\mathfrak{h}$-component of $[e_s, e_t]$.

PROOF. See Corollary 1 in [2].

LEMMA 3. *Let G be a Lie group with a left invariant metric $\langle\ ,\ \rangle$ and $\mathfrak{g}$ its Lie algebra. If $\mathfrak{g}=\mathfrak{s}_1+\mathfrak{g}_1$ (direct sum) where $\mathfrak{s}_1$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$. Then there is an orthonormal base $\{e_1,\ldots, e_n\}$ of $\mathfrak{g}$ such that e_α $(4\leqq\alpha\leqq n)$ are in $\mathfrak{g}_1$ and $\langle[e_t, e_s], e_s\rangle=0$ for $1\leqq s, t\leqq 3$.*

PROOF. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{g}_1$ in $\mathfrak{g}$. By (4.2) in [1], there is an orthonormal base $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of $\mathfrak{g}/\mathfrak{g}_1$ such that $\langle[\bar{e}_t, \bar{e}_s], \bar{e}_s\rangle_{G/G_1}=0$ for $1\leqq s, t\leqq 3$. The restriction of π_* to $\mathfrak{p}$ is an isometry. So we can choose an orthonormal base $\{e_1, e_2, e_3\}$ of $\mathfrak{p}$, with $\pi_* e_s=\bar{e}_s$. Let $\{e_4,\ldots, e_n\}$ be an orthonormal base of $\mathfrak{g}_1$. Then $\{e_1,\ldots, e_n\}$ is an orthonormal base of $\mathfrak{g}$ and $\langle[e_t, e_s], e_s\rangle=0$ for $1\leqq s, t\leqq 3$. Q.E.D.

PROOF OF THEOREM. Let G be a Lie group homeomorphic to Euclidean space, and $\mathfrak{g}$ its Lie algebra. If G is solvable, then the theorem was proved by J. Milnor ([1]). Assume G is not solvable. Then, by Lemma 1, $\mathfrak{g}=\mathfrak{s}_1+\mathfrak{g}_0$ (direct sum) where $\mathfrak{s}_1$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$. Let G_0 be the analytic subgroup of G whose Lie algebra is $\mathfrak{g}_0$. Then G_0 is a closed normal subgroup of G and homeomorphic to Euclidean space. We give a left invariant metric on G, and choose the left invariant metrics on G/G_0 and G_0 described before. By Lemma 3, let $\{e_1,\ldots, e_n\}$ be an orthonormal base of $\mathfrak{g}$ such that e_α $(4\leqq\alpha\leqq n)$ are in $\mathfrak{g}_0$ and $\langle[e_t, e_s], e_s\rangle=0$ for $1\leqq s, t\leqq 3$. Let L_i denote the linear transformation $\mathrm{ad}(e_i)$ on $\mathfrak{g}$, so that $L_i x=[e_i, x]$ for x in $\mathfrak{g}$. Let L_i^* denote the adjoint transformation of L_i. Then using the equations

$$\kappa(e_i, e_j) = \langle\nabla_{[e_i,e_j]}e_i - \nabla_{e_i}\nabla_{e_j}e_i + \nabla_{e_j}\nabla_{e_i}e_i, e_j\rangle,$$

$$\langle\nabla_x y, z\rangle = \frac{1}{2}(\langle[x, y], z\rangle - \langle[y, z], x\rangle + \langle[z, x], y\rangle),$$

we get the following equation:

$$\kappa(e_i, e_j) = -\frac{3}{4}\langle L_i e_j, L_i e_j\rangle - \frac{1}{2}\langle L_i e_j, L_i^* e_j\rangle - \frac{1}{2}\langle L_j e_i, L_j^* e_i\rangle$$

$$+\frac{1}{4}\langle L_i^* e_j + L_j^* e_i, L_i^* e_j + L_j^* e_i\rangle - \langle L_i^* e_i, L_j^* e_j\rangle .$$

Hence, for $4\leqq\alpha, \beta\leqq n$ $(\alpha\neq\beta)$,

$$\begin{aligned}
&\kappa(e_\alpha, e_\beta)\\
&= -\frac{3}{4}\langle L_\alpha e_\beta, L_\alpha e_\beta\rangle - \frac{1}{2}\langle L_\alpha e_\beta, L_\alpha^* e_\beta\rangle - \frac{1}{2}\langle L_\beta e_\alpha, L_\beta^* e_\alpha\rangle\\
&\quad + \frac{1}{4}\langle L_\alpha^* e_\beta + L_\beta^* e_\alpha, L_\alpha^* e_\beta + L_\beta^* e_\alpha\rangle - \langle L_\alpha^* e_\alpha, L_\beta^* e_\beta\rangle\\
&= -\frac{3}{4}\langle \bar{L}_\alpha e_\beta, \bar{L}_\alpha e_\beta\rangle_{G_0} - \frac{1}{2}\langle \bar{L}_\alpha e_\beta, \bar{L}_\alpha^* e_\beta\rangle_{G_0} - \frac{1}{2}\langle \bar{L}_\beta e_\alpha, \bar{L}_\beta^* e_\alpha\rangle_{G_0}\\
&\quad + \frac{1}{4}\langle \bar{L}_\alpha^* e_\beta + \bar{L}_\beta^* e_\alpha, \bar{L}_\alpha^* e_\beta + \bar{L}_\beta^* e_\alpha\rangle_{G_0} - \langle \bar{L}_\alpha^* e_\alpha, \bar{L}_\beta^* e_\beta\rangle_{G_0}\\
&\quad + \frac{1}{4}\textstyle\sum_{s=1}^3 \langle L_\alpha^* e_\beta + L_\beta^* e_\alpha, e_s\rangle^2 - \sum_{s=1}^3 \langle L_\alpha^* e_\alpha, e_s\rangle\langle L_\beta^* e_\beta, e_s\rangle\\
&= \bar{\kappa}(e_\alpha, e_\beta) + \frac{1}{4}\textstyle\sum_{s=1}^3 \langle L_\alpha^* e_\beta + L_\beta^* e_\alpha, e_s\rangle^2 - \sum_{s=1}^3 \langle L_\alpha^* e_\alpha, e_s\rangle\langle L_\beta^* e_\beta, e_s\rangle ,
\end{aligned}$$

and, for $1\leqq s\leqq 3$ and $4\leqq\alpha\leqq n$,

$$\begin{aligned}
\kappa(e_s, e_\alpha) = &-\frac{3}{4}\langle \bar{L}_s e_\alpha, \bar{L}_s e_\alpha\rangle_{G_0} - \frac{1}{2}\langle \bar{L}_s e_\alpha, \bar{L}_s^* e_\alpha\rangle_{G_0} + \frac{1}{4}\langle \bar{L}_s^* e_\alpha, \bar{L}_s^* e_\alpha\rangle_{G_0}\\
&+ \frac{1}{4}\langle L_s^* e_\alpha + L_\alpha^* e_s, e_s\rangle^2 + \frac{1}{4}\textstyle\sum_{t=1,t\neq s}^3\langle L_s^* e_\alpha + L_\alpha^* e_s, e_t\rangle^2\\
&- \textstyle\sum_{t=1,t\neq s}^3\langle L_s^* e_s, e_t\rangle\langle L_\alpha^* e_\alpha, e_t\rangle ,
\end{aligned}$$

where $\bar{L}_i$ is the linear transformation ad(e_i) restricted to $\mathfrak{g}_0$ and $\bar{L}_i^*$ is the adjoint transformation of $\bar{L}_i$. Put $\mathfrak{g}_i=\mathbf{R}\cdot e_i+\mathfrak{g}_0$ $(i=1, 2, 3)$. Then $\mathfrak{g}_i$ $(i=1, 2, 3)$ are subalgebras of $\mathfrak{g}$. Let G_1, G_2 and G_3 be the analytic subgroups of G corresponding to $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ respectively. We choose the induced metrics from G on G_1, G_2 and G_3. Let $\rho(G_1)$, $\rho(G_2)$ and $\rho(G_3)$ denote the scalar curvatures of G_1, G_2 and G_3 respectively. Then

$$\rho(G_s) = \rho(G_0) + \frac{1}{4} \sum_{\alpha,\beta=4,\alpha\neq\beta}^{n} \langle L_\alpha^* e_\beta + L_\beta^* e_\alpha, e_s\rangle^2$$

$$- \sum_{\alpha,\beta=4,\alpha\neq\beta}^{n} \langle L_\alpha^* e_\alpha, e_s\rangle \langle L_\beta^* e_\beta, e_s\rangle$$

$$+ 2 \sum_{\alpha=4}^{n} \kappa(e_s, e_\alpha) - \frac{1}{2} \sum_{\alpha=4}^{n} \sum_{t=1,t\neq s}^{3} \langle L_s^* e_\alpha + L_\alpha^* e_s, e_t\rangle^2$$

$$+ \sum_{\alpha=4}^{n} \sum_{t=1,t\neq s}^{3} \langle L_s^* e_s, e_t\rangle \langle L_\alpha^* e_\alpha, e_t\rangle .$$

Hence

$$\sum_{s=1}^{3} \rho(G_s)$$

$$= 2\rho(G_0) + \sum_{\alpha,\beta=4,\alpha\neq\beta}^{n} \bar{\kappa}(e_\alpha, e_\beta) + \frac{1}{4} \sum_{s=1}^{3} \sum_{\alpha,\beta=4,\alpha\neq\beta}^{n} \langle L_\alpha^* e_\beta + L_\beta^* e_\alpha, e_s\rangle^2$$

$$- \sum_{s=1}^{3} \sum_{\alpha,\beta=4,\alpha\neq\beta}^{n} \langle L_\alpha^* e_\alpha, e_s\rangle \langle L_\beta^* e_\beta, e_s\rangle + 2 \sum_{s=1}^{3} \sum_{\alpha=4}^{n} \kappa(e_s, e_\alpha)$$

$$- \frac{1}{2} \sum_{\alpha=4}^{n} \sum_{s,t=1,s\neq t}^{3} \langle L_s^* e_\alpha + L_\alpha^* e_s, e_t\rangle^2$$

$$+ 2\sum_{\alpha=4}^{n} \sum_{s,t=1,s\neq t}^{3} \langle L_s^* e_s, e_t\rangle \langle L_\alpha^* e_\alpha, e_t\rangle .$$

Since $\langle L_s^* e_s, e_t\rangle = 0$, we have

$$\sum_{\alpha,\beta=4,\alpha\neq\beta}^{n} \kappa(e_\alpha, e_\beta) + 2 \sum_{s=1}^{3} \sum_{\alpha=4}^{n} \kappa(e_s, e_\alpha)$$

$$= \sum_{s=1}^{3} \rho(G_s) - 2\rho(G_0) + \frac{1}{2} \sum_{\alpha=4}^{n} \sum_{s,t=1,s\neq t}^{3} \langle L_s^* e_\alpha + L_\alpha^* e_s, e_t\rangle^2$$

$$= \sum_{s=1}^{3} \rho(G_s) - 2\rho(G_0) + \frac{1}{2} \sum_{s,t=1,s\neq t}^{3} \| [e_s, e_t]_{\mathfrak{g}_0} \|^2 .$$

We put $S_i = \frac{1}{2}(\bar{L}_i + \bar{L}_i^*)$ $(i=1, 2, 3)$. Then, by Lemma 5.6 in [1],

$$\rho(G_i) = \rho(G_0) - \operatorname{trace}(S_i^2) - (\operatorname{trace} S_i)^2 .$$

Using Lemma 2, we obtain the following equality:

$$\rho(G) = \rho(G_0) + \rho(G/G_0) - \sum_{i=1}^{3} \operatorname{trace}(S_i^2) - \sum_{i=1}^{3} (\operatorname{trace} S_i)^2$$

$$- \frac{1}{4} \sum_{s,t=1,s\neq t}^{3} \| [e_s, e_t]_{\mathfrak{g}_0} \|^2 .$$

Hence

$$\rho(G) \leqq \rho(G_0) + \rho(G/G_0),$$

where the equality holds if and only if the space spanned by $\{e_1, e_2, e_3\}$ is a subalgebra of $\mathfrak{g}$ and $\bar{L}_i$ ($i=1, 2, 3$) are skew adjoint. By Corollary 4.7 in [1], $\rho(G/G_0)$ is strictly negative and by the induction hypothesis $\rho(G_0)$ is non-positive. So we have $\rho(G)$ is strictly negative. Q. E. D.

References

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