HIROSHIMA MATH. J.
**10** (1980), 311–322

# Modularity in Lie algebras

Ralph K. AMAYO and Jochen SCHWARZ

(Received October 31, 1979)

A subalgebra $M$ of a Lie algebra $L$ is termed modular in $L$ ($M \mathrm{m} L$) if $M$ is a modular element in the lattice formed by the subalgebras of $L$, i.e., if

( $*$ ) $\langle M, U\rangle \cap V = \langle U, M \cap V\rangle$ for all $U, V \leqq L$ with $U \leqq V$ and

($**$) $\langle M, U\rangle \cap V = \langle U \cap V, M\rangle$ for all $U, V \leqq L$ with $M \leqq V$ hold.

Simple examples for modular subalgebras of a Lie algebra $L$ are the quasi-ideals of $L$—$Q \leqq L$ is called a quasi-ideal of $L$ ($Q \mathrm{q} L$) if $Q$ is permutable with every subspace $R$ of $L$, i.e., if $[Q, R] \subseteq Q+R$ for all $R \subseteq L$ ([1], p. 28).

That the reverse implication is not true is shown by the Lie algebra $L$ ($L= \langle e\rangle + \langle f\rangle + \langle g\rangle$) defined over a field containing no pair of elements $\alpha$, $\beta$ such that $\alpha^2+\beta^2=-1$, with the following multiplication: $[e,f]=g$, $[f,g]=e$, $[g,e]=f$. $L$ is simple, and every one-dimensional subalgebra of $L$ is maximal and modular in $L$, but not a quasi-ideal of $L$.

We prove the following ($M_L$ denotes the core of $M$ in $L$):

(i) A modular subalgebra $M$ of a Lie algebra $L$ permutable with a solvable subalgebra $A$ of $L$ is a quasi-ideal of $M+A$ — in particular $M$ is a quasi-ideal of $L$ if $L$ is solvable.

(ii) A modular subalgebra $M$ of a finite-dimensional Lie algebra $L$ over any field of characteristic zero is either

a) an ideal of $L$; or

b) $L/M_L$ is metabelian, every subalgebra of $L/M_L$ is a quasi-ideal, $M/M_L$ is one-dimensional and is spanned by an element which acts as the identity map on $([L, L]+M_L)/M_L$; and $L/([L, L]+M_L)$ is one-dimensional; or

c) $M/M_L$ is two-dimensional and $L/M_L$ is the three-dimensional split simple Lie algebra; or

d) $M/M_L$ is a one-dimensional maximal subalgebra of $L/M_L$ and $L/M_L$ is a three-dimensional non-split simple Lie algebra.

## 1. Elementary properties of modular subalgebras

The properties 1.1–1.3 hold for modular elements in more general lattices; proofs can be found in [9], where the modular elements are called "Dedekind elements."

PROPOSITION 1.1. *Let $M$ be modular in a Lie algebra $L$ and let $U$ be a* *subalgebra of L. Then* $M \cap U$ *is modular in* $U$ ([9], III, p. 74).

PROPOSITION 1.2. *Let M be modular in L and let I be an ideal of L with* $I \leqq M$. *Then* $M/I$ *is modular in* $L/I$ ([9], IV, p. 75).

PROPOSITION 1.3. *Let M and N be modular in a Lie algebra L. Then* $\langle M, N\rangle$ *is modular in L* ([9], V, p. 75).

LEMMA 1.4. *Let M be modular in L, Q be a quasi-ideal of L, and* $\phi: L \to L'$ *a homomorphism of Lie algebras. Then* $\phi(M)$ *is modular in* $\phi(L)$, *and* $\phi(Q)$ *is a quasi-ideal of* $\phi(L)$.

LEMMA 1.5. *Let M be modular in a Lie algebra L. Then*

a) *M is a maximal subalgebra of* $\langle M, x\rangle$ *for all* $x \in L\backslash M$.

b) $I_L(M)$, *the idealizer of M in L, is either L or M.*

PROOF. a) If $M \leqq N \leqq \langle M, x\rangle$, then by (**) $N = N \cap \langle M, x\rangle = \langle N \cap \langle x\rangle, M\rangle$, hence $N = M$ or $N = \langle M, x\rangle$.

b) Suppose $M \lneqq I_L(M) \lneqq L$ and let $x \in I_L(M)\backslash M$, $y \in L\backslash I_L(M)$. Then $M \neq M + [y, M] = M + [x+y, M]$ and $M$ is a maximal subalgebra of $\langle y, M\rangle$ and of $\langle x+y, M\rangle$ due to a). Now

$$M \lneqq \langle M, [x+y, M]\rangle \leqq \langle x+y, M\rangle \quad \text{and}$$

$$M \lneqq \langle M, [y, M]\rangle \leqq \langle y, M\rangle .$$

Using a) we get

$$\langle x+y, M\rangle = \langle M, [x+y, M]\rangle = \langle M, [y, M]\rangle = \langle y, M\rangle .$$

This means $x+y \in \langle y, M\rangle$, hence $x \in \langle y, M\rangle$. Therefore, $x \in I_L(M) \cap \langle y, M\rangle = \langle I_L(M) \cap \langle y\rangle, M\rangle = M$ by (**), which contradicts our assumption. □

LEMMA 1.6. *Let M be modular in a Lie algebra L and let U be a subalgebra of L with* $\langle M, U\rangle = M + U$. *Then*

a) *M is permutable with all quasi-ideals W of U.*

b) *Every* $V \leqq L$ *with* $U \cap M \leqq V \leqq U$ *is permutable with M.*

c) *If in addition,* $U \cap M$ *is a quasi-ideal of U, then M is a quasi-ideal of* $M + U$.

PROOF. a) As $W \mathrm{q} U$ and $\langle U, M\rangle = U + M$ we have $\langle W, M\rangle = U \cap \langle W, M\rangle + M = \langle W, M \cap U\rangle + M = W + M \cap U + M = W + M$ by (*).

b) $\langle V, M\rangle = U \cap \langle V, M\rangle + M = \langle V, U \cap M\rangle + M = V + M$ due to (*).

c) Let $x = u + m$; $u \in U$, $m \in M$. As $(U \cap M) \mathrm{q} M$ we have by (*) $\langle x, M\rangle = \langle u, M\rangle = \langle u, U \cap M\rangle + M = \langle u\rangle + U \cap M + M = \langle u\rangle + M = \langle x\rangle + M$. □

PROPOSITION 1.7. *Let $M$ be modular in a Lie algebra $L$ and $A$ be a solvable subalgebra of $L$. Then $A \cap M$ is a quasi-ideal of $A$; more precisely: $A \cap M$ is an ideal of $A$ or* $\dim A/A \cap M = 1$ *or* $A = [A, A] + A \cap M$, $[[A, A], [A, A]] \leqq A \cap M = \langle m \rangle + (A \cap M)_A$ *with* $m \in M$ *and* $[a, m] = a \pmod{(A \cap M)_A}$ *for all* $a \in [A, A]$.

PROOF. Let $a \in A \setminus A \cap M$. $B := \langle a, A \cap M \rangle$ is solvable, and by (*) it follows that

$$B = B \cap (A \cap \langle a, M \rangle) = B \cap \langle a, M \rangle = \langle a, B \cap M \rangle.$$

If we pick $r \geqq 1$, $r \in \mathbf{N}$, maximal with respect to $B^{(r)} \nleqq M$, we have $M \lneqq \langle B^{(r)}, M \rangle \leqq \langle B, M \rangle \leqq \langle a, M \rangle$ and $\langle B^{(r)}, M \rangle = \langle a, M \rangle$ by Lemma 1.5a); now (using (*))

$$B = B \cap \langle B^{(r)}, M \rangle = \langle B^{(r)}, M \cap B \rangle = B^{(r)} + M \cap B \quad \text{and}$$
$$B^{(r+1)} \leqq B^{(r)} \cap M \lhd B^{(r)}.$$

Thus $A \cap M = B \cap M$ is a quasi-ideal of $B$ due to Lemma 1.6c). In particular, $B = \langle a, B \cap M \rangle = \langle a \rangle + B \cap M = \langle a \rangle + A \cap M$. As $a$ was arbitrary, we have $(A \cap M)\, \mathrm{q}\, A$. The additional remarks of the proposition are a direct consequence of Theorem 3.6 in [1]. □

COROLLARY 1.8. *Let $M$ be modular in $L$ and let $A$ be a solvable subalgebra of $L$ permutable with $M$. Then $M$ is a quasi-ideal of $M + A$.*

The following are two convenient technical lemmas:

LEMMA 1.9. *Let $M$ be a modular and maximal subalgebra of a Lie algebra $L$. Then* $\dim V \leqq 1$ *for every subalgebra $V$ of $L$ with $M \cap V = 0$.*

PROOF. Suppose $M \cap V = 0$ for $V \leqq L$ with $\dim V > 1$, and let $0 \neq u \in V$. But now (*) does not hold, which contradicts $M\, \mathrm{m}\, L$:

$$\langle u, M \rangle \cap V = L \cap V = V \neq \langle u \rangle = \langle u, M \cap V \rangle. \qquad \square$$

LEMMA 1.10. *Let $M$ be a modular and maximal subalgebra of a Lie algebra $L$. Then $M \cap U$ is a modular and maximal subalgebra of $U$ for every subalgebra $U$ of $L$ with $U \nleqq M$.*

PROOF. We have $(M \cap U)\, \mathrm{m}\, U$ by Proposition 1.1. If $x \in U \setminus (M \cap U)$, $\langle x, M \rangle = L$ holds and by (*) it follows that $U = U \cap L = U \cap \langle x, M \rangle = \langle x, U \cap M \rangle$. □

## 2. The finite-dimensional case in characteristic zero

All Lie algebras in this section are finite dimensional and defined over an arbitrary field of characteristic zero. The following lemma is a slight extension of a theorem of Chevalley, Tuck and Towers, which occurs for $V=A$ ([8], pp. 443–444).

LEMMA 2.1. *Let $U$ and $V$ be subspaces of a finite-dimensional algebra $A$ over any field of characteristic zero with $U\subseteq V$. If $U$ is invariant under all automorphisms $\alpha$ of $A$ with $\alpha(V)=V$, then $U$ is invariant under all derivations $d$ of $A$ with $d(V)\subseteq V$. In particular if $A$ is a Lie algebra and $V$ a subalgebra of $A$, then $U$ is an ideal of $V$.*

PROOF. The Lie algebra $\mathrm{L}(\mathrm{Aut}(A))$ of the algebraic group $\mathrm{Aut}(A)$ of the automorphisms of $A$ coincides with the derivation algebra $\mathrm{D}(A)$ of $A$, a subalgebra of $\mathfrak{gl}(A)$ ([4], p. 179 and p. 128). The automorphisms of $A$ which leave $V$ invariant form an algebraic group ($:=\mathrm{AUT}_V(A)$), whose Lie algebra $\mathrm{L}(\mathrm{AUT}_V(A))$ consists of all endomorphisms of the vector space $A$ mapping $V$ into $V$ ([4], pp. 144–145). Thus $\mathrm{Aut}(A)\cap\mathrm{AUT}_V(A)$ is an algebraic group ([4], p. 79); these are the algebra-automorphisms of $A$ which leave $V$ invariant. Since $\mathrm{L}(\mathrm{Aut}(A)\cap\mathrm{AUT}_V(A))=\mathrm{L}(\mathrm{Aut}(A))\cap\mathrm{L}(\mathrm{AUT}_V(A))$ ([4], p. 171, 172), we have $\mathrm{L}(\mathrm{Aut}(A)\cap\mathrm{AUT}_V(A))=\mathrm{D}_V(A):=\{d\in\mathrm{D}(A);\ d(V)\subseteq V\}\leqq\mathrm{D}(A)$. Hence $\mathrm{Aut}(A)\cap\mathrm{AUT}_V(A)$ is contained in $\mathfrak{g}$, where

$$\mathfrak{g}:=\{G;\ G \text{ is an algebraic group of automorphisms of the vector space } A \text{ with } \mathrm{L}(G)\geqq\mathrm{D}_V(A)\}.$$

Let $\mathrm{G}(A)=\cap\{G;\ G\in\mathfrak{g}\}$. Then $\mathrm{G}(A)\leqq\mathrm{Aut}(A)\cap\mathrm{AUT}_V(A)$. $\mathrm{G}(A)$ is an irreducible algebraic group with $\mathrm{L}(\mathrm{G}(A))\geqq\mathrm{D}_V(A)$ ([4], Definition 1, p. 86; Theorem 10, p. 165, and Theorem 14, p. 175). As $\mathrm{G}(A)$ is contained in $\mathrm{Aut}(A)\cap\mathrm{AUT}_V(A)$, thus leaving $U$ invariant by construction, $U$ is invariant under $\mathrm{L}(\mathrm{G}(A))\geqq\mathrm{D}_V(A)$ ([8], pp. 443–444). □

THEOREM 2.2. *If a Lie algebra $L$ over any field of characteristic zero contains a core-free subalgebra of codimension 1, then $L$ is either one-dimensional, or two-dimensional non-abelian, or $L$ is the three-dimensional split simple Lie algebra.*

The proof of the theorem can be found in [7], pp. 105–107.

NOTATION. We call a modular subalgebra $M$ of a Lie algebra $L$ *maximal modular* in $L$, if there is no modular subalgebra $N$ of $L$ such that $M\lneqq N\lneqq L$.

Using Lemma 2.1, the proof of the next lemma is analogous to part (ii) of the proof of R. Schmidt's Lemma 1 for groups ([6], p. 361):

LEMMA 2.3. *Let $M$ be maximal modular in a Lie algebra $L$. If $M$ is* *not an ideal of L, then M is a maximal subalgebra of L.*

THEOREM 2.4. *Let M be maximal modular in a Lie algebra L. If M is not an ideal of L, one of the following holds:*

a) *M is of codimension 1 in L and* $\dim L/M_L \leqq 3$,

b) *M is a maximal subalgebra of L with* $\dim M/M_L = 1$, *and* $L/M_L$ *is a three-dimensional non-split simple Lie algebra.*

PROOF. Let $L$ be a counterexample of minimal dimension. $M$ is a maximal subalgebra of $L$ due to Lemma 2.3, and we can assume that $M_L = 0$ (w.l.o.g.) by Lemma 1.2.

If the largest solvable ideal of $L$, the radical of $L$ (rad$(L)$), is non-trivial, $\mathrm{rad}(L) \nleqq M$ holds, hence $M$ is a quasi-ideal of $L = M + \mathrm{rad}(L)$ (Corollary 1.8). As $M$ is maximal in $L$, we have $\dim L/M = 1$, and by Theorem 2.2 it follows that $\dim L/M_L \leqq 3$. Now let $\mathrm{rad}(L) = 0$ and $L = \oplus_{i=1}^{n} E_i$, $1 \leqq n \in \mathbf{N}$, with simple ideals $E_i$ of $L$.

If $n \geqq 2$, then $\langle M, E_1 \rangle = M + E_1 = L$ and $M \cap E_2 \lhd M$, $E_1 \leqq I_L(M \cap E_2)$ (since $[E_1, E_2] = 0$); thus $M \cap E_2 \lhd L$, i.e., $M \cap E_2 = 0$. By Lemma 1.9 the contradiction $\dim E_2 \leqq 1$ follows.

Therefore $n = 1$ and $L$ is simple.

An easy calculation shows that $L$ is not a counterexample for $\dim L = 3$, because a) holds if $L$ is split, and b) holds if $L$ is non-split. Hence we may assume that

A) $L$ is a simple Lie algebra with $\dim L > 3$.

If we look at $L$ over an algebraically closed extension field $\mathfrak{k}'$ of $\mathfrak{k}$, $L \otimes_{\mathfrak{k}} \mathfrak{k}'$ is semi-simple and $\dim_{\mathfrak{k}'}(L \otimes_{\mathfrak{k}} \mathfrak{k}') = \dim L$ ([3], p. 95). $L$ has non-trivial Cartan-subalgebras ([2], p. 21), every Cartan-subalgebra $H$ of $L$ remains (as $H \otimes_{\mathfrak{k}} \mathfrak{k}'$) Cartan, and by ([2], p. 36) $\dim_{\mathfrak{k}'}(H \otimes_{\mathfrak{k}} \mathfrak{k}') = \dim H$ holds. As $\dim L > 3$ it follows by 1.9 that

A1) $\dim H \geqq 2$ and $M \cap H \neq 0$ for all $H \in \mathrm{Cart}(L)$ and $\dim(H/(M \cap H)) \leqq 1$.

As $\alpha(M) \mathrm{m} L$ for all $\alpha \in \mathrm{Aut}(L)$, Lemma 2.1 leads to (recall that $M_L = 0$)

A2) $\cap \{\alpha(M);\ \alpha \in \mathrm{Aut}(L)\} = 0$.

By A1) and the maximality of $M$ in $L$ we have

B1) $\dim M \geqq 2$.

Now we show that

B2) $M$ is not solvable and $M$ is not three-dimensional split simple.

Assume the contrary. Take $H \in \mathrm{Cart}(L)$ such that $H \nleqq M$. Due to A2) there exists a $\beta \in \mathrm{Aut}(L)$ such that $H \cap M \nleqq H \cap \beta(M)$. $M$ and $\beta(M)$ are modular and maximal in $L$ (Lemma 1.4). A1), B1) and Lemma 1.9 imply that $H \cap \beta(M) \neq 0$, $M \cap \beta(M) \neq 0$. Furthermore

$$H \cap M \nleqq \beta(M), \text{ i.e., } H \cap M \nleqq M \cap \beta(M), \quad \text{and}$$

$$H \cap \beta(M) \nleqq M, \text{ i.e., } H \cap \beta(M) \nleqq M \cap \beta(M).$$

$M \cap \beta(M)$ is modular and maximal in $M$, rsp. in $\beta(M)$ by Lemma 1.10. So we have $\dim M/(M \cap \beta(M)) = 1 = \dim \beta(M)/(M \cap \beta(M))$ by the choice of $L$, and therefore $M = H \cap M + M \cap \beta(M)$, and $\beta(M) = H \cap \beta(M) + M \cap \beta(M)$. As $H$ is nilpotent (hence solvable) the maximal quasi-ideal $M \cap H$ of $H$ is of codimension 1 in $H$ (Lemma 1.10, Proposition 1.7); thus $H = H \cap M + H \cap \beta(M)$. Now we have

$$[M, \beta(M)] \leqq M + \beta(M), \text{ i.e., } M + \beta(M) = \langle M, \beta(M) \rangle = L.$$

Hence $\dim L/M = \dim \beta(M)/(M \cap \beta(M)) = 1$, and by Theorem 2.2 it now follows that $\dim L \leqq 3$ — which is a contradiction to A).

Next we prove the following:

B3) $M$ is semi-simple.

Suppose $\operatorname{rad}(M) \neq 0$. By A2) and (*) there exists a $\beta(M) \ngeqq \operatorname{rad}(M)$, $\beta \in \operatorname{Aut}(L)$, such that $M = M \cap L = M \cap \langle \operatorname{rad}(M), \beta(M) \rangle = \operatorname{rad}(M) + M \cap \beta(M)$. $M \cap \beta(M)$ cannot be solvable; if it were, $M$ would be solvable. $M \cap \beta(M)$ is a quasi-ideal of $M$ (Corollary 1.8), and $\dim M/(M \cap \beta(M)) = 1$ by the maximality of $M \cap \beta(M)$ in $M$ (Lemma 1.10). So $(M \cap \beta(M))\ \mathrm{q}\ \beta(M)$ since $\dim M \cap \beta(M) = \dim M - 1 = \dim \beta(M) - 1$.

Let $(M \cap \beta(M))^{(\omega)}$ denote $\cap_{n=1}^{\infty} (M \cap \beta(M))^{(n)}$; then $(M \cap \beta(M))^{(\omega)} \neq 0$ is an ideal of $M$ and of $\beta(M)$ due to Corollary 3.3 in [1]. Thus the contradiction $(M \cap \beta(M))^{(\omega)} \lhd \langle M, \beta(M) \rangle = L$ follows.

Now we can show that

B) $M$ is three-dimensional non-split simple.

Suppose $M$ is not simple. Let $E$ be one of the simple ideals of $M$ not contained in $\alpha(M) \neq M$, $\alpha \in \operatorname{Aut}(L)$. Then (by (*))

$$M = L \cap M = \langle \alpha(M), E \rangle \cap M = \langle E, M \cap \alpha(M) \rangle = E + M \cap \alpha(M).$$

$M \cap \alpha(M)$ cannot be solvable since $M = M^{(n)} = (M \cap \alpha(M))^{(n)} + E$ for all $n \in \mathrm{N}$. Let $A = (M \cap \alpha(M))_M$ and $B = (M \cap \alpha(M))_{\alpha(M)}$. The choice of $L$ and Lemma 1.10 lead to

$$\dim (M \cap \alpha(M))/A \leqq 2 \quad \text{and} \quad \dim (M \cap \alpha(M))/B \leqq 2.$$

Hence $(M \cap \alpha(M))^{(2)} \leqq A \cap B$, and because $A$, rsp. $B$, is a semi-simple ideal of $M$, rsp. $\alpha(M)$, we have

$$0 \neq (M \cap \alpha(M))^{(\omega)} = A^{(\omega)} = A = B = B^{(\omega)} = (M \cap \alpha(M))^{(\omega)},$$

where $(M \cap \alpha(M))^{(\omega)}$ denotes $\cap_{n=1}^{\infty} (M \cap \alpha(M))^{(n)}$. But now $0 \neq (M \cap \alpha(M))^{(\omega)} \lhd \langle M, \alpha(M) \rangle = L$ contradicts the simplicity of $L$ — $M$ is therefore simple. Lemma 1.10 and the choice of $L$ lead to $\dim M/(M \cap \beta(M))_M = \dim M \leqq 3$ for all

$\beta(M)\neq M$, $\beta\in \mathrm{Aut}(L)$, i.e., $M$ is a three dimensional non-split simple subalgebra of $L$ by B2).

We have the following deductions from B):

B4) $\dim M \cap \beta(M) = 1$ for all $\beta(M) \neq M$, $\beta \in \mathrm{Aut}(L)$

B5) $\dim M \cap H = 1$ for all $H \in \mathrm{Cart}(L)$ (by A1))

B6) All Cartan-subalgebras of $L$ are two-dimensional, and $\dim A \leqq 2$ holds for every solvable subalgebra $A$ of $L$ (since $\dim A/(M \cap A) \leqq 1$ and $\dim A = \dim M \cap A + \dim A/(M \cap A)$).

Now we show the following:

C1) Let $E$ be a three-dimensional non-split simple Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Then for any two independent elements $a$ and $z$ of $E$ (i.e., $z \in E\setminus\langle a\rangle$), there exist scalars $\tau_j(z)$, $j=1, 2, 3$, such that

$$[[a, z], a] = \tau_1(z)\cdot a + \tau_2(z)\cdot z \qquad \text{and}$$

$$[[a, z], z] = \tau_3(z)\cdot a - \tau_1(z)\cdot z \quad \text{and} \quad \tau_2(z)\cdot\tau_3(z) \neq 0.$$

It is clear that $a$, $z$ and $[a, z]$ span $E$. Thus fixing $a$ we have for each $z \in E\setminus\langle a\rangle$ that

(i) $[[a, z], a] = \mu_1 z + \mu_2[a, z] + \mu_3 a$, $\mu_1 \neq 0$, and

(ii) $[[a, z], z] = \lambda_1 z + \lambda_2[a, z] + \lambda_3 a$, $\lambda_3 \neq 0$,

where $\mu_i = \mu_i(z) \in \mathfrak{k}$ and $\lambda_i = \lambda_i(z) \in \mathfrak{k}$ for $i=1, 2, 3$. As $[[[a, z], a], z] = [[[a, z], z], a]$ holds, (i) and (ii) imply

(iii) $\mu_3 = -\lambda_1$, $\mu_2\lambda_1 = \lambda_2\mu_1$, and $\mu_2\lambda_3 = \lambda_2\mu_3$.

Using (i) and setting $\hat{z} = \mu_3 a + \mu_1 z$ we have $[[a, \hat{z}], a] = \mu_1\hat{z} + \mu_2[a, \hat{z}]$, i.e., $\hat{\mu}_1 = \hat{\mu}_1(\hat{z}) = \mu_1$, $\hat{\mu}_2 = \hat{\mu}_2(\hat{z}) = \mu_2$, and $\hat{\mu}_3 = \hat{\mu}_3(\hat{z}) = 0$. By (iii) we have

$$[[a, \hat{z}], \hat{z}] = 0\cdot\hat{z} + 0\cdot[a, \hat{z}] + \hat{\lambda}_3 a,$$

where $\hat{\lambda}_3 = \hat{\lambda}_3(\hat{z})$ and $\hat{\lambda}_1 = \hat{\lambda}_1(\hat{z}) = -\hat{\mu}_3 = 0$, $\hat{\lambda}_2 = \hat{\lambda}_2(\hat{z}) = 0$; hence

$$\hat{\mu}_2 = \mu_2 = 0.$$

Now

$$[[a, \hat{z}], a] = \mu_1\hat{z} \quad \text{and} \quad [[a, \hat{z}], \hat{z}] = \hat{\lambda}_3 a$$

with $\mu_1 \neq 0$, $\hat{\lambda}_3 \neq 0$. Replacing $\hat{z}$ by $\mu_3 a + \mu_1 z$ the result follows.

Now we claim that

C2) $M + \alpha(M) \neq L$ for all $\alpha \in \mathrm{Aut}(L)$.

Suppose $M + \alpha(M) = L$ for an $\alpha(M) \neq M$, $\alpha \in \mathrm{Aut}(L)$. Then $L$ is five-dimensional by B) and B4). But this is not possible — if we look at $L$ over an algebraically closed extension field $\mathfrak{k}'$ of $\mathfrak{k}$, $L\otimes_{\mathfrak{k}}\mathfrak{k}'$ is semi-simple ([3], p. 95), $\dim_{\mathfrak{k}'}(L\otimes_{\mathfrak{k}}\mathfrak{k}')=\dim L=5$ holds ([3], p. 95) and $H\otimes_{\mathfrak{k}}\mathfrak{k}'$ is two-dimensional for every $H\in\mathrm{Cart}(L)$ ([3], p. 36). Looking now at the Cartan-decomposition of $L\otimes_{\mathfrak{k}}\mathfrak{k}'$ rsp. $H\otimes_{\mathfrak{k}}\mathfrak{k}'$, $\dim_{\mathfrak{k}'}(L\otimes_{\mathfrak{k}}\mathfrak{k}')=\dim L$ cannot be an odd number. (Using C1) and the Jacobi-identity we can give an elementary proof of C2) showing the elementary character of the result.)

Next we prove the following:

C3) Let $P$ and $Q$ be elements of $\Omega:=\{\alpha(M);\ \alpha\in\mathrm{Aut}(L)\}$ such that $H\cap P\neq H\cap Q$ for a Cartan-subalgebra $H$ of $L$. Then if $\langle a\rangle=P\cap Q$, $\langle x\rangle=H\cap P$ and $\langle y\rangle=H\cap Q$ we have

$$L = P + Q + \langle[[a, x], y]\rangle,$$

$$P = \langle x\rangle + \langle[a, x]\rangle + \langle a\rangle, \quad \text{and}$$

$$Q = \langle y\rangle + \langle[a, y]\rangle + \langle a\rangle,$$

and for $z\in\{x, y\}$,

(1) $[[a, z], a] = \tau_2(z)\cdot z$

$[[a, z], z] = \tau_3(z)\cdot a$ with $\tau_2(z)\cdot\tau_3(z)\neq 0$, and

(2) $[[[a, x], y], a] = 0 = [[a, x], [a, y]]$,

(3) $[[a, x], [[a, x], y]] = \tau_3(x)\cdot\tau_2(y)y$,

(4) $[[a, y], [[a, x], y]] = \tau_3(y)\cdot\tau_2(x)x$, and

$J=\langle[[a, x], y], a\rangle$ is a Cartan-subalgebra of $L$ with $P\cap J=Q\cap J=P\cap Q=\langle a\rangle$.

As a consequence of B4) and B5) we get $\dim P\cap Q=\dim H\cap P=\dim H\cap Q=1$. By A2) there exist $P, Q\in\Omega$ such that $H\cap P=\langle x\rangle\neq\langle y\rangle=H\cap Q$ for an $H\in\mathrm{Cart}(L)$. Setting $\langle a\rangle=P\cap Q$, $P$ and $Q$ are of the above form. For $z\in\{x, y\}$ we have by C1)

$$\text{(i)}\qquad \begin{aligned} [[a, z], a] &= \tau_1(z)\cdot a + \tau_2(z)\cdot z,\\ [[a, z], z] &= \tau_3(z)\cdot a - \tau_1(z)\cdot z \quad\text{with}\quad \tau_2(z)\cdot\tau_3(z)\neq 0. \end{aligned}$$

$H$ must be abelian by B6), hence $[x, y]=0$. Using (i) we have

$$[[[a, x], y], a] + [[a, x], [a, y]] = [[[a, x], a], y] = \tau_1(x)[a, y],$$

$$[[[a, y], x], a] + [[a, y], [a, x]] = [[[a, y], a], x] = \tau_1(y)[a, x].$$

Thus setting $u=[[a, x], y]=[[a, y], x]$ we get

$$\text{(ii)}\qquad \begin{aligned}&[u, a] = \delta_1[a, x] + \delta_2[a, y], \quad \text{and}\\ &[[a, x], [a, y]] = -\delta_1[a, x] + \delta_2[a, y] \quad \text{with}\end{aligned}$$

$$\text{(iii)}\qquad \delta_1 = \frac{1}{2}\tau_1(y) \quad \text{and} \quad \delta_2 = \frac{1}{2}\tau_1(x).$$

Now $[u, x]=\tau_3(x)[a, y]$ and $[u, y]=\tau_3(y)[a, x]$ by (i); hence

$$[u, a + \beta_1 x + \beta_2 y] = 0, \quad \text{where}$$

$$\text{(iv)}\qquad \beta_1 = -\delta_2\cdot\tau_3(x)^{-1}, \quad \beta_2 = -\delta_1\cdot\tau_3(y)^{-1}.$$

$u$ is linearly independent of $v:=a+\beta_1 x+\beta_2 y$ by C2). Therefore $J:=\langle u, v\rangle =\langle u\rangle+\langle v\rangle$ is two-dimensional and abelian and by B6) a Cartan-subalgebra of $L$. Now

$$0 \neq \lambda_1 u + \lambda_2 v \in P \cap J, \quad \text{and}$$

$$0 \neq \mu_1 u + \mu_2 v \in Q \cap J, \quad \text{where} \quad \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathfrak{k}.$$

So we obtain $\lambda_1=0=\mu_1$, as $u\notin P+Q$ and $v\in P+Q$, and $\beta_1=0=\beta_2$, as $y\notin P$ and $x\notin Q$, whence $\tau_1(x)=\tau_1(y)=0$ by (iii) and (iv). Thus $J=\langle[[a, x], y], a\rangle$ and $P\cap J=Q\cap J=P\cap Q=\langle a\rangle$. The equations (1)–(4) follow from (i)–(iv) and $L=P+Q+\langle[[a, x], y]\rangle$.

Next we show that

C) $M$ is not three-dimensional non-split simple.

Assume the contrary. Let $P$, $Q$, $H$, $J$ be as in C3), $\langle l\rangle \subsetneqq I_L(\langle l\rangle)$ holds for all $l\in L$ by B6). Let $N=\langle[[a, x], y]\rangle+\langle[a, y]\rangle+\langle y\rangle$; then $L$ is the vector space direct sum $L=N+P$, where $P=\langle x\rangle+\langle[a, x]\rangle+\langle a\rangle$. By the equations (1)–(4) of C3) we have $[N, P]\subseteq N$, and $N$ is not a subalgebra of $L$.

Let $u=a+x$ and let $v\in I_L(\langle u\rangle)\setminus\langle u\rangle$; then $v=p+n$ with $p\in P$ and $n\in N\setminus 0$ (as $I_P(\langle u\rangle)=\langle u\rangle$). Now

$$[u, v] = [a + x, p] + [a + x, n] \in \langle u\rangle,$$

and since $[P, N]\subseteq N$ we have

$$[a + x, n] \in P \cap N = 0.$$

Let $n=\gamma_1[[a, x], y]+\gamma_2[a, y]+\gamma_3 y$, where $\gamma_i\in\mathfrak{k}$, $i=1, 2, 3$. By C3), by $[x, y] =0$, and since $[[[a, x], y], x]=[[[a, x], x], y]$, we obtain

$$\begin{aligned}0 &= [n, a + x]\\ &= \gamma_1(0 + \tau_3(x)[a, y]) + \gamma_2(\tau_2(y)y + [[a, y], x]) + \gamma_3(-[a, y] + 0).\end{aligned}$$

Thus $\gamma_2=0$ (as $\tau_2\neq 0$) and $\gamma_1\cdot\tau_3=\gamma_3$. Now $n=\gamma_1([[a, x], y]+\tau_3(x)y)$ and $J':=$

$\langle [[a, x], y] + \tau_3(x)y, a+x \rangle$ is abelian, whence a Cartan-subalgebra of $L$.

So $Q \cap J' \neq 0$ for $Q = \langle y \rangle + \langle [a, y] \rangle + \langle a \rangle \in \Omega$ by A1). Hence there exist elements $q \in Q$ and $j \in J'$ such that

$$0 \neq j = \lambda_1([[a, x], y] + \tau_3(x)y) + \lambda_2(a + x)$$
$$= q = \mu_1 y + \mu_2[a, y] + \mu_3 a, \quad \text{where} \quad \lambda_i, \mu_j \in \mathfrak{k}; \; i, j \in \{1, 2, 3\}.$$

Now

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 0,$$

whence $j = 0$, which is a contradiction.

But now C) contradicts B) and therefore $L$ cannot be a counterexample. □

COROLLARY 2.5. *A simple Lie algebra $L$ over any field of characteristic zero has proper, non-trivial modular subalgebras if and only if $L$ is three-dimensional.*

COROLLARY 2.6. *Let $L$ be a Lie algebra over an arbitrary field $\mathfrak{k}$ of characteristic zero. Then a modular subalgebra $M$ of $L$ is either a quasi-ideal of $L$ or else $M/M_L$ is a one-dimensional, maximal subalgebra of the three-dimensional non-split simple Lie algebra $L/M_L$.*

REMARK. The complete result referred to in the introduction is an immediate consequence of Corollary 2.6 and Theorem 3.6 in [1].

PROOF. W.l.o.g. we may assume that $M_L = 0$. We use induction on $\dim L/M$. If $\dim L/M = 1$, then $M$ is a quasi-ideal of $L$. Let $\mathscr{S} = \{N \,\mathrm{m}\, L;\ M \lneqq N \lneqq L\}$. If $\mathscr{S} = \phi$, there is nothing to prove due to Theorem 2.4. Now we pick an $M' \in \mathscr{S}$ such that $\dim M'/M$ is minimal. The corollary holds for $M'$ by induction and we are left with two cases:

A) $M' \,\mathrm{q}\, L$.

Let $x \in L \setminus M'$ and $u \in M'$ be arbitrary. By (**) we have $M = \langle M, M' \cap \langle x+u \rangle \rangle = \langle M, x+u \rangle \cap M' \,\mathrm{q}\, \langle M, x+u \rangle$. So for an arbitrary $m \in M$ the following holds: $[m, x+u] = \lambda(x+u) \pmod M$, and $[m, x+u] = (\mu x + [m, u]) \pmod M$ with $\lambda, \mu \in \mathfrak{k}$. Hence $[m, u] = \lambda u \pmod M$ holds and $M \,\mathrm{q}\, L$.

B) $M' \,\bar{\mathrm{q}}\, L$, i.e., $M'$ is a maximal subalgebra of $L$ with $\dim M'/M'_L = 1$, and $L/M'_L$ is a three dimensional non-split simple Lie algebra.

We may assume that $M'_L$ is non-trivial, otherwise $M = 0$. Let $x \in L \setminus M'$ and $u \in M'$ be arbitrary. By (**) we have $M \cap M'_L = \langle M, M' \cap \langle x+u \rangle \rangle \cap M'_L = \langle M, x+u \rangle \cap M'_L \lhd \langle M, x+u \rangle$. So $M \cap M'_L$ is idealized by $x$ and $x+u$, hence by $u = (x+u) - x$. Therefore $M \cap M'_L \lhd L$, which implies $M \cap M'_L = 0$. By the minimality of $M'$, $M' = M + M'_L$ holds (see Prop. 1.3), and $M$ is one-dimensional (so let $M = \langle m \rangle$). Let $v \in M'_L$ be arbitrary. Then by (*) we have

$$\langle v\rangle = \langle v, M \cap M'_L\rangle = \langle M, v\rangle \cap M'_L \lhd \langle M, v\rangle .$$

Now, for arbitrary $v, w \in M'_L$,

$$[m, v] = \lambda_v v,\ [m, w] = \lambda_w w \text{ and } [m, v + w] = \lambda_{(v+w)}(v + w)$$

with $\lambda_v, \lambda_w, \lambda_{(v+w)} \in \mathfrak{k}$. So $\lambda_v = \lambda_w = \lambda_{(v+w)}$ and there exists a $\lambda \in \mathfrak{k}$ such that

$$[m, v] = \lambda v \qquad \text{for all} \quad v \in M'_L.$$

This also implies that $M'_L \subseteq [M, L]$ for $\lambda \neq 0$ — for $\lambda = 0$, $M'_L(\nsubseteq M)$ idealizes $M$, and hence $M$ is an ideal of $L$ by Lemma 1.5b), i.e., the contradiction $M=0$ follows.

Next let $x \in L\setminus M'$, $u \in M'$, and $v \in M'_L$ be arbitrary. Then

$$[m, [x + u, v]] = \lambda[x + u, v], \quad \text{and}$$

$$[m, [x + u, v]] = [[m, x + u], v] + \lambda[x + u, v] .$$

Therefore $[[m, x+u], v]=0$ for all choices of $x$, $u$ and $v$, hence $[[M, L], M'_L]=0$. Thus $[[M, L]^L, M'_L]=0$, where $[M, L]^L$ denotes the smallest ideal of $L$ containing $[M, L]$. As $L/M'_L$ is simple and $M'_L \subseteq [M, L]$, $[M, L]^L = L$ holds. So $[L, M'_L]=0$, which implies that $M'_L$ ($\nsubseteq M$) is contained in $I_L(M)$, hence $M \lhd L$ by Lemma 1.5b) and the contradiction $M=0$ follows (as $M_L=0$). □

COROLLARY 2.7. *Let L be a Lie algebra over any field of characteristic zero. If every subalgebra of L is modular in L, then L is either metabelian or else three-dimensional non-split simple.*

PROOF. If $U \,\mathrm{q}\, L$ for all $U \leqq L$, then $L$ is metabelian ([1], Theorem 3.8). If there exists a subalgebra $U$ of $L$, which is not a quasi-ideal of $L$, then $L$ contains a three-dimensional non-split simple subalgebra $E = \langle e\rangle + \langle f\rangle + \langle g\rangle$. Suppose $L \neq E$; then $\dim L > 3$.

But now $\langle e\rangle$ has to be a quasi-ideal of $L$ since $\dim L/\langle e\rangle \geqq 3$ by Corollary 2.6, which is not possible, because $\langle e, f\rangle = E \neq \langle e\rangle + \langle f\rangle$ — hence $L = E$. □

ACKNOWLEDGMENT. A portion of this paper is based on the second author's Diplomarbeit written under the supervision of Prof. O. H. Kegel.

## References

[1] R. K. Amayo: Quasi-ideals of Lie algebras I, Proc. London Math. Soc. (3) **33** (1976), 28–36.
[2] D. W. Barnes: Lie algebras (lecture notes), University of Tübingen, 1968/69.
[3] N. Bourbaki: Groupes et algèbres de Lie, Chap. 1, Hermann, Paris, 1960.
[4] C. Chevalley: Théorie des groupes de Lie, Tome II, Hermann, Paris, 1951.
[5] N. Jacobson: Lie algebras, Interscience, New York, 1966.

[6] R. Schmidt: Modulare Untergruppen endlicher Gruppen, Illionis J. Math. **13** (1969), 358–377.

[7] J. Tits: Sur une classe de groupes de Lie résolubles, Bull. Soc. Math. Belg. **11** (1959), 100–115.

[8] D. A. Towers: A Frattini theory for Lie algebras, Proc. London Math. Soc. (3) **27** (1973), 440–462.

[9] H. Zassenhaus: The theory of groups, Vandenhoeck & Ruprecht, 2. Aufl., Göttingen, 1958.

*Department of Mathematics,*
*Southern Illinois University,*
*Carbondale, Illinois* 62901,
*U.S.A.*