HIROSHIMA MATH. J.
10 (1980), 233–262

On the covering properties of certain exceptional sets in a half-space

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(Received January 8, 1979)
(Revised January 16, 1980)

§1. Introduction

Let $D=\{x\in\mathbf{R}^p: x_1>0\}$ where $x=(x_1,\dots, x_p)$ and $p\geq 2$. We shall say that a set $E\subset\mathbf{R}^p$ has a covering $\{r_n, R_n\}$ if there exists a sequence of balls $\{B_n\}$ in $\mathbf{R}^p$ such that $E\subset\bigcup_{n=1}^{\infty} B_n$, where r_n is the radius of B_n, and R_n is the distance between the origin and the centre of B_n. On the other hand, we shall say that $E\subset D$ has a covering $\{t_n, r_n, R_n\}$ if there exists a sequence of balls $\{B_n\}$ with centres in D such that $E\subset\bigcup_{n=1}^{\infty} B_n$, where r_n and R_n are defined as above and where t_n is the distance between the centre of B_n and the Euclidean boundary of D, to be denoted by ∂D.

The motivation of this work stems from two classical questions which are concerned with the behaviour at ∞ of a suitably restricted subharmonic function u on D. If u is subharmonic on D and $y\in\partial D$, we define $u(y)=\limsup_{x\to y} u(x)$, where $x\in D$. If $u(y)\leq 0$ for all $y\in\partial D$, and if $\sup_{x\in D}(u(x)/x_1)<\infty$, then it is generally known that u can be uniquely decomposed as

$$u(x) = \alpha x_1 - G\mu(x) - w(x),$$

where α is a real number, $G\mu$ is the Green potential of a mass distribution μ on D, and w is a positive harmonic function on D which can be represented as

$$w(x) = \int_{\partial D} K(y, x)d\nu(y)$$

where K is the Poisson kernel on $\partial D\times D$ and ν is a suitable mass distribution on ∂D.

The first question, to be designated by (I), is concerned with an analysis of $u(x)/x_1$ as $x\to\infty$, $x\in D$. By introducing the idea of a minimally thin set at ∞ with respect to D, J. Lelong-Ferrand ([12], pp. 134–143) presented a solution of (I) by showing that for any $\varepsilon>0$, there exists a set $E_\varepsilon\subset D$, minimally thin at ∞ in D, such that

$$|u(x)/x_1 - \alpha| < \varepsilon, \quad x\in D\setminus E_\varepsilon, \quad |x|\geq 1.$$

She also proved that her results are best possible in the sense that if $E\subset D$ is unbounded and minimally thin at ∞ in D, then there exists a subharmonic function u on D, restricted as above, such that $\liminf_{x\to\infty} |u(x)/x_1-\alpha|>0$, $x\in E$. With the introduction of the fine topology of Naïm (see Brelot [3], second part, for details) onto the minimal Martin boundary of a Green space as well as later generalizations, Lelong-Ferrand's result is now interpreted as meaning that

$$\text{fine}\lim_{x\to\infty} u(x)/x_1 = \alpha. \tag{1.1}$$

In analogy with a theorem of H. Cartan (cf. Helms [10], p. 216) which states that the limit of a function in the classical fine topology amounts to the ordinary limit of that function on a suitably chosen fine neighbourhood, M. Brelot ([4], Lemma 9) has pointed out that (1.1) is equivalent to the existence of a minimally thin set E at ∞ in D such that

$$\lim_{x\to\infty} (u(x)/x_1 - \alpha) = 0, \qquad x\in D\setminus E.$$

This result is best possible and it only remains to give a deeper geometrical description of a minimally thin set at ∞ in D than has been accomplished up to the present time.

The second question, to be denoted by (II), is involved with the behaviour of $(u(x)-\alpha x_1)/|x|$ as $x\to\infty$, $x\in D$ (cf. Azarin [1], Essén [7], p. 18 and Essén and Lewis [9] for details). The main idea here is to characterize an exceptional set $S\subset D$ in some best possible sense, so that

$$(u(x) - \alpha x_1)/|x| \longrightarrow 0, \qquad x\longrightarrow\infty, \quad x\in D\setminus S.$$

Question (II) was initiated by Ahlfors and Heins in the special case where $p=2$. They showed that if $\mathbf{K}\subset D$ is any Stolz domain, then $S\cap\mathbf{K}$ has certain properties in terms of radial and circular projections. Since (I) and (II) are equivalent questions when one is restricted to a Stolz domain $\mathbf{K}$, it follows that $S\cap\mathbf{K}$ must be minimally thin at ∞ in D for every such $\mathbf{K}$. A significant result of Ahlfors and Heins which is not subsumed by the work of Lelong-Ferrand, is the theorem which states that if $S\cap\mathbf{K}$ is projected onto the positive real axis by circular projection, then the projected set has finite logarithmic length. Hayman strengthened this result by showing that the circular projection of S itself onto the positive real axis has finite logarithmic length and Azarin [1], in turn, strengthened and generalized Hayman's result by demonstrating that if $p\geq 2$, then S can be covered by a sequence of balls $\{r_n, R_n\}$ such that $\sum_n (r_n/R_n)^{p-1}<\infty$. If we know that there exist a subharmonic function u in D which is restricted as above and an unbounded set $E\subset D$ such that $\liminf_{x\to\infty} |u(x)-\alpha x_1|/|x|>0$, $x\in E$, it is obvious that E is minimally thin at ∞ in D. The following example shows that the set E is not characterized by this condition. The vertical strip $\{x\in D\colon 0<x_1<1\}$ is minimally thin at ∞ in D but does not satisfy Azarin's condition if $p\geq 2$, or Hayman's condition if $p=2$.

Our work started as an effort to relate Azarin's condition to that of minimal thinness. In our first work [8], we assumed that the exceptional set in question was restricted to a Stolz domain, and found that if $\alpha>p-2$, a minimally thin set E at ∞ in D can always be covered by a sequence of balls $\{r_n, R_n\}$ such that $\sum_n (r_n/R_n)^\alpha<\infty$. For an exceptional set restricted to a Stolz domain, our work in [8] shows that the critical value of the exponent α is $p-2$ rather than $p-1$. This strictly improves the result of Ahlfors and Heins and also shows that Azarin's condition does not characterize the exceptional set in (II). Our earlier results, however, do not contain all of those of Azarin because we restricted ourselves to a Stolz domain, whereas Azarin makes no such requirement.

In the present work, we wish to investigate the case when the exceptional set is no longer restricted to a Stolz domain. Let us first discuss exceptional sets of type (II). We wish to give a precise potential-theoretic characterization of these sets as well as detailed covering theorems for them. We shall do this by introducing a new type of exceptional set that we shall call a *rarefied set* (cf. Definition 3.2), and will demonstrate that a rarefied set plays the same role in (II) as a minimally thin set does in (I). We mention that Lelong-Ferrand ([12], p. 134) has also introduced the term "rarefied set". We shall define a set to be *semi-rarefied* iff it is rarefied according to her definition. We introduce this change of terminology because it appears to us that a rarefied set according to Lelong-Ferrand resembles more closely a minimally semi-thin set than it does a thin set. The introduction of a rarefied set will lead to strict improvements of the results of Hayman and Azarin as well as those of Essén and Lewis (cf. [9], also our Remark 4.8).

Let us state two of our covering results. By $B=(t, r, R)$, we mean a ball of radius r, centre $P=(t, x_2,\ldots, x_p)$, where $t>0$ and $R=|P|$. We also introduce $\mathbf{H}$ to be the collection of all sets of the form $B\cap D$ where $0<r\leq t\sqrt{p}$. This means that if $F\subset D\cap\partial D$ is a closed cube with sides parallel to the coordinate axis, then the ball B whose centre is at the centre of F and whose diameter is that of F is such that $B\cap D\in\mathbf{H}$. In both theorems below, we consider sequences $\{H_n\}$ in $\mathbf{H}$ with $H_n=B_n\cap D$ where $B_n=(t_n, r_n, R_n)$.

THEOREM 1.1. *Let $p\geq 3$. Suppose that $E\subset D\subset\mathbf{R}^p$ can be covered by a sequence $\{H_n\}$ so that*

$$\sum_n (t_n/R_n)(r_n/R_n)^{p-2} < \infty. \tag{1.2}$$

Then E is rarefied at ∞ in D.

The proof is given in Section 4. Conversely, we have

THEOREM 1.2. *Let $p\geq 3$ and suppose that $E\subset D\subset\mathbf{R}^p$ is rarefied at ∞ in*

D. For each $\alpha > p-2$, there exists a covering of E by a sequence $\{H_n\}$ so that

$$(1.3) \qquad \sum_n (t_n/R_n)(r_n/R_n)^{\alpha} < \infty.$$

This result is not true if $\alpha \leq p-2$.

Theorem 1.2 is a corollary of the more general Theorem 5.1. The analogues of Theorems 1.1 and 1.2 for a minimally thin set are given by Theorem 4.3 and the Corollary of Theorem 5.1. They are similar to the results stated above except that (1.2) and (1.3) are replaced by

$$(1.2)' \qquad \sum_n (t_n/R_n)^2(r_n/R_n)^{p-2} < \infty,$$

and

$$(1.3) \qquad \sum_n (t_n/R_n)^2(r_n/R_n)^{\alpha} < \infty.$$

It follows from these results that the imposition of the dual conditions of Azarin and minimal thinness simultaneously will not characterize a set which is rarefied at ∞ in D.

An important tool used in our work is Lemma 5.2 which gives a connection between two different types of capacities introduced in Section 2 and the covering measure L_h introduced in Definition 5.2. In the proof of Lemma 5.2, we need Lemma 5.1 which generalizes a classical result of Frostman to the new situation considered here.

In the present paper, we consider only the case $p \geq 3$. The case $p=2$ will be covered in a separate paper.

The authors wish to express their appreciation to Prof. M. Ohtsuka and Mr. H. Aikawa for their many suggestions concerning the preparation of the final version of this work.

§2. Outer charge, Green energy and Green mass

In the whole paper, we assume that the dimension $p \geq 3$. We shall also introduce the following notation.

(i) As in §1, D shall denote the half-space $\{x \in \mathbf{R}^p : x_1 > 0\}$, and $\mathbf{H}$ is defined as in the end of the introduction.

(ii) Let $s > 1$ be fixed and define $I_n = \{x \in D \cup \partial D : s^n \leq |x| < s^{n+1}\}$. We call I_n the n^{th} intersphere with respect to s. In most cases, we can choose $s=2$.

(iii) Let $\phi_{p-2}(|x-y|) = |x-y|^{2-p}$ be the fundamental kernel on $\mathbf{R}^p$. We shall abbreviate ϕ_{p-2} to ϕ if the dimension of the space is understood from the context.

(iv) If $E \subset \mathbf{R}^p$, we let $E_n = E \cap I_n$. Let $c(E_n)$ be the outer (ordinary) capacity of E_n. We also let $E'_n = \{x/s^n : x \in E_n\}$, or $E'_n = s^{-n}E_n$.

(v) If $x = (x_1, x_2, \dots, x_p)$, then $x' = (-x_1, x_2, \dots, x_p)$ shall denote the reflection of x about the Euclidean boundary ∂D of D.

(vi) Let $G(x, y) = \phi(|x-y|) - \phi(|x-y'|)$ be the Green kernel for D, and let $G\mu(x) = \int_{S(\mu)} G(x, y)d\mu(y)$ be the Green potential at x of the Radon measure μ whose support is $S(\mu)$. The mutual Green energy of μ and ν shall be written $(\mu, \nu) = \int_{S(\nu)} G\mu(x)d\nu(x)$, and the Green energy of μ shall simply be $(\mu, \mu) = \|\mu\|^2$.

(vii) We introduce a Martin type kernel on $(D \cup \partial D \cup \{\infty\}) \times D$ as follows:

$$K(y, x) = \begin{cases} G(y, x)/y_1 & \text{on } D \times D, \\ 2x_1 c_p |x-y|^{-p} & \text{on } \partial D \times D, \\ x_1 & \text{on } \{\infty\} \times D. \end{cases}$$

Here $c_p = p-2$. We note that if $x \in D$, $K(\cdot, x)$ will be continuous on $(D \cup \partial D) \setminus \{x\}$.

(viii) If f and g are positive real-valued functions on a set X, we shall say that f is comparable to g, and write $f \approx g$ iff there exist constants A, B, $0 < A \le B$, such that $Ag \le f \le Bg$ everywhere on X.

(ix) If v is a non-negative superharmonic function on D and if H is the greatest harmonic minorant of v on D, then the F. Riesz decomposition theorem indicates that $v - H$ is the Green potential of a unique ν on D which we shall call the measure associated with v on D.

DEFINITION 2.1. Given $E \subset D$, suppose there exists a Radon measure λ_E on D whose Green potential is $G\lambda_E = \hat{R}^E_{x_1}$, where $\hat{R}^E_{x_1}$ is the regularized reduced function (balayage) of x_1 on E with respect to the cone of positive superharmonic functions on D. Following Lelong-Ferrand ([12], p. 129), we shall call λ_E the *fundamental distribution* on E and $\lambda_E(D)$ the *outer charge* of E. If $\hat{R}^E_{x_1}$ is not a Green potential, we shall define the outer charge of E to be infinite.

If E is understood from the context, we shall often abbreviate the fundamental distribution λ_E to simply λ.

DEFINITION 2.2. In case $\hat{R}^E_{x_1} = G\lambda_E$, the Green energy of λ_E, namely (λ_E, λ_E), shall in the future be called the *Green energy* of E and denoted by $\gamma(E)$. Otherwise, we set $\gamma(E) = \infty$.

REMARK 2.1. Lelong-Ferrand referred to $\gamma(E)$ as the outer power of E.

We shall derive some properties of γ which will be used later.

LEMMA 2.1. *γ is monotone, $\gamma(E_n) \uparrow \gamma(E)$ if $E_n \uparrow E$, γ is countably subadditive and*

(2.1) $$\gamma(E) = \inf\{\gamma(O)\colon O \supset E,\ O \text{ is open}\}.$$

PROOF. Let $E_1 \subset E_2$ with $\gamma(E_2) < \infty$. Since $\hat{R}_{x_1}^{E_1} \leq \hat{R}_{x_1}^{E_2}$ in D, $\gamma(E_1) = (\lambda_{E_1}, \lambda_{E_1}) \leq (\lambda_{E_2}, \lambda_{E_1}) = (\lambda_{E_1}, \lambda_{E_2}) \leq (\lambda_{E_2}, \lambda_{E_2}) = \gamma(E_2)$. Let $E_n \uparrow E$ with $\gamma(E) < \infty$. Then $\hat{R}_{x_1}^{E_n} \uparrow \hat{R}_{x_1}^{E}$; see (e) in p. 49 of [3]. For any $a < \gamma(E)$, there exists n_0 such that $a < (\lambda_{E_n}, \lambda_E)$ for any $n \geq n_0$. Next there exists $m \geq n_0$ such that $a < (\lambda_{E_m}, \lambda_{E_{n_0}})$. It follows that $a < (\lambda_{E_m}, \lambda_{E_m}) = \gamma(E_m)$. This proves that $\gamma(E_n) \uparrow \gamma(E)$ as $n \to \infty$.

Before proving the countable subadditivity of γ we establish (2.1). We may assume that $\gamma(E)$ is finite. Let $\hat{R}_{x_1}^{E} = G\lambda_E$. According to a topological lemma due to Choquet (see [2], p. 3) there is a sequence $\{O_n\}$ of open subsets of D such that $O_1 \supset O_2 \supset \cdots \supset E$ and if g is any lower semicontinuous function on D with $g \leq \lim_{n\to\infty} \hat{R}_{x_1}^{O_n}$, then $g \leq \inf_{O\supset E} \hat{R}_{x_1}^{O}$. It is known that $\hat{R}_{x_1}^{E}$ is equal to $\inf_{O\supset E} \hat{R}_{x_1}^{O}$ q.e. in D; see p. 49, (f) of [3]. Take the regularization of $\lim_{n\to\infty} \hat{R}_{x_1}^{O_n}$ as g. It follows that $\lim_{n\to\infty} \hat{R}_{x_1}^{O_n} = \hat{R}_{x_1}^{E}$ q.e. so that $\lim_{n\to\infty} G\lambda_{O_n} = G\lambda_E$ q.e. in D. Since (λ_E, λ_E) is finite, λ_E vanishes on any set of capacity zero. Hence, by Lebesgue's theorem

$$\gamma(E) = (\lambda_E, \lambda_E) = \lim_{n\to\infty} (\lambda_{O_n}, \lambda_E) = \lim_{n\to\infty} (\lambda_E, \lambda_{O_n}).$$

Choose n_0 so that $(\lambda_E, \lambda_{O_n})$ is close to $\gamma(E)$ if $n \geq n_0$. Then choose $m \geq n_0$ so that $(\lambda_{O_m}, \lambda_{O_{n_0}})$ is close to $\gamma(E)$. It follows that $(\gamma_{O_m}, \gamma_{O_m}) = \gamma(O_m)$ is close to $\gamma(E)$. This gives (2.1).

In order to prove that γ is countably subadditive we note that

$$\hat{R}_{x_1}^{F_1 \cup F_2} \leq \hat{R}_{x_1}^{F_1} + \hat{R}_{x_1}^{F_2}$$

for any compact subsets F_1 and F_2 of D. We have

$$\begin{aligned}(\lambda_{F_1\cup F_2}, \lambda_{F_1\cup F_2}) &\leq (\lambda_{F_1}, \lambda_{F_1\cup F_2}) + (\lambda_{F_2}, \lambda_{F_1\cup F_2})\\ &= (\lambda_{F_1\cup F_2}, \lambda_{F_1}) + (\lambda_{F_1\cup F_2}, \lambda_{F_2})\\ &= \int x_1 d\lambda_{F_1} + \int x_1 d\lambda_{F_2}\\ &= (\lambda_{F_1}, \lambda_{F_1}) + (\lambda_{F_2}, \lambda_{F_2}).\end{aligned}$$

Thus $\gamma(F_1 \cup F_2) \leq \gamma(F_1) + \gamma(F_2)$. For any open subsets O_1 and O_2 of D we choose compact sets $\{F_n\}$ and $\{F'_n\}$ so that $F_n \uparrow O_1$ and $F'_n \uparrow O_2$, and obtain $\gamma(O_1 \cup O_2) \leq \gamma(O_1) + \gamma(O_2)$. For arbitrary sets $E_1, E_2 \subset D$ we apply (2.1) and derive $\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2)$. Finally take $\{E_n\}$ and set $A_k = \cup_{n=1}^{k} E_n$. As $k \to \infty$ $\gamma(A_k) \uparrow \gamma(\cup_{n=1}^{\infty} E_n)$. It follows that

$$\gamma(A_k) \leq \textstyle\sum_{n=1}^{k} \gamma(E_n) \leq \sum_{n=1}^{\infty} \gamma(E_n)$$

and the countable subadditivity of γ is easily established.

DEFINITION 2.3. Let $E \subset D$ and let λ'_E be the Radon measure on $\bar{D} = D \cup \partial D$ which is such that

$$\hat{R}_1^E(x) = \int_{\bar{D}} K(y, x) d\lambda'_E(y) = K\lambda'_E(x),$$

where $\hat{R}_1^E$ is the regularized reduced function with respect to the cone of positive superharmonic functions on D. The total mass $\lambda'_E(\bar{D})$ of the measure λ'_E, sometimes abbreviated to $\lambda'(E)$, will be called the *Green mass* of E.

REMARK 2.2. If $\hat{R}_1^E$ has a representation

$$\hat{R}_1^E(x) = \int_{\partial D} K(y, x) d\mu_1(y) + \int_D G(y, x) d\mu_2(y),$$

we have

$$\lambda'(E) = \mu_1(\partial D) + \int_D y_1 d\mu_2(y).$$

The Green mass of any bounded set is finite, and may or may not be finite in the case of unbounded sets. If $F \subset D$ is compact, we have $\lambda'(F) = \lambda'_F(F)$. If $E \subset D$ is relatively compact in D, then the support of λ'_E is contained in D. Otherwise, it is possible that $\lambda'_E(\partial D) > 0$.

We shall use

LEMMA 2.2. *Let $E \subset D$. Let u, v be positive superharmonic functions in D, and μ_v^E be the measure associated with $\hat{R}_v^E$ in D. Then*

$$\mu_v^E(\{x \in D : \hat{R}_u^E(x) < u(x)\}) = 0.$$

PROOF. Let $\mathbf{B}_E$ be the set of points x of D such that E is not thin at x. If v is a Green potential on D, then $\mu_v^E(D \setminus \mathbf{B}_E) = 0$ by Lemma VIII, 2 in p. 61 of [3]. If v is a positive harmonic function on D, then $\mu_v^E(D \setminus \mathbf{B}_E) = 0$ by Theorem VIII, 11 in p. 65 of [3]. The relation holds for general v because of (c′) in p. 51 of [3].

Suppose there exists $x \in \mathbf{B}_E$ such that $\hat{R}_u^E(x) < u(x)$. Set

$$e = \{y \in E : \hat{R}_u^E(y) < u(y)\}.$$

This is a polar set (see p. 49 of [3]) and hence $E \setminus e$ is not thin at any point of $\mathbf{B}_E$. We have

$$\hat{R}_u^E(x) < u(x) = \liminf_{y \to x, y \in E \setminus e} u(y) = \liminf_{y \to x, y \in E \setminus e} \hat{R}_u^E(y).$$

Therefore $E \setminus e$ is thin at x. This contradiction shows that $\hat{R}_u^E = u$ on $\mathbf{B}_E$. Accordingly the set $\{x \in D\colon \hat{R}_u^E(x) < u(x)\}$ is contained in $D \setminus \mathbf{B}_E$, and hence μ_v^E vanishes on it. This proves our lemma.

Definition 2.4. Following Brelot [2], p. 31, we let K^*, defined so that $K^*(x, y) = K(y, x)$, be the associated kernel of K. If λ is a mass distribution on D, we define

$$K^*\lambda(x) = \int_D K^*(y, x)d\lambda(y) = \int_D K(x, y)d\lambda(y).$$

If $x \in D$, then $K^*\lambda(x) = G\lambda(x)/x_1$. Let us now prove that if $x \in \partial D$, then

$$K^*\lambda(x) = \liminf_{z \to x, z \in D} K^*\lambda(z).$$

We want to use the classical Martin theory as presented in chapters XIV and XV of Brelot [3]. For $y_0 \in D$ fixed, consider the Martin kernel $\tilde{K}(x, y) = G(x, y)/G(x, y_0)$ (cf. [3], p. 111, where this kernel is referred to as $K(x, y)$). It is clear that $K(x, y) = K(x, y_0)\tilde{K}(x, y)$. Furthermore every point of $\Delta = \partial D \cup \{\infty\}$ is a minimal point. A set $E \subset D$ is defined to be minimally thin at $X \in \Delta$ iff $R_{\tilde{K}_X}^E \not\equiv \tilde{K}_X$ where $\tilde{K}_X(\cdot) = \tilde{K}(X, \cdot)$ (cf. [3], p. 122). Evidently D is not minimally thin at any point of Δ. It is known that $E \subset D$ is minimally thin at $X \in \Delta$ if and only if there exists a measure μ_X in D such that

$$\int \tilde{K}(X, y)d\mu_X(y) < \liminf_{x \to X, x \in E} \int \tilde{K}(x, y)d\mu_X(y),$$

where $x \to X$ is considered in the Martin topology (cf. Theorem XV, 6 in p. 125 of [3]). We observe that this inequality is equivalent to $K^*\mu_X(X) < \liminf_{x \to X, x \in E} K^*\mu_X(x)$. It follows that $K^*\lambda(x) = \liminf_{z \to x, z \in D} K^*\lambda(z)$ for $x \in \partial D$.

Next we prove

Lemma 2.3. *Let $E \cap D$ and suppose $\hat{R}_{x_1}^E = G\lambda_E$. Let $\mathbf{B}_E$ be the set of points of ∂D at which E is not minimally thin. Then $K^*\lambda_E = 1$ on $\mathbf{B}_E$ and $\lambda_E'(\partial D \setminus \mathbf{B}_E) = 0$, where $\hat{R}_1^E = K\lambda_E'$.*

Proof. Let e be the set of points on $E \cap D$ at which $\hat{R}_{x_1}^E(x) < x_1$. It is a polar set, and since it is thin at every $x \in \partial D$ in the ordinary sense, it must also be minimally thin there (cf. [11], théorème 1, also [3], p. 151). Now let $x \in \mathbf{B}_E$. Then $E \setminus e$ is not minimally thin at x. Therefore

$$K^*\lambda_E(x) = \liminf_{y \to x, y \in E \setminus e} K^*\lambda_E(y) = \liminf_{y \to x, y \in E \setminus e} \frac{1}{y_1} \hat{R}_{x_1}^E(y) = 1.$$

Let u be the greatest harmonic minorant of $\hat{R}_1^E = K\lambda_E'$ on D. If μ is the restriction to ∂D of λ_E', then $u = K\mu$. Set $v = \hat{R}_1^E - u$. Then $\hat{R}_{u+v}^E = \hat{R}_1^E$ (see [3], p. 49, d)), $\hat{R}_u^E \le u$ and $\hat{R}_v^E \le v$. Hence $u + v = \hat{R}_{u+v}^E \le \hat{R}_u^E + \hat{R}_v^E \le u + v$. It follows that $\hat{R}_u^E = u$ on D. By Theorem XV, 11 at the bottom of p. 129 of [3] therefore $\mu(\partial D \setminus \mathbf{B}_E) = \lambda'_E(\partial D \setminus \mathbf{B}_E) = 0$.

In the proof of the next lemma we let $\Omega = \Omega(a)$ be a set of the form $\{x \in \mathbf{R}^p;\ |x| < a\} \cap D$ for some $a > 0$ and let $\hat{R}_{x_1}^{\Omega} = G\lambda_{\Omega}$.

LEMMA 2.4. *λ' is monotone, $\lambda'(E_n) \uparrow \lambda'(E)$ if $E_n \uparrow E$, λ' is countably subadditive and*

$$\lambda'(E) = \inf\{\lambda'(O);\ O \supset E,\ O \text{ is open}\}. \tag{2.2}$$

PROOF. Let $E_1 \subset E_2$. Then $\hat{R}_1^{E_1} \le \hat{R}_1^{E_2}$ and hence $K\lambda'_{E_1} \le K\lambda'_{E_2}$ on $\bar{D}$. Let $(\lambda, \nu)_K = \int K\lambda(x)d\nu(x)$. We have

$$\lambda'_{E_1}(\bar{\Omega}) \le (\lambda_\Omega, \lambda'_{E_1})_{K^*} = (\lambda'_{E_1}, \lambda_\Omega)_K \le (\lambda'_{E_2}, \lambda_\Omega)_K$$
$$= (\lambda_\Omega, \lambda'_{E_2})_{K^*} \le \lambda'_{E_2}(\bar{D}).$$

The arbitrariness of Ω yields $\lambda'(E_1) \le \lambda'(E_2)$.

Let $E_n \uparrow E$. By (e) in p. 49 of [3] $\hat{R}_1^{E_n} \uparrow \hat{R}_1^E$. We see that

$$(\lambda_\Omega, \lambda'_{E_n})_{K^*} = (\lambda'_{E_n}, \lambda_\Omega)_K \uparrow (\lambda'_E, \lambda_\Omega)_K = (\lambda_\Omega, \lambda'_E)_{K^*}.$$

It follows that $\lim_{n\to\infty} \lambda'_{E_n}(\bar{D}) \ge \lambda_E(\bar{D} \cap \bar{\Omega})$. The arbitrariness of Ω yields $\lim_{n\to\infty} \lambda'_{E_n}(\bar{D}) \ge \lambda'_E(\bar{D})$. Since the inverse inequality is also valid, the equality follows.

Next we prove (2.2). By Brelot [3] (p. 49, (f)), there exists a decreasing sequence $\{O_n\}$ of open sets such that $O_n \supset E$ for each n and $\hat{R}_1^{O_n} \downarrow \hat{R}_1^E$ q.e. in D. Evidently $\mathbf{B}_{O_n} \supset \mathbf{B}_{O_m}$ if $m \ge n$. By Lemma 2.3 $K^*\lambda_{O_n} = 1$ on $\mathbf{B}_{O_m}$. Since $\lambda'_{O_m}(\partial D \setminus \mathbf{B}_{O_m}) = 0$ by the same lemma,

$$\int_{\partial D} K^*\lambda_{O_n} d\lambda'_{O_m} = \int_{\partial D \setminus \mathbf{B}_{O_m}} K^*\lambda_{O_n} d\lambda'_{O_m} + \int_{\mathbf{B}_{O_m}} K^*\lambda_{O_n} d\lambda'_{O_m} = \lambda'_{O_m}(\partial D).$$

Therefore, by Lemma 2.2

$$\int_{\bar{D}} K^*\lambda_{O_n} d\lambda'_{O_m} = \int_D K^*\lambda_{O_n} d\lambda'_{O_m} + \int_{\partial D} K^*\lambda_{O_n} d\lambda'_{O_m} = \lambda'_{O_m}(\bar{D}).$$

Similarly $\int_{\bar{D}} K^*\lambda_{O_n} d\lambda'_E = \lambda'_E(\bar{D})$. We have, if n is fixed,

$$\lim_{m\to\infty} \lambda'_{O_m}(\bar{D}) = \lim_{m\to\infty} \int_{\bar{D}} K^*\lambda_{O_n} d\lambda'_{O_m} = \lim_{m\to\infty} \int_D K\lambda'_{O_m} d\lambda_{O_n}$$
$$= \int_D K\lambda'_E d\lambda_{O_n} = \int_{\bar{D}} K^*\lambda_{O_n} d\lambda'_E = \lambda'_E(\bar{D}).$$

This proves (2.2).

To prove the countable subadditivity we note that

$$\hat{R}_1^{F_1 \cup F_2} \leq \hat{R}_1^{F_1} + \hat{R}_1^{F_2}$$

for any compact sets F_1 and F_2 in D. If $\Omega \supset F_1 \cup F_2$, then

$$(\lambda'_{F_1 \cup F_2}, \lambda_\Omega)_K \leq (\lambda'_{F_1}, \lambda_\Omega)_K + (\lambda'_{F_2}, \lambda_\Omega)_K .$$

Inverting the order of integration, we have

$$\lambda'(F_1 \cup F_2) \leq \lambda'(F_1) + \lambda'(F_2).$$

The rest of the proof is carried out in exactly the same way as in the proof of Lemma 2.1.

Now we prove

LEMMA 2.5. *The outer charge and the Green mass coincide for any set* $E \subset D$.

PROOF. First we consider the case when $\hat{R}_{x_1}^E$ is equal to a potential $G\lambda_E$. We observe that $\int_D d\lambda_E = \int_D \hat{R}_1^E d\lambda_E$ and $\int_D d\lambda'_E = \int_D \hat{R}_{x_1}^E(y)/y_1 d\lambda'_E(y)$ by Lemma 2.2. Hence

$$\begin{aligned}\lambda_E(D) &= \int_D \hat{R}_1^E d\lambda_E = \int_D K\lambda'_E d\lambda_E = \int_{\bar{D}} K^*\lambda_E d\lambda'_E \\ &= \int_D K^*\lambda_E d\lambda'_E + \int_{\partial D} K^*\lambda_E d\lambda'_E\end{aligned}$$

and

$$\lambda'_E(\bar{D}) = \int_D \frac{1}{y_1} \hat{R}_{x_1}^E(y) d\lambda'_E(y) + \lambda'_E(\partial D) = \int_D K^*\lambda_E d\lambda'_E + \lambda'_E(\partial D).$$

By Lemma 2.3

$$\int_{\partial D} K^*\lambda_E d\lambda'_E = \int_{\mathbf{B}_E} K^*\lambda_E d\lambda'_E = \lambda'_E(\partial D).$$

Therefore $\lambda_E(D) = \lambda'_E(\bar{D})$.

Next we consider the case when $\hat{R}_{x_1}^E$ is not expressed as a Green potential. Then $\hat{R}_{x_1}^E = x_1$ by Theorem XII, 3 in p. 102 of [3]. Denote by B_n the closed ball with center at $(n, 0, \ldots, 0)$ and radius $n - 1/n$. Let $\hat{R}_{x_1}^{B_n \cap E} = G\lambda_{B_n \cap E}$ and $\hat{R}_1^{B_n} = K\lambda'_{B_n}$. By (e) in p. 49 of [3] $\hat{R}_{x_1}^{B_n \cap E} \uparrow \hat{R}_{x_1}^E = x_1$ as $n \to \infty$. For any m we have

$$\int K^*\lambda_{B_n \cap E} d\lambda'_{B_m} = \int \frac{G\lambda_{B_n \cap E}}{x_1} d\lambda'_{B_m} \uparrow \lambda'_{B_m}(B_m) \qquad \text{as} \quad n \longrightarrow \infty$$

and

$$\int K^*\lambda_{B_n\cap E}d\lambda'_{B_m} = \int K\lambda'_{B_m}d\lambda_{B_n\cap E} \leq \lambda_{B_n\cap E}(D) = \lambda'_{B_n\cap E}(\bar{D}).$$

Since $\lambda'_{B_n\cap E}(\bar{D})\leq\lambda'_E(\bar{D})$ by Lemma 2.4, $\lambda'(B_m)=\lambda'_{B_m}(B_m)\leq\lambda'_E(\bar{D})$. If $\lambda'(B_m)\approx m^{p-1}$, then $\lambda'_E(\bar{D})=\infty$ follows. Since the outer charge of E is infinite when $\hat{R}^E_{x_1}$ is not a Green potential (cf. Def. 2.1) therefore the outer charge always coincides with the Green mass.

What remains is to show that $\lambda'(B_m)\approx m^{p-1}$. Let ω_m be the Green capacitary distribution on B_m, i.e. $G\omega_m=\hat{R}^{B_m}_1$ and note that $\operatorname{supp}\omega_m\subset\partial B_m$. We recall from Jackson ([11], Lemme 1) that

$$G(x, y) \approx x_1y_1\phi(|x-y|)\,|x'-y|^{-2}. \tag{2.3}$$

The constants of comparison depend only on the dimension p. We have

$$G\omega_m(x_m) = 1 \approx \frac{1}{m}\int_{\partial B_m} y_1\phi(|x_m-y|)d\omega_m(y)$$
$$= \frac{1}{m^{p-1}}(\omega_m, \lambda_{B_m}) = \frac{\lambda(B_m)}{m^{p-1}}.$$

Hence $\lambda'(B_m)=\lambda(B_m)\approx m^{p-1}$. Our proof is now completed.

REMARK 2.3. The set functions γ and λ (resp. λ') can be extended from the compact subsets of D (resp. $\bar{D}$) to all the subsets of D (resp. $\bar{D}$) in a standard way using the theory of capacity. We take γ as an example.

We regard γ as a set function on the class of compact sets F in D. It satisfies (i) $\gamma(\varnothing)=0$, $\gamma(F_1)\leq\gamma(F_2)$ if $F_1\subset F_2$, (ii) $\gamma(F_1\cup F_2)+\gamma(F_1\cap F_2)\leq\gamma(F_1)+\gamma(F_2)$, (iii) $\gamma(F_n)\downarrow\gamma(F)$ if $F_n\downarrow F$. In fact, (i) and (iii) are proved in our Lemma 2.1 and (ii) can be proved as in the proof of Lemma 2.1. Therefore γ is a strong capacity (cf. [2], p. 17) or Choquet capacity (cf. [10], Theorem 7.20). Then we extend γ to open sets $O\subset D$ by

$$\gamma_*(O) = \sup\{\gamma(F)\colon F\subset O, \text{ is compact}\}.$$

To show $\gamma_*(O)=\gamma(O)$, take a sequence of compact subsets $\{F_n\}$ of O such that the interior of F_n increases to O. Then given $F\subset O$, there exists $F_n\supset F$ and hence $\gamma(F_n)\geq\gamma(F)$. Since $\lim_{n\to\infty}\gamma(F_n)=\gamma(O)$ by Lemma 2.1, both $\gamma(O)\leq\gamma_*(O)$ and $\gamma(O)\geq\gamma_*(O)$ follow. This gives $\gamma(O)=\gamma_*(O)$. We extend γ further to an arbitrary set $E\subset D$ by

$$\gamma^*(E) = \inf\{\gamma(O)\colon O\supset E, O \text{ is open}\}.$$

We know that $\gamma^*(E)=\gamma(E)$ by (2.1).

Thus our γ is a general capacity (cf. [3], p. 66) which is sometimes called a true capacity (cf. [2], pp. 6 and 18) or an outer capacity (cf. [10], pp. 145–146). Accordingly any K-analytic set is γ-capacitable. We shall not use this fact in this paper.

§ 3. Some definitions and lemmas

We now observe how the different capacities that we have introduced transform under a homothetic transformation on D of the form

$$T(x) = kx, \quad \text{where} \quad k > 0.$$

We will have

$$G(T(x), T(y)) = k^{2-p}G(x, y).$$

If $E \subset D$ is bounded, it follows that

$$\lambda(T(E)) = \lambda'(T(E)) = k^{p-1}\lambda'(E),$$

$$\gamma(T(E)) = k^p\gamma(E).$$

DEFINITION 3.1. Following Lelong-Ferrand, we shall define $E \subset D$ to be *minimally thin* at ∞ in D provided that $\sum_n \gamma(E_n)s^{-np} < \infty$ for some $s > 1$. If $E'_n = s^{-n}E_n$, then E is minimally thin at ∞ in D iff $\sum_n \gamma(E'_n) < \infty$.

REMARK 3.1. One normally defines $E \subset D$ to be minimally thin at ∞ in D iff $\hat{R}^E_{x_1} \not\equiv x_1$, in which case $\hat{R}^E_{x_1}$ is a Green potential on D. Brelot ([3], p. 152) has pointed out that $\hat{R}^E_{x_1}$ is a Green potential on D iff for any $x_0 \in D$, we have $\sum_n \hat{R}^{E_n}_{x_1}(x_0) < \infty$ for some $s > 1$. The convergence of the series $\sum_n \hat{R}^{E_n}_{x_1}(x_0)$ is independent of s, and it is easy to show (see § 4) that $\hat{R}^{E_n}_{x_1}(x_0) \approx s^{-np}\gamma(E_n)$ which gives the equivalence of the two definitions. Lelong-Ferrand also noticed that the definition of minimal thinness at ∞ in D is independent of the choice of s by demonstrating that if $\gamma(r)$ is the Green energy of $E \cap \{x \in D: |x| < r\}$, then E is minimally thin at ∞ in D iff $\int_1^\infty r^{-p-1}\gamma(r)dr < \infty$, or equivalently, iff $\int_1^\infty r^{-p}d\gamma(r) < \infty$. We are using Lelong-Ferrand's definition of minimal thinness here rather than the standard one because it is easier for us to adapt our covering theorems to it.

DEFINITION 3.2. We shall define $E \subset D$ to be *rarefied* at ∞ in D iff $\sum_n \lambda'(E_n)s^{-n(p-1)} < \infty$ for some $s > 1$. We recall that $\lambda'(E_n)$ is the Green mass of E_n. This is equivalent to the condition that $\sum_n \lambda'(E'_n) < \infty$, where $E'_n = s^{-n}E_n$.

LEMMA 3.1. *Let $\lambda'(r)$ be the Green mass of $E \cap \{x \in D: |x| < r\}$. Then E is rarefied at ∞ in D iff $\int_1^\infty r^{-p}\lambda'(r)dr < \infty$, or equivalently, iff $\int_1^\infty r^{1-p}d\lambda'(r) < \infty$.*

PROOF. Since the Green mass is subadditive, we have $\lambda'(s^{n+1})-\lambda'(s^n)\leq \lambda'(E_n)\leq\lambda'(s^{n+1})$. We note that the Green mass and the outer charge of $\{x\in D: |x|<r\}$ are equal and dominated by r^{p-1} (cf. Lelong-Ferrand [12], p. 130) up to a constant factor. Hence $\lambda'(r)$ is also dominated by Const. r^{p-1} and an integration by parts shows that the two integrals in the lemma are co-convergent.

If E is rarefied at ∞ in D, then $\sum_n(\lambda'(s^{n+1})-\lambda'(s^n))s^{-n(p-1)}<\infty$ which implies that $\int_1^\infty r^{1-p}d\lambda'(r)$ converges. Conversely, if $\int_1^\infty r^{-p}\lambda'(r)dr<\infty$, we have

$$\sum_n \lambda'(E_n)s^{-(n+2)p}(s^{n+2}-s^{n+1})\leq\sum_n\int_{s^{n+1}}^{s^{n+2}} r^{-p}\lambda'(r)dr<\infty,$$

and it follows that E is rarefied at ∞ in D.

REMARK 3.2. It is clear from Lemma 3.2 that the definition of a rarefied set is independent of the choice of $s>1$. Later we shall show that a rarefied set has other properties that are parallel to those of a minimally thin set. Since $\gamma(E_n)\leq s^{n+1}\lambda(E_n)=s^{n+1}\lambda'(E_n)$, it is evident that a rarefied set at ∞ in D is necessarily minimally thin there. In general the implication is strict, but if E is restricted to a Stolz domain with vertex at the origin, then $\gamma(E_n)\approx s^n\lambda'(E_n)$ and we conclude that E is rarefied at ∞ iff it is minimally thin there.

DEFINITION 3.3. We define $E\subset D$ to be *semirarefied* at ∞ in D iff $\lim_{n\to\infty}\lambda'(E_n)s^{-n(p-1)}=0$, or equivalently, iff $\lim_{n\to\infty}\lambda'(E_n')=0$.

DEFINITION 3.4. Following Lelong-Ferrand, we define $E\subset D$ to be *minimally semithin* at ∞ in D iff $\lim_{n\to\infty}\gamma(E_n)s^{-np}=0$.

REMARK 3.3. Our definition of a semirarefied set coincides with Lelong-Ferrand's definition of a rarefied set ([12], p. 134). She has demonstrated that $E\subset D$ is minimally semithin at ∞ (resp. semirarefied at ∞) in D iff $\lim_{r\to\infty}r^{-p}\gamma(r)=0$ (resp. $\lim_{r\to\infty}r^{1-p}\lambda'(r)=0$) which indicates that these definitions do not depend on the choice of $s>1$. A semirarefied set is always minimally semithin, and the two concepts coincide in a Stolz domain. On the other hand, a semirarefied set is in general non-comparable with a minimally thin one.

LEMMA 3.2. *Let $H=B\cap D\in\mathbf{H}$ where we recall that t is the first coordinate of the centre of B, r is its radius, and $0<r\leq t\sqrt{p}$. Then $\gamma(H)\approx t^2c(H)\approx t^2r^{p-2}$ where the constants of comparison depend only on the dimension p.*

PROOF. If $t\geq r$, then $c(H)=c(B)=r^{p-2}$. If $t\leq r$, we consider the ball B' concentric with B and with radius t. We clearly have $c(B')\leq c(H)\leq c(B)$. Since $t\approx r$, therefore $c(B')\approx c(B)$ which implies that $c(H)\approx r^{p-2}$ in all cases.

We first let $B\subset D$ and define λ to be the fundamental distribution on B. If we choose x to be the centre of B, then

$$G\lambda(x) = t \approx t^{-1}\int_{\partial B} y_1\phi(|x-y|)d\lambda(y)$$

by (2.3). We note that supp $\lambda \subset \partial B$ because $\hat{R}^B_{x_1}$ is harmonic on $D\setminus\partial B$. Therefore $t^2 \approx \phi(r)\gamma(B)$, or

$$\gamma(H) = \gamma(B) \approx t^2 r^{p-2}.$$

If $t<r$, then $t\approx r$ and we know that $\gamma(B')\leq\gamma(H)$. We now embed H in a half ball D' of radius $2r$ with face on ∂D and recall from Lelong-Ferrand ([12], p. 130) that $\gamma(D')\approx r^p$. It follows that $\gamma(B')\approx\gamma(D')$ and hence $\gamma(H)\approx t^2r^{p-2}$ in all cases.

LEMMA 3.3. *If H is defined as in Lemma 3.2 then $\lambda'(H)\approx tc(H)\approx tr^{p-2}$, where the constants of comparison depend only on the dimension p.*

PROOF. The case, where $r<t$ so that $H=B$ is relatively compact in D, is treated as in the proof of Lemma 2.4.

If $t\leq r$, we construct $B'\subset H\subset D'$ as in the proof of Lemma 3.2. In this case, $\lambda'(D')=\lambda(D')$ which in turn is comparable to r^{p-1} (cf. Lelong-Ferrand [12], p. 130) so that $\lambda(D')\approx\lambda(B')$ since $t\approx r$. The lemma follows.

§ 4. Some characterizations of exceptional sets in D

We shall start with two preliminary covering theorems.

THEOREM 4.1. *Let $E=\cup_n H_n\subset D\subset\mathbf{R}^p$, where $H_n=B_n\cap D$, $H_n\in\mathbf{H}$, $B_n=(t_n, r_n, R_n)$ and $H_n\subset I_n$ for some given $s>1$. Then E is rarefied or minimally thin at ∞ in D iff*

$$\sum_n (t_n/R_n)(r_n/R_n)^{p-2} < \infty, \tag{4.1}$$

$$\sum_n (t_n/R_n)^2(r_n/R_n)^{p-2} < \infty,$$

respectively.

PROOF. Since $H_n\subset I_n$, we have $R_n\approx s^n$, $n=1, 2,\dots$. From Lemmas 3.2 and 3.3, we see that $\gamma(H_n)\approx t_n^2r_n^{p-2}$ and $\lambda(H_n)\approx t_nr_n^{p-2}$. Theorem 4.1 follows from Definitions 3.1 and 3.2 respectively.

REMARK 4.1. If $H_n'=s^{-n}H_n$, then $E=\cup_n H_n$ is rarefied or minimally thin at ∞ in D iff

$$\sum_n (t_n/R_n)c(H_n') < \infty, \quad \text{or} \quad \sum_n (t_n/R_n)^2c(H_n') < \infty,$$

respectively, where $c(H_n')$ is the ordinary capacity of H_n'. Such a set is semirarefied or semithin at ∞ in D iff

$$\lim_{n\to\infty}(t_n/R_n)c(H'_n) = 0, \quad \text{or} \quad \lim_{n\to\infty}(t_n/R_n)^2 c(H'_n) = 0,$$

respectively. Since $t_n \geq r_n/\sqrt{p}$, it is clear from (4.1) that we have $\sum_n (r_n/R_n)^{p-1} < \infty$ in the rarefied case, i.e., E satisfies Azarin's condition when $p \geq 3$. In the case where $t_n \approx r_n$ for all n, E is rarefied at ∞ in D iff E satisfies Azarin's condition. We point out here that the twin conditions of minimal thinness and Azarin's condition do not characterize a rarefied set at ∞ in D. In order to see this, consider $E = \cup_n H_n$ constructed so that $(r_n/R_n)^{p-2} \approx n^{-1}$ and $t_n/R_n \approx (\log n)^{-1}$.

DEFINITION 4.1. Let β be given, $0 \leq \beta \leq 1$. Let $\mathbf{S}_\beta$ be the class of all positive superharmonic functions u on $D \subset \mathbf{R}^p$, which are such that there exists a nonnegative Radon measure μ on $\bar{D}$ such that

$$u(x) = \int_{\bar{D}} K(y, x)d\mu(y), \quad \text{where} \quad \int_{\bar{D}} \frac{d\mu(y)}{(1+|y|)^{p-1+\beta}} < \infty .$$

REMARK 4.2. If $0 \leq \alpha \leq \beta \leq 1$, it is evident that $\mathbf{S}_\alpha \subset \mathbf{S}_\beta$. A positive superharmonic function u on D belongs to $\mathbf{S}_1$ iff for any $\rho > 0$, $u(x)$ fails to dominate ρx_1 everywhere on D. In other words, the canonical measure of u fails to charge $\{\infty\}$. If $v \in \mathbf{S}_1$, and u is superharmonic on D such that $0 \leq u \leq v$ on D, then it is well-known (and follows from the statement above) that $u \in \mathbf{S}_1$ also.

We shall now develop some of the fundamental properties of rarefied sets.

LEMMA 4.1. *E is rarefied at ∞ in $D \subset \mathbf{R}^p$ iff for any $x_0 \in D$ and $s > 1$, we have* $\sum_n \hat{R}_r^{E_n}(x_0) < \infty$. (We recall that $E_n = E \cap I_n$ and $r = |x|$.)

PROOF. Since $|x| = r \approx s^n$ in I_n, it follows that $\hat{R}_r^{E_n}(x_0) \approx s^n \hat{R}_1^{E_n}(x_0)$. Let $\hat{R}_1^{E_n} = \int_{\bar{D}} K(y, \cdot)d\lambda'_n(y)$ and note that

$$K(y, x_0) \approx s^{-np}, \quad y \in \operatorname{supp} \lambda'_n \subset \bar{E}_n.$$

It follows that $\hat{R}_1^{E_n}(x_0) \approx s^{-np}\lambda'(E_n)$ where $\lambda'(E_n)$ is the Green mass of E_n and that

$$\hat{R}_r^{E_n}(x_0) \approx s^{-n(p-1)}\lambda'(E_n).$$

This proves Lemma 4.1 (cf. Definition 3.1).

REMARK 4.3. A set $E \subset D$ is semirarefied at ∞ in D iff $\lim_{n\to\infty} \hat{R}_r^{E_n}(x_0) = 0$ where $E_n = E \cap I_n$ as before. One can modify the reasoning in the proof of Lemma 4.1 to show that $\hat{R}_{x_1}^{E_n}(x_0) \approx s^{-np}\gamma(E_n)$. A set $E \subset D$ is therefore minimally thin or semithin at ∞ in D iff

$$\sum_n \hat{R}_{x_1}^{E_n}(x_0) < \infty, \quad \text{or} \quad \lim_{n\to\infty} \hat{R}_{x_1}^{E_n}(x_0) = 0,$$

respectively.

LEMMA 4.2. *Let $\{A_n\}$ be a sequence of sets in D such that $\hat{R}_r^{A_n} \in \mathbf{S}_1$, and set $S = \sum_{n=1}^{\infty} \hat{R}_r^{A_n}$. If $S \not\equiv \infty$, then $S \in \mathbf{S}_1$.*

PROOF. We note that $\sum_{n=1}^{k} \hat{R}_r^{A_n}(x) \in \mathbf{S}_1$. We can write it as

$$\int_D G(\cdot, x) d\mu_k + \int_{\partial D} K(\cdot, x) d\mu_k.$$

Set $T_k(x) = \sum_{n=k+1}^{\infty} \hat{R}_r^{A_n}(x)$ and express it as

$$\int_D G(\cdot, x) d\nu_k + \int_{\partial D} K(\cdot, x) d\nu_k + \nu_k(\{\infty\}) x_1.$$

Since $S(x)$ is superharmonic in D, $T_k(x) \to 0$ as $k \to \infty$ q.e. on D. Therefore $\nu_k(\{\infty\}) \to 0$ as $k \to \infty$. We have

$$S(x) = \int_D G(\cdot, x) d(\mu_k + \nu_k) + \int_{\partial D} K(\cdot, x) d(\mu_k + \nu_k) + \nu_k(\{\infty\}) x_1.$$

The uniqueness of the expression implies that $\nu_k(\{\infty\})$ is a constant which is in fact zero.

THEOREM 4.2. *A set $E \subset D$ is rarefied at ∞ in D iff $\hat{R}_r^E \in \mathbf{S}_1$.*

PROOF. We assume that E is rarefied at ∞ in D, and that $S = \sum_{n=1}^{\infty} \hat{R}_r^{E_n}$. Then $S \not\equiv \infty$ so that S is necessarily superharmonic on D. Since S dominates r q.e. on $E = \cup_n E_n$, we have $S \geq \hat{R}_r^E$. Since each $\hat{R}_r^{E_n} \in \mathbf{S}_1$, Lemma 4.2 shows $S \in \mathbf{S}_1$ and therefore $\hat{R}_r^E \in \mathbf{S}_1$ follows.

For the converse, we modify the reasoning in Lelong-Ferrand [12] (p. 135) as follows: Let $\hat{R}_r^E = \int_{\bar{D}} K(y, \cdot) d\nu(y)$ and recall that (if $c_p = p-2$)

$$K(y, x) = \begin{cases} G(y, x)/y_1, & (y, x) \in D \times D, \\ 2x_1 c_p |x-y|^{-p}, & (y, x) \in \partial D \times D. \end{cases}$$

We note that

$$K(y, x) \approx x_1 |y|^{-p}, \quad |y| \geq 2|x|,$$

so that $\int_{|y| \geq 1} |y|^{-p} d\nu(y) = A < \infty$. Now choose $s \geq 2$ and let $J_n = I_{n-1} \cup I_n \cup I_{n+1}$. We see that if $x \in \bar{E}_n$,

$$C^{-1} \int_{\bar{D} \setminus J_n} K(y, x) d\nu(y) \leq \int_{|y| \leq s^{n-1}} x_1 |x|^{-p} d\nu(y) + \int_{|y| \geq s^{n+2}} x_1 |y|^{-p} d\nu(y)$$

with a constant C. If s is chosen so that $4AC < s^p$, then $C \int_{|y| \leq s^{n-1}} x_1 |x|^{-p} d\nu(y)$

$\leq r/4$. For large n, say for $n \geq n_0$, we will have

$$\int_{D \setminus J_n} K(y, x) dv(y) \leq r/2, \quad x \in \bar{E}_n$$

which implies that $r \leq \hat{R}_r^E(x) \leq \int_{J_n} K(y, x) dv(y) + r/2$ q.e. on $\bar{E}_n \cap D$. Hence $2\int_{J_n} K(y, x) dv(y) \geq r$ q.e. on E_n. Therefore

$$\hat{R}_r^{E_n} \leq 2\int_{J_n} K(y, x) dv(y) \quad \text{everywhere on} \quad D,$$

by the definition of $\hat{R}_r^{E_n}$ (cf. Brelot [3], p. 49(d)). If we now sum over $n \geq n_0$, we obtain

$$\sum_{n=n_0}^{\infty} \hat{R}_r^{E_n} \leq 6\hat{R}_r^E.$$

Applying Lemma 4.1, we obtain Theorem 4.2.

REMARK 4.4. The argument in the proof of Theorem 4.2 also leads to the conclusion that $\hat{R}_r^E \in \mathbf{S}_1$ iff $\sum_n \hat{R}_r^{E_n} \in \mathbf{S}_1$. We could define $E \subset D$ to be rarefied at ∞ in D iff $\hat{R}_r^E \in \mathbf{S}_1$ or equivalently, iff there exists $u \in \mathbf{S}_1$ which dominates $r = |x|$ q.e. on E. If $y_0 \in \partial D$ and $\rho = |x - y_0|$, we can define $E \subset D \subset \mathbf{R}^p$ to be rarefied at y_0 in D iff $\hat{R}_{\rho^{1-p}}^E$ is a positive superharmonic function on D whose canonical measure does not charge $\{y_0\}$. In other words, $\hat{R}_{\rho^{1-p}}^E$ fails to dominate any minimal harmonic function on D whose pole is at y_0.

We shall now establish some elementary covering results for minimally thin and rarefied sets, respectively.

THEOREM 4.3. *Suppose that $E \subset D \subset \mathbf{R}^p$ can be covered by a sequence $\{H_n\}$, $H_n \in \mathbf{H}$, $H_n = B_n \cap D$ where $B_n = (t_n, r_n, R_n)$ such that*

$$\sum_n (t_n/R_n)^2 (r_n/R_n)^{p-2} < \infty. \tag{4.2}$$

Then E is minimally thin at ∞ with respect to D.

PROOF. If $x_0 \in D$, we have

$$\hat{R}_{x_1}^{H_n}(x_0) \approx \gamma(H_n)/R_n^p \approx (t_n/R_n)^2 (r_n/R_n)^{p-2} \qquad \text{(Lemma 3.2)}.$$

Therefore, (4.2) is equivalent to the condition that $\sum_n \hat{R}_{x_1}^{H_n}(x_0) < \infty$. Lemma 4.2 shows that $\sum_n \hat{R}_{x_1}^{H_n} \in \mathbf{S}_1$. Hence $\hat{R}_{x_1}^E \in \mathbf{S}_1$ also, since it is dominated by $\sum_n \hat{R}_{x_1}^{H_n}$, so that $\hat{R}_{x_1}^E \not\equiv x_1$. But this implies that E is minimally thin at ∞ (see Brelot [3], p. 103 and p. 152).

The proof of Theorem 1.1 is similar.

Proof of Theorem 1.1.
For any $x_0 \in D$ and n sufficiently large, it follows from Lemma 3.3 that

$$\hat{R}_r^{H_n}(x_0) \approx \lambda(H_n)/R_n^{p-1} \approx (t_n/R_n)(r_n/R_n)^{p-2}.$$

Lemma 4.2 shows that $\sum_n \hat{R}_r^{H_n} \in \mathbf{S}_1$ since we have $\sum \hat{R}_r^{H_n}(x_0) < \infty$. It follows that $\hat{R}_r^E \in \mathbf{S}_1$. This completes the proof.

Remark 4.5. We can obtain analogous covering results for minimally semithin or semirarefied sets at ∞ in D. We omit the details.

Let us now introduce some further properties of rarefied and minimally thin sets in D.

Lemma 4.3. *If $E \subset D$, then E is rarefied at ∞ in D iff $\sum_n \hat{R}_{r\beta}^{E_n} \in \mathbf{S}_\beta$ for any $\beta \in [0, 1]$.*

Proof. Let $\hat{R}_{r\beta}^{E_n} = \int_D K(y, \cdot) d\mu_n(y)$ and $\mu = \sum_{n=1}^\infty \mu_n$. Then $\sum_n \hat{R}_{r\beta}^{E_n} \in \mathbf{S}_\beta$ iff $\int_{|y| \geq 1} |y|^{1-p-\beta} d\mu(y) < \infty$ or equivalently, iff

$$\sum_n \mu(I_n) s^{-n(p+\beta-1)} < \infty. \tag{4.3}$$

If $x_0 \in D$ and n is sufficiently large, we have

$$\mu_n(\bar{I}_n) s^{-np} \approx \hat{R}_{r\beta}^{E_n}(x_0) \approx s^{n\beta} \hat{R}_1^{E_n}(x_0) \approx s^{n(\beta-p)} \lambda(E_n),$$

so that E is rarefied at ∞ in D iff

$$\sum_n \mu_n(\bar{I}_n) s^{-n(p+\beta-1)} < \infty. \tag{4.4}$$

Elementary calculations show that the series (4.3) and (4.4) are co-convergent.

Theorem 4.4. *$E \subset D \subset \mathbf{R}^p$ is rarefied at ∞ in D iff for any $\beta \in [0, 1]$, we have $\hat{R}_{r\beta}^E \in \mathbf{S}_\beta$.*

Proof. If E is rarefied at ∞ in D, then it follows from Lemma 4.3 that $\sum_n \hat{R}_{r\beta}^{E_n} \in \mathbf{S}_\beta$. If we let $\hat{R}_{r\beta}^E = \int_D K(y, \cdot) d\nu(y) = K\nu$ and $\sum_n \hat{R}_{r\beta}^{E_n} = \int_D K(y, \cdot) d\mu(y) = K\mu$, then $K\nu \leqq K\mu$. For each $\beta \in [0, 1]$ the function $x_1 |x|^{1-p-\beta}$ is a positive superharmonic function on D, and $\min\{x_1, x_1|x|^{1-p-\beta}\}$ is a Green potential on D whose canonical measure shall be denoted by λ. Since $K\nu \leq K\mu$, $(\nu, \lambda)_K \leq (\mu, \lambda)_K$ and hence $(\lambda, \nu)_{K^*} \leq (\lambda, \mu)_{K^*}$. We recall that $K^*\lambda(x) = G\lambda(x)/x_1$ for $x \in D$ and therefore $K^*\lambda(x) = |x|^{1-p-\beta}$ if $|x| \geqq 1$ and $x \in D$. Hence

$$(\lambda, \nu)_{K^*} \approx \int_{|y|\geq 1} |y|^{1-p-\beta} d\nu(y).$$

Since

$$(\lambda, \mu)_{K^*} \approx \int_{|y|\geq 1} |y|^{1-p-\beta} d\mu(y) < \infty,$$

we obtain $\hat{R}^E_{r\beta} \in \mathbf{S}_\beta$.

For the converse, we assume that $\int_{|y|\geq 1} |y|^{1-p-\beta} d\nu(y) = A < \infty$. The argument in the second half of the proof of Theorem 4.2 can be used with minor modifications. Let $x \in \bar{E}_n$, $s \geq 2$ and J_n be defined as before. Then for n sufficiently large, we have

$$\begin{aligned} &C\int_{\bar{D}\setminus J_n} K(y, x) d\nu(y) \\ &\leq \int_{|y|\leq s^{n-1}} x_1 |x|^{-p} d\nu(y) + \int_{|y|\geq s^{n+2}} x_1 |y|^{-p} d\nu(y) \\ &\leq |x|^\beta \Big\{ s^{1-p-\beta} \int_{|y|\leq s^{n-1}} + \int_{|y|\geq s^{n+2}} \Big\} |y|^{1-p-\beta} d\nu(y) \leq C r^\beta / 2 \end{aligned}$$

with a constant $C>0$, if s is suitably chosen. Therefore, we have $\int_{J_n} K(y, x) d\nu(y) \geq r^\beta/2$ q.e. on E_n which implies that

$$2\int_{J_n} K(y, x) d\nu(y) \geq \hat{R}^{E_n}_{r\beta} \quad \text{everywhere on} \quad D.$$

We conclude that

$$\mu_n(\bar{E}_n) \leq 2\nu(J_n).$$

An elementary calculation leads to the convergence of the series $\sum_n s^{-n(p+\beta-1)} \cdot \mu_n(\bar{E}_n)$, and it follows that $\sum_n \hat{R}^{E_n}_{r\beta} \in \mathbf{S}_\beta$. Applying Lemma 4.3, we obtain Theorem 4.4.

Remark 4.6. Using exactly the same arguments as in the proof of Theorem 4.4, we can prove an apparently more general version by replacing $\hat{R}^E_{r\beta}$ by any $u \in \mathbf{S}_\beta$ which dominates $\hat{R}^E_{r\beta}$.

For completeness, we state analogous results for minimally thin sets. The proofs are analogous to those of Lemma 4.3 and Theorem 4.4. We do not include the details.

Lemma 4.4. *If $E \subset D$, then E is minimally thin at ∞ in D, iff $\sum_n \hat{R}^{E_n}_{x_1 r^{\beta-1}} \in \mathbf{S}_\beta$ for any $\beta \in [0, 1]$.*

THEOREM 4.5. *$E \subset D$ is minimally thin at ∞ in D if $\hat{R}^E_{x_1 r^{\beta-1}} \in \mathbf{S}_\beta$ for any $\beta \in [0, 1]$.*

REMARK 4.7. Naturally, $\hat{R}^E_{x_1 r^{\beta-1}}$ is also a Green potential.

We shall now prove a theorem for rarefied sets that is similar to a theorem of H. Cartan ([10], p. 216) for classically thin sets, and one of M. Brelot ([4], Lemma 9) for minimally thin sets. Brelot indicates his result under very general conditions.

THEOREM 4.6. *If $u \in \mathbf{S}_1$ on $D \subset \mathbf{R}^p$, then there exists a set $E \subset D$ such that E is rarefied at ∞ in D where*

$$\lim u(x)/|x| = 0, \qquad x \longrightarrow \infty, \quad x \in D \setminus E.$$

Conversely, if $E \subset D$ is rarefied at ∞ in D, there exists $u \in \mathbf{S}_1$ such that E is contained in the exceptional set for u.

PROOF. We start with the converse. If $E \subset D$ is rarefied at ∞ in D, then there exists an open set $O \subset D$ which contains E such that O is also rarefied at ∞ in D. We now choose $u = \hat{R}^O_r$ and recall from Theorem 4.2 that $u \in \mathbf{S}_1$. Furthermore, u dominates r everywhere on $O \supset E$ so that $\liminf u(x)/|x| \geq 1$, $x \to \infty$, $x \in O$.

For the direct part, we choose $\varepsilon > 0$ and define $A_\varepsilon = \{x \in D \colon u(x)/|x| \geq \varepsilon\}$. Then $\hat{R}^{A_\varepsilon}_r$ is dominated by $u/\varepsilon \in \mathbf{S}_1$. It follows from Theorem 4.2 that A_ε is rarefied at ∞ in D for each $\varepsilon > 0$. Choose $x_0 \in D$ and $\varepsilon_n = 1/n$, $n = 1, 2, \ldots$. We also define $W_m = \bigcup_{j=m}^\infty I_j$ and the double sequence of sets $E_{n,m} = A_{1/n} \cap W_m$. Since $A_{1/n}$ is rarefied at ∞ in D for each n, it follows from Lemma 4.1 that

$$\lim \hat{R}^{E_{n,m}}_r(x_0) = 0, \qquad m \longrightarrow \infty, \quad n \text{ fixed}.$$

We now choose an increasing sequence $\{m_n\}$ so that $\hat{R}^{E_{n,m_n}}_r(x_0) \leq 2^{-n}$. Let $V_n = E_{n,m_n}$ and $E = \bigcup_{n=1}^\infty V_n$. We note that $R^E_r(x_0) \leq \sum_n \hat{R}^{V_n}_r(x_0) < \infty$. Lemma 4.2 shows that $\sum \hat{R}^{V_n}_r \in \mathbf{S}_1$, and hence $\hat{R}^E_r \in \mathbf{S}_1$, i.e. E is rarefied at ∞ in D. What happens if $x \in (D \setminus E) \cap W_{m_n}$? Then $x \notin A_{1/n}$ so that $u(x)/|x| \leq 1/n$. Thus $u(x)/|x| \to 0$, $x \to \infty$, $x \in D \setminus E$ which terminates the proof.

THEOREM 4.7. *Let $\beta \in [0, 1]$ be given and assume that $D \subset \mathbf{R}^p$. If $u \in \mathbf{S}_\beta$, there exists a set $E \subset D$ such that E is rarefied or minimally thin at ∞ in D such that*

$$\lim u(x)/|x|^\beta = 0, \qquad x \longrightarrow \infty, \quad x \in D \setminus E,$$

$$\lim u(x)/(x_1 r^{\beta-1}) = 0, \quad x \longrightarrow \infty, \quad x \in D \setminus E,$$

respectively.

Conversely, this result is best possible in the sense that if E is rarefied or minimally thin at ∞ *in D, then there exists* $u \in \mathbf{S}_\beta$ *such that E is contained in the exceptional set for u.*

PROOF. We use results from Theorems 4.4 and 4.5 to construct a proof that is analogous to the proof of Theorem 4.6.

REMARK 4.8. The results of Theorem 4.7 are closely related to recent results of Essén and Lewis (see [9]). An essential part of their work is a study of superharmonic functions of the form $u \in \mathbf{S}_\beta$. Their conclusion is that $u(x)/|x|^\beta \to 0$, $x \to \infty$, $x \in D \setminus E$, where E is an exceptional set that satisfies Azarin's condition. One can conclude from this work along with Azarin's, that a rarefied set $E \subset D \subset \mathbf{R}^p$ can be covered by a sequence of balls $\{r_n, R_n\}$ such that $\sum_n (r_n/R_n)^{p-1} < \infty$. This condition does not, of course, characterize a rarefied set. In fact, it should only be viewed as a good approximation for that part of the exceptional set that is near (or even on) the boundary. For that part of an exceptional set that is restricted to a Stolz domain, for example, our earlier work [8] indicates that the critical value for the convergence exponent is $p-2$ rather than $p-1$. In the next section, we shall consider these questions further.

§5. The main covering theorems

In order to obtain covering results which are converse to Theorems 1.1 and 4.3, we shall require a result, to be named Lemma 5.1, that resembles Lemma 1 in [8].

DEFINITION 5.1. Let $h: [0, \infty) \to [0, \infty)$ be a non-decreasing continuous function such that $h(0)=0$. If $H = B \cap D \in \mathbf{H}$ where B has radius r and the first coordinate of the centre of B is t, then we define the premeasures (see Rogers [13], p. 9) of the form

$$\tau_h(H) = t^{1+a} h(r), \quad H \in \mathbf{H},$$

for each of the numbers $a=0$, $a=1$.

DEFINITION 5.2. For each α of some index set, let $\mathbf{H}_\alpha \subset \mathbf{H}$ be a countable cover of $E \subset D$. If $\mathbf{H}_\alpha = \{H_{\alpha(n)}\}$, and if $\{\mathbf{H}_\alpha\}$ is the set of all coverings of E defined above, then we define

$$L_h(E) = \inf_{\{\mathbf{H}_\alpha\}} \{\textstyle\sum_n \tau_h(H_{\alpha(n)})\} \qquad \text{for} \quad a = 0 \text{ or } 1.$$

REMARK 5.1. For each $a=0$ or 1, the set function L_h is an outer measure, or a countably sub-additive (c.s.a.) weight (Brelot [3], p. 22), and is constructed from the premeasure τ_h by Method I in Rogers [13], p. 9. By Theorem 13 in

[13], p. 24, L_h is an $\mathbf{H}_{\sigma\delta}$-regular outer measure. Therefore, it is a continuous weight on D ([3], pp. 22–25). The outer measure L_h is constructed from the pre-measure τ_h in the same way as the outer measure M_h in Carleson ([5], p. 6) is constructed from the premeasure $\rho_h(B)=h(r)$. Dellacherie ([6], p. 92, Cor. 19) has pointed out that M_h is, in fact, a general capacity. This also follows from Rogers ([13], p. 90, Theorem 47) and we suspect, but shall not prove, that L_h is also a capacity in the general sense. The plausibility of this assertion is indicated by the fact that the metric outer measure constructed from τ_h using method II in Rogers [13] appears to be of "Hausdorff type" as developed by Davies (cf. Dellacherie [6], pp. 100–101). Such a measure would relate to L_h as the Hausdorff measure Λ_h relates to the capacity M_h (cf. [5], p. 6).

Remark 5.2. If $h(r)=r^{p-2}$, we have $\tau_h(H)\approx\gamma(H)$ if $a=1$ (Lemma 3.2) and $\tau_h(H)\approx\lambda(H)$ if $a=0$ (Lemma 3.3). By using only the monotone and countably subadditive properties of the Green energy γ and the Green mass λ' (cf. Lemmas 2.1 and 2.3), respectively, we have $\gamma(E)\leq k(p)L_h(E)$ if $a=1$, and $\lambda'(E)\leq k(p)L_h(E)$ if $a=0$, where k is a constant depending only on the dimension of the space.

Remark 5.3. In order to analyze L_h further, we shall also find it convenient to cover $E\subset D$ by dyadic cubes instead of balls or elements of H. Let us say that a cube in $\mathbf{R}^p$ is half-open if it is of the form $\{x\in\mathbf{R}^p\colon a_i\leq x_i<a_i+b,\ i=1,2,\dots,p\}$. Let $\mathbf{G}_n$ be a net of half-open cubes in D similar to those constructed in Carleson ([5], pp. 6–7) and let $\mathbf{G}=\cup_n\mathbf{G}_n$. We recall that all cubes have their sides parallel to the coordinate axis and that the length of a side of each cube in $\mathbf{G}_n$ is 2^{-n}. Furthermore, the cubes in $\mathbf{G}_n$ are obtained by dividing each side of every cube in $\mathbf{G}_{n-1}$ into halves so that every cube in $\mathbf{G}_{n-1}$ will be subdivided into 2^p equal subcubes. In addition, we arrange each net $\mathbf{G}_n$ so that the first coordinate of any vertex of each member of $\mathbf{G}_n$ is either 0 or of the form $m2^{-n}$ $(m\in N,\ n\in N)$.

Definition 5.3. Let $\omega\in\mathbf{G}_n$ and let t be the first coordinate of the centre of ω. We define the pre-measure τ_h^* on $\mathbf{G}$ so that if $a=0$ or 1, we have

$$\tau_h^*(\omega)=t^{1+a}h(2^{-n}),\quad \omega\in\mathbf{G}_n.$$

Definition 5.4. Let $\{\mathbf{C}_\alpha\}\subset\mathbf{G}$ be a countable cover of $E\subset D$. If $\mathbf{C}_\alpha=\{\omega_{\alpha(i)}\}$ and if $\{\mathbf{C}_\alpha\}$ is the collection of all such coverings of E, we define for each $a=0,1$,

$$L_h^*(E)=\inf_{\{\mathbf{C}_\alpha\}}\left(\textstyle\sum_i\tau_h^*(\omega_{\alpha(i)})\right).$$

Remark 5.4. The set function L_h^* is an outer measure (or a c.s.a. weight) on the subsets of D which is constructed from the pre-measure τ_h^* by Method I in Rogers [13; p. 9]. Furthermore, L_h^* is $\mathbf{G}_{\sigma\delta}$-regular. Since $\mathbf{G}$ is a net on D (cf. [13], p. 101), L_h^* satisfies the following monotone sequence property:

$$L_h^*(\cup_j S_j) = \sup_j \{L_h^*(S_j)\}, \tag{5.1}$$

for any increasing sequence $\{S_j\}$ of subsets of D ([13], Theorem 52, p. 107) and is therefore a precapacity in the sense of Dellacherie ([6], p. 19). The function L_h^* bears a relationship with L_h which is analogous to the relationship between the set functions m_h (or m_h') and M_h that are constructed in Carleson ([5], pp. 6–11). An elementary argument shows that

$$L_h(E) \approx L_h^*(E), \tag{5.2}$$

where the constants of comparison depend only on the dimension of the space. For similar discussions, we refer to Carleson ([5], pp. 6, 7, 11) or Rogers ([13], p. 102).

In the proof of the relation

$$L_h(E) \leq \text{Const.}\, L_h^*(E),$$

(which in fact is the only one used in this work), it is essential that we have used the sets in $\mathbf{H}$ in the definition of L_h.

We shall now prove a result which resembles the converse part of Theorem 1, p. 7, in Carleson ([5]).

LEMMA 5.1. *Let F be a compact set contained in $D \cap \{|x|<b\}$, and h: $[0, \infty) \to [0, \infty)$ be defined as in Definition 5.1. For each $a=0, 1$, there exist a mass distribution (non-negative Radon measure) μ supported by F, a constant $C_1>0$ depending only on the dimension p of the space and a constant C_2 depending only on p and b such that if $H \in \mathbf{H}$, then*

$$L_h(F) \leq C_1(\lambda, \mu) = C_1 \int_F x_1 d\mu(x), \tag{5.3}$$

$$(\lambda, \mu|_H) = \int_H x_1 d\mu(x) \leq C_2 \tau_h(H), \tag{5.4}$$

where λ is the fundamental distribution on F.

PROOF. We shall follow an argument initiated by Frostman, and similar to the one outlined by Carleson ([5], p. 8). We will construct a sequence $\{\mu_n\}$ of mass distributions on D that satisfy the following conditions:

(i) $\operatorname{supp} \mu_n = F_n$ is the closure of the union of those cubes $\omega \in \mathbf{G}_n$ which are such that $\omega \in F \neq \varnothing$. We note that $F_{n+1} \subset F_n$ and that $\bigcap_{n=1}^{\infty} F_n = F$.

(ii) If $\omega \in \bigcup_{i=0}^{n} \mathbf{G}_i$, then

$$(\lambda, \mu_n|_\omega) \le \int_\omega x_1 d\mu_n(x) \le \tau_h^*(\omega).$$

(iii) $$\int_{F_n} x_1 d\mu_n(x) \ge L_h^*(F), \quad n = 1, 2, \dots .$$

Let n be a given natural number. We first define a preliminary sequence $\{\mu_n^{(n)}\}$ of mass distributions on D so that $\{\mu_n^{(n)}\}$ has constant density on each $\omega \in \mathbf{G}_n$ and is defined so that the total measure is

$$\mu_n^{(n)}(\omega) = \begin{cases} t^a h(2^{-n}), & \text{if } \omega \cap F \ne \emptyset, \\ 0, & \text{if } \omega \cap F = \emptyset. \end{cases}$$

Therefore supp $\mu_n^{(n)} = F_n$ and

$$\int_\omega x_1 d\mu_n^{(n)}(x) = t\mu_n^{(n)}(\omega) \le \tau_h^*(\omega),$$

if $\omega \in \mathbf{G}_n$. Equality holds if $\omega \subset F_n$. It is clear that $\int_{F_n} x_1 d\mu_n^{(n)}(x) \ge L_h^*(F)$ so that the preliminary sequence $\{\mu_u^{(n)}\}$ satisfies (i) and (iii). In order to obtain (ii) as well, we follow an inductive argument of Frostman which progressively reduces the density of each $\mu_n^{(n)}$ where necessary, according to the following procedure: if for some $\omega \in \mathbf{G}_{n-1}$, $\omega \cap F \ne \emptyset$, we have $\int_\omega x_1 d\mu_n^{(n)}(x) > t^{1+a} h(2^{-n+1}) = \tau_h^*(\omega)$, we reduce the density of $\mu_n^{(n)}$ on ω (by multiplying it by a positive number <1) to obtain a new measure whose corresponding integral over ω becomes equal to $\tau_h^*(\omega)$. We continue this procedure for all such cubes in $\mathbf{G}_{n-1}$ and obtain a new measure, denoted by $\mu_n^{(n-1)}$, such that (ii) holds for any $\omega \in \mathbf{G}_n \cup \mathbf{G}_{n-1}$, and such that (i) and (iii) are still valid. After n such steps, we obtain a mass distribution $\mu_n^{(0)}$, denoted by μ_n, for which (ii) holds as well as (i) and (iii). It follows from (ii) that the sequence $\left\{\int_{F_n} x_1 d\mu_n(x)\right\}$ is bounded. Since the distance from F to ∂D is positive, the sequence $\{\mu_n(F_n)\}$ will also be bounded. It follows that there exists a subsequence $\{\mu_{n_k}\}$ that converges in the weak* topology to a mass distribution μ of support F. Therefore we have

(5.5) $$(\lambda, \mu) = \int_F x_1 d\mu(x) = \lim_k \int_{F_{n_k}} x_1 d\mu_{n_k}(x) \ge L_h^*(F).$$

Combining (5.5) and (5.2), we obtain (5.3).

Finally we shall show (5.4). Let $H = B \cap D = B(t, r, R) \cap D$. First we restrict $r<1$ and choose an integer n_0 so that $2^{-n_0} \le r < 2^{-n_0+1}$. Since 5 dyadic intervals, each of length 2^{-n_0}, are sufficient to cover an interval of length $2r$ $=\operatorname{diam} H$, 5^p dyadic cubes in $\mathbf{G}_{n_0}$ are sufficient to cover H. If $\omega_1, \dots, \omega_{5^p}$ with the first coordinates of centres $t_1, \dots, t_{5^p}$ cover H, then for any $n \ge n_0$

$$\int_H x_1 d\mu_n(x) \le \sum_j \int_{\omega_j} x_1 d\mu_n(x)$$
$$\le \sum_j \tau_h^*(\omega_j) = \sum_j t_j^{1+a} h(2^{-n_0}) \le h(r) \sum_j t_j^{1+a}$$

by (ii), where the summation is taken for $j=1,\dots, 5^p$. Since $t_i \le 2(t+r) \le 2t(1+\sqrt{p})$,

$$\int_H x_1 d\mu_n(x) \le 5^p 2^{1+a}(1+\sqrt{p})^{1+a} t^{1+a} h(r) = C_2 \tau_h(H).$$

Since F is compact in D, we may assume that the supports of $\{\mu_n\}$ are contained in a compact set $F' \supset F$ in D for large n. Let χ_H be the characteristic function of H, which is considered to be open. We have

$$(\lambda, \mu|_H) = \int_H x_1 d\mu(x) = \int \chi_H x_1 d\mu(x) \le \liminf_{k\to\infty} \int_H x_1 d\mu_{n_k}(x)$$

because $\chi_H x_1$ is lower semicontinuous. Hence $(\lambda, \mu|_H) \le C_2 \tau_h(H)$.

Next we assume $1 \le r$. Let m_0 be the smallest integer such that $m_0 \ge b$, and denote by A the set $\{(x_1,\dots, x_p); 0 \le x_1 < m_0, |x_i| < m_0, k=2,\dots, p\}$, consisting of m_0^p half open squares $\{\omega_j\}$. Evidently A contains $D \cap \{|x|<b\}$. For any $H = B(t, r, R) \cap D$ with $r \ge 1$ and for $n_k \ge m_0$,

$$\int_H x_1 d\mu_{n_k}(x) = \int_F x_1 d\mu_{n_k}(x) = \int_A x_1 d\mu_{n_k}(x) = \sum_j \int_{\omega_j} x_1 d\mu_{n_k}(x)$$
$$\le \sum_j \tau_h^*(\omega_j) = \sum_j t_j^{1+a} h(1) \le h(r) \sum_j t_j^{1+a}$$

by (ii), where $\{t_j\}$ are the first coordinates of the centres of $\{\omega_j\}$. Since $t_j < m_0$ and $t \ge r/\sqrt{p} > 1/\sqrt{p}$, $t_j < m_0\sqrt{p}\, t$ so that

$$\int_H x_1 d\mu_{n_k}(x) \le 2m_0^p (m_0\sqrt{p})^{1+a} t^{1+a} h(r) = C_2 \tau_h(H),$$

where $C_2 = 2m_0^p(m_0\sqrt{p})^{1+a}$ depends only on p and b. By letting $k\to\infty$ we obtain $(\lambda, \mu|_H) \le C_2 \tau_h(H)$. This completes the proof of Lemma 5.1.

LEMMA 5.2. *If $F \subset D \cap \{|x|<b\}$, where F is compact, and if h is defined as in Definition 5.1 and such that $\int_0^\infty \phi(r) dh(r) = A(p) < \infty$, then there exists a positive constant $C(p)$ (which only depends on the dimension and b) such that*

$$L_h(F) \le \begin{cases} C(p)\gamma(F), & \text{if } a=1, \\ C(p)\lambda(F), & \text{if } a=0. \end{cases}$$

PROOF. Let μ be the measure depending on h, F and a that was constructed in Lemma 5.1. We shall demonstrate that

$$(5.6) \qquad (\mu, \lambda) \leq \begin{cases} \text{Const.}\,(\lambda, \lambda), & \text{if} \quad a = 1, \\ \text{Const.}\,\lambda(F), & \text{if} \quad a = 0. \end{cases}$$

Elementary calculations ([11], Lemma 1) imply that

$$G(x, y) \leq 4(p-2)x_1 y_1 \phi(|x-y|)\,|x-y'|^{-2}, \qquad (x, y) \in D \times D.$$

Since $|x-y'| \geq x_1$ and $|x-y'| \geq |x-y|$, it is clear that

$$(5.7) \qquad G(x, y) \leq 4(p-2)x_1^{-1}\phi(|x-y|)y_1,$$

$$(5.7)' \qquad G(x, y) \leq 4(p-2)x_1 y_1 |x-y|^{-p}.$$

Now define $\Phi_x(r) = (\lambda, \mu|_B)$, where B is a ball of centre x and radius r. For $x \in D$ fixed, we can say that Φ_x is a non-decreasing function of r and is well defined for all $r > 0$ since $\operatorname{supp} \mu \subset F \subset D$. By (5.4) in Lemma 5.1, we have (if $a = 0$ or 1)

$$(5.8) \quad \Phi_x(r) \leq C_1 \tau_h(B) = C_1 x_1^{1+a} h(r), \quad B \cap D \in \mathbf{H}, \qquad (\text{i.e. if } r \leq x_1\sqrt{p}).$$

From (5.7) and (5.7)′ we see that

$$G\mu(x) \leq 4(p-2)\left\{x_1^{-1}\int_0^{x_1\sqrt{p}} \phi(r)\,d\Phi_x(r) + x_1 \int_{x_1\sqrt{p}}^{\infty} r^{-p}\,d\Phi_x(r)\right\}$$
$$= 4(p-2)(J_1 + J_2).$$

We first discuss J_1 and note that

$$t^{2-p}h(t) \leq \int_0^t s^{2-p}\,dh(s) \longrightarrow 0, \quad t \longrightarrow 0+,$$

$$t^{2-p}h(t) \longrightarrow 0, \quad t \longrightarrow \infty.$$

The last relation holds since

$$t^{2-p}(h(t) - h(t_0)) \leq \int_{t_0}^t s^{2-p}\,dh(s) \longrightarrow 0, \quad t_0 < t, \; t_0 \longrightarrow \infty.$$

Integrating by parts (cf. Carleson [5], p. 29), we obtain from (5.8) that

$$(5.9) \qquad J_1 \leq \text{Const.}\, x_1^a\{A(p) + \sup_{t>0} t^{2-p}h(t)\}, \quad a = 0 \text{ or } 1.$$

We now turn to J_2 and note that

$$J_2 \leq \begin{cases} x_1 \int_{x_1\sqrt{p}}^{\infty} r^{-p} d\Phi_x(r), & a = 1, \\ \int_{x_1\sqrt{p}}^{\infty} r^{1-p} d\Phi_x(r), & a = 0. \end{cases}$$

We choose $s=2$. If $s^{n+1} > x_1\sqrt{p}$, we cover $I_n = \{y \in D: 2^n \leq |x-y| < 2^{n+1}\}$ by finitely many sets from $\mathbf{H}$, of the form $B \cap D$ where the radius of B is 2^n. It is clear that we can do this in such a way that the number of balls needed has an upper bound only depending on the dimension p. Once more using (5.4), we see that

$$\Phi_x(2^{n+1}) - \Phi_x(2^n) \leq \text{Const.}\, 2^{n(1+a)} h(2^n), \qquad (a = 0, 1).$$

It is now easy to see that (if $\sum$ denotes summation over those indices n which are such that $2^{n+1} > x_1\sqrt{p}$) we have, if $a=0$ or 1,

$$(5.10) \qquad \begin{aligned} J_2 &\leq x_1^a \int_{x_1\sqrt{p}}^{\infty} r^{1-a-p} d\Phi_x(r) \\ &\leq x_1^a \sum \int_{2^n}^{2^{n+1}} d\Phi_x(r) 2^{n(1-a-p)} \\ &\leq \text{Const.}\, x_1^a \sum 2^{n(1+a)} h(2^n) 2^{n(1-a-p)} \\ &\leq \text{Const.}\, x_1^a \int_0^{\infty} r^{1-p} h(r) dr \leq \text{Const.}\, x_1^a. \end{aligned}$$

Combining (5.9) and (5.10), we see that $G\mu(x) \leq \text{Const.}\, x_1^a$, $x \in D$. Integrating with respect to λ, we obtain (5.6) and therefore also our lemma by applying (5.3) in Lemma 5.1.

LEMMA 5.3. *Let $E \subset D \cap \{|x| < b\}$, and let h be defined as in Lemma 5.2. Then there exists a positive constant $C(p)$ (which only depends on the dimension and b) such that*

$$L_h(E) \leq \begin{cases} C(p)\gamma(E), & a = 1, \\ C(p)\lambda(E), & a = 0. \end{cases}$$

PROOF. We first consider the case when $E = O$ is an open subset of D. Then O is σ-compact and we can construct an increasing sequence $\{F_n\}$ of compact subsets whose union is O. Therefore,

$$\lim_n L_h(F_n) \leq \begin{cases} \text{Const.}\, \gamma(O), & a = 1, \\ \text{Const.}\, \lambda(O), & a = 0, \end{cases}$$

by Lemma 5.2. We now apply the results of Remark 5.4. From (5.2) and the monotone sequence property of L_h^* described in (5.1), we obtain

$$L_h(O) \approx L_h^*(O) = \lim_n L_h^*(F_n) \approx \lim_n L_h(F_n).$$

Thus our lemma holds when E is an open subset of D. Since L_h is monotone and $\gamma(E)=\inf\{\gamma(O)\colon O\supset E,\ O \text{ open}\}$ (cf. (2.1)), Lemma 5.3 follows in the case $a=1$. When $a=0$, the same argument works (cf. (2.2)), and the lemma is proved.

We are now able to prove our main covering theorems.

THEOREM 5.1. *Let $E\subset D\subset \mathbf{R}^p$, be minimally thin or rarefied at ∞ in D. If the function h: $[0, \infty)\to[0, \infty)$ is non-decreasing, continuous and such that $h(0)=0$, and if ϕ is defined as in* (iii), § 2 *and if*

$$\int_0^\infty \phi(r)dh(r) = A(p) < \infty,$$

then there exists a covering of E by a sequence $\{H_n\}$ in $\mathbf{H}$, where $H_n=B\cap D$, $B_n=(t_n, r_n, R_n)$ such that

$$\sum_n (t_n/R_n)^2 h(r_n/R_n) < \infty,$$

$$\sum_n (t_n/R_n) h(r_n/R_n) < \infty,$$

respectively.

PROOF. As in Section 3, we let $E_n'=s^{-n}E_n$, and recall from Definition 3.1 or Definition 3.2 that $E\subset D$ is minimally thin or rarefied at ∞ in D iff

$$\sum_n \gamma(E_n') < \infty,$$

$$\sum_n \lambda(E_n') < \infty,$$

respectively. We apply Lemma 5.3 to obtain

$$\sum_n L_h(E_n') < \infty, \quad a = 1 \quad \text{or} \quad 0, \quad \text{respectively.} \tag{5.11}$$

For each n, we can cover E_n' by a sequence $\{H_{nj}'\}=\{B_{nj}'\cap D\}$ in $\mathbf{H}$, where $B_{nj}'=(t_{nj}', r_{nj}', R_{nj}')$, such that

$$\sum_j (t_{nj}')^{1+a} h(r_{nj}') = L_h(E_n') + \varepsilon_n, \quad 0 \le \varepsilon_n < 2^{-n}, \quad a = 1 \text{ or } 0.$$

The convergence of the double series $\sum_{n,j}(t_{nj}')^{1+a}h(r_{nj}')$ in the two cases considered here follows from (5.11). Defining $B_{nj}=s^n B_{nj}'$ so that $B_{nj}=(t_{nj}, r_{nj}, R_{nj})$, we see that the double sequence $\{B_{nj}\cap D\}$ is a covering of the required type. This completes the proof of Theorem 5.1.

COROLLARY. *Let $E\subset D\subset \mathbf{R}^p$ be minimally thin or rarefied at ∞ in D.*

For each $\beta > p-2$, *E can be covered by a sequence* $\{B_n \cap D\}$ *in* $\mathbf{H}$ *such that*

$$\sum_n (t_n/R_n)^{1+a}(r_n/R_n)^{\beta} < \infty, \quad a = 1 \text{ or } 0, \text{ respectively.}$$

Proof. In Theorem 5.1, we choose $h(r)=r^{\beta}$ for $0 \leqq r \leqq 1$ and $h(r)=1$ for $r>1$. If $r_n/R_n>1$, then $R_n<r_n\leq\sqrt{p}\,t_n$ and hence $(t_n/R_n)^{1+a}(r_n/R_n)^{\beta}\geq p^{-(1+a)/2}$. Therefore there exists $n_0\geq 0$ such that $r_n/R_n\leq 1$ if $n>n_0$. Hence $h(r_n/R_n)=(r_n/R_n)^{\beta}$ if $n>n_0$. Our Corollary now follows.

Remark 5.5. One cannot choose $\beta\leq p-2$ in the corollary to Theorem 5.1. This is clear from our earlier work ([8] p. 338) when the exceptional set is restricted to a Stolz domain. A direct proof that a set E which is rarefied at ∞ in D satisfies Azarin's condition starts from Theorem 4.2 which gives the result that $\hat{R}_r^E \in \mathbf{S}_1$. From this point, we can follow Azarin's proof and study the exceptional set for the positive superharmonic function $\hat{R}_r^E$.

Remark 5.6. In Remark 4.8, we discussed certain results of Essén and Lewis. It follows from Theorem 4.7 that their exceptional set E is rarefied at ∞ in D. Therefore, the covering results in Theorem 5.1 hold also for this set E. Strictly speaking, we have proved this only in a half-space and Essén and Lewis work in circular cones. The generalization to this more general situation is technical but causes no essential difficulties.

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