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J-groups of lens spaces modulo powers of two

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§1. Introduction

Let $J(X)$ be the J-group of a CW-complex X of finite dimension. Then by J. F. Adams [2] and D. Quillen [10], it is shown that

$$(1.1)\qquad J(X) = KO(X)/\mathrm{Ker}\,J,\quad \mathrm{Ker}\,J = \textstyle\sum_k(\bigcap_e k^e(\Psi^k-1)KO(X)),$$

where $KO(X)$ is the KO-group of X, $J\colon KO(X)\to J(X)$ is the natural epimorphism and Ψ^k is the Adams operation.

In this paper, we study the J-group of the standard lens space modulo 2^r $(r\geqq 2)$:

$$L^n(2^r) = S^{2n+1}/Z_{2^r},\quad Z_{2^r} = \{z\in S^1\colon z^{2^r}=1\},$$

which is the orbit manifold of the unit $(2n+1)$-sphere S^{2n+1} in C^{n+1} by the diagonal action $z(z_0,\dots,z_n)=(zz_0,\dots,zz_n)$. In the case $r=1$, $L^n(2)$ is the real projective space RP^{2n+1}, and its J-group $J(L^n(2))$ is determined by J. F. Adams ([1, Th. 7.4], [2, II, Ex. (6.3)]).

Let η be the canonical complex line bundle over $L^n(2^r)$, i.e., the induced bundle of the canonical complex line bundle over the complex projective space $CP^n=S^{2n+1}/S^1$ by the natural projection $L^n(2^r)\to CP^n$. Then, the main purpose of this paper is to prove the following

THEOREM 1.2. *Let $r\geqq 2$ and let $r(\eta^i-1)\in\widetilde{KO}(L^n(2^r))$ be the real restriction of the stable class of the i-fold tensor product $\eta^i=\eta\otimes\cdots\otimes\eta$ of the canonical complex line bundle η over $L^n(2^r)$. Then the order of the J-image*

$$Jr(\eta^i-1)\in\tilde{J}(L^n(2^r))$$

is equal to

$$2^{f(n,r;v)},\quad f(n,r;v) = \max\{s-v+[n/2^s]2^{s-v}\colon v\leqq s<r \text{ and } 2^s\leqq n\},$$

where $v=v_2(i)$ is the exponent of 2 in the prime power decomposition of i and $\max\emptyset=0$.

Recently, we have proved in [5, Th. 1.1, 3.1] that the above theorem is valid also for any odd prime p instead of 2 and any $r \geqq 1$, by replacing 2 with p and 2^s with $p^s(p-1)$.

On the group structure of the reduced J-group $\tilde{J}(L^n(2^r))$ $(r \geqq 2)$, we have the following theorem, where

$$(1.3) \qquad a_s = [n/2^s], \quad b_s = n - 2^s a_s \qquad (0 \leqq s < r),$$

$$(1.4) \qquad X(d,v) = \sum_{j \in Z} (-1)^{j(2^v+1)} \binom{2d}{d+2^v j},$$

$$(1.5) \qquad Y(d,v) = \sum_{j \in Z} \binom{2d-1}{d+2^v(2j+1)}.$$

THEOREM 1.6. (i) *$\tilde{J}(L^n(2^r))$ $(r \geqq 2)$ is generated by*

$$J\kappa \quad and \quad \alpha_s = Jr(\eta^{2^s}-1) \qquad (0 \leqq s \leqq r-2),$$

where $\kappa = \rho - 1$ and ρ is the non-trivial real line bundle over $L^n(2^r)$.

(ii) ([6, Th. 4.5]) $J\colon \widetilde{KO}(L^n(4)) \cong \tilde{J}(L^n(4))$.

(iii) *The relations of $\tilde{J}(L^n(2^r))$ for $r \geqq 3$ are given as follows:*

(a) *The case $n \not\equiv 1 \bmod 4$:*

$$(1.6.1) \quad 2^{1+a_{r-1}} J\kappa = 0, \qquad 2^{r-1+2a_1}\alpha_0 = 0, \qquad 2^{r-1-s+a_s}\alpha_s = 0 \quad (1 \leqq s \leqq r-2).$$

$$(1.6.2) \quad 2^{a_{r-1}} J\kappa + \sum_{v=0}^{r-2} 2^{2^{r-1-v}(1+a_{r-1})-2}\alpha_v = 0 \quad if \quad a_1 \geqq 2^{r-2}.$$

$$(1.6.3) \quad 2^{r-s-2+a_s}\alpha_s + \sum_{v=0}^{s-1} 2^{r-s-3+2^{s-v}(1+a_s)}\alpha_v = 0 \qquad (1 \leqq s \leqq r-2,\ 2^s \leqq a_1).$$

$$(1.6.4) \quad \sum_{v=0}^{s} (-1)^{2^{s-v}} 2^{r-s-4+2^{s+1-v}(a_{s+1}+\delta)} X(d,v)\alpha_v = 0$$
$$(1 \leqq s \leqq r-2,\ 1 \leqq d < 2^s,\ 2^s + d \leqq a_1),$$

where $\delta = 1$ if $2d \leqq b_{s+1}$, $=0$ otherwise.

$$(1.6.5) \quad 2^{2i-2}\alpha_0 - \sum_{v=1}^{t} Y(i,v)\alpha_v = 0 \quad where \quad 2^t \leqq i < 2^{t+1} \qquad (a_1 < i < 2^{r-1}).$$

(b) *The case $n \equiv 1 \bmod 4$: The relations in* (a), *excluded the one in* (1.6.4) *for $s = r-2$, $2d = 1 + b_{r-1}$ and the one in* (1.6.5) *for $i = a_1 + 1$, and in addition,*

$$(1.6.6) \quad 2^{a_0}\alpha_0 - \sum_{v=1}^{t} 2Y(a_1+1,v)\alpha_v = 0 \quad where \quad 2^t \leqq a_1 + 1 < 2^{t+1} \quad if \quad a_1 < 2^{r-2}.$$

For the special case that $n = 2^{r-1}a$ or $2^{r-1}a - 1$, we can reduce the relations of $\tilde{J}(L^n(2^r))$ in (iii) of the above theorem to more simple ones, and $\tilde{J}(L^n(2^r))$ is given by the following explicit form, where $Z_h\langle x \rangle$ denotes the cyclic group of order h generated by the element x.

THEOREM 1.7. (i) *If $n = 2^{r-1}a - 1$ $(r \geqq 3,\ a \geqq 2)$, then $\tilde{J}(L^n(2^r))$ is the direct sum*

$$\oplus_{s=0}^{r-2} Z_{h(s)}\langle\alpha_s\rangle \oplus Z_{h(r-1)}\langle J\kappa+\textstyle\sum_{s=0}^{r-1} 2^{a_s-a_{r-1}-1}\alpha_s\rangle,$$

where $h(s)=2^{a_s}=2^{2^{r-s-1}a-1}$ *for* $0\leqq s\leqq r-1$.

(ii) *If* $n=2^{r-1}a$ $(r\geqq 3,\ a\geqq 2)$, *then* $\tilde{J}(L^n(2^r))$ *is the direct sum*

$$Z_{k(0)}\langle\alpha_0\rangle \oplus \oplus_{s=1}^{r-1} Z_{k(s)}\langle\alpha_s-2^{a_{s-1}-a_s+1}\alpha_{s-1}\rangle \oplus Z_{k(r-1)}\langle J\kappa+2^{a_{r-2}-a_{r-1}}\alpha_{r-2}\rangle,$$

where $k(0)=2^{r-1+n}$, $k(s)=2^{a_s-1}=2^{2^{r-s-1}a-1}$ *for* $1\leqq s\leqq r-2$ *and* $k(r-1)=2^{a_{r-1}}=2^a$.

By using the above theorem, we can determine the kernel of the homomorphism

$$(1.8)\qquad i^*\colon \tilde{J}(L^n(2^r)) \longrightarrow \tilde{J}(L^{n-1}(2^r))$$

induced by the inclusion $i\colon L^{n-1}(2^r)\subset L^n(2^r)$ as follows:

PROPOSITION 1.9. *i^* in* (1.8) *is isomorphic if* $n\equiv 3 \bmod 4$, *epimorphic otherwise, and*

$$\operatorname{Ker} i^* = \begin{cases} Z_4\langle 2J(\bar{\sigma}^{2m+1})\rangle & \text{if } n=4m+2\\ Z_2\langle J(\bar{\sigma}^{2m+1})\rangle & \text{if } n=4m+1\\ Z_u\langle J(\bar{\sigma}^{2m})\rangle & \text{if } n=4m>0,\end{cases}$$

where $\bar{\sigma}=r(\eta-1)\in\widetilde{KO}(L^n(2^r))$ *and*

$$u=2^{\min\{r+1,l+2\}}\quad \textit{for } n=4m=2^l q \textit{ with } (2,q)=1.$$

By this proposition, we see immediately the following

THEOREM 1.10. *The order of the reduced J-group* $\tilde{J}(L^n(2^r))$ *is equal to*

$$2^{\varphi(n,r)},\quad \varphi(n,r)=(r+1)a_{r-1}+\textstyle\sum_{s=1}^{r-2}(s+2)\,[(a_s+1)/2]+1+\varepsilon,$$

where a_s *is the integer given by* (1.3) *and*

$$(1.11)\qquad \varepsilon=1\quad \textit{if}\quad n\equiv 1 \bmod 4,\qquad =0\quad \textit{otherwise}.$$

By using Proposition 1.9 and Theorem 1.7 (ii), we can prove Theorem 1.2 by the induction on n and r.

We prepare in § 2 some known results on the K- and KO-groups of $L^n(2^r)$ given in [4], and determine in § 3 the generators of Ker J in (1.1) for $X=L^n(2^r)$ explicitly. Some lemmas for the coefficient $X(d,v)$ in (1.4) are prepared in § 4.

By using these results, we prove Theorem 1.6 in § 5, and Theorem 1.7 in § 6. In § 7, we prove Proposition 1.9 in Corollary 7.11 and Theorem 1.10 in Proposition 7.9 (ii) by using the results on Ker $\{i^*\colon \widetilde{KO}(L^n(2^r))\to\widetilde{KO}(L^{n-1}(2^r))\}$ ([4, Prop. 4.4]) and by studying the Adams operation Ψ^3 on $\widetilde{KO}(L^n(2^r))$. Theorem 1.2 is proved in § 8.

For the special case that $r \leqq 5$, we give the direct sum decomposition of $\tilde{J}(L^n(2^r))$ in Proposition 9.3.

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§ 2. The structures of $\tilde{K}(L^n(2^r))$ and $\widetilde{KO}(L^n(2^r))$

In this section, we prepare some known results on the K- and KO-rings of the standard lens space $L^n(2^r)$.

Let η be the canonical complex line bundle over $L^n(2^r)$. Then,

(2.1) (N. Mahammed [9]) *the K-ring of $L^n(2^r)$ is the quotient ring*

$$K(L^n(2^r)) = Z[\eta]/\langle \eta^{2^r}-1, (\eta-1)^{n+1}\rangle$$

of the integral polynomial ring $Z[\eta]$ by the ideal generated by $\eta^{2^r}-1$ and $(\eta-1)^{n+1}$, and the order of the reduced K-group $\tilde{K}(L(2^r))$ is equal to 2^{rn}.

Moreover, consider the elements

(2.2) $$\sigma = \eta-1 = \sigma(0), \quad \sigma(s) = \eta^{2^s}-1 = (1+\sigma)^{2^s}-1 \qquad (s \geqq 0)$$

in $\tilde{K}(L^n(2^r))$. Then, (2.1) implies that

(2.3) $$\sigma(s) = 0 \quad \text{for} \quad s \geqq r, \qquad \sigma^i = 0 \quad \text{for} \quad i > n,$$

and by [8, Lemma 2.3], we see that

(2.4) $$2^{r-s-1+a_s}\sigma^{b_s}\sigma(s) = 0 \qquad (s \geqq 0)$$

where a_s and b_s are the integers in (1.3), i.e.,

(2.5) $$n = 2^s a_s + b_s, \quad 0 \leqq b_s < 2^s.$$

We notice that the group structure of $\tilde{K}(L^n(2^r))$ is given explicitly in [4, Th. 3.1].

For the reduced KO-group $\widetilde{KO}(L^n(2^r))$, consider the elements

(2.6) $$\kappa = \rho - 1, \quad \bar{\sigma} = r\sigma = \bar{\sigma}(0), \quad \bar{\sigma}(s) = r(\eta^{2^s}-1) = r\sigma(s),$$

where ρ is the non-trivial real line bundle over $L^n(2^r)$ and $r\colon K \to KO$ is the real restriction. Then, the equalities

(2.7) ([4, Prop. 6.3 (i)]) $\bar{\sigma}(s) = 4\bar{\sigma}(s-1) + \bar{\sigma}(s-1)^2 = \bar{\sigma}^{2^s} + \sum_{j=1}^{2^s-1} y_{sj}\bar{\sigma}^j \quad (s>0)$

hold, and we have the following

THEOREM 2.8 ([4, Th. 1.9]). *In the reduced KO-ring* $\widetilde{KO}(L^n(2^r))$, *there hold the relations*

(2.9) $\bar{\sigma}^i = 0 \quad$ *for* $\quad i > a_1+\varepsilon, \quad a_1 = [n/2], \quad \varepsilon = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ 0 & \text{otherwise}, \end{cases}$

(2.10) $$\bar{\sigma}(r-1) = 2\kappa;$$

and $\widetilde{KO}(L^n(2^r))$ $(r \geqq 2)$ *is the direct sum*

(2.11) $\widetilde{KO}(L^n(2^r)) = \oplus_{i=0}^{N'} Z_{u(i)}\langle\bar{\sigma}_i\rangle, \quad N' = \min\{2^{r-1}-1, a_1+\varepsilon\}$,

where the order $u(i)$ *and the generator* $\bar{\sigma}_i$ *are given by using* a_s *and* b_s *in* (2.5) *and* κ, $\bar{\sigma}$ *and* $\bar{\sigma}(s)$ *in* (2.6) *as follows*:

(i) $r=2$: $u(0) = 2, \quad \bar{\sigma}_0 = \kappa \quad (n=0)$,
$u(0) = 2^{a_1+\varepsilon}, \quad \bar{\sigma}_0 = \kappa + 2^{a_1}\bar{\sigma} \quad (n \geqq 1)$;
$u(1) = 2^{2a_1+1}, \quad \bar{\sigma}_1 = \bar{\sigma} \quad (n \geqq 1)$.

(ii) $r \geqq 3$: (a) *The case* $n \not\equiv 1 \bmod 4$: *For* $i=0$,

$u(0) = 2^{a_{r-1}}, \quad \bar{\sigma}_0 = \kappa + \sum_{t=1}^{r-1} 2^{(2^t-1)(a_{r-1}+1)-1}\bar{\sigma}(r-1-t) \quad (n \geqq 2^{r-1})$,
$u(0) = 2, \quad \bar{\sigma}_0 = \kappa \quad (n < 2^{r-1})$;

and for $i = 2^s+d \leqq a_1$ *with* $0 \leqq s \leqq r-2$ *and* $0 \leqq d < 2^s$,

$u(1) = 2^{r-1+2a_1}, \quad \bar{\sigma}_1 = \bar{\sigma}$;
$u(i) = 2^{r-s-2+a_s}, \quad \bar{\sigma}_i = \bar{\sigma}(s) + \sum_{t=1}^{s} 2^{(2^t-1)(a_s+1)}\bar{\sigma}(s-t) \quad$ *if* $\quad i = 2^s \geqq 2$;
$u(i) = 2^{r-s-3+a'(i)}, \quad a'(i) = \begin{cases} a_{s+1}+1 & \text{for } 2d \leqq b_{s+1}, \\ a_{s+1} & \text{for } 2d > b_{s+1}, \end{cases}$
$\bar{\sigma}_i = \bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{s-1}(2+\bar{\sigma}(t)) - 2^{a'(i)-1}\bar{\sigma}^d\bar{\sigma}(s) + \sum_{t=2}^{s+1} 2^{(2^t-1)a'(i)-1}\bar{\sigma}^d\bar{\sigma}(s+1-t)$
if $\quad i = 2^s+d \geqq 3, \ d \geqq 1$.

(b) *The case* $n \equiv 1 \bmod 4$: $u(i)$ *and* $\bar{\sigma}_i$ *are the same as* (a) *if* $i \neq a_1+1-2^{r-2}(a_{r-1}-1)$*), *and*

$u(i) = 2^{a_{r-1}}, \quad \bar{\sigma}_i = \bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{r-3}(2+\bar{\sigma}(t))$
if $\quad i = a_1+1-2^{r-2}(a_{r-1}-1) = 2^{r-2}+d, \ 2d = b_{r-1}+1 \quad (n \geqq 2^{r-1})$;
$u(i) = 2, \quad \bar{\sigma}_i = \bar{\sigma}^i \quad$ *if* $\quad i = a_1+1 \quad (n < 2^{r-1})$.

We notice the following lemma for the real restriction $r\colon K(L^n(2^r)) \to KO(L^n(2^r))$.

*) The condition $i \not\equiv a_1+1 \bmod 2^{r-2}$ in (b) on p. 471 of [4, Th. 1.9] is incomplete. It should be replaced by $i \neq a_1+1-2^{r-2}(a_{r-1}-1)$ of above.

LEMMA 2.12. (i) $r(\eta^i - \eta^{-i}) = 0 \quad (i \geqq 1)$.

(ii) $\bar{\sigma}^k \bar{\sigma}(s)^l = r(\sigma^{2k-1}\sigma(s)^{2l}/(1+\sigma)^{k-1}(1+\sigma(s))^l) \quad (s>0,\ k>0,\ l\geqq 0)$,

$2\bar{\sigma}^k \bar{\sigma}(s)^l = r(\sigma^{2k}\sigma(s)^{2l}/(1+\sigma)^{k}(1+\sigma(s))^l) \quad (s>0,\ k>0,\ l>0)$,

$\bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{s-1}(2+\bar{\sigma}(t)) = r(\sigma^{2d-1}\sigma(s+1)/(1+\sigma)^d(1+\sigma(s)))$

$(d>0,\ s\geqq 1)$.

(iii) $$\bar{\sigma}^k = r\left\{\sum_{i=1}^{k}\left\{\sum_{j=0}^{k-i}(-1)^j\binom{2k}{j}\binom{k-j}{i}\right\}\sigma^i\right\}.$$

PROOF. (i) Consider the complexification $c\colon KO \to K$ and the conjugation $t\colon K \to K$. Then $cr = 1+t$ and $t\eta = \eta^{-1}$ by [1, Th. 5.1], and hence $cr(\eta^i - \eta^{-i}) = 0$. Since $c\colon \widetilde{KO}(L^n(2^r)) \to \tilde{K}(L^n(2^r))$ is monomorphic if $n \equiv 3 \bmod 4$ by [11, (A.13)] (cf. [4, Prop. 5.3]), we see (i) for $n \equiv 3 \bmod 4$ and so for any n by the naturality.

(ii) By [4, Lemma 6.2 (i)], we see easily that

$$c(\bar{\sigma}^k \bar{\sigma}(s)^l) = cr(\sigma^{2k-1}\sigma(s)^{2l}/(1+\sigma)^{k-1}(1+\sigma(s))^l),$$
$$2c(\bar{\sigma}^k \bar{\sigma}(s)^l) = cr(\sigma^{2k}\sigma(s)^{2l}/(1+\sigma)^{k}(1+\sigma(s))^l),$$
$$c(\bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{s-1}(2+\bar{\sigma}(t))) = cr(\sigma^{2d-1}\sigma(s+1)/(1+\sigma)^d(1+\sigma(s))),$$

and these imply (ii) by the same way as the above proof.

(iii) By the first equality of (ii), we see that

$$\begin{aligned}\bar{\sigma}^k &= r(\sigma^{2k-1}/(1+\sigma)^{k-1}) = r((\eta-1)^{2k-1}\eta^{-k+1}) && \text{(by (2.2))}\\ &= r\left\{\sum_{i=0}^{2k-1}\binom{2k-1}{i}(-1)^{i+1}\eta^{i-k+1}\right\}\\ &= r\left\{(-1)^k\binom{2k-1}{k-1} + \sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}\eta^{k-j}\right\} && \text{(by (i))}\\ &= r\left\{\sum_{i=1}^{k}\left\{\sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}\binom{k-j}{i}\right\}\sigma^i\right\}. && \text{q.e.d.}\end{aligned}$$

§3. Some relations in $\tilde{J}(L^n(2^r))$

Now, consider the real restriction and the J-homomorphism

$$\tilde{K}(L^n(2^r)) \xrightarrow{r} \widetilde{KO}(L^n(2^r)) \xrightarrow{J} \tilde{J}(L^n(2^r)) \qquad (r \geqq 2),$$

where J is an epimorphism and

(3.1) $$\operatorname{Ker} J = \sum_k L_k, \quad L_k = \cap_e k^e(\Psi^k - 1)\widetilde{KO}(L^n(2^r))$$

by (1.1). Furthermore, consider the subgroup W of $\tilde{K}(L^n(2^r))$ defined by

(3.2) $$W = \sum_k W_k, \quad W_k = \cap_e k^e(\Psi_C^k - 1)\tilde{K}(L^n(2^r)),$$

where Ψ_C^k is the Adams operation on $\tilde{K}(L^n(2^r))$. Then, we have

LEMMA 3.3. (i) *W is the subgroup of $\tilde{K}(L^n(2^r))$ generated by*

$$\sigma^d(1+\sigma)\sigma(s) \qquad (0 \leqq s \leqq r-1,\ 0 \leqq d < 2^s - 1).$$

(ii)
$$\operatorname{Ker} J = rW.$$

(iii) $\operatorname{Ker} J$ *is generated by*

$$r(\sigma^d(1+\sigma)\sigma(s)) \qquad (0 \leqq s \leqq r-2,\ 0 \leqq d < 2^s - 1).$$

(iv) *Consider the elements*

$$\alpha_s = J\bar{\sigma}(s) = Jr\sigma(s) \in \tilde{J}(L^n(2^r)) \qquad (\alpha_s = 0 \text{ if } s \geqq r) \tag{3.4}$$

given in Theorem 1.6. *Then* (iii) *means the equalities*

$$Jr(\sigma^d\sigma(s)) = (-1)^d\alpha_s \qquad (0 \leqq s \leqq r-2,\ 0 \leqq d < 2^s). \tag{3.5}$$

(v) *The equalities* (3.4) *for* $s=r-1$ *and* (3.5) *imply*

$$Jr(\sigma^d\sigma(s)) = (-1)^d\alpha_s \qquad (0 \leqq s \leqq r-1,\ 0 \leqq d < 2^s). \tag{3.6}$$

PROOF. (i) Since $\Psi_C^k\eta^i = \eta^{ki}$ by [1, Th. 5.1], the last half of (2.1) shows that $W_k = 0$ if $k \equiv 0 \bmod 2$ and W_k is generated by $\{\eta^{kj} - \eta^j\}$ otherwise. By these facts and the relation $\eta^{2^r} = 1$ in (2.1), we see that W is generated by the elements

$$(*) \qquad \alpha(s, k) = \eta^{k2^s} - \eta^{2^s}, \qquad 0 \leqq s < r, \quad 1 \leqq k < 2^{r-s}, \quad (2,k) = 1.$$

Since $\alpha(t,1) = 0$ and $\alpha(t, k+2^{s-t}) - \alpha(t,k) = \eta^{k2^t}\sigma(s)$ for $0 \leqq t \leqq s$, the elements

$$(**) \qquad \eta^j\sigma(s), \qquad 0 \leqq s < r, \quad 1 \leqq j < 2^s,$$

are the linear combinations of the elements of $(*)$ and the converse is also true. Further it is easy to see that the elements in (i) are the linear combinations of the elements of $(**)$ and the converse is true.

(ii) Since the order of $\widetilde{KO}(L^n(2^r))$ is a power of 2 by Theorem 2.8, L_k in (3.1) is 0 if $k \equiv 0 \bmod 2$. Also the group $\widetilde{KO}(L^n(2^r))$ is generated by κ and $\bar{\sigma}^i$ $(i \geqq 1)$ by Theorem 2.8. We see easily that $\Psi^k\rho = \rho^k = \rho$ if $k \equiv 1 \bmod 2$ by [1, Th. 5.1] and $\rho^2 = 1$ ([4, (1.4)]), and so $(\Psi^k - 1)\kappa = 0$ if $k \equiv 1 \bmod 2$. Therefore $\operatorname{Ker} J$ is generated by the elements $(\Psi^k - 1)\bar{\sigma}^i$ $(i \geqq 1)$. By Lemma 2.12 (iii), we see that $\bar{\sigma}^i = rx$ for some $x \in \tilde{K}(L^n(2^r))$. Since $\Psi^k \circ r = r \circ \Psi_C^k$ by [3, Lemma A2], we have $(\Psi^k - 1)\bar{\sigma}^i = (\Psi^k - 1)rx = r(\Psi_C^k - 1)x$. Therefore $\operatorname{Ker} J \subset rW$ holds. Also the converse is easily seen by the equality $\Psi^k \circ r = r \circ \Psi_C^k$.

(iii) Consider the elements $r(\sigma^d(1+\sigma)\sigma(r-1))$ $(0 \leqq d < 2^{r-1} - 1)$. Then we have

$$r(\sigma^d(1+\sigma)\sigma(r-1)) = r(\eta(\eta-1)^d(\eta^{2^{r-1}}-1)) \qquad \text{(by (2.2))}$$
$$= r\left\{\sum_{i=1}^{d+1}(-1)^{d-i+1}\binom{d}{i-1}(\eta^{2^{r-1}-i}-\eta^i)\right\}$$
$$\text{(by } \eta^{2^{r-1}}=\eta^{-2^{r-1}} \text{ and Lemma 2.12(i))}.$$

In the above equalities, we see easily that each element $\eta^{2^{r-1}-i}-\eta^i$ is a linear combination of the elements $\eta^{k2^s}-\eta^{2^s}$ $(0\leqq s<r-2,\ 1\leqq k<2^{r-s-1},\ (2,k)=1)$ and hence that of the elements $\sigma^d(1+\sigma)\sigma(s)$ $(0\leqq s<r-1,\ 0\leqq d<2^s-1)$ in the same way as the proof of (i). Thus we have (iii).

(iv) is an immediate consequence of (iii) and (1.1).

(v) follows from (i)–(iv). q.e.d.

For any non-negative integers a, b, u and v, consider the integers $\theta(a,b;u,v)$ and $\theta(a;v)$ defined by

(3.7.1) $$\theta(a,b;u,v) = \sum_{i\geqq 0}(-1)^{i2^v}\sum_{c=0}^{b}(-1)^{c(2^u+1)}\binom{a}{i2^v-c2^u}\binom{b}{c},$$

(3.7.2) $$\theta(a;v) = \theta(a,0;u,v) = \sum_{i\geqq 0}(-1)^{i2^v}\binom{a}{i2^v}.$$

Then, we have the following

LEMMA 3.8. *The equalities* (3.4) *for $s=r-1$ and* (3.5) *imply the equalities*

(3.8.1) $$Jr(\sigma^a\sigma(u)^b) = (-1)^{a+b}\sum_{v=0}^{r-1}\theta(a,b;u,v+1)(\alpha_{v+1}-\alpha_v) \qquad (a+b2^u>0),$$

(3.8.2) $$Jr(\sigma^a) = (-1)^{a+1}\alpha_s + (-1)^a\sum_{v=0}^{s-1}\theta(a;v+1)(\alpha_{v+1}-\alpha_v) \quad (0<a<2^{s+1})$$
$$= (-1)^a\sum_{v=0}^{r-1}\theta(a;v+1)(\alpha_{v+1}-\alpha_v),$$

where $\alpha_s=0$ for $s\geqq r$.

To prove this lemma, we prepare two lemmas for the integers in (3.7.1–2).

LEMMA 3.9. (i) *$\theta(a,b;u,v)$ is the constant term q_0 of the right hand side of*

$$(1-x)^a(1-x^{2^u})^b \equiv \sum_{i=0}^{2^v-1}q_ix^i \mod 1-x^{2^v}.$$

(ii) $\theta(a,b;u,v)=0$ *if* $b\geqq 1,\ u\geqq v$.

(iii) $\theta(a,b;u,v)=1$ *if* $a+b2^u<2^v$.

PROOF. (i) follows immediately from the definition (3.7.1). (ii) and (iii) are seen easily by (i). q.e.d.

LEMMA 3.10. (i) $\sum_{j=1}^{2^u}(-1)^j\binom{2^u}{j}\theta(a+j,b;u,v) = -\theta(a,b+1;u,v).$

(ii) $\sum_{j=1}^{2^u}(-1)^j\binom{2^u}{j}\theta(a+j;v)=0 \quad$ *if* $u\geqq v$.

PROOF. We notice that $((1+x)-1)^k(1+x)^a=x^k(1+x)^a$ shows the equality

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{a+j}{l}=(-1)^k\binom{a}{l-k}.$$

(i) By (3.7.1) and the above equality, the left hand side of (i) is equal to

$$\sum_{i\geqq 0}(-1)^{i2^v}\sum_{c=0}^{b}(-1)^{c(2^u+1)}\sum_{j=1}^{2^u}(-1)^j\binom{2^u}{j}\binom{a+j}{i2^v-c2^u}\binom{b}{c}$$
$$=\sum_{i\geqq 0}(-1)^{i2^v}\sum_{c=0}^{b}(-1)^{c(2^u+1)}\left\{(-1)^{2^u}\binom{a}{i2^v-(c+1)2^u}-\binom{a}{i2^v-c2^u}\right\}\binom{b}{c}$$

and the last is equal to the right hand side, since $\binom{b}{c}+\binom{b}{c-1}=\binom{b+1}{c}$.

(ii) The result follows from (i) for $b=0$ and Lemma 3.9 (ii). q.e.d.

PROOF OF LEMMA 3.8. By Lemma 3.3 (v), it is sufficient to show that (3.6) implies (3.8.1–2).

We show the first equality of (3.8.2) by the induction on a. For $a=1$, the desired equality is the definition (3.4). Let $2^s\leqq a<2^{s+1}$, $s\geqq 1$ and $d=a-2^s$. Then we have

$$\begin{aligned}
Jr(\sigma^a) &= Jr\{\sigma^d\sigma(s)-(\sigma^d\sigma(s)-\sigma^a)\}\\
&=(-1)^d\alpha_s-Jr\Big(\sum_{j=1}^{2^s-1}\binom{2^s}{j}\sigma^{d+j}\Big) && \text{(by (3.6) and (2.2))}\\
&=(-1)^d\alpha_s-\sum_{j=1}^{2^s-1}(-1)^{d+j+1}\binom{2^s}{j}\alpha_s\\
&\quad -\sum_{v=0}^{s-1}\sum_{j=1}^{2^s-1}(-1)^{d+j}\theta(d+j;v+1)\binom{2^s}{j}(\alpha_{v+1}-\alpha_v)\\
&&& \text{(by the inductive assumption)}\\
&=(-1)^{d+1}\alpha_s+(-1)^d\sum_{v=0}^{s-1}\theta(d+2^s;v+1)(\alpha_{v+1}-\alpha_v) && \text{(by Lemma 3.10 (ii))}.
\end{aligned}$$

Thus the first equality of (3.8.2) holds, and so the last one of (3.8.2) by Lemma 3.9 (iii). Since (3.8.1) for $b=0$ is (3.8.2), we show (3.8.1) by the induction on b. Let $b\geqq 1$. Then

$$\begin{aligned}
Jr(\sigma^a\sigma(u)^b) &= \sum_{i=1}^{2^u}\binom{2^u}{i}Jr(\sigma^{a+i}\sigma(u)^{b-1}) && \text{(by (2.2))}\\
&=\sum_{v=0}^{r-1}\sum_{i=1}^{2^u}(-1)^{a+b+i-1}\binom{2^u}{i}\theta(a+i,b-1;u,v+1)(\alpha_{v+1}-\alpha_v)\\
&&& \text{(by the inductive assumption)}\\
&=(-1)^{a+b+1}\sum_{v=0}^{r-1}-\theta(a,b;u,v+1)(\alpha_{v+1}-\alpha_v) && \text{(by Lemma 3.10 (i))}
\end{aligned}$$

Therefore we have (3.8.1). q.e.d.

By the above results, we have the following

PROPOSITION 3.11. *$\tilde{J}(L^n(2^r))$ $(r\geqq 2)$ is generated by*

$$J\kappa \quad and \quad \alpha_s \qquad (0\leqq s\leqq r-2),$$

where $J\kappa$ is the J-image of κ in (2.6) and α_s is the element of (3.4). Furthermore, $J\colon \widetilde{KO}(L^n(4))\cong\tilde{J}(L^n(4))$, and the relations between these generators for $r\geqq 3$ are given by the J-images

$$(3.11.1) \qquad J(\bar{\sigma}^i)=0 \quad for \quad a_1+\varepsilon<i<2^{r-1}, \quad \varepsilon=\begin{cases}1 & if\ n\equiv 1 \bmod 4,\\ 0 & otherwise,\end{cases}$$

$$(3.11.2) \qquad u(i)J(\bar{\sigma}_i)=0 \quad for \quad 0\leqq i\leqq N'=\min\{2^{r-1}-1, a_1+\varepsilon\},$$

of the relations (2.9) for $i<2^{r-1}$ and $u(i)\bar{\sigma}_i=0$ of (2.11) in $\widetilde{KO}(L^n(2^r))$. Here, the left hand sides of (3.11.1–2) can be written by $J\kappa$ and α_s $(0\leqq s\leqq r-2)$ by using Lemma 2.12 (ii), (3.8.1–2) and the equality

$$(3.11.3) \qquad \alpha_{r-1}=J(\bar{\sigma}(r-1))=2J\kappa \qquad (cf.\ (2.10)\ and\ (3.4)).$$

PROOF. By Theorem 2.8 and (2.7), $\widetilde{KO}(L^n(2^r))$ $(r\geqq 2)$ is an abelian group generated by the elements

$$\bar{\sigma}^i \ \ (1\leqq i<2^{r-1}) \qquad \text{and} \qquad \kappa$$

with the relations (2.9) for $a_1+\varepsilon<i<2^{r-1}$ and $u(i)\bar{\sigma}_i=0$ in (2.11). Furthermore, by Lemma 2.12 (iii) and (2.2), the subgroup generated by $\bar{\sigma}^i$ $(1\leqq i<2^{r-1})$ coincides with the one generated by

$$r(\sigma^d\sigma(s)) \qquad (0\leqq s<r-1,\ 0\leqq d<2^s);$$

and it contains $\operatorname{Ker} J$, which is generated by

$$r(\sigma^d\sigma(s)+\sigma^{d+1}\sigma(s)) \qquad (0\leqq s\leqq r-2,\ 0\leqq d<2^s-1)$$

and is 0 if $r=2$, by Lemma 3.3 (iii). Thus, we see the proposition for $\tilde{J}(L^n(2^r))$ $=\widetilde{KO}(L^n(2^r))/\operatorname{Ker} J$, by Lemmas 3.3 (iv)–(v) and 3.8. q.e.d.

We notice that there hold the relations

$$(3.12) \qquad 2^{r-s-1+a_s}\alpha_s=0 \quad (0\leqq s<r) \qquad \text{in} \quad \tilde{J}(L^n(2^r)),$$

where a_s is the integer in (2.5). In fact, (3.12) is the Jr-images of the relations (2.4) in $\tilde{K}(L^n(2^r))$ by (3.6). In § 5, we use these relations to represent the left hand sides of (3.11.1–2) by $J\kappa$ and α_s.

§4. Some preliminary lemmas for binomial coefficients

In this section, we prepare some lemmas for the integers $\theta(a,b;u,v)$ and $\theta(a;v)$ given by (3.7.1–2).

LEMMA 4.1. *Let* $0\leqq v<r$. *Then*

$$(4.2)\quad \sum_{k=0}^{2^r-d}(-1)^k\binom{2^r-d}{k}\theta(2d+1+k;v+1)$$
$$=(-1)^d\sum_{j\in Z}\binom{2d+1}{d+2^{v+1}j}=(-1)^d\sum_{j\in Z}\binom{2d+1}{d+1+2^{v+1}j}\qquad \text{if } d\geqq 0;$$

$$(4.3)\quad \sum_{k=0}^{2^r-2^{v-1}-d}(-1)^k\binom{2^r-2^{v-1}-d}{k}\theta(2d-1+k,1;v,v+1)=0$$
$$\text{if } 1\leqq d<2^{v-1};$$

$$(4.4)\quad \sum_{k=0}^{2^r-2^u-d}(-1)^k\binom{2^r-2^u-d}{k}\theta(2d+k,2;u,v+1)$$
$$=2(-1)^{d+1}\sum_{j\in Z}\left\{(-1)^{2^u+1}\binom{2d}{d+2^u+2^{v+1}j}+\binom{2d}{d+2^{v+1}j}\right\}$$
$$\text{if } 0\leqq u\leqq v \text{ and } d\geqq 1,$$

and the last is equal to $2(-1)^{d+1}X(d,v)$ *if* $u=v$, *where*

$$(4.5)\qquad X(d,v)=\sum_{j\in Z}(-1)^{j(2^v+1)}\binom{2d}{d+2^v j}\qquad (d>0,\ v\geqq 0)$$

is the integer given by (1.4).

PROOF. By Lemma 3.9 (i), we see easily that the left hand sides of (4.2–4) are the constant terms of the polynomials of degree less than 2^{v+1} obtained from

$$(4.2)'\qquad (1-(1-x))^{2^r-d}(1-x)^{2d+1}=x^{2^r-d}(1-x)^{2d+1},$$
$$(4.3)'\qquad x^{2^r-2^{v-1}-d}(1-x)^{2d-1}(1-x^{2^v}),$$
$$(4.4)'\qquad x^{2^r-2^u-d}(1-x)^{2d}(1-x^{2^u})^2$$

by reducing mod $1-x^{2^{v+1}}$, respectively.

Thus we see the first equality in (4.2). The second equality in (4.2) is clear.

Since $r>v$, (4.3)′ is congruent to

$$x^{2^v+2^{v-1}-d}(1-x)^{2d-1}(1-x^{2^v})\equiv x^{2^{v-1}-d}(1-x)^{2d-1}(x^{2^v}-1)$$

mod $1-x^{2^{v+1}}$. The last is a polynomial in x with degree less than 2^{v+1} by the assumption $1\leqq d<2^{v+1}$, and its constant term is 0. Thus we see (4.3).

(4.4)′ is equal to

$$(1-x)^{2d}(x^{2^r-2^u-d}+x^{2^r+2^u-d}-2x^{2^r-d}).$$

Thus the left hand side of (4.4) is equal to

$$\sum_{j\in Z}\left\{(-1)^{d+2^u}\binom{2d}{d+2^u+2^{v+1}j}+(-1)^{d-2^u}\binom{2d}{d-2^u+2^{v+1}j}\right.$$
$$\left.+2(-1)^{d+1}\binom{2d}{d+2^{v+1}j}\right\},$$

which is clearly the right hand side of (4.4). The desired result for $u=v$ is clear. q. e. d.

Now, in the rest of this section, we give some lemmas for the integers $X(d,v)$ given in (4.5).

By (4.5), we see immediately the following

LEMMA 4.6. *$X(d,v)$ is the constant term p_0 of the right hand side of*

$$(-1)^d x^{2^u-2^v-d}(1-x)^{2d}(y-1)\equiv\sum_{i=0}^{2^{v+1}-1}p_i x^i \mod y^2-1 \quad (y=x^{2^v}),$$

where u is a sufficiently large integer with $2^u\geqq\max\{2^{v+1},\ 2^v+d\}$.

From now on, we denote by

$$\nu(n)=\nu_2(n) \quad\text{and}\quad \mu(n)=\mu_2(n) \qquad \text{for any positive integer} \quad n$$

the exponent of 2 in the prime power decomposition of n and the number of terms in the dyadic expansion of n, respectively. Also, we regard that $\mu(0)=0$.

LEMMA 4.7 (M. Sugawara). (i) $\mu(d)+\nu(d)\leqq m$ *if* $d<2^m$.

(ii) $\mu(d+c)+\mu(d-c)\geqq 2\mu(d)+\nu(c)-m$ *if* $0<c\leqq d<2^m$.

PROOF. (i) Let $d=2^{d_1}+\cdots+2^{d_t}$ $(d_1>\cdots>d_t\geqq 0)$. Then $\mu(d)=t$ and $\nu(d)=d_t$ by definition, and we see easily (i).

(ii) Let $c=2^{c_1}+\cdots+2^{c_l}$ $(c_1>\cdots>c_l\geqq 0)$. If $c=d$, then (ii) is seen easily by (i). If $t=1$ and $c<d$, then (ii) holds since the right hand side is equal to $2+c_l-m$ which is non-positive. Thus, we assume $c<d$ and prove (ii) by the induction on t.

Suppose $t\geqq 2$ and $d_1=c_1$. Then

$$\mu(d+c)=\mu(d-2^{d_1}+c-2^{c_1})+1,\quad \mu(d-c)=\mu(d-2^{d_1}-(c-2^{c_1})).$$

Thus we see (ii) for $l=1$ easily since $c-2^{c_1}=0$, and for $l\geqq 2$ by the inductive assumption since $d-2^{d_1}<2^{m-1}$.

Suppose $t\geqq 2$ and $d_1>c_1>\cdots>c_s\geqq d_2>c_{s+1}$, and put

$$d' = d - 2^{d_1} = 2^{d_2} + \cdots + 2^{d_t} < 2^{d_2+1},$$

$$c' = 2^{c_{s+1}} + \cdots + 2^{c_l} = c - c'' < d', \quad c'' = 2^{c_1} + \cdots + 2^{c_s};$$

and consider the non-negative integers α and β such that

$$(*) \qquad \begin{aligned} \mu(d+c) &= \mu(d'+c'+2^{d_1}+c'') = \mu(d'+c') + \alpha, \\ \mu(d-c) &= \mu(d'-c'+2^{d_1}-c'') = \mu(d'-c') + \beta. \end{aligned}$$

If $s=0$, then $c''=0$, $c'=c$ and $\beta=1$, and hence we see (ii) by the inductive assumption. If $s=l$, then $c''=c$, $c'=0$ and we see (ii) easily. Let $0<s<l$. Then, $(*)$ and the inductive assumption imply that

$$\mu(d+c) + \mu(d-c) \geqq 2t - 2 + c_l - (d_2+1) + \alpha + \beta.$$

If $\alpha+\beta\geqq 1$, then this implies (ii) easily. If $\alpha+\beta=0$, i.e., if $\alpha=0=\beta$, then the definition $(*)$ implies that

$$2^{d_2} \leqq d'-c' < d'+c' < 2^{d_2+1}, \quad (d_1,c_1,\dots,c_s) = (d_2+s, d_2+s-1,\dots,d_2).$$

Thus, we see that $c'<2^{d_2-1}$ and so $c_{s+1}\leqq d_2-2$, and that

$$\begin{aligned} \mu(d+c) &= \mu(d'+2^{c_s}+c'+2^{d_1}+(c''-2^{c_s})) = \mu(d'+2^{c_s}+c'), \\ \mu(d-c) &= \mu(d'-(2^{c_s}+c')+2^{d_1}-(c''-2^{c_s})) \geqq \mu(d'-2^{c_s}+c')) + 1. \end{aligned}$$

By the inductive assumption or (ii) for $d=c$, these equalities imply easily (ii).

q. e. d.

LEMMA 4.8. $\nu(n!) = \sum_{i\geqq 1} [n/2^i] = n - \mu(n)$.

PROOF. The desired equalities follow immediately from the definitions of $\nu(n!)$ and $\mu(n)$. q. e. d.

By the above lemmas, we can study the exponent of 2 in the prime power decomposition of $X(d,v)$.

LEMMA 4.9. *Put*

$$X(d,v) = 2^{\nu(d,v)}\xi(d,v) \qquad (\xi(d,v)\text{: odd integer})$$

for the integer $X(d,v)$ $(d>0,\ v\geqq 0)$ *in* (4.5). *Then,*

(i) $\nu(d,0) = 2d, \quad \xi(d,0) = 1;$

(ii) $\nu(d,v) = [d/2^{v-1}] + \mu(d-2^{v-1}[d/2^{v-1}]) \qquad (v>0)$.

PROOF. (i) is obvious by the definition of $X(d,0)$.

(ii) Put $d=2^{v-1}a+b$, $0\leqq b<2^{v-1}$.

The case $a=0$: $X(d,v)=\binom{2b}{b}$ by (4.5), and

$$(4.10)\qquad \nu\left(\binom{2b}{b}\right)=\nu((2b)!)-2\nu(b!)=2b-\mu(2b)-2(b-\mu(b))=\mu(b)$$

by Lemma 4.8. Thus the desired equality is obtained.

The case $a>0$: Put $(1-x)^{2^v}=1+y+2B(x)$ $(y=x^{2^v})$. Then

$$\begin{aligned}(4.11)\qquad (1-x)^{2d}x^c(y-1) &= ((1-x)^{2^v})^a(1-x)^{2b}x^c(y-1)\\ &= \textstyle\sum_{i=0}^{a}\binom{a}{i}(2B(x))^{a-i}(1+y)^i(y-1)(1-x)^{2b}x^c\\ &\equiv 2^aB(x)^a(y-1)(1-x)^{2b}x^c \mod 1-y^2\end{aligned}$$

Let $b=0$. Then, since

$$B(x)^a=\textstyle\sum_{k=a}^{(2^v-1)a}c_kx^k \quad \text{where } c_k \text{ is odd if and only if } k=2^{v-1}a,$$

(4.11) for $c=2^u-2^v-d=2^u-2^v-2^{v-1}a$ $(2^u\geqq\max\{2^{v+1},2^v+d\})$ implies that

$$(-1)^d(1-x)^{2d}x^c(y-1)\equiv 2^a(1-y)+2^{a+1}P(x) \mod 1-y^2$$

for some polynomial $P(x)$. Thus, we see (ii) by Lemma 4.6.

Let $0<b<2^{v-1}$. Consider the set

$$\Delta=\{(i,j)\colon 1\leqq i\leqq v,\ 1\leqq j\leqq 2^{v-i}\},$$

and the involution $\sigma\colon \Delta\to\Delta$ given by $\sigma(i,j)=(i,2^{v-i}-j+1)$. Put

$$\beta(i,j)=2^{i-1}(2j-1) \quad\text{and}\quad \alpha(i,j)=(-1)^{2^{i-1}}2^{-(v-i+1)}\binom{2^v}{\beta(i,j)}$$

for $(i,j)\in\Delta$. Then,

$$(4.12.1)\qquad \beta(i,j)=2^v-\beta\sigma(i,j),\quad \alpha(i,j)=\alpha\sigma(i,j)\equiv 1 \mod 2;$$

and $B(x)=((1-x)^{2^v}-1-y)/2$ is given by

$$(4.12.2)\qquad B(x)=\textstyle\sum_{(i,j)\in\Delta}A(i,j),\quad A(i,j)=2^{v-i}\alpha(i,j)x^{\beta(i,j)}.$$

To study $B(x)^a$, we consider the set

$$F=\{f\colon \{1,\dots,a\}\to\Delta\}$$

and the involution $\sigma\colon F\to F$ given by $\sigma f=\sigma\circ f$. Then σ has only one fixed point g, the constant map to $(v,1)$, and

$$(*)\qquad F=\{g\}\cup G\cup\sigma G \qquad \text{(disjoint union)}$$

for some $G\subset F$. For any $f\in F$, let $f(i,j)$ $((i,j)\in\Delta)$ be the number of elements in $f^{-1}((i,j))$, which satisfies $\sum_{(i,j)\in\Delta}f(i,j)=a$. Then by (4.12.1–2) and (*), we see easily that

$$\text{(3.13)}\quad \begin{aligned} B(x)^a &= \textstyle\sum_{f\in F}\prod_{t=1}^a Af(t) = A(v,1)^a + \sum_{f\in G}(\prod_{t=1}^a Af(t) + \prod_{t=1}^a A\sigma f(t)) \\ &= \alpha(v,1)^a x^{2^{v-1}a} + \textstyle\sum_{f\in G} 2^{p(f)}\alpha(f)(x^{k(f)} + x^{2^v a-k(f)}), \end{aligned}$$

where $p(f)$, $\alpha(f)$ and $k(f)$ for $f\in G$ are given by

$$\begin{aligned} p(f) &= \textstyle\sum_{(i,j)\in\Delta}(v-i)f(i,j) = \sum_{i=1}^v (v-i)f_i \qquad (f_i=\sum_{j=1}^{2^{v-1}}f(i,j)), \\ \alpha(f) &= \textstyle\prod_{(i,j)\in\Delta}\alpha(i,j)^{f(i,j)} \equiv 1 \mod 2, \\ k(f) &= \textstyle\sum_{(i,j)\in\Delta}\beta(i,j)f(i,j) = \sum_{(i,j)\in\Delta}2^{i-1}(2j-1)f(i,j). \end{aligned}$$

Now, by Lemma 4.6, (4.11) for $c=2^u-2^v-d$ $(d=2^{v-1}a+b,\ 2^u\geqq\max\{2^{v+1}, 2^v+d\})$ and (4.13), we obtain easily the equality

$$\begin{aligned} (-1)^d X(d,v) = 2^a\Big\{&(-1)^b\alpha(v,1)^a\binom{2b}{b} \\ &+ \textstyle\sum_{f\in G}2^{p(f)+1}\alpha(f)\sum_{l\in Z}(-1)^{d-k(f)+l}\dbinom{2b}{d-k(f)+2^v l}\Big\}. \end{aligned}$$

In this equality, $\alpha(v,1)$ and $\alpha(f)$ are odd, and $\nu\left(\binom{2b}{b}\right)=\mu(b)$ by (4.10). Thus, we see the desired result $\nu(d,v)=a+\mu(b)$ by showing that

$$\text{(4.14)}\qquad p(f)+1+\nu\left(\binom{2b}{b-m}\right) > \mu(b) \qquad (m=|k(f)-2^{v-1}a-2^v l|)$$

for any $f\in G$ and l with $m\leqq b$. By Lemma 4.8, this is equivalent to

$$\text{(4.14)}'\qquad p(f)+1+\mu(b+m)+\mu(b-m) > 2\mu(b).$$

If $m=0$, then (4.14)′ is trivial. Suppose $m>0$. Then by Lemma 4.7,

$$\mu(b+m)+\mu(b-m) \geqq 2\mu(b)+\nu(m)-v+1,$$

since $0<m\leqq b<2^{v-1}$. On the other hand, by the definitions of $p(f)$, $k(f)$ in (4.13) and m in (4.14), we see easily that

$$p(f)\geqq v-i_0,\quad \nu(m)\geqq i_0-1 \qquad (i_0=\min\{i: f_i\neq 0,\ 1\leqq i\leqq v\}).$$

These inequalities imply (4.14)′, and we obtain (4.14) as desired. q.e.d.

LEMMA 4.15. *Let $\xi(d,v)$ be the odd integer given in Lemma 4.9. Then*

$$\text{(i)}\qquad \xi(2^{s-1},v) = 2^{-1}\binom{2^s}{2^{s-1}} \quad \textit{for} \quad v\geqq s,$$

and $\xi(2^{s-1},s)=1$ if $s=1$, $\equiv 3 \bmod 8$ if $s\geqq 2$.

(ii) $\xi(2^{s-1},s-1) = 2^{-2}\left\{\binom{2^s}{2^{s-1}} - 2\right\} \equiv 1 \mod 4 \quad$ *for* $\quad s \geqq 2.$

Proof. The first equalities in (i) and (ii) follow from the definition (4.5) and Lemma 4.9. For the rest, it is sufficient to show that

$$(*) \qquad 2^{-1}\binom{2^s}{2^{s-1}} \equiv 3 \mod 8 \qquad \text{if} \quad s \geqq 2.$$

The left hand side of $(*)$ is the product of

$$(2^s - 2^{s-k}q)/(2^{s-1} - 2^{s-k}q) = (2^k - q)/(2^{k-1} - q)$$

for $2 \leqq k \leqq s$, $1 \leqq q < 2^{k-1}$ and $(2,q)=1$. In the group Z_8^* of reduced residue classes mod 8, $(2^k-q)/(2^{k-1}-q) \equiv 1$ if $k \geqq 4$, and $(7/3)(5/1)(3/1) \equiv 3$. Thus, we see $(*)$.

Lemma 4.16. *Let* $0 < d < 2^s$. *Then*

$$\sum_{i=0}^{s} (-1)^{2^i} 2^{-i} X(d,s-i) = 0.$$

Proof. Set $X'(d,v) = \sum_{j \in Z}\binom{2d}{d+2^v j}$. Then, we can show that

$$(4.17) \qquad \sum_{i=1}^{v} 2^{-i} X(d,v-i) = X'(d,v) \qquad (v \geqq 1)$$

by the induction on v as follows. The desired equality is (4.17) for $v=s$, since $X'(d,v) = \binom{2d}{d} = X(d,v)$ by the assumption $0 < d < 2^s$.

By the definition, we see that $X(d,0) = X'(d,0) = 2X'(d,1)$, which is (4.17) for $v=1$. Assume (4.17) for v. Then, we see that

$$\sum_{i=1}^{v} 2^{-i} X(d,v-i) + X(d,v) = X'(d,v) + X(d,v) = 2X'(d,v+1),$$

which is (4.17) for $v+1$. Thus (4.17) holds for $v \geqq 1$. q.e.d.

§5. Proof of Theorem 1.6

By using (3.8.1–2), Lemma 2.12 (ii) and the results obtained in the previous section, $J(\bar{\sigma}^i)$ in (3.11.1) and $u(i)J(\bar{\sigma}_i)$ in (3.11.2) can be represented by $J\kappa$ and α_s as follows.

Lemma 5.1. *If* $2^t \leqq i < 2^{t+1} \leqq 2^{r-1}$, *then*

$$(-1)^{i+1} J(\bar{\sigma}^i) = 2^{2i-2}\alpha_0 - \sum_{v=1}^{t} Y(i,v)\alpha_v \qquad \text{in} \quad \tilde{J}(L^n(2^r)),$$

where $Y(i,v) = \sum_{j \in Z}\binom{2i-1}{i+2^v(2j+1)}$ *is the integer given in* (1.5).

Proof. By Lemma 2.12 (ii), we see that

$$\bar{\sigma}^i = r(\sigma^{2i-1}/(1+\sigma)^{i-1}) = \sum_{k=0}^{2^r-i+1}\binom{2^r-i+1}{k} r(\sigma^{2i-1+k}),$$

since $(1+\sigma)^{2^r}=\eta^{2^r}=1$ by (2.1). Therefore

$$J(\bar{\sigma}^i) = \sum_{v=0}^{r-1}\sum_{k=0}^{2^r-i+1}(-1)^{k+1}\binom{2^{r-i+1}}{k}\theta(2i-1+k;v+1)(\alpha_{v+1}-\alpha_v).$$

The coefficient of $\alpha_{v+1}-\alpha_v$ in the right hand side of this equality is given by (4.2). Thus, we see the desired equality

$$(-1)^{i+1}J(\bar{\sigma}^i) = \sum_{v=0}^{r-1}\sum_{j\in Z}\binom{2i-1}{i+2^{v+1}j}(\alpha_v-\alpha_{v+1}) = 2^{2i-2}\alpha_0 - \sum_{v=1}^{t} Y(i,v)\alpha_v,$$

by noticing $\sum_{j\in Z}\binom{2i-1}{i+2j}=2^{2i-2}$ and $Y(i,v)=0$ for $v\geqq t+1$. q.e.d.

In the following lemmas, we use the relation

(5.2) $\quad 2^{r-v-1+a_v}\alpha_v = 0 \quad (0\leqq v<r) \quad$ *in* $\tilde{J}(L^n(2^r))$ $\quad$ (cf. (3.12)).

LEMMA 5.3. *Let* $0\leqq s\leqq r-2$, $1\leqq d<2^s$ *and* $a_{s+1}\geqq 1$. *Then, in* $\tilde{J}(L^n(2^r))$,

$$\sum_{k=0}^{2^r-2^s-d}(-1)^k\binom{2^r-2^s-d}{k}\cdot$$
$$\sum_{v=0}^{r-1}\theta(2d-1+k,1;s+1,v+1)2^{r-s-3+a_{s+1}}(\alpha_{v+1}-\alpha_v) = 0.$$

PROOF. $\quad \theta(2d-1+k,1;s+1,v+1) = 0 \quad$ for $\quad v\leqq s$

by Lemma 3.9 (ii). Also

$$2^{r-s-3+a_{s+1}}\alpha_v = 0 = 2^{r-s-3+a_{s+1}}\alpha_{v+1} \qquad \text{for} \quad v\geqq s+2$$

by $\alpha_r=0$ and (5.2), since $a_t\geqq a_{t'}$ if $t<t'$ by the definition of a_t in (2.5).
Furthermore (4.3) shows that

$$\sum_{k=0}^{2^r-2^s-d}(-1)^k\binom{2^r-2^s-d}{k}\theta(2d-1+k,1;s+1,v+1) = 0 \quad \text{for} \quad v=s+1.$$

Thus, we see the lemma. q.e.d.

LEMMA 5.4. $u(i)J(\bar{\sigma}_i)$ *in* (3.11.2) *for* $r\geqq 3$ *can be written as follows*:

(a) *The case* $n\not\equiv 1 \bmod 4$:

$$\text{(5.4.1)} \quad u(0)J(\bar{\sigma}_0) = \begin{cases} 2^{a_{r-1}}J\kappa + \sum_{v=0}^{r-2} 2^{\,2^{r-1-v}(a_{r-1}+1)-2}\alpha_v & (a_1\geqq 2^{r-2}), \\ 2J\kappa & (a_1<2^{r-2}). \end{cases}$$

For $i=2^s+d\leqq a_1$ *with* $0\leqq s\leqq r-2$ *and* $0\leqq d<2^s$, (5.4.2) $u(1)J(\bar{\sigma}_1) = 2^{r-1+2a_1}\alpha_0$,

(5.4.3) $u(i)J(\bar{\sigma}_i) = 2^{r-s-2+a_s}\alpha_s + \sum_{v=0}^{s-1} 2^{r-s-3+2^{s-v}(a_s+1)}\alpha_v$ *if* $i=2^s \geqq 2$,

(5.4.4) $u(i)J(\bar{\sigma}_i) = \sum_{v=0}^{s} (-1)^{2^{s-v}+d} 2^{r-s-4+2^{s+1-v}a'(i)} X(d,v)\alpha_v$ *if* $i \geqq 3$, $d \geqq 1$.

(b) *The case* $n \equiv 1 \bmod 4$: $u(i)J(\bar{\sigma}_i)$ *are the same as* (a) *if* $i \neq a_1+1-2^{r-2}(a_{r-1}-1)$, *and*

(5.4.5) $u(i)J(\bar{\sigma}_i) = 0$

if $i=a_1+1-2^{r-2}(a_{r-1}-1)=2^{r-2}+d$, $2d=b_{s+1}+1$ $(a_1 \geqq 2^{r-2})$,

(5.4.6) $u(i)J(\bar{\sigma}_i) = 2^{a_0}\alpha_0 - \sum_{v=1}^{t} 2Y(i,v)\alpha_v$

if $i=a_1+1$ $(a_1<2^{r-2}, 2^t \leqq a_1+1<2^{t+1})$.

Here a_s, b_s *and* $a'(i)$ *are the integers in* (1.3) *and Theorem* 2.8(ii), *and* $X(d,v)$ *and* $Y(d,v)$ *are the ones in* (1.4–5).

PROOF. We see (5.4.1–3, 6) immediately from the definitions of $u(i)$ and $\bar{\sigma}_i$ in Theorem 2.8 (ii), by (3.4) and Lemma 5.1.

Consider $u(i)$ and $\bar{\sigma}_i$ for the case that

$$i=2^s+d \leqq a_1, \quad 1 \leqq s \leqq r-2, \quad 1 \leqq d < 2^s,$$

in Theorem 2.8 (ii), (a), i.e.,

$$u(i) = 2^{r-s-3+a'(i)}, \quad a'(i) = \begin{cases} a_{s+1}+1 & \text{for } 2d \leqq b_{s+1}, \\ a_{s+1} & \text{for } 2d > b_{s+1}, \end{cases}$$

$$\bar{\sigma}_i = \bar{\sigma}^{d-1}\bar{\sigma}(1)\prod_{t=0}^{s-1}(2+\bar{\sigma}(t)) + \sum_{t=1}^{s+1}(-1)^{2^{t-1}}2^{(2^t-1)a'(i)-1}\bar{\sigma}^d\bar{\sigma}(s+1-t).$$

Then, by noticing that the condition $a_1 \geqq i$ implies $a'(i) \geqq 2$, and by using the last two equalities in Lemma 2.12 (ii), (2.2) and $(1+\sigma)^{2^r}=1$ in (2.1), we see that

$$\bar{\sigma}_i = \zeta + \sum_{t=1}^{s+1}(-1)^{2^{t-1}}2^{(2^t-1)a'(i)-2}\zeta_{s+1-t}, \quad \text{where}$$
$$\zeta = r(\sigma^{2d-1}(1+\sigma)^{2^r-2^s-d}\sigma(s+1)), \quad \zeta_u = r(\sigma^{2d}(1+\sigma)^{2^r-2^u-d}\sigma(u)^2).$$

By expanding $(1+\sigma)^i$ and by using (3.8.1), we have

$$J(\zeta) = \sum_{k=0}^{2^r-2^s-d}(-1)^k\binom{2^r-2^s-d}{k}\sum_{v=0}^{r-1}\theta(2d-1+k,1;s+1,v+1)(\alpha_{v+1}-\alpha_v),$$

$$J(\zeta_u) = \sum_{k=0}^{2^r-2^u-d}(-1)^k\binom{2^r-2^u-d}{k}\sum_{v=0}^{r-1}\theta(2d+k,2;u,v+1)(\alpha_{v+1}-\alpha_v).$$

Lemma 5.3 means that $u(i)J(\zeta)=0$. These and Lemma 3.9 (ii) imply that

$$u(i)J(\bar{\sigma}_i) = \sum_{v=0}^{r-1}\sum_{u=0}^{\min\{s,v\}}(-1)^{2^{s-u}}2^{r-s-5+2^{s-u+1}a'(i)}p_{v,u}(\alpha_{v+1}-\alpha_v),$$

where the coefficient $p_{v,u}$ is equal to

$$p_{v,u} = \sum_{k=0}^{2^r-2^u-d}(-1)^k\binom{2^r-2^u-d}{k}\theta(2d+k,2;u,v+1)$$
$$= 2(-1)^{d+1}\sum_{j\in Z}\left\{(-1)^{2^u+1}\binom{2d}{d+2^u+2^{v+1}j}+\binom{2d}{d+2^{v+1}j}\right\} \quad \text{(by (4.4))}.$$

If $v-1\geqq s\geqq u$ or $s\geqq v\geqq u$, then we see easily that

$$r-s-4+2^{s-u+1}a'(i) \geqq r-1-v+a_v > r-1-(v+1)+a_{v+1},$$

by noticing $a'(i)\geqq 2$, $a'(i)\geqq a_{s+1}$ and that the definitions of a_t and b_t imply

(5.5) $$a_v = 2^{t-v}a_t + [b_t/2^v] \geqq a_t \qquad \text{if} \quad t\geqq v.$$

Therefore, by (5.2) and the last half in (4.4),

(*) $$u(i)J(\bar{\sigma}_i) = \sum_{v=0}^{s}(-1)^{2^{s-v}+d+1}2^{r-s-4+2^{s+1-v}a'(i)}X(d,v)(\alpha_{v+1}-\alpha_v).$$

Furthermore, we see by (5.5) that

$$r-s-4+2^{s+1-v}a'(i)+\nu(d,v)$$
$$= r-s-4+a_{v+1}-[b_{s+1}/2^{v+1}]+2^{s-v}a_{s+1}+\nu(d,v)+\begin{cases}2^{s+1-v} & \text{if } 2d\leqq b_{s+1}\\ 0 & \text{if } 2d>b_{s+1}\end{cases}$$
$$\geqq r-1-(v+1)+a_{v+1}+2^{s+1-v}-(s-v+2) \geqq r-1-(v+1)+a_{v+1},$$

because $a'(i)=a_{s+1}+1\geqq 2$ if $2d\leqq b_{s+1}$, and

$$a'(i) = a_{s+1} \geqq 2 \quad \text{and} \quad \nu(d,v) \geqq [d/2^{v-1}] \geqq [b_{s+1}/2^{v+1}] \qquad \text{if} \quad 2d>b_{s+1}$$

by Lemma 4.9. Thus, by the definition $X(d,v)=2^{\nu(d,v)}\xi(d,v)$ in Lemma 4.9, we see that $2^{r-s-4+2^{s+1-v}a'(i)}X(d,v)\alpha_{v+1}=0$ in (*), and (5.4.4) is shown.

Finally, (5.4.5) is shown in the above proof of $u(i)J(\zeta)=0$ for $i=2^{r-2}+d$, $2d=b_{r-1}+1$. q.e.d.

Now, we are ready to prove Theorem 1.6 in § 1.

Proof of Theorem 1.6. Based on Proposition 3.11, we complete the proof of Theorem 1.6 by combining (5.2), (3.11.3), Lemmas 5.1 and 5.4. q.e.d.

§ 6. Proof of Theorem 1.7

Let $r\geqq 3$, $n\not\equiv 1 \bmod 4$ and $n\geqq 2^r-1$. Then, the relations (1.6.1–4) of $\tilde{J}(L^n(2^r))$ in Theorem 1.6 (iii) are written as follows:

(6.1) $$2^{1+a_{r-1}}J\kappa = 0, \quad 2^{r-1+a_0-b_1}\alpha_0 = 0, \quad 2^{r-1-v+a_v}\alpha_v = 0 \quad (1\leqq v\leqq r-2),$$

(6.2) $$2^{a_{r-1}}J\kappa + \sum_{v=0}^{r-2} 2^{a_v-2-[b_{r-1}/2^v]+2^{r-1-v}}\alpha_v = 0,$$

(6.3) $\sum_{v=0}^{s} 2^{r-s-3+a_v-[b_s/2^v]+2^{s-v}}\alpha_v = 0 \quad (1 \leqq s \leqq r-2)$,

(6.4.1) $\sum_{v=0}^{s} (-1)^{2^{s-v}} 2^{r-s-4+a_v-[b_{s+1}/2^v]+2^{s+1-v}+\nu(d,v)}\xi(d,v)\alpha_0 = 0$

$(1 \leqq s \leqq r-2,\ 2 \leqq 2d \leqq b_{s+1})$,

(6.4.2) $\sum_{v=0}^{s} (-1)^{2^{s-v}} 2^{r-s-4+a_v-[b_{s+1}/2^v]+\nu(d,v)}\xi(d,v)\alpha_v = 0$

$(1 \leqq s \leqq r-2,\ b_{s+1} < 2d < 2^{s+1})$,

by (5.5) and $X(d,v) = 2^{\nu(d,v)}\xi(d,v)$ in Lemma 4.9.

LEMMA 6.5. *The relations* (6.1), (6.3) *and* (6.4.1) *are equivalent to* (6.1), (6.3) *and*

(6.6) $$2^{r-s-3+a_v-[b_{s+1}/2^v]+2^{s+1-v}}\alpha_v = 0 \qquad (1 \leqq s \leqq r-2,\ 0 \leqq v \leqq s).$$

PROOF. By Lemma 4.9, we see that $\nu(d,v) \geqq 1$ ($v \geqq 0$, $d \geqq 1$). Thus (6.6) implies (6.4.1). Also we notice that (6.1) implies (6.6) for s with $b_{s+1} < 2^s$. In fact, if $b_{s+1} < 2^s$, then

$$r-s-3+a_v-[b_{s+1}/2^v]+2^{s+1-v} \geqq r-1-v+a_v \qquad \text{for} \quad v \leqq s,$$

since $2^{s-v} \geqq 2-v+1$.

Now suppose that (6.1), (6.3) and (6.4.1) hold. Then, we can prove (6.6) by the induction on s as follows:

Let $s=1$. If $b_2 < 2$, then (6.6) for $s=1$ holds by the above notice. Assume $b_2 \geqq 2$. Then $b_2 = 2+b_1$ and $[b_2/2]=1$, and (6.4.1) for $d=1$ is the following form:

$$2^{r-3+a_1}\alpha_1 - 2^{r-1+a_0-b_1}\alpha_0 = 0,$$

because $\nu(1,1) = 1 = \xi(1,1)$ and $\nu(1,0) = 2$, $\xi(1,0) = 1$ by $X(1,1) = 2$ and $X(1,0) = 2^2$. On the other hand, (6.3) implies

$$2^{r-3+a_1}\alpha_1 + 2^{r-2+a_0-b_1}\alpha_0 = 0.$$

Therefore $2^{r-2+a_0-b_1}\alpha_0 = 0 = 2^{r-3+a_1}\alpha_1$, which are (6.6) for $s=1$.

Let $s > 1$, and assume inductively (6.6) for $s-1$, i.e.,

(*) $$2^{r-s-2+a_v-[b_s/2^v]+2^{s-v}}\alpha_v = 0 \qquad (0 \leqq v \leqq s-1).$$

If $b_{s+1} < 2^s$, then (6.6) holds for s by the above notice. Assume $b_{s+1} \geqq 2^s$. Then $b_{s+1} = 2^s + b_s$. Consider (6.4.1) for s and $d = 2^k$ ($0 \leqq k < s$);

(**) $$\sum_{v=0}^{s} (-1)^{2^{s-v}} 2^{r-s-4+a_v-[b_{s+1}/2^v]+2^{s+1-v}+\nu(2^k,v)}\xi(2^k,v)\alpha_v = 0.$$

Here, $2^{s+1-v} - [b_{s+1}/2^v] = 2^{s-v} - [b_s/2^v]$, and $\xi(2^k,v)$ is odd and

$$\nu(2^k,v)=1 \quad \text{if} \quad k<v, \qquad =2^{k-v+1} \quad \text{if} \quad k\geqq v,$$

by Lemma 4.9. Thus, by (6.1) for $v=s$ and (*), (**) is

$$\textstyle\sum_{v=k+1}^{s} 2^{r-s-3+a_v-[b_{s+1}/2^v]+2^{s+1-v}}\alpha_v=0 \qquad (0\leqq k<s).$$

These equalities and (6.3) imply (6.6) for s, as desired. q.e.d.

Now, we are ready to prove Theorem 1.7(i).

PROOF OF THEOREM 1.7(i). Let $n=2^{r-1}a-1$ ($r\geqq 3$, $a\geqq 2$). Then $b_{s+1}=2^{s+1}-1$ ($0\leqq s\leqq r-2$). Thus there is no relation in (6.4.2), and (6.6) for $s=r-2$ is the following form:

$$(*) \qquad 2^{a_v}\alpha_v=0 \qquad (0\leqq v\leqq r-2).$$

Furthermore, (*) and (6.2) imply

$$2^{a_{r-1}}J\kappa+\textstyle\sum_{v=0}^{r-2}2^{a_v-1}\alpha_v=0.$$

Conversely, it is easily seen that (*) and (**) imply (6.6) for $s<r-2$, (6.1), (6.2) and (6.3).

Thus, Theorem 1.7(i) is proved by Theorem 1.6(iii) and the above lemma. q.e.d.

To prove Theorem 1.7(ii), we use the following

LEMMA 6.7. *Assume $b_{t+1}=0$. Then the relations* (6.1), (6.3) *for $s=t$ and* (6.4.2) *for $s=1$ and $2^{t-1}\leqq d<2^t$ are equivalent to the relations* (6.1) *and*

$$(6.8) \qquad \begin{aligned} &2^{r-4+a_1}\alpha_1=2^{r-3+a_0}\alpha_0 && (\text{if } t=1),\\ &2^{r-t-3+a_t}\alpha_t+2^{r-t-2+a_{t-1}}\alpha_{t-1}+2^{r-t+a_{t-2}}\alpha_{t-2}=0 && (\text{if } t\geqq 2).\end{aligned}$$

PROOF. Let $t=1$ and assume $b_2=b_1=0$. Then, the relation (6.8) is (6.4.2) for $s=1=d$, since $\nu(1,1)=1=\xi(1,1)$ and $\nu(1,0)=2$, $\xi(1,0)=1$. Also, (6.3) for $s=1$ follows from (6.8).

Let $t\geqq 2$ and assume $b_{t+1}=0$. Consider (6.4.2) for $s=t$ and $d=2^{t-1}$:

$$(*) \qquad \textstyle\sum_{v=0}^{t}(-1)^{2^{t-v}}2^{r-t-4+a_v+\nu(2^{t-1},v)}\xi(2^{t-1},v)\alpha_v=0.$$

Here, $\xi(2^{t-1},v)$ is odd and

$$\nu(2^{t-1},v)=2^{t-v}, \quad \xi(2^{t-1},v)\equiv\begin{cases}-1 \mod 4 & \text{if } v=t,\\ 1 \mod 4 & \text{if } v=t-1,\end{cases}$$

by Lemmas 4.9 and 4.15. Thus, (6.1) and (*) imply (6.8), since $2^k>k+3$ if $k\geqq 3$.

Conversely, assume (6.1) and (6.8). Then (6.3) holds for $s=t$, since

$$\sum_{v=0}^{t} 2^{r-t-3+a_v+2^{t-v}}\alpha_v = 2^{r-t-2+a_t}\alpha_t + 2^{r-t-1+a_{t-1}}\alpha_{t-1} = 0.$$

Furthermore, we can show the equality

$$(**) \qquad \sum_{v=0}^{t}(-1)^{2^{t-v}} 2^{r-t-4+a_v+\nu(d,v)}\xi(d,v)\alpha_v = 0 \qquad (2^{t-1}<d<2^t)$$

in (6.4.2) for $s=t$ as follows: Let $d=2^{t-1}+2^k$ $(0\leqq k<t-1)$. Then, $\xi(d,v)$ is odd and

$$\nu(d,v) = 2^{t-v}+1 \quad \text{if} \quad k<v\leqq t, \qquad = 2^{t-v}+2^{k-v+1} \quad \text{if} \quad 0\leqq v\leqq k,$$

by Lemma 4.9. Thus $(**)$ holds by (6.1) and (6.3), since $2^{t-v}\geqq t-v+2$ if $t-v \geqq 2$. Let $d=2^{t-1}+d'$ with $\mu(d')\geqq 2$. Then,

$$\nu(d,v) \geqq 2^{t-v}+2 \qquad \text{(by Lemma 4.9)},$$

and we see $(**)$ by (6.1). q.e.d.

LEMMA 6.9. *Assume $b_{t+1}=0$. Then, the relations* (6.1), (6.3) *for $1\leqq s\leqq t$ and* (6.4.1–2) *for $1\leqq s\leqq t$ are equivalent to the relations* (6.1) *and*

$$(6.10) \qquad 2^{r-t-3+a_v}\alpha_v = 2^{r-t-2+a_{v-1}}\alpha_{v-1} \qquad (1\leqq v\leqq t).$$

PROOF. The assumption $b_{t+1}=0$ implies $b_{s+1}=0$ $(1\leqq s\leqq t)$. Therefore, there is no relation in (6.4.1).

Now, suppose that (6.1) and (6.3), (6.4.2) for $1\leqq s\leqq t$ hold. Then, by the above lemma, there hold the relations

$$(6.8)' \qquad \begin{aligned} &2^{r-4+a_1}\alpha_1 = 2^{r-3+a_0}\alpha_0, \\ &2^{r-s-3+a_s}\alpha_s + 2^{r-s-2+a_{s-1}}\alpha_{s-1} + 2^{r-s+a_{s-2}}\alpha_{s-2} = 0 \qquad (1<s\leqq t). \end{aligned}$$

Thus (6.10) for $t=1$ is the first equality in (6.8)$'$.

Assume inductively that (6.10) holds for $t-1\,(\geqq 1)$, i.e., that

$$(6.10)' \qquad 2^{r-t-2+a_v}\alpha_v = 2^{r-t-1+a_{v-1}}\alpha_{v-1} \qquad (1\leqq v<t).$$

Then, (6.10) for $v=t$ follows easily from (6.8)$'$ for $s=t$, (6.10)$'$ for $v=t-1$ and (6.1) for $v=t-1$. Let $1\leqq s<t$ and assume inductively that (6.10) holds for $s<v \leqq t$. Consider the equality

$$(*) \qquad \sum_{v=0}^{t}(-1)^{2^{t-v}} 2^{r-t-4+a_v+\nu(d,v)}\xi(d,v)\alpha_v = 0 \qquad \text{for} \quad 2^{s-1}\leqq d<2^s$$

in (6.4.2). Then, by (4.5), Lemma 4.9 and the condition $2^{s-1}\leqq d<2^s$, we see that $\nu(d,v)=\nu(d,s)$ and $\xi(d,v)=\xi(d,s)$ for $s\leqq v\leqq t$, since $X(d,v)=\binom{2d}{d}=X(d,s)$. Therefore

(a) $\sum_{v=s}^{t}$ in $(*)$ is equal to

$-2^{r-t-4+a_s+\nu(d,s)}\xi(d,s)\alpha_s$ (by the inductive assumption (6.10) for $s<v\leqq t$).

Furthermore, if $v<s$, then $\nu(d,v)\geqq 2^{s-v}(\geqq s-v+1)$ by Lemma 4.9, and hence $r-t-4+a_v+\nu(d,v)\geqq r-t-3+s-v+a_v$. Therefore

(b) $\sum_{v=0}^{s-1}$ in (∗) is equal to

$$\sum_{v=0}^{s-1} 2^{r-t-4+a_{s-1}-(s-1-v)+\nu(d,v)}\xi(d,v)\alpha_{s-1} \qquad \text{(by (6.10)')}$$
$$= 2^{r-t-3+a_{s-1}+\nu(d,s)}\xi(d,s)\alpha_{s-1} \qquad \text{(by Lemma 4.16)}.$$

Thus, (∗) is the following form:

$$(**) \qquad 2^{r-t-4+a_s+\nu(d,s)}\alpha_s = 2^{r-t-3+a_{s-1}+\nu(d,s)}\alpha_{s-1} \qquad \text{for} \quad 2^{s-1}\leqq d<2^s,$$

since $\xi(d,s)$ is odd. (∗∗) for $d=2^{s-1}$ is (6.10) for $v=s$, since $\nu(2^{s-1},s)=1$ by Lemma 4.9. Therefore, (6.10) holds for $1\leqq v\leqq t$ by the induction on v; and hence (6.10) is shown by the induction on t.

Conversely, we see easily that (6.1) and (6.10) imply (6.8)′. Furthermore (∗∗) follows from (6.10), since $\nu(d,s)\geqq 1$ for $2^{s-1}\leqq d<2^s$ by Lemma 4.9. Therefore we see that (6.1) and (6.10) imply (6.3) and (6.4.2) by the above lemma and the above proof. q.e.d.

Proof of Theorem 1.7 (ii). Let $n=2^{r-1}a$ $(r\geqq 3,\ a\geqq 2)$. Then $b_{r-1}=0$. Thus (6.10) for $t=r-2$ is the following form:

$$(*) \qquad 2^{a_v-1}\alpha_v = 2^{a_{v-1}}\alpha_{v-1} \qquad (1\leqq v\leqq r-2).$$

Furthermore, by (6.2) and (6.1), we see that

$$(**) \qquad 2^{a_{r-1}}J\kappa + 2^{a_{r-2}}\alpha_{r-2} = 0.$$

Conversely, it is easily seen that (∗), (∗∗) and $2^{r-1+a_0}\alpha_0=0$ imply (6.10) for $t<r-2$, (6.1), (6.2) and (6.3).

Thus, Theorem 1.7 (ii) is proved by Theorem 1.6 (iii) and the above lemma. q.e.d.

Finally, we notice the following

Remark 6.11. *In $\tilde{J}(L^{2^{r-1}}(2^r))$ $(r\geqq 3)$, there hold the relations*

$$2^{a_v}\alpha_v = 2^{a_{v-1}+1}\alpha_{v-1} \qquad (1\leqq v\leqq r-3),$$
$$2^2\alpha_{r-2} + 2^5\alpha_{r-3} = 0 = 2J\kappa + 2^2\alpha_{r-2}.$$

In fact, the last two relations are (6.2) and (6.3) for $s=r-2$, respectively, by (6.1). The first one is (6.10) for $t=r-3$, which is valid by Lemma 6.9 since $b_{r-2}=0$ and (6.4.1–2) holds for $s\leqq r-3$ by (1.6.4).

§7. The induced homomorphism on the J-groups of the inclusion $L^{n-1}(2^r)\subset L^n(2^r)$

Throughout this section we assume $r\geqq 2$, and we use the following notation:

(7.1) $$L_r^{2n+1} = L^n(2^r), \quad L_r^{2n} = L_0^n(2^r),$$

where $L_0^n(2^r)=\{[z_0,\dots, z_n]\in L^n(2^r)\colon z_n \text{ is real } \geqq 0\}\subset L^n(2^r)$. Then we have

(7.2) $$L_r^k/L_r^{k-1} = S^k.$$

For the induced homomorphism

$$i_k^*\colon \widetilde{KO}(L_r^k) \longrightarrow \widetilde{KO}(L_r^{k-1}) \qquad (i_k\colon L_r^{k-1}\subset L_r^k),$$

we have the following proposition, where the elements

(7.3) $$\bar\sigma = r\sigma = r\eta - 2 \quad\text{and}\quad \kappa = \rho - 1 \qquad \text{in } \widetilde{KO}(L_r^k) \quad (k>0)$$

are the ones in (2.6) for $k=2n+1$, and are defined to be the images $i_{2n+1}^*\bar\sigma$ and $i_{2n+1}^*\kappa$ for $k=2n$.

PROPOSITION 7.4 ([4, Prop. 4.4]). *i_k^* is isomorphic if $k\equiv 7$, 6, 5 or 3 mod 8, and epimorphic otherwise. Furthermore,*

(7.5) $$\operatorname{Ker} i_k^* = \begin{cases} Z_{2^r}\langle 2\bar\sigma^{2m+1}\rangle & \text{if } k=8m+4,\\ Z_2\langle \bar\sigma^{2m+1}\rangle & \text{if } k=8m+2,\\ Z_2\langle \kappa\bar\sigma^{2m}\rangle & \text{if } k=8m+1,\\ Z_{2^r}\langle \bar\sigma^{2m}\rangle & \text{if } k=8m>0.\end{cases}$$

LEMMA 7.6. *The equality $\kappa\bar\sigma^{2m}=2^r\bar\sigma^{2m}$ holds in $\widetilde{KO}(L_r^{8m+1})$.*

PROOF. Consider the c-images of $2^r\bar\sigma^{2m}$ and $\bar\sigma^{2m+1}$ in $\widetilde{KO}(L_r^{8m+3})$, where c is the complexification. Then $c(2^r\bar\sigma^{2m})=2^r\sigma^{4m}=-2^{r-1}\sigma^{4m+1}\neq 0$ and $c(\bar\sigma^{2m+1})=\sigma^{4m+2}=0$ in $\widetilde{K}(L_r^{8m+3})$ by [4, Lemmas 4.3 and 2.9 (ii)] and (2.4). Thus $\bar\sigma^{2m+1}\neq 2^r\bar\sigma^{2m}\neq 0$ in $\widetilde{KO}(L_r^{8m+3})$, and so $2^r\bar\sigma^{2m}\neq 0$ in $\widetilde{KO}(L_r^{8m+1})$ by the above proposition. Therefore by the above proposition, we have $\kappa\bar\sigma^{2m}=2^r\bar\sigma^{2m}$ in $\widetilde{KO}(L_r^{8m+1})$.

q.e.d.

To study the induced homomorphism $i_k^*\colon \tilde J(L_r^k)\to\tilde J(L_r^{k-1})$, we use the following

(7.7) ([2, II, (3.12)] and [10]) *Let $X \xrightarrow{i} Y \xrightarrow{\pi} Z$ be a cofibering of finite connected CW-complexes and assume that the upper sequence in the commutative diagram*

$$\begin{array}{ccccccc}
\widetilde{KO}(Z) & \xrightarrow{\pi^*} & \widetilde{KO}(Y) & \xrightarrow{i^*} & \widetilde{KO}(X) & \longrightarrow & 0 \\
\downarrow J & & \downarrow J & & \downarrow J & & \\
\tilde{J}(Z) & \xrightarrow{\pi^*} & \tilde{J}(Y) & \xrightarrow{i^*} & \tilde{J}(X) & \longrightarrow & 0
\end{array}$$

is exact. Then, the lower sequence is also exact.

LEMMA 7.8. *Let Ψ^3 be the Adams operation on $\widetilde{KO}(L_r^k)$. Then*

$$(\Psi^3-1)\bar{\sigma}^i = (3^{2i}-1)\bar{\sigma}^i + \sum_{j=1}^{2i}\binom{2i}{j}3^{2i-j}\bar{\sigma}^{i+j} \qquad (i\geqq 1),$$

and $3^{2i}-1\equiv 2^{v+3} \bmod 2^{v+4}$, where $v=v_2(i)$.

PROOF. For the first half, it is sufficient to show $\Psi^3\bar{\sigma}=\bar{\sigma}(\bar{\sigma}+3)^2$, since Ψ^3 is a ring homomorphism. By the complexification $c\colon KO(L_r^k)\to K(L_r^k)$, we see that

$$\begin{aligned}
c\Psi^3\bar{\sigma} &= (1+t)\Psi_C^3(\eta-1) = \eta^3-2+\eta^{-3} = (\eta-2+\eta^{-1})(\eta+1+\eta^{-1})^2 \\
&= (1+t)(\eta-1)\{(1+t)(\eta-1)+3\}^2 = c(\bar{\sigma}(\bar{\sigma}+3)^2),
\end{aligned}$$

since $\bar{\sigma}=r(\eta-1)$, $cr=1+t$ and $t\eta=\eta^{-1}$ (t is the conjugation). By [4, Prop. 5.3], $c\colon \widetilde{KO}(L_r^k)\to\tilde{K}(L_r^k)$ is monomorphic if $k\equiv 7 \bmod 8$. Thus $\Psi^3\bar{\sigma}=\bar{\sigma}(\bar{\sigma}+3)^2$ in $\widetilde{KO}(L_r^k)$ for $k\equiv 7 \bmod 8$, and also so for any k by the naturality.

The last half can be shown by the induction on v. If $v=0$ (i is odd), then $3^{2i}-1=(2^3+1)^i-1\equiv 2^3 \bmod 2^4$. Let $v\geqq 1$ and assume $3^{2^v u}-1\equiv 2^{v+2} \bmod 2^{v+3}$ for any positive odd integer u. Then $3^{2^{v+1}u}-1=(3^{2^v u})^2-1=(1+2^{v+2}+2^{v+3}a)^2-1\equiv 2^{v+3} \bmod 2^{v+4}$. Therefore we have the desired result. q.e.d.

By using the above results and Theorem 1.7, we see the following proposition, where (ii) is Theorem 1.10:

PROPOSITION 7.9. (i) *The induced homomorphism*

$$i_k^*\colon \tilde{J}(L_r^k) \longrightarrow \tilde{J}(L_r^{k-1}) \qquad (i_k\colon L_r^{k-1}\subset L_r^k,\ r\geqq 2)$$

is isomorphic if $k\equiv 7$, 6, 5 or 3 mod 8, epimorphic otherwise, and

$$\text{(7.10)} \qquad \operatorname{Ker} i_k^* = \begin{cases}
Z_4\langle 2J(\bar{\sigma}^{2m+1})\rangle & \text{if } k=8m+4, \\
Z_2\langle J(\bar{\sigma}^{2m+1})\rangle & \text{if } k=8m+2, \\
Z_2\langle 2^r J(\bar{\sigma}^{2m})\rangle & \text{if } k=8m+1, \quad r<l+2, \\
0 & \text{if } k=8m+1, \quad r\geqq l+2, \\
Z_{2^h}\langle J(\bar{\sigma}^{2m})\rangle & \text{if } k=8m>0,
\end{cases}$$

where $l=v_2(4m)$, i.e., $4m=2^l q$ with odd q, and $h=\min\{r,l+2\}$.

(ii) $\#\tilde{J}(L^n(2^r)) = 2^{\varphi(n,r)}$,

$$\varphi(n,r) = (r+1)a_{r-1} + \sum_{s=1}^{r-2}(s+2)[(a_s+1)/2]+1+\varepsilon,$$

where $\#G$ is the order of a group G, a_s and ε are the integers in (1.3) *and* (1.11), *respectively.*

Proof. Consider (7.7) for the cofibering $L_r^{k-1}\subset L_r^k\to S^k$ in (7.2). Then, the first half of (i) is obvious by the first half of Proposition 7.4.

Furthermore, by (7.5) and Lemma 7.6, it is easy to see that Ker i_k^* is a cyclic group generated by the generator of the group given in the right hand side of (7.10).

Now, we can show that

$$(*)\qquad \#\mathrm{Ker}\, i_k^* \leqq \begin{cases} 4 & \text{if } k=8m+4, \\ 2 & \text{if } k=8m+2, \text{ or } k=8m+1 \text{ and } r<l+2, \\ 1 & \text{if } k=8m+1 \text{ and } r\geqq l+2, \\ 2^h & \text{if } k=8m>0. \end{cases}$$

In fact, Ker i_{8m+4}^* is generated by $2J(\bar{\sigma}^{2m+1})$. On the other hand, $(\Psi^3-1)(\bar{\sigma}^{2m+1}) =(3^{4m+2}-1)\bar{\sigma}^{2m+1}=2^3a\bar{\sigma}^{2m+1}$ (a: odd) in $\widetilde{KO}(L_r^{8m+5})$ by Lemma 7.8 and (2.9). Therefore $2^3J(\bar{\sigma}^{2m+1})=0$ in $\tilde{J}(L_r^{8m+5})=\tilde{J}(L_r^{8m+4})$ by (1.1), since $\#\tilde{J}(L_r^k)$ is a power of 2 by Theorem 2.8 and (1.1). Thus, $(*)$ for $k=8m+4$ holds. $(*)$ for the second case is easily seen by (7.5) and (7.7) for the cofibering $L_r^{k-1}\subset L_r^k\to S^k$. Now, the generator of Ker i_{8m+1}^* is $2^rJ(\bar{\sigma}^{2m})$. On the other hand, by Lemma 7.8 and (2.9), $(\Psi^3-1)\bar{\sigma}^{2m}=(3^{4m}-1)\bar{\sigma}^{2m}=2^{l+2}b\bar{\sigma}^{2m}$ (b: odd) in $\widetilde{KO}(L_r^{8m+1})$. Thus $2^{l+2}J(\bar{\sigma}^{2m}) =0$ in $\tilde{J}(L_r^{8m+1})$ by (1.1), and $(*)$ for the third case is valid. Finally, Ker i_{8m}^* is generated by $J(\bar{\sigma}^{2m})$ and $2^rJ(\bar{\sigma}^{2m})=0=2^{l+2}J(\bar{\sigma}^{2m})$ in $\tilde{J}(L_r^{8m})$ by the above proof. Thus $(*)$ holds for $k=8m$.

Now, $(*)$ implies that

$$\prod_{m=1}^{[n/4]}\#\mathrm{Ker}\, i_{8m}^* \leqq 2^{\psi(n,r)},\qquad \psi(n,r) = \sum_{l=2}^{r-1}(l+2)[(a_l+1)/2]+ra_{r-1},$$
$$\prod_{m=0}^{[n/4]}\#\mathrm{Ker}\, i_{8m+1}^* \leqq 2^{a_{r-1}+1},\quad \prod_{m=0}^{[(n-1)/4]}\#\mathrm{Ker}\, i_{8m+2}^* \leqq 2^{[(n-1)/4]+1},$$
$$\prod_{m=0}^{[(n-2)/4]}\#\mathrm{Ker}\, i_{8m+4}^* \leqq 2^{2[(n-2)/4]+2};$$

and hence we see by the routine calculations that

$(**)$ $(*)$ implies $\#\tilde{J}(L^n(2^r))\leqq 2^{\varphi(n,r)}$ and the equality holds if and only if the equality holds in $(*)$ for any $k\leqq 2n+1$.

On the other hand, by Theorems 1.6(ii), 2.8(i) and 1.7(ii), we see easily that

$$\#\tilde{J}(L^n(2^r)) = 2^{\varphi(n,r)} \qquad \text{for}\quad n=2^{r-1}a-1,\quad a\geqq 2.$$

Thus, we see the proposition by $(**)$. q.e.d.

Proposition 7.9 (i) implies immediately the following corollary, which is Proposition 1.9:

COROLLARY 7.11. *For the induced homomorphism*

$$i^*: \tilde{J}(L^n(2^r)) \longrightarrow \tilde{J}(L^{n-1}(2^r)) \qquad (i: L^{n-1}(2^r) \subset L^n(2^r),\ r \geqq 2),$$

i^* is isomorphic if $n \equiv 3 \bmod 4$, epimorphic otherwise, and

$$\text{(7.12)} \qquad \operatorname{Ker} i^* = \begin{cases} Z_4\langle 2J(\bar{\sigma}^{2m+1})\rangle & \text{if } n=4m+2, \\ Z_2\langle J(\bar{\sigma}^{2m+1})\rangle & \text{if } n=4m+1, \\ Z_u\langle J(\bar{\sigma}^{2m})\rangle & \text{if } n=4m>0, \end{cases}$$

where $u=2^{\min\{r+1,l+2\}}$ $(l=\nu_2(4m))$.

§8. Proof of Theorem 1.2

To prove Theorem 1.2, we prepare some lemmas.

LEMMA 8.1. *The following equality holds in $\tilde{J}(L^n(2^r))$ $(r \geqq 2)$:*

$$Jr(\eta^i - 1) = Jr\sigma(\nu) = \alpha_\nu \qquad \textit{for} \quad i \geqq 1,$$

where $\nu = \nu_2(i)$ is the exponent of 2 in the prime power decomposition of i.

PROOF. By the proof of Lemma 3.3, we notice that the kernel of J: $\widetilde{KO}(L^n(2^r)) \to \tilde{J}(L^n(2^r))$ is generated additively by the elements

$$r(\eta^j \sigma(s)) \qquad (0 \leqq s < r,\ 1 \leqq j < 2^s).$$

If $2^s \leqq i < 2^{s+1}$, then $\eta^i - 1 = \eta^j \sigma(s) + \eta^j - 1$ where $j = i - 2^s$ by (2.2). If $j > 0$ in addition, then $Jr(\eta^i - 1) = Jr(\eta^j - 1)$ by the above notice and $\sigma(s) = 0$ $(s \geqq r)$. By continuing this process, we have the desired equality by the definitions of $\nu_2(i)$ and α_s in (3.4). q.e.d.

Now, let $f(n,r;\nu)$ be the non-negative integer such that

$$\text{(8.2)} \qquad \#Jr\sigma(\nu) = \#\alpha_\nu = 2^{f(n,r;\nu)} \quad \text{in} \quad \tilde{J}(L^n(2^r)) \qquad (n \geqq 0,\ r \geqq 2)$$

by Proposition 7.9 (ii), where $\#\alpha$ denotes the order of α. Then by the definition of α_ν in (3.4) and (2.9),

$$\text{(8.3)} \qquad f(n,r;\nu) = 0 \quad \text{if} \quad n=0 \quad \text{or} \quad \nu \geqq r.$$

LEMMA 8.4. *If $n = 2^{r-1}a$ and $r \geqq 3$, then*

$$f(n,r;\nu) = r - 1 - \nu + 2^{r-1-\nu}a \qquad \textit{for} \quad n>0, \quad 0 \leqq \nu < r.$$

PROOF. The equality for $a \geqq 2$ is easily seen from Theorem 1.7 (ii) and $\alpha_{r-1} = 2J\kappa$ of (3.11.3).

Consider the case $n = 2^{r-1}$. Then, by Corollary 7.11,

$$\# J(\bar{\sigma}^{2m}) = 2^{r+1} \quad \text{in} \quad \tilde{J}(L^{2^{r-1}}(2^r)) \qquad (4m = 2^{r-1}).$$

On the other hand, $2^r \bar{\sigma}^{2m} = 2^{r+4m-2} \bar{\sigma}$ in $\widetilde{KO}(L^{2^{r-1}}(2^r))$ by [8, Lemma 2.3]. Thus, we obtain

$$\# \alpha_0 = \# J(\bar{\sigma}) = 2^{r-1+2^{r-1}}.$$

Furthermore, this relation, the ones in $\tilde{J}(L^{2^{r-1}}(2^r))$ given in Remark 6.11 and $\alpha_{r-1} = 2\kappa$ imply immediately

$$\# \alpha_\nu = r - 1 - \nu + 2^{r-1-\nu} \qquad (0 \leqq \nu < r),$$

which is the equality for $a = 1$. q.e.d.

Consider the commutative diagram ($r \geqq 3$)

$$\begin{array}{ccc} \operatorname{Ker} i^* \subset \tilde{J}(L^n(2^r)) & \xrightarrow{i^*} & \tilde{J}(L^{n-1}(2^r)) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi'^*} \\ \operatorname{Ker} i'^* \subset \tilde{J}(L^n(2^{r-1})) & \xrightarrow{i'^*} & \tilde{J}(L^{n-1}(2^{r-1})) \end{array} \tag{8.5}$$

of the induced homomorphisms, where i and i' are the inclusions and π and π' are the natural projections induced by the inclusion $Z_{2^{r-1}} \subset Z_{2^r}$. Then we have the following

LEMMA 8.6. *If $n \not\equiv 0 \bmod 2^{r-1}$ ($r \geqq 3$), then*

$$\pi^* \mid \operatorname{Ker} i^* \colon \operatorname{Ker} i^* \longrightarrow \operatorname{Ker} i'^*$$

is isomorphic.

PROOF. If $n = 4m = 2^l q$ (q: odd), then the assumption $n \not\equiv 0 \bmod 2^{r-1}$ implies $r - 1 > l$ and so $\min\{r+1, l+2\} = l+2 = \min\{r, l+2\}$. Thus, we see immediately the lemma by Corollary 7.11, by noticing that $\pi^* r\eta = r\pi^*\eta = r\eta$ and hence $\pi^* J(\bar{\sigma}^i) = J(\bar{\sigma}^i)$. q.e.d.

LEMMA 8.7. *If $n \not\equiv 0 \bmod 2^{r-1}$ ($r \geqq 3$), then*

$$f(n,r;\nu) = \max\{f(n-1,r;\nu), f(n,r-1;\nu)\}.$$

PROOF. Consider the diagram (8.5). Then the definition (8.2) implies that

$$f(n,r;\nu) \geqq \max\{f(n-1,r;\nu), f(n,r-1;\nu)\},$$

since $i^*\alpha_\nu=\alpha_\nu$ and $\pi^*\alpha_\nu=\alpha_\nu$. Moreover, if $f(n,r;\nu)>\max\{f(n-1,r;\nu), f(n,r-1;\nu)\}$, then the non-zero element $2^{f(n,r;\nu)-1}\alpha_\nu$ in $\tilde{J}(L^n(2^r))$ is mapped to 0 by i^* and π^*. This contradicts Lemma 8.6. Thus we have the lemma. q.e.d.

PROOF OF THEOREM 1.2. By (8.3), it is sufficient to show that

(8.8) $$f(n,r;\nu) = \max\{s-\nu+[n/2^s]2^{s-\nu}: \nu\leqq s<r \text{ and } 2^s\leqq n\} \quad (0\leqq\nu<r).$$

(8.8) for $r=2$ is an easy consequence of Theorems 1.6 (ii), 2.8 (i), (3.11.3) and (3.4). By Lemma 8.4, (8.8) holds if $r\geqq 3$ and $n\equiv 0 \bmod 2^{r-1}$.

For the case $r\geqq 3$ and $2^{r-1}a<n<2^{r-1}(a+1)$, assume inductively that (8.8) holds for $(n-1,r;\nu)$ and $(n,r-1;\nu)$ instead of $(n,r;\nu)$. Then, we see easily that the right hand side of the equality in Lemma 8.7 is equal to

$$\begin{cases} f(n,r-1;\nu) & \text{if } a=0, \\ \max\{f(n,r-1;\nu),\ r-1-\nu+[(n-1)/2^{r-1}]2^{r-1-\nu}\} & \text{if } a>0, \end{cases}$$

and hence to the right hand side of (8.8). Thus Lemma 8.7 implies (8.8) by the induction on n and r.

These complete the proof of Theorem 1.2. q.e.d.

§9. $\tilde{J}(L^n(2^r))$ for $r\leqq 5$

$\tilde{J}(L^n(4))$ is given by Theorems 1.6 (ii) and 2.8 (i).

In this section, we present the direct sum decomposition of $\tilde{J}(L^n(2^r))$ for $r=3$, 4 or 5 explicitly in Proposition 9.3 without proof, which is obtained from Theorem 1.6 by the direct computations of the integers $X(d,v)$ and $Y(d,v)$ for $v\leqq 3$ and the routine calculations.

Before we state the result, we notice the following

PROPOSITION 9.1. (i) *In Theorem* 1.6, $\tilde{J}(L^n(2^r))$ $(r\geqq 3)$ *is the direct sum of the subgroup* $Z_{m(r-1)}\langle J\kappa+\alpha(r-1)\rangle$ *and the one generated by* α_s $(0\leqq s\leqq r-2)$, *where*

$$\begin{aligned} &m(r-1) = 2, && \alpha(r-1) = 0 && \text{if } n<2^{r-1}, \\ &m(r-1) = 2^{a_{r-1}}, && \alpha(r-1) = \textstyle\sum_{s=0}^{r-2} 2^{(2^{r-1-s}-1)(1+a_{r-1})-1}\alpha_s && \text{if } n\geqq 2^{r-1}. \end{aligned}$$

(ii) *Let* $n<2^r$. *Then there exists an isomorphism*

$$f\colon \tilde{J}(L^n(2^{r+1})) \cong \tilde{J}(L^n(2^r)) \qquad (r\geqq 3),$$

which is given by

(9.2) $$f(J\kappa) = J\kappa + \alpha(r-1), \quad f(\alpha_s) = \alpha_s \qquad (0\leqq s<r).$$

Proof. In the relations (1.6.1–6) of Theorem 1.6, $J\kappa$ appears only in the first one of (1.6.1) and (1.6.2). Thus (i) follows immediately from Theorem 1.6.

(ii) The assumption $n<2^r$ implies that $m(r-1)=2=m(r)$ in (i) and that $\#\tilde{J}(L^n(2^{r+1}))=\#\tilde{J}(L^n(2^r))$ by Proposition 7.9 (ii). On the other hand, $\pi^*(\alpha_s)=\alpha_s$ and $\pi^*(J\kappa)=0$ for the homomorphism $\pi^*\colon \tilde{J}(L^n(2^{r+1}))\to\tilde{J}(L^n(2^r))$ induced by the natural projection $\pi\colon L^n(2^r)\to L^n(2^{r+1})$. Thus, we obtain the desired isomorphism f by (9.2). q.e.d.

Proposition 9.3.*) *Let $r=3$, 4 or 5. Then $\tilde{J}(L^n(2^r))$ is the direct sum*

$$Z_{m(0)}\langle\alpha_0\rangle \oplus \oplus_{i=1}^{r-2} Z_{m(i)}\langle\alpha_i+\alpha(i)\rangle \oplus Z_{m(r-1)}\langle J\kappa+\alpha(r-1)\rangle,$$

and the last summand is the one given in (i) *of the above proposition, and the order $m(i)$ $(0\leqq i\leqq r-2)$ and the element $\alpha(i)$ $(1\leqq i\leqq r-2)$ are given in Table* 1, 2 *or* 3 *for $r=3$, 4 or 5, respectively, where $\tilde{J}(L^n(2^r))$ for $n<2^{r-1}$ $(r=4$ or 5) is isomorphic to $\tilde{J}(L^n(2^{r-1}))$ by* (ii) *of the above proposition.*

Table 1 $(r=3)$

<table>
<tr><th>n $(t\geqq 1)$</th><th>$m(0)$</th><th>$m(1)$</th><th>$\alpha(1)$</th></tr>
<tr><td>0</td><td>1</td><td rowspan="3">1</td><td rowspan="3"></td></tr>
<tr><td>1</td><td>2</td></tr>
<tr><td>2, 3</td><td>2^3</td></tr>
<tr><td>$4t$</td><td rowspan="2">2^{4t+2}</td><td>2^{2t-1}</td><td>$2^3\alpha_0$ $(t=1)$, $-2^{2t+1}\alpha_0$ $(t>1)$</td></tr>
<tr><td>$4t+1$</td><td>2^{2t}</td><td>$2^{2t+1}\alpha_0$</td></tr>
<tr><td>$4t+2$, 3</td><td>2^{4t+3}</td><td>2^{2t+1}</td><td>0</td></tr>
</table>

Table 2 $(r=4)$

<table>
<tr><th>n $(t\geqq 1)$</th><th>$m(0)$</th><th>$m(1)$</th><th>$\alpha(1)$</th><th>$m(2)$</th><th>$\alpha(2)$</th></tr>
<tr><td>$8t$</td><td rowspan="3">2^{8t+3}</td><td>2^{4t-1}</td><td>$2^5 3\alpha_0$ $(t=1)$,
$-2^{4t+1}\alpha_0$ $(t>1)$</td><td rowspan="3">2^{2t-1}</td><td rowspan="3">$-2^3\alpha_1-2^9\alpha_0$ $(t=1)$,
$2^{2t+1}\alpha_1+2^{6t+3}\alpha_0$ $(t>1)$</td></tr>
<tr><td>$8t+1$</td><td>2^{4t}</td><td>$-2^{4t+1}\alpha_0$</td></tr>
<tr><td>$8t+2$, 3</td><td rowspan="3">2^{4t+2}</td><td rowspan="3">$2^{4t+3}\alpha_0$</td></tr>
<tr><td>$8t+4$</td><td rowspan="2">2^{8t+6}</td><td>2^{2t}</td><td>$2^{2t+1}\alpha_1+2^{6t+4}\alpha_0$</td></tr>
<tr><td>$8t+5$</td><td rowspan="2">2^{2t+1}</td><td rowspan="2">0</td></tr>
<tr><td>$8t+6$, 7</td><td>2^{8t+7}</td><td>2^{4t+3}</td><td>0</td></tr>
</table>

*) In [7, Prop. 5.3], T. Kobayashi and M. Sugawara have already computed $\tilde{J}(L^n(8))$, and $\tilde{J}(L^n(16))$ has been computed by T. Kobayashi.

TABLE 3 $(r=5)$

<table>
<tr><th>$n\ (t\geqq 1)$</th><th>$m(0)$</th><th>$m(1)$</th><th>$\alpha(1)$</th><th>$m(2)$</th><th>$\alpha(2)$</th></tr>
<tr><td>$16t$</td><td rowspan="3">2^{16t+4}</td><td>2^{8t-1}</td><td>$-2^{8t+1}\alpha_0$</td><td rowspan="3">2^{4t-1}</td><td rowspan="3">$2^5\alpha_1+2^{15}\alpha_0$ $(t=1)$,
$-2^{4t+1}3\alpha_1+2^{12t+3}\alpha_0$ $(t>1)$</td></tr>
<tr><td>$16t+1$</td><td>2^{8t}</td><td>$-2^{8t+1}\alpha_0$</td></tr>
<tr><td>$16t+2$, 3</td><td>2^{8t+2}</td><td>$2^{8t+1}\alpha_0$</td></tr>
<tr><td>$16t+4$</td><td rowspan="2">2^{16t+6}</td><td rowspan="4">2^{8t+3}</td><td rowspan="4">$-2^{8t+5}\alpha_0$</td><td>2^{4t}</td><td>$-2^{4t+1}\alpha_1-2^{12t+4}\alpha_0$</td></tr>
<tr><td>$16t+5$</td><td>2^{4t+1}</td><td>$2^{4t+1}\alpha_1+2^{12t+4}\alpha_0$</td></tr>
<tr><td>$16t+6$, 7</td><td>2^{16t+7}</td><td rowspan="5">2^{4t+2}</td><td rowspan="5">$2^{4t+3}\alpha_1+2^{12t+10}\alpha_0$</td></tr>
<tr><td>$16t+8$</td><td rowspan="3">2^{16t+11}</td></tr>
<tr><td>$16t+9$</td><td>2^{8t+4}</td><td>$-2^{8t+5}\alpha_0$</td></tr>
<tr><td>$16t+10$, 11</td><td rowspan="3">2^{8t+6}</td><td rowspan="3">$2^{8t+7}\alpha_0$</td></tr>
<tr><td>$16t+12$</td><td rowspan="2">2^{16t+14}</td></tr>
<tr><td>$16t+13$</td><td rowspan="2">2^{4t+3}</td><td rowspan="2">0</td></tr>
<tr><td>$16t+14$, 15</td><td>2^{16t+15}</td><td>2^{8t+7}</td><td>0</td></tr>
</table>

$n\ (t\geqq 1)$	$m(3)$	$\alpha(3)$
$16t\leqq n\leqq 16t+7$	2^{2t-1}	$-2^3\alpha_2-2^9\alpha_1-2^{21}\alpha_0$ $(t=1)$, $2^{2t+1}\alpha_2+2^{6t+3}\alpha_1+2^{14t+7}\alpha_0$ $(t>1)$
$16t+8$, 9, 10, 11	2^{2t}	$-2^{2t+1}\alpha_2+2^{6t+4}\alpha_1+2^{14t+10}\alpha_0$
$16t+12$, 13, 14, 15	2^{2t+1}	0

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