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On the behavior at infinity of Green potentials in a half space

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1. Introduction

Let u be a Green potential in the half space $D=\{x=(x_1,\dots, x_n);\ x_n>0\}$, $n\geqq 2$. Then it is known that $x_n^{-1}u(x)$ tends to zero as $|x|\to\infty$, $x\in D-E$, where E is minimally thin at infinity (cf. [2]). Recently Essén-Jackson [2] have proved that $|x|^{-1}u(x)$ tends to zero as $|x|\to\infty$, $x\in D-E$, where the exceptional set E is called rarefied at infinity. Our aim in this note is to extend these results to Green potentials of general order.

Let k be a non-negative Borel measurable function on $R^n\times R^n$, and set

$$k(x, \mu) = \int_E k(x, y)d\mu(y) \quad \text{and} \quad k(\mu, y) = \int_E k(x, y)d\mu(x)$$

for a non-negative measure μ on a Borel set $E\subset R^n$. We define a capacity C_k by

$$C_k(E) = \sup \mu(R^n), \qquad E\subset D,$$

where the supremum is taken over all non-negative measures μ such that S_μ (the support of μ) is contained in E and

$$k(x, \mu) \leqq 1 \qquad \text{for every} \quad x\in D.$$

Let G_α be the Green function of order α for D, i.e.,

$$G_\alpha(x, y) = \begin{cases} |x-y|^{\alpha-n} - |\bar{x}-y|^{\alpha-n} & \text{in case} \quad 0<\alpha<n, \\ \log(|\bar{x}-y|/|x-y|) & \text{in case} \quad \alpha=n, \end{cases}$$

where $\bar{x}=(x_1,\dots, -x_n)$ for $x=(x_1,\dots, x_n)$. For $0\leqq\beta\leqq 1$, we consider the function $k_{\alpha,\beta}$ defined by

$$k_{\alpha,\beta}(x, y) = \begin{cases} x_n^{-1}y_n^{-\beta}G_\alpha(x, y) & \text{for } x, y\in D, \\ \lim_{z\to x, z\in D} z_n^{-1}y_n^{-\beta}G_\alpha(z, y) = a_\alpha y_n^{1-\beta}|x-y|^{\alpha-n-2} & \\ \qquad\qquad \text{for } x\in\partial D \text{ and } y\in D, & \end{cases}$$

where $a_\alpha=2(n-\alpha)$ if $\alpha<n$ and $=2$ if $\alpha=n$. In case $\beta=1$, $k_{\alpha,1}$ is extended to be continuous on $\bar{D}\times\bar{D}$ in the extended sense.

Our first aim is to establish the following theorem.

THEOREM 1. *Let λ be a non-negative measure on $\bar{D}$ which satisfies*

$$(1) \qquad \int_{\bar{D}} (1 + |y|)^{\alpha+\gamma-n-1} d\lambda(y) < \infty, \qquad -1 \leqq \gamma < n - \alpha + 1.$$

Then there exists a Borel set $E \subset D$ such that

$$\lim_{|x|\to\infty, x\in D-E} x_n^{1-\beta}|x|^{\beta+\gamma} k_{\alpha,1}(x, \lambda) = 0,$$

$$(\mathrm{a}) \qquad \sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta}}(E_i) < \infty,$$

where $E_i = \{x \in E; 2^i \leqq |x| < 2^{i+1}\}$.

COROLLARY. *Let μ be a non-negative measure on D satisfying*

$$(2) \qquad \int_D y_n(1 + |y|)^{\alpha+\gamma-n-1} d\mu(y) < \infty, \qquad -1 \leqq \gamma < n - \alpha + 1.$$

Then there exists a Borel set $E \subset D$ satisfying (a) such that

$$\lim_{|x|\to\infty, x\in D-E} x_n^{-\beta}|x|^{\beta+\gamma} G_\alpha(x, \mu) = 0.$$

A similar result was obtained in [4] for Riesz potentials in R^n.

Theorem 1 is the best possible as to the size of the exceptional set. In fact we shall prove the next result.

THEOREM 2. *Let $E \subset D$ be a Borel set satisfying (a). Then we can find a non-negative measure λ satisfying (1) such that*

$$\lim_{|x|\to\infty, x\in E} x_n^{1-\beta}|x|^{\beta+\gamma} k_{\alpha,1}(x, \lambda) = \infty.$$

REMARK. In the case $\alpha=2$, by using Lemma 3 below, we can easily show that $C_{k_{2,\beta}} = \lambda^\beta$ in the notation of [1]. Thus, Condition (a) with $\alpha=2$ means that E is β-rarefied at infinity in the sense of [1], so that [1; Theorem 4.2] is a corollary to our Theorems 1 and 2. In particular, Condition (a) with $\alpha=2$, $\beta=1$ (resp. $\alpha=2$, $\beta=0$) means that E is minimally thin at infinity (resp. rarefied at infinity in the sense of [2]).

2. Proof of Theorem 1

First we consider the case $0<\alpha<n$; in this case our theorem is an easy consequence of the next lemmas.

LEMMA 1. *Let $0<\alpha<n$. Then there exist positive constants c_1 and c_2 such that*

$$c_1 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2} \leqq G_\alpha(x, y) \leqq c_2 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2}$$

for $x=(x_1,\dots, x_n)$ and $y=(y_1,\dots, y_n)$ in D.

This can be proved by elementary calculation.

LEMMA 2. *Let $0<\alpha<n$. For a non-negative measure λ on $\bar{D}$ satisfying (1), we set*

$$U(x) = x_n \int_{\bar{D}} |x-y|^{\alpha-n}|\bar{x}-y|^{-2} d\lambda(y), \qquad x \in D.$$

Then there exists a Borel set $E \subset D$ satisyfing (a) such that

$$\lim_{|x|\to\infty, x\in D-E} x_n^{-\beta}|x|^{\beta+\gamma}U(x) = 0.$$

PROOF. Write U as the sum of U_1 and U_2, where

$$U_1(x) = x_n \int_{\{y\in\bar{D}; |x-y|\geqq|x|/2\}} |x-y|^{\alpha-n}|\bar{x}-y|^{-2} d\lambda(y),$$

$$U_2(x) = x_n \int_{\{y\in\bar{D}; |x-y|<|x|/2\}} |x-y|^{\alpha-n}|\bar{x}-y|^{-2} d\lambda(y).$$

We see that if $x, y \in D$ and $|x-y| \geqq |x|/2 \geqq 1$, then

$$x_n^{1-\beta}|x|^{\beta+\gamma}|x-y|^{\alpha-n}|\bar{x}-y|^{-2} \leqq \text{const.}\,(1+|y|)^{\alpha+\gamma-n-1}.$$

Hence, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{|x|\to\infty} x_n^{-\beta}|x|^{\beta+\gamma}U_1(x) = 0.$$

By assumption (1) we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i=\infty$ and $\sum_{i=1}^{\infty} a_i b_i<\infty$, where

$$b_i = \int_{2^{i-1}<|y|<2^{i+2}} |y|^{\alpha+\gamma-n-1} d\lambda(y).$$

Consider the sets

$$E_i = \{x \in D;\ 2^i \leqq |x| < 2^{i+1},\ x_n^{-\beta}U_2(x) \geqq a_i^{-1}2^{-i(\beta+\gamma)}\},$$

for $i=1, 2,\dots$. If μ is a non-negative measure on D such that $S_\mu \subset E_i$ and $k_{\alpha,\beta}(x, \mu) \leqq 1$ for $x \in D$, or equivalently for $x \in \bar{D}$, then we have

$$\int_D d\mu \leqq a_i 2^{i(\beta+\gamma)} \int x_n^{-\beta} U_2(x) d\mu(x)$$
$$\leqq a_i 2^{i(\beta+\gamma)} \int_{2^{i-1}<|y|<2^{i+2}} c_1^{-1} k_{\alpha,\beta}(y, \mu) d\lambda(y)$$
$$\leqq c_1^{-1} a_i 2^{i(\beta+\gamma)} \int_{2^{i-1}<|y|<2^{i+2}} d\lambda(y)$$
$$\leqq c_1^{-1} 4^{n-\alpha-\gamma+1} 2^{i(n-\alpha+\beta+1)} a_i b_i,$$

so that

$$C_{k_{\alpha,\beta}}(E_i) \leqq c_1^{-1} 4^{n-\alpha-\gamma+1} 2^{i(n-\alpha+\beta+1)} a_i b_i,$$

which yields

$$\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta}}(E_i) < \infty.$$

Setting $E=\cup_{i=1}^{\infty} E_i$, we see that (a) is satisfied and

$$\limsup_{|x|\to\infty, x\in D-E} x_n^{-\beta}|x|^{\beta+\gamma} U_2(x) \leqq \limsup_{i\to\infty} 2^{|\beta+\gamma|} 2^{i(\beta+\gamma)} a_i^{-1} 2^{-i(\beta+\gamma)} = 0.$$

Next we consider the case $\alpha=n$. In this case our theorem can be proved similarly; we have only to use the following lemma instead of Lemma 1.

LEMMA 1′. *There exist* c_1' *and* $c_2'>0$ *such that*

$$c_1' \frac{x_n y_n}{|\bar{x}-y|^2} \leqq \log \frac{|\bar{x}-y|}{|x-y|} \leqq c_2' \frac{x_n y_n}{|x-y||\bar{x}-y|}$$

for $x=(x_1,\ldots, x_n)$ *and* $y=(y_1,\ldots, y_n)$ *in* D.

3. Proof of Theorem 2

To show Theorem 2, we need the following lemma which is a consequence of [3; Théorème 7.8].

LEMMA 3. *For any Borel set* $E\subset D$, *we have*

$$C_{k_{\alpha,\beta}}(E) = \hat{C}_{k_{\alpha,\beta}}(E),$$

where $\hat{C}_{k_{\alpha,\beta}}=\inf\lambda(\bar{D})$, *the infimum being taken over all non-negative measures* λ *on* $\bar{D}$ *such that* $k_{\alpha,\beta}(\lambda, y)\geqq 1$ *for* $y\in E$.

PROOF OF THEOREM 2. We prove only the case $\alpha<n$, because the remaining case can be proved similarly. By Lemma 3, for each i we can find a non-negative measure λ_i on $\bar{D}$ such that $\lambda_i(\bar{D})<C_{k_{\alpha,\beta}}(E_i)+1$ and $k_{\alpha,\beta}(\lambda_i, z)\geqq 1$ on E_i. Denote by λ_i' the restriction of λ_i to the set $\{x\in\bar{D};\ 2^{i-1}<|x|<2^{i+2}\}$, and set

$$\lambda = \sum_{i=1}^{\infty} a_i 2^{-i(\beta+\gamma)} \lambda'_i,$$

where $\{a_i\}$ is a sequence of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ but $\sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta+1)}\{C_{k_{\alpha,\beta}}(E_i)+1\} < \infty$. If $z \in E_i$, then

$$k_{\alpha,\beta}(\lambda'_i, z) \geqq 1 - \int_{\{|x| \leqq 2^{i-1}\} \cup \{|x| \geqq 2^{i+2}\}} k_{\alpha,\beta}(x, z) d\lambda'_i(x)$$
$$\geqq 1 - c_2 2^{-(i-1)(n-\alpha+\beta+1)}\{C_{k_{\alpha,\beta}}(E_i) + 1\}.$$

We also obtain

$$z_n^{1-\beta}|z|^{\beta+\gamma} k_{\alpha,1}(z, \lambda) = |z|^{\beta+\gamma} k_{\alpha,\beta}(\lambda, z) \geqq 2^{-|\beta+\gamma|} a_i k_{\alpha,\beta}(\lambda'_i, z)$$

for $z \in E_i$. Consequently it follows that

$$\lim_{|z|\to\infty, z \in E} z_n^{1-\beta}|z|^{\beta+\gamma} k_{\alpha,1}(z, \lambda) = \infty.$$

On the other hand,

$$\int |x|^{\alpha+\gamma-n-1} d\lambda(x) = \sum_{i=1}^{\infty} a_i 2^{-i(\beta+\gamma)} \int |x|^{\alpha+\gamma-n-1} d\lambda'_i(x)$$
$$\leqq \sum_{i=1}^{\infty} 2^{n-\alpha-\gamma+1} a_i 2^{-i(n-\alpha+\beta+1)}\{C_{k_{\alpha,\beta}}(E_i) + 1\} < \infty.$$

4. Radial limits

If a Borel set $E \subset D$ is contained in a cone

$$\{x = (x_1,\dots, x_n) \in D;\ |x| < ax_n\}, \qquad a > 0,$$

then there exists a positive constant M independent of E such that

$$M^{-1} C_{k_{\alpha,\beta}}(E_i) \leqq 2^{i(\beta+1)} C_{R_\alpha}(E_i) \leqq M C_{k_{\alpha,\beta}}(E_i),$$

where $E_i = \{x \in E;\ 2^i \leqq |x| < 2^{i+1}\}$ and

$$R_\alpha(x, y) = \begin{cases} |x-y|^{\alpha-n} & \text{in case } \alpha < n, \\ \log(|\bar{x}-y|/|x-y|) & \text{in case } \alpha = n; \end{cases}$$

this implies that in case $n \geqq 3$, E is minimally thin or rarefied at infinity if and only if E is thin at infinity in the sense of [4].

In the same way as [4; Corollary 3.4] we can prove the following theorem.

THEOREM 4. *Let λ be a non-negative measure on $\bar{D}$ satisfying* (1). *Then there exists a Borel set $E \subset S_+ = \{x \in D;\ |x| = 1\}$ such that $C_{R_\alpha}(E) = 0$ and*

$$\lim_{r\to\infty} r^{\gamma+1}k_{\alpha,1}(rx, \lambda) = 0 \qquad \textit{for every} \quad x\in S_+-E.$$

COROLLARY 1. *Let μ be a non-negative measure on D satisfying* (2). *Then there exists a Borel set $E\subset S_+$ with $C_{R_\alpha}(E)=0$ such that*

$$\lim_{r\to\infty} r^{\gamma}G_\alpha(rx, \mu) = 0 \qquad \textit{for every} \quad x\in S_+-E.$$

Denote by S_γ (cf. [2]) the family of all superharmonic functions u on D which are of the form

$$u(x) = G_2(x, \mu) + P(x, \nu),$$

where P is the Poisson kernel for D and μ (resp. ν) is a non-negative measure on D (resp. ∂D) such that

$$\int_D y_n(1+|y|)^{1+\gamma-n}d\mu(y) < \infty \qquad \left(\text{resp.} \int_{\partial D}(1+|y|)^{1+\gamma-n}d\nu(y) < \infty\right).$$

COROLLARY 2. *Let $u\in S_\gamma$, $-1\leqq\gamma<n-1$. Then there exists a Borel set $E\subset S_+$ with $C_{R_2}(E)=0$ such that*

$$\lim_{r\to\infty} r^{\gamma}u(rx) = 0 \qquad \textit{for every} \quad x\in S_+-E.$$

5. Green potential of a measure with density in Lebesgue classes

In the case where μ in Corollary to Theorem 1 has a density in Lebesgue classes, we can state the following result:

Let f be a non-negative measurable function on D such that

$$\int_D y_n^{\gamma}(1+|y|)^{\delta} f(y)^p dy < \infty.$$

If $p>1$, $p(\alpha-1)-\delta-n<\gamma<2p-1$ and $p(1-\beta)<n+\gamma+\delta-p(\alpha-1)$, then there exists a set $E\subset D$ such that

$$\lim_{|x|\to\infty, x\in D-E} x_n^{-\beta}|x|^{(n+\gamma+\delta-(\alpha-\beta)p)/p}G_\alpha(x, \mu) = 0,$$

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-(\alpha-\beta)p)}C_{k,p}(E_i) < \infty,$$

where $d\mu(y)=f(y)dy$, $E_i=\{x\in E;\, 2^i\leqq|x|<2^{i+1}\}$, $k=k_{\alpha,\gamma/p}$ and

$$C_{k,p}(E_i) = \inf\left\{\int_D |g(y)|^p dy;\ x_n^{1-\beta}\int_D k(x, y)g(y)dy \geqq 1 \quad \textit{for} \quad x\in E_i\right\}.$$

This fact can be proved in the same way as Theorem 1 and [4; Theorem 4.5].

References

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