HIROSHIMA MATH. J.
10 (1980), 597–605

# Subnormality and ascendancy in groups

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(Received April 15, 1980)

## Introduction

Recently subnormality in groups was investigated by Wielandt [7], Peng [4, 5], McCaughan and McDougall [3], and subnormality and ascendancy in groups were investigated by Hartley and Peng [1]. On the other hand, subideality and ascendancy in Lie algebras were examined by Kawamoto [2] and the author [6].

In this paper, following the paper [6] we shall introduce two notions of weak subnormality and weak ascendancy for subgroups, study their properties, and investigate several criteria for subnormality and ascendancy of subgroups.

Let $H$ be a subgroup of a group $G$. We shall show that when either (a) $G$ is hyperabelian, (b) $G$ has an ascending abelian series and $H$ is finite, or (c) $G$ is finite-by-hyperabelian and $H$ is finite, $H$ is ascendant in $G$ if and only if $H$ is weakly ascendant in $G$ (Theorems 3 and 7). Similar results for subnormality will be shown in Theorems 3 and 6. We shall also give characterizations of weak subnormality and $\omega$-step weak ascendancy (Theorem 4), and show that every finite, weakly ascendant subgroup of a group is at most of $\omega$-step (Theorem 5).

## 1.

Let $G$ be a group. If $x$, $y$ are elements of $G$, then $[x, y]=x^{-1}y^{-1}xy$ and we write $[x, {}_0y]=x$, $[x, {}_{n+1}y]=[[x, {}_ny], y]$ for an integer $n\geq 0$. If $X$, $Y$ are non-empty subsets of $G$, $[X, Y]$ is the set of all $[x, y]$ with $x\in X$ and $y\in Y$ and we write $[X, {}_0Y]=X$, $[X, {}_{n+1}Y]=[[X, {}_nY], Y]$.

We write $H\leq G$ if $H$ is a subgroup of $G$ and $H\lhd G$ if $H$ is a normal subgroup of $G$. For any ordinal $\lambda$, a subgroup $H$ of $G$ is a $\lambda$-step ascendant subgroup of $G$, denoted by $H\lhd^{\lambda}G$, if there is a series $(S_\alpha)_{\alpha\leq\lambda}$ of subgroups of $G$ such that

(a) $S_0=H$ and $S_\lambda=G$,

(b) $S_\alpha\lhd S_{\alpha+1}$ for any ordinal $\alpha<\lambda$,

(c) $S_\beta=\bigcup_{\alpha<\beta}S_\alpha$ for any limit ordinal $\beta\leq\lambda$.

$H$ is an ascendant subgroup of $G$ if $H\lhd^{\lambda}G$ for some ordinal $\lambda$. When $\lambda<\omega$, $H$ is a subnormal subgroup of $G$, denoted by $H$ sn $G$.

We say a subgroup $H$ of $G$ to be a $\lambda$-step weakly ascendant subgroup of $G$, if there is an ascending series $(S_\alpha)_{\alpha\leq\lambda}$ of subsets of $G$ satisfying the above conditions (a), (c) and the following condition:

(b′) *If* $\alpha$ *is any ordinal* $<\lambda$, *then* $u^{-1}Hu \subseteq S_\alpha$ *for any* $u \in S_{\alpha+1}$.

We then write $H \leq^\lambda G$. We simply call such a series $(S_\alpha)_{\alpha\leq\lambda}$ a weakly ascending series from $H$ to $G$. We call $H$ a weakly ascendant subgroup of $G$ if $H \leq^\lambda G$ for some ordinal $\lambda$. We then write $H$ wasc $G$. When $\lambda < \omega$, we call $H$ a weakly subnormal subgroup of $G$ and write $H$ wsn $G$.

As usual, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{A}$, $\mathrm{E}\mathfrak{A}$ and $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ are the classes of finite, finitely generated, abelian, solvable and hyperabelian groups respectively, and Min is the class of groups satisfying the minimal condition for subgroups. $\acute{\mathrm{E}}_\lambda\mathfrak{A}$ is the class of groups $G$ having an ascending abelian series $(G_\alpha)_{\alpha\leq\lambda}$ of subgroups, and $\acute{\mathrm{E}}\mathfrak{A} = \bigcup_\lambda \acute{\mathrm{E}}_\lambda\mathfrak{A}$. $G \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ means that there is an ascending abelian series of normal subgroups of $G$. ($\mathrm{E}\mathfrak{A}$, $\acute{\mathrm{E}}\mathfrak{A}$ and $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ are also denoted by $\mathrm{P}\mathfrak{A}$, $\acute{\mathrm{P}}\mathfrak{A}$ and $\acute{\mathrm{P}}_n\mathfrak{A}$ respectively.)

For $K, L \leq G$, we denote $K \in \acute{\mathrm{E}}_\lambda(L)\mathfrak{A}$ if there is an ascending abelian series $(K_\alpha)_{\alpha\leq\lambda}$ of $L$-invariant subgroups of $G$, and denote $\acute{\mathrm{E}}(L)\mathfrak{A} = \bigcup_\lambda \acute{\mathrm{E}}_\lambda(L)\mathfrak{A}$.

## 2.

We first show the following two lemmas which will be used in the subsequent sections.

LEMMA 1. *Let* $G$ *be a group and let* $H \leq^\lambda G$. *Then there exists a weakly ascending series* $(S_\alpha)_{\alpha\leq\lambda}$ *from* $H$ *to* $G$ *such that for any ordinal* $\alpha \leq \lambda$

$$HS_\alpha H = S_\alpha \quad \textit{and} \quad S_\alpha^{-1} = S_\alpha.$$

PROOF. Let $(S_\alpha)_{\alpha\leq\lambda}$ be a weakly ascending series from $H$ to $G$. Put

$$M_\alpha = H(S_\alpha \cap S_\alpha^{-1})H.$$

Then $M_0 = H$ and $M_\lambda = G$. Let $\alpha$ be any ordinal $<\lambda$. Any element $u$ of $M_{\alpha+1}$ is expressed as

$$u = axb \quad \text{with} \quad a, b \in H \quad \text{and} \quad x \in S_{\alpha+1} \cap S_{\alpha+1}^{-1}.$$

Hence

$$\begin{aligned} u^{-1}Hu &= (axb)^{-1}H(axb) \\ &= b^{-1}(x^{-1}a^{-1}Hax)b. \end{aligned}$$

Since $x^{-1}a^{-1}Hax \subseteq S_\alpha \cap S_\alpha^{-1}$, it follows that

$$u^{-1}Hu \subseteq H(S_\alpha \cap S_\alpha^{-1})H = M_\alpha.$$

For any limit ordinal $\beta \leq \lambda$,

$$M_\beta = H(S_\beta \cap S_\beta^{-1})H$$
$$= H(\cup_{\alpha<\beta}(S_\alpha \cap S_\alpha^{-1}))H$$
$$= \cup_{\alpha<\beta} H(S_\alpha \cap S_\alpha^{-1})H$$
$$= \cup_{\alpha<\beta} M_\alpha.$$

Therefore $(M_\alpha)_{\alpha\leq\lambda}$ is a weakly ascending series from $H$ to $G$ satisfying the conditions.

LEMMA 2. *Let $G$ be a group.*

(a) *If $H\leq^\lambda G$ and $K\leq G$, then $H\cap K\leq^\lambda K$.*

(b) *Let $f$ be a homomorphism of $G$ onto a group $\bar{G}$. If $H\leq^\lambda G$, then $f(H)\leq^\lambda\bar{G}$. If $\bar{H}\leq^\lambda\bar{G}$, then $f^{-1}(\bar{H})\leq^\lambda G$.*

PROOF. Assume that $H\leq^\lambda G$ and let $(S_\alpha)_{\alpha\leq\lambda}$ be a weakly ascending series from $H$ to $G$. Then $(S_\alpha\cap K)_{\alpha\leq\lambda}$ is a weakly ascending series from $H\cap K$ to $K$ and $H\cap K\leq^\lambda K$. $(f(S_\alpha))_{\alpha\leq\lambda}$ is also a weakly ascending series from $f(H)$ to $\bar{G}$ and $f(H)\leq^\lambda\bar{G}$.

Assume that $\bar{H}\leq^\lambda\bar{G}$ and let $(\bar{S}_\alpha)_{\alpha\leq\lambda}$ be a weakly ascending series from $\bar{H}$ to $\bar{G}$. Then $(f^{-1}(\bar{S}_\alpha))_{\alpha\leq\lambda}$ is a weakly ascending series from $f^{-1}(\bar{H})$ to $G$ and $f^{-1}(\bar{H})$ $\leq^\lambda G$.

## 3.

We begin this section with

LEMMA 3. *Let $G$ be a group such that $G=HK$ with $H\leq G$, $K\lhd G$ and $K\in\mathfrak{A}$. If $H\leq^\lambda G$, then $H\lhd^\lambda G$.*

PROOF. Let $(S_\alpha)_{\alpha\leq\lambda}$ be a weakly ascending series from $H$ to $G$. By Lemma 1 we may assume that $HS_\alpha H=S_\alpha$ and $S_\alpha^{-1}=S_\alpha$ for any ordinal $\alpha\leq\lambda$. Then

$$S_\alpha = S_\alpha \cap (KH) = (S_\alpha \cap K)H.$$

Let $\alpha$ be any ordinal $<\lambda$. Let $u, v$ be any elements of $S_{\alpha+1}$, $S_\alpha$ respectively. Then

$$u = xa \quad \text{with} \quad x\in S_{\alpha+1}\cap K \quad \text{and} \quad a\in H,$$
$$v = yb \quad \text{with} \quad y\in S_\alpha\cap K \quad \text{and} \quad b\in H.$$

By using the fact that $K\in\mathfrak{A}$, we have

$$u^{-1}vu = (xa)^{-1}(yb)(xa)$$

$$= a^{-1}y(x^{-1}bx)a$$
$$\in HS_\alpha^2 H = S_\alpha^2.$$

It follows that for any $u \in S_{\alpha+1}$

$$u^{-1}S_\alpha^n u \subseteq S_\alpha^{2n} \qquad (n \geq 0).$$

We put $H_\alpha = \langle S_\alpha \rangle$ for any ordinal $\alpha \leq \lambda$. Then $H_0 = H$, $H_\lambda = G$ and $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \lambda$. For any limit ordinal $\beta \leq \lambda$, we have $H_\beta = \cup_{\alpha<\beta} H_\alpha$. In fact, any $u \in H_\beta$ is expressed as

$$u = x_1 x_2 \cdots x_n \qquad \text{with} \quad x_i \in S_\beta.$$

Then $x_i \in S_{\alpha_i}$ for some ordinal $\alpha_i < \beta$. Take $\alpha = \max(\alpha_1, \dots, \alpha_n)$. Then each $x_i$ belongs to $S_\alpha$ and therefore $u \in H_\alpha$. Thus $H \lhd^\lambda G$.

THEOREM 1. *Let $G$ be a group such that $G = HK$ with $H \leq G$ and $K \lhd G$.*
(a) *Let $K \in \acute{E}_\lambda(H)\mathfrak{A}$. If $H \leq^\mu G$, then $H \lhd^{\mu\lambda} G$.*
(b) *Let $K \in \mathfrak{A}^n$. If $H \leq^\mu G$, then $H \lhd^{\mu n} G$.*

PROOF. (a) Let $(K_\alpha)_{\alpha\leq\lambda}$ be an ascending abelian series of $H$-invariant subgroups of $K$. Then for any ordinal $\alpha \leq \lambda$

$$K_\alpha \lhd HK_\alpha \leq G.$$

Assume that $H \leq^\mu G$. For each ordinal $\alpha < \lambda$, put

$$\bar{H} = HK_\alpha/K_\alpha \quad \text{and} \quad \bar{K}_{\alpha+1} = K_{\alpha+1}/K_\alpha.$$

Then $\bar{K}_{\alpha+1} \lhd \bar{H}\bar{K}_{\alpha+1}$ and $\bar{K}_{\alpha+1} \in \mathfrak{A}$. Since $H \leq^\mu HK_{\alpha+1}$ by Lemma 2 (a), we have $\bar{H} \leq^\mu \bar{H}\bar{K}_{\alpha+1}$ by Lemma 2 (b). Hence by making use of Lemma 3 we see that

$$\bar{H} \lhd^\mu \bar{H}\bar{K}_{\alpha+1}.$$

If follows that

$$HK_\alpha \lhd^\mu HK_{\alpha+1}.$$

For any limit ordinal $\beta \leq \lambda$

$$HK_\beta = H(\cup_{\alpha<\beta} K_\alpha) = \cup_{\alpha<\beta} HK_\alpha.$$

Thus we conclude that $H \lhd^{\mu\lambda} G$.

(b) Put $K_i = K^{(n-i)}$ for $0 \leq i \leq n$. Then $(K_i)_{i\leq n}$ is an ascending series of $H$-invariant subgroups of $K$. Therefore the assertion in (b) follows from (a).

For elements $x$, $y$ of a group $G$, $x^y = y^{-1}xy$ and for $H \leq G$, $\langle x^H \rangle = \langle x^a | a \in H \rangle$. With this notation we show the following

**Lemma 4.** *Let $G \in \acute{\mathrm{E}}_\lambda \mathfrak{A}$ and let $H \leq G$. If $\langle x^H \rangle \in \mathfrak{G}$ for any $x \in G$, then $G \in \acute{\mathrm{E}}_\lambda(H)\mathfrak{A}$.*

**Proof.** Let $(G_\alpha)_{\alpha \leq \lambda}$ be an ascending abelian series of $G$. Let $M_\alpha$ be the subgroup generated by all $H$-invariant subgroups of $G_\alpha$. Then $M_\alpha$ is the largest $H$-invariant subgroup of $G_\alpha$. Obviously $M_0 = 1$ and $M_\lambda = G$. For any ordinal $\alpha < \lambda$,

$$[M_{\alpha+1}, M_{\alpha+1}] \subseteq [G_{\alpha+1}, G_{\alpha+1}] \subseteq G_\alpha.$$

Hence $\langle [M_{\alpha+1}, M_{\alpha+1}] \rangle$ is an $H$-invariant subgroup of $G_\alpha$ and therefore

$$[M_{\alpha+1}, M_{\alpha+1}] \subseteq M_\alpha.$$

It follows that $M_\alpha \lhd M_{\alpha+1}$ and $M_{\alpha+1}/M_\alpha \in \mathfrak{A}$.

Let $\beta$ be any limit ordinal $\leq \lambda$. For any $x \in M_\beta$,

$$\langle x^H \rangle \leq M_\beta \leq G_\beta.$$

It follows that $\langle x^H \rangle \leq G_\alpha$ for some $\alpha < \beta$. Since $\langle x^H \rangle$ is $H$-invariant,

$$x \in \langle x^H \rangle \leq M_\alpha.$$

Hence $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$. Thus $G \in \acute{\mathrm{E}}_\lambda(H)\mathfrak{A}$.

**Theorem 2.** *Let $G \in \acute{\mathrm{E}}_\lambda \mathfrak{A}$, let $H \leq G$ and assume that $\langle x^H \rangle \in \mathfrak{G}$ for any $x \in G$. If $H \leq^\mu G$, then $H \lhd^{\mu\lambda} G$.*

**Proof.** By Lemma 4, $G \in \acute{\mathrm{E}}_\lambda(H)\mathfrak{A}$. Hence the assertion follows from Theorem 1.

As consequences of Theorems 1 and 2 we have the following

**Theorem 3.** (a) *Let $G \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. If $H$* wasc *$G$, then $H$* asc *$G$.*
(b) *Let $G \in \acute{\mathrm{E}}\mathfrak{A}$, let $H \leq G$ and $H \in \mathfrak{F}$. If $H$* wasc *$G$, then $H$* asc *$G$.*
(c) *Let $G \in \mathrm{E}\mathfrak{A}$. If $H$* wsn *$G$, then $H$* sn *$G$.*

**Proof.** (a) and (c) follow from Theorem 1 where we take $K = G$. If $H \in \mathfrak{F}$, we have $\langle x^H \rangle \in G$ for any $x \in G$. Hence (b) follows from Theorem 2.

## 4.

In this section we shall mainly investigate some properties of weakly ascendant subgroups of step $\leq \omega$.

**Theorem 4.** *Let $G$ be a group and let $H \leq G$. Let $0 \leq n < \omega$. Then*
(a) *$H \leq^n G$ if and only if $[G, {}_nH] \subseteq H$.*

(b) *$H \leq^{\omega} G$ if and only if for any $x \in G$ there exists an integer $n = n(x) \geq 0$ such that $[x, {}_nH] \subseteq H$.*

Proof. (a) Assume that $H \leq^{n} G$ and let $(S_i)_{i \leq n}$ be a weakly ascending series from $H$ to $G$ such that $HS_iH = S_i$ for $0 \leq i \leq n$. Then by induction we have

$$[G, {}_iH] \subseteq S_{n-i} \qquad (0 \leq i \leq n).$$

Taking $i = n$, we have $[G, {}_nH] \subseteq H$.

Conversely, assume that the condition is satisfied and put

$$S_i = H\{x \in G \mid [x, {}_iH] \subseteq H\}H \qquad (0 \leq i \leq n).$$

Then $S_0 = H$ and $S_n = G$. For any $i < n$, every element of $S_{i+1}$ is expressed as $u = axb$ where

$$[x, {}_{i+1}H] \subseteq H \quad \text{and} \quad a, b \in H.$$

For any $h \in H$,

$$u^{-1}hu = (axb)^{-1}h(axb)$$

$$= b^{-1}[x, a^{-1}h^{-1}a]a^{-1}hab \in S_i.$$

Hence $H \leq^{n} G$.

(b) Assume that $H \leq^{\omega} G$ and let $(S_\alpha)_{\alpha \leq \omega}$ be a weakly ascending series from $H$ to $G$ such that $HS_\alpha H = S_\alpha$ for $0 \leq \alpha \leq \omega$. Let $x \in G$. Then there exists an integer $n \geq 0$ such that $x \in S_n$. By induction we have

$$[x, {}_iH] \subseteq S_{n-i} \qquad (0 \leq i \leq n).$$

Taking $i = n$, we have $[x, {}_nH] \subseteq H$.

Conversely, assume that the condition is satisfied and put

$$S_i = H\{x \in G \mid [x, {}_iH] \subseteq H\}H \qquad (0 \leq i \leq n),$$

$$S_\omega = G.$$

Then $S_0 = H$. If $i < n$, we have

$$u^{-1}Hu \subseteq S_i \qquad \text{for any } u \in S_{i+1},$$

as in the proof of (a). Furthermore

$$\cup_{i<\omega} S_i = H(\cup_{i<\omega} \{x \in G \mid [x, {}_iH] \subseteq H\})H$$

$$= HGH = G = S_\omega.$$

Therefore $H \leq^{\omega} G$.

COROLLARY. (a) *Let* $G \in \mathfrak{F}$. *If* $H$ wsn $G$, *then* $H$ sn $G$.

(b) *Let* $G \in \text{Min}$. *If* $H$ wsn $G$, *then* $H$ sn $G$. *If* $H \leq^{\omega} G$, *then* $H$ asc $G$.

PROOF. (a) follows from Theorem 4 (a) and Theorem 3 in [7]. (b) follows from Theorem 4 and Theorem A in [1].

LEMMA 5. *Let* $G$ *be a group, let* $H$ wasc $G$ *and let* $H = \langle U \rangle$ *with* $U$ *a finite set. Then for any* $x \in G$ *there exists an integer* $n = n(x) \geq 0$ *such that* $[x, {}_nU] \subseteq H$.

PROOF. Let $(S_\alpha)_{\alpha \leq \lambda}$ be a weakly ascending series from $H$ to $G$ such that $HS_\alpha H = S_\alpha$ for $0 \leq \alpha \leq \lambda$. For each integer $n \geq 0$, let $\mu_n$ be the least ordinal such that

$$[x, {}_nU] \subseteq S_{\mu_n}.$$

Then $\mu_n$ is not a limit ordinal. Since $[S_{\alpha+1}, U] \subseteq S_\alpha$ for any ordinal $\alpha < \lambda$, $\mu_{n+1} < \mu_n$ unless $\mu_n = 0$. Since the ordinals $\leq \lambda$ are well-ordered, $\mu_n = 0$ for some $n$. Hence $[x, {}_nH] \subseteq H$.

COROLLARY. *Let* $G$ *be a group and let* $x$ *be an element of* $G$ *such that* $\langle x \rangle$ wasc $G$. *Then* $x$ *is a left Engel element of* $G$.

PROOF. For any $y \in G$, $[y, {}_nx] \in \langle x \rangle$ for some $n$ by Lemma 5. It follows that $[y, {}_{n+1}x] = 1$.

THEOREM 5. *Let* $G$ *be a group. Then every finite, weakly ascendant subgroup of* $G$ *is at most of* $\omega$*-step.*

PROOF. The statement follows from Theorem 4(b) and Lemma 5.

## 5.

Let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be an {E, Q}-closed class of groups $G$ such that $H$ wsn $G$ implies $H$ sn $G$ (resp. $H$ wasc $G$ implies $H$ asc $G$), and let $\mathfrak{X}'$ (resp. $\mathfrak{Y}'$) be a class of groups $G$ such that $H \in \mathfrak{X}$ and $H$ wsn $G$ imply $H$ sn $G$ (resp. $H \in \mathfrak{Y}$ and $H$ wasc $G$ imply $H$ asc $G$). Using these notations, we show the following

LEMMA 6. *Let* $G$ *be a group and let* $H \leq G$.

(a) *Assume that* $G \in \mathfrak{X}\mathfrak{X}'$ *and* $H \in \mathfrak{X}$. *If* $H$ wsn $G$, *then* $H$ sn $G$.

(b) *Assume that* $G \in \mathfrak{Y}\mathfrak{Y}'$ *and* $H \in \mathfrak{Y}$. *If* $H$ wasc $G$, *then* $H$ asc $G$.

PROOF. We shall only give the proof of (b), since the other is similar to this. Take a normal subgroup $K$ of $G$ such that $K \in \mathfrak{Y}$ and $G/K \in \mathfrak{Y}'$. If $H$ wasc $G$, then by Lemma 2

$$H \text{ wasc } HK \quad \text{and} \quad (HK)/K \text{ wasc } G/K.$$

Since $\mathfrak{Y}$ is E-closed and

$$(HK)/K \cong H/(H \cap K) \in \mathrm{Q}\mathfrak{Y} = \mathfrak{Y},$$

we have $HK \in \mathfrak{Y}$. It follows that $H$ asc $HK$. Since $G/K \in \mathfrak{Y}'$ and $(HK)/K \in \mathfrak{Y}$, $(HK)/K$ asc $G/K$ and therefore $HK$ asc $G$. Thus we have $H$ asc $G$.

THEOREM 6. *Let $G$ be a group and let $H \leq G$. Assume that one of the following conditions is satisfied*:

(a) $G \in \mathfrak{F}(\mathrm{E}\mathfrak{A}) \cup \mathfrak{F}\,\mathrm{Min}$ *and* $H \in \mathfrak{F}$.

(b) $G \in (\mathrm{E}\mathfrak{A})\mathfrak{F} \cup (\mathrm{E}\mathfrak{A})\,\mathrm{Min}$ *and* $H \in \mathrm{E}\mathfrak{A}$.

(c) $G \in \mathrm{Min}\,\mathfrak{F} \cup \mathrm{Min}\,(\mathrm{E}\mathfrak{A})$ *and* $H \in \mathrm{Min}$.

*If $H$* wsn *$G$, then $H$* sn *$G$.*

PROOF. It is easy to see that Min is $\{\mathrm{E}, \mathrm{Q}\}$-closed. Hence the statement follows from Theorem 3 (c), Corollary to Theorem 4 and Lemma 6.

THEOREM 7. *Let $G$ be a group and let $H \leq G$. Assume that*

$$G \in \mathfrak{F}(\acute{\mathrm{E}}\mathfrak{A}) \quad and \quad H \in \mathfrak{F}.$$

*If $H$* wasc *$G$, then $H$* asc *$G$.*

PROOF. The statement follows from Theorem 3 (a) and (b), Corollary to Theorem 4 and Lemma 6.

## 6.

A weakly ascendant subgroup of a group is not an ascendant subgroup in general (T. Ikeda).

The statements of Theorem 1 and Lemma 6 hold for Lie algebras. The statements and proofs are obtained replacing the terms in groups by the corresponding terms in Lie algebras. The result corresponding to Theorem 1 generalizes Theorem 1 in [6].

## References

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