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Wu classes and unoriented bordism classes of certain manifolds

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§1. Introduction

Let M be a closed manifold, and let w_i and v_i be the ith Stiefel-Whitney class and the ith Wu class of M, respectively. Then, the Wu formula means that they are related by the equality

$$v_n = \sum_{i=1}^{n} \theta^{n-i} w_i \tag{1.1}$$

(cf. Proposition 3.2), *where* $\theta^l = c(Sq^l) \in \mathscr{A}(2)$ *is the conjugation of* Sq^l *given in* [7, II, §4] *and is defined inductively by*

$$\theta^l = Sq^l + \sum_{i=1}^{l-1} Sq^i \theta^{l-i} = Sq^l + \sum_{j=1}^{l-1} \theta^{l-j} Sq^j \qquad (l \geqq 0).$$

The main purpose of this paper is to study the Wu classes by using (1.1).

To do this, we study the element θ^l in §2, and prove the following basic formula (Theorem 2.4), where we use always the notation

$$t' = 2^{t-1} \quad \text{for any positive integer } t:$$

(1.2) *If* $n = 2^k - 1$, *then*

$$\theta^n = Sq^{k'} Sq^{(k-1)'} \cdots Sq^1;$$

and if $n = 2^k - 1 - t'_1 - \cdots - t'_l$ *with* $k \geqq t_1 > \cdots > t_l \geqq 1$, *then*

$$\theta^n = \sum_{1 \leqq p_1 < \cdots < p_l \leqq k} Sq^{I(p_1, \ldots, p_l)},$$

where $I(p_1, \ldots, p_l) = (i_1, \ldots, i_k)$ *is given by*

$$i_{p_s} = (k - p_s + 1)' - t'_s \quad (s = 1, \ldots, l), \qquad i_p = (k - p + 1)' \quad (p \neq p_1, \ldots, p_l),$$

and $Sq^{(i_1, \ldots, i_k)} = Sq^{i_1} \cdots Sq^{i_k}$ *with* $Sq^0 = 1$ *and* $Sq^i = 0$ *for* $i < 0$.

As an application of this formula, we see the well known formula

$$\theta^{2n+1} = \theta^{2n} Sq^1$$

(Corollary 2.14) and the one given by D. M. Davis [2, Th. 2] (Corollary 2.16). By using the former, we can reduce the equality (1.1) to the form given in Theorem 3.9, and we obtain the equality

$$v_{2n+1} = \sum_{i\geqq 1} (w_1)^{2^i-1} v_{2n+2-2^i}$$

(Theorem 3.10). We notice that this equality implies immediately the well known result that the odd dimensional Wu class v_{2n+1} of an oriented manifold M vanishes.

In § 4, we are concerned with a closed manifold M whose total Stiefel-Whitney class wM satisfies the condition

$$(1.3) \qquad wM = 1 + \sum_{b\geqq 1} w_{b'} \qquad (b'=2^{b-1}).$$

For such a manifold, by noticing that $w_{b'}w_{c'}=0$ if $c\geqq b+2$ (Proposition 4.2) and by using (1.2), we can reduce (1.1) to the following explicit form (Theorem 4.3):

$$(1.4) \quad \begin{array}{ll} v_i = \sum_{b=1}^{a} (w_{b'})^{(a-b+1)'} & \textit{if } i=a'\geqq 1, \\ v_i = \sum_{b=1}^{a_2} \sum_{j=a_2+1}^{a_1} (w_{b'})^{(i-j')/b'} (w_{2b'})^{(j-b)'} & \textit{if } i=a_1'+a_2' \textit{ with } a_1>a_2\geqq 1, \\ v_i = 0 & \textit{otherwise.} \end{array}$$

Some examples of manifolds satisfying (1.3) are given at the end of § 4.

These equalities are applied in § 5 to study some sufficient conditions that the unoriented bordism class of M with (1.3) vanishes. In fact, under (1.3) and the condition that $\dim M$ is not equal to a power of 2, we can show that almost all the Stiefel-Whitney numbers of M vanish by using (1.4) and the fact that $v_i=0$ for $i>\dim M/2$; and we obtain the following results (Theorems 5.1 and 5.4):

THEOREM. *Let M be a closed manifold. Then, the unoriented bordism class $[M]$ of M is 0, if one of the following three conditions holds:*

(1) *The total Stiefel-Whitney class wM satisfies* (1.3), *and*

$$\dim M = p_1'+\cdots+p_k'+1 \quad \textit{with} \quad p_1>\cdots>p_k>1 \quad \textit{and} \quad k\geqq 2, \quad (p'=2^{p-1}).$$

(2) $wM=1+w_{b'}+w_{c'}$ *for some b and c with $c>b\geqq 1$ in* (1.3), *and* $\dim M$ *is not a power of* 2.

(3) $wM=1+w_i$ *for some $i\geqq 1$.*

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§ 2. Some relations in the mod 2 Steenrod algebra

Let $\mathscr{A}(2)$ be the mod 2 Steenrod algebra. For any sequence $I=(i_1,\dots, i_k)$ of positive integers, put

$$Sq^I = Sq^{i_1}\cdots Sq^{i_k} \in \mathscr{A}(2), \quad |I| = i_1+\cdots+i_k;$$

and define the element $\theta^n \in \mathscr{A}(2)$ by

(2.1) $$\theta^0 = 1, \quad \theta^n = \sum_{|I|=n} Sq^I \qquad (n \geqq 1).$$

Then, we have clearly the relations

(2.1)′ $$\theta^n = Sq^n + \sum_{i=1}^{n-1} Sq^i \theta^{n-i} = Sq^n + \sum_{j=1}^{n-1} \theta^{n-j} Sq^j \qquad (n \geqq 0),$$

which give the inductive definition of θ^n. Thus, it is easily seen that θ^n is equal to $c(Sq^n)$ in [7, p. 26] or $\chi(Sq^n)$ in [2].

To study θ^n, we use the following notation:

Let $I=(i_1,\dots, i_k)$ and $T=(t_1,\dots, t_l)$ be sequences of positive integers. Put

(2.2) $$Sq^I - (T) = \sum_{1 \leqq p_1 < \cdots < p_l \leqq k} Sq^{\,I-T(p_1,\dots,p_l)}$$

where $I-T(p_1,\dots, p_l)=(j_1,\dots, j_k)$ is given by

$$j_{p_s} = i_{p_s} - t_s \quad (s = 1,\dots, l), \qquad j_p = i_p \quad (p \neq p_1,\dots, p_l),$$

and $Sq^{(j_1,\dots,j_k)} = Sq^{j_1} \cdots Sq^{j_k}$ under the convention that

(*) $$Sq^0 = 1 \quad \text{and} \quad Sq^j = 0 \quad \text{if} \quad j<0.$$

Then, $Sq^I - (T)$ can be defined inductively on the lengths k of I and l of T by

(2.2)′ $$\begin{aligned} Sq^I - (T) &= Sq^I \ \text{ if } \ l=0, \qquad Sq^I - (T) = 0 \ \text{ if } \ l>k; \\ Sq^I - (T) &= Sq^{i_1 - t_1}\{Sq^{I_1} - (T_1)\} + Sq^{i_1}\{Sq^{I_1} - (T)\} \\ &= \{Sq^{I_k} - (T_l)\} Sq^{i_k - t_l} + \{Sq^{I_k} - (T)\} Sq^{i_k} \end{aligned}$$

under the convention (*), where $J_s=(j_1,\dots, j_{s-1}, j_{s+1},\dots, j_m)$ for $J=(j_1,\dots, j_m)$.

Furthermore, put $Sq^I - (t) = Sq^I - (J(t))$ and

$$\theta^n - (t) = \sum_{|I|=n} \{Sq^I - (t)\} \qquad \text{for} \quad n, t \geqq 0,$$

where $J(t)=(2^{t-1}, 2^{t-2},\dots, 1)$. Then we see the following

PROPOSITION 2.3. $$\theta^n - (t) = \begin{cases} 0 & \text{for} \quad n<2^t-1, \\ \theta^{n-2^t+1} & \text{for} \quad n \geqq 2^t - 1 \geqq 0. \end{cases}$$

PROOF. The equality for $n<2^t-1$ or $t=0$ is seen immediately by definition.

We prove the equality for $n \geqq 2^t - 1 \geqq 1$ by the induction on n. By (2.1)′, (2.2)′ and the above definition, we see that

$$\theta^n - (t) = \sum_{i=1}^{n} Sq^{i-t'}(\theta^{n-i} - (t-1)) + \sum_{i=1}^{n-1} Sq^i(\theta^{n-i} - (t)) \quad (t'=2^{t-1}).$$

By the equality for $n<2^t-1$, the inductive assumption and (2.1)′, this is equal to

$$\sum_{i=t'}^{n-t'+1} Sq^{i-t'} \theta^{n-i-t'+1} + \sum_{i=1}^{n-2t'+1} Sq^i \theta^{n-i-2t'+1} = \theta^{n-2t'+1},$$

as desired. q.e.d.

Now, the main purpose in this section is to prove the following theorem, where we use always the notation

$$t' = 2^{t-1} \qquad \text{for any positive integer} \quad t.$$

THEOREM 2.4. (i) *Let* $n=2^k-1$. *Then*

$$\theta^n = Sq^{J(k)} \qquad (J(k) = (k', (k-1)', \dots, 1)).$$

(ii) *Let*

$$n = 2^k - 1 - t_1' - \cdots - t_l' = 2^k - 1 - |T|$$

for $T=(t_1', \dots, t_l')$ *with* $k \geqq t_1 > \cdots > t_l \geqq 1$ *and* $l \geqq 1$. *Then,*

$$\theta^n = Sq^{J(k)} - (T) \qquad (J(k) = (k', (k-1)', \dots, 1)),$$

where the right hand side is given by (2.2).

By this theorem and (2.2)', we have the following

COROLLARY 2.5. *For n in* (ii) *of the above theorem with* $k>t_1$,

$$\theta^n = Sq^a\theta^{n-a} + Sq^{k'}\theta^{n-k'} \quad \text{where} \quad a = k'-t_1'.$$

PROOF. By the above theorem and (2.2)', θ^n is equal to

$$\begin{aligned} Sq^{J(k)} - (T) &= Sq^a\{Sq^{J(k-1)} - (T_1)\} + Sq^{k'}\{Sq^{J(k-1)} - (T)\} \\ &= Sq^a\theta^{k'-1-|T_1|} + Sq^{k'}\theta^{k'-1-|T|} \qquad (T_1=(t_2', \dots, t_l')), \end{aligned}$$

which is equal to the right hand side of the desired equality. q.e.d.

To prove Theorem 2.4, we prepare several results.

Let $P(=RP^\infty)$ be the ∞-dimensional real projective space and P^m be the m-fold Cartesian product of P. Let u be the generator of $H^1(P; Z_2)=Z_2$, and consider the cohomology class

$$u_1 \times \cdots \times u_m \in H^m(P^m; Z_2) \qquad (u_1 = \cdots = u_m = u).$$

Furthermore for any sequence $A=(a_1, \dots, a_m)$ of positive integers, we consider the cohomology class

$$u(A) = u_1(a_1) \times \cdots \times u_m(a_m) \in H^*(P^m; Z_2) \qquad (u(a)=u^{a'},\ a'=2^{a-1}).$$

Then, we have the following proposition, where $\varepsilon=(\varepsilon_1, \dots, \varepsilon_m)$ is a sequence with $\varepsilon_i=0$ or 1 and $A+\varepsilon=(a_1+\varepsilon_1, \dots, a_m+\varepsilon_m)$ and $\|A\|=a_1'+\cdots+a_m'$ for $A=(a_1, \dots, a_m)$:

PROPOSITION 2.6. *In* $H^*(P^m; Z_2)$, *there hold the equalities*

(i) $\quad Sq^a u(A) = \sum_{\|A+\varepsilon\|=\|A\|+a} u(A+\varepsilon)$,

(ii) $\quad \theta^n(u_1\times\cdots\times u_m) = \sum_{\|A\|=n+m} u(A)$.

PROOF. Let x be any 1-dimensional cohomology class. Then, the equality $Sq^i x^k = \binom{k}{i} x^{k+i}$ of [7, I, Lemma 2.4] implies

$$(2.7) \qquad Sq^i(x(a)) = \begin{cases} x(a+\varepsilon) & \text{if} \quad i=\varepsilon a', \quad \varepsilon=0 \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $x(b)=x^{b'}$ $(b'=2^{b-1})$. Thus, we see by definition that

$$(2.7)' \qquad Sq^I x = \begin{cases} x(l) & \text{if} \quad I=J(l-1), \\ 0 & \text{otherwise;} \end{cases}$$

$$(2.8) \qquad \theta^n x = \begin{cases} x^{n+1} = x(l) & \text{if} \quad n=l'-1\geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) follows immediately from (2.7) and the Cartan formula.

(ii) By the Cartan formula and (2.7)′, we see easily that

$$(2.9) \quad Sq^I(u_1\times u_2\times\cdots\times u_m) = \sum_{t\geqq 1} u_1(t)\times(Sq^I-(t-1))(u_2\times\cdots\times u_m).$$

Therefore, by (2.1) and Proposition 2.3,

$$\begin{aligned} \theta^n(u_1\times u_2\times\cdots\times u_m) &= \sum_{t\geqq 1} u_1(t)\times\theta^{n-t'+1}(u_2\times\cdots\times u_m) \\ &=\cdots= \sum u_1(a_1)\times\cdots\times u_{m-1}(a_{m-1})\times\theta^{n-a}(u_m), \end{aligned}$$

where $a=(a_1'-1)+\cdots+(a_{m-1}'-1)$. Hence, we see the equality (ii) by (2.8)

q.e.d.

For the case $m=n$ in (ii) of the above proposition, we have the following lemma, where A and B are sequences of n positive integers and ε and ρ are sequences of n integers consisting of 0 or 1:

LEMMA 2.10. (i) *If* $n=2^k-1\geqq 1$, *then*

$$\theta^n(u_1\times\cdots\times u_n) = \sum_{\|A\|=k'+n-1}\sum_{\|A+\varepsilon\|=2n} u(A+\varepsilon).$$

(ii) *If* $n=2^k-t'-s$ *with* $k>t\geqq 1$ *and* $1\leqq s\leqq t'$, *then*

$$\begin{aligned} \theta^n(u_1\times\cdots\times u_n) = &\sum_{\|A\|=k'-s+n}\sum_{\|A+\varepsilon\|=2n} u(A+\varepsilon) \\ &+ \sum_{\|B\|=k'-t'-s+n}\sum_{\|B+\rho\|=2n} u(B+\rho). \end{aligned}$$

PROOF. (ii) Let $C=(c_1,\ldots, c_n)$ be a sequence of positive integers with $\|C\|=2n$, and assume that $u(C)$ appears a and b times in the first and the second summations in the right hand side of the equality in (ii), respectively. Then, by (ii) of the above proposition, it is sufficient to prove that

$$a + b = \text{odd}.$$

Assume that a positive integer l appears α_l times in C. Then

$$\alpha_l \geqq 0, \quad \sum_{l\geqq 1} \alpha_l = n \quad \text{and} \quad \sum_{l\geqq 1} l'\alpha_l = 2n. \tag{2.11}$$

Furthermore, in the first summation in the right hand side of the equality in (ii), the equality $A+\varepsilon=C$ holds if and only if $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$ satisfies the condition that

$$(*) \qquad 0 \leqq p_l \leqq \alpha_l \ (l\geqq 2), \quad \sum_{l\geqq 2} (l-1)'p_l = 2n-(k'-s+n) = k'-t',$$

where p_l is the number of elements of $\{i \mid c_i=l,\ \varepsilon_i=1\}$. Thus

$$a = \sum_{(*)} \binom{\alpha_2}{p_2} \cdots \binom{\alpha_l}{p_l} \cdots,$$

which is equal to the coefficient of $x^{k'-t'}$ in the polynomial

$$(1+x)^{\alpha_2} \cdots (1+x^{(l-1)'})^{\alpha_l} \cdots .$$

By (2.11), this polynomial is congruent to $(1+x)^{n-\alpha_1/2} \bmod 2$. Hence

$$a \equiv \binom{n-\alpha_1/2}{k'-t'} \qquad \bmod 2.$$

By the same way, since $2n-(k'-t'-s+n)=k'$, we see that

$$b \equiv \binom{n-\alpha_1/2}{k'} \qquad \bmod 2.$$

On the other hand, by using the well known formula

$$\binom{\beta}{\alpha} \equiv \prod_i \binom{b_i}{a_i} \quad \bmod 2 \quad \text{for} \quad \alpha = \sum_i a_i 2^i,\ \beta = \sum_i b_i 2^i\ (0\leqq a_i,\ b_i\leqq 1), \tag{2.12}$$

we see easily that

$$\binom{n-\alpha_1/2}{k'-t'} \equiv \begin{cases} 1 & \bmod 2 \quad (n/2\leqq n-\alpha_1/2\leqq k'-1) \\ 0 & \bmod 2 \quad (k'\leqq n-\alpha_1/2\leqq n), \end{cases}$$

$$\binom{n-\alpha_1/2}{k'} \equiv \begin{cases} 0 & \bmod 2 \quad (n/2\leqq n-\alpha_1/2\leqq k'-1) \\ 1 & \bmod 2 \quad (k'\leqq n-\alpha_1/2\leqq n), \end{cases}$$

since $n=2k'-t'-s\geqq\alpha_1\geqq 0$ with $k'>t'\geqq s\geqq 1$. Thus $a+b\equiv 1 \bmod 2$, and (ii) is proved.

(i) can be proved similarly by noticing $2n-(k'+n-1)=k'$ and $\binom{n-\alpha_1/2}{k'}$ $\equiv 1 \bmod 2$ for $n=2k'-1 \geqq \alpha_1 \geqq 0$. q.e.d.

By using the above results, we can prove Theorem 2.4.

Proof of Theorem 2.4. (i) Since $\theta^1=Sq^1$, we see (i) for $k=1$. Assume inductively that (i) holds for $k-1$. Then

$$Sq^{J(k)} = Sq^{k'}\theta^{k'-1}.$$

On the other hand, by Proposition 2.6 and Lemma 2.10(i), we see that

$$\begin{aligned} Sq^{k'}\theta^{k'-1}(u_1\times\cdots\times u_n) &= Sq^{k'}\textstyle\sum_{\|A\|=k'-1+n} u(A) \\ &= \textstyle\sum_{\|A\|=k'-1+n}\sum_{\|A+\varepsilon\|=2n} u(A+\varepsilon) = \theta^n(u_1\times\cdots\times u_n) \qquad (n=2k'-1). \end{aligned}$$

Therefore $Sq^{k'}\theta^{k'-1}=\theta^n$ by the following fundamental result in [7, I, Cor. 3.3]:

(2.13) *The homomorphism $\mathscr{A}(2)\to H^*(P^m; Z_2)$ given by $Sq^I\to Sq^I(u_1\times\cdots\times u_m)$ is a monomorphism in degree $\leqq m$.*

Thus, we obtain $\theta^n=Sq^{J(k)}$ as desired.

(ii) We prove (ii) by the induction on k. If $k=1$, then (ii) is clear, since $\theta^0=1=Sq^0$. Assume inductively that (ii) holds for $k-1$. Then, by (2.2)′, (i) and the inductive assumption, we see that

$$\begin{aligned} Sq^{J(k)}-(T) &= Sq^a\{Sq^{J(k-1)}-(T_1)\} + Sq^{k'}\{Sq^{J(k-1)}-(T)\} \\ &= Sq^a\theta^{n-a} + Sq^{k'}\theta^{n-k'} \qquad (a=k'-t'_1,\ T_1=(t'_2,\dots,t'_l)), \end{aligned}$$

where the second terms do not appear if $k=t_1$ by the convention (*) in (2.2)′.

If $k=t_1$, then $a=0$ and we have the desired equality.

Let $k>t_1$. Then, by Proposition 2.6 and Lemma 2.10(ii), we see that

$$\begin{aligned} &(Sq^a\theta^{n-a} + Sq^{k'}\theta^{n-k'})(u_1\times\cdots\times u_n) \\ &= Sq^a\textstyle\sum_{\|A\|=2n-a} u(A) + Sq^{k'}\sum_{\|B\|=2n-k'} u(B) \\ &= \textstyle\sum_{\|A\|=2n-a}\sum_{\|A+\varepsilon\|=2n} u(A+\varepsilon) + \sum_{\|B\|=2n-k'}\sum_{\|B+\rho\|=2n} u(B+\rho) \\ &= \theta^n(u_1\times\cdots\times u_n), \qquad (n=2k'-t'_1-s,\ s=t'_2+\cdots+t'_l+1). \end{aligned}$$

Therefore $Sq^a\theta^{n-a}+Sq^{k'}\theta^{n-k'}=\theta^n$ by (2.13).

Thus $Sq^{J(k)}-(T)=\theta^n$, and the theorem is proved completely. q.e.d.

As applications of Theorem 2.4, we have the following known results:

Corollary 2.14. $\theta^{2n+1}=\theta^{2n}Sq^1$.

Proof. We notice that

$$(2.15)\qquad Sq^{2a-1}Sq^{a} = \sum_{j=0}^{a-1}\binom{a-1-j}{2a-1-2j}Sq^{3a-1-j}Sq^{j} = 0$$

by the Adam relation [7, p. 2].

If $n=0$, then the equality holds since $\theta^1=Sq^1$.

Let $2n=2^k-1-|T|>0$ for $T=(t'_1,\dots,t'_l)$ with $k\geqq t_1>\dots>t_l\geqq 1$. Then $t'_l=1$. Thus, in the summation of the equality

$$Sq^{J(k)}-(T)=\sum_{1\leqq p_1<\dots<p_l\leqq k}Sq^{J(k)-T(p_1,\dots,p_l)}$$

of (2.2), the term for $p_l=k-a<k$ contains $Sq^{2a'-1}Sq^{a'}$ and is 0 by (2.15). Therefore, the above sum is equal to

$$Sq^{J(k)_k}-(T_l)\quad (J(k)_k=(k',(k-1)',\dots,2),\ T_l=(t'_1,\dots,t'_{l-1})).$$

On the other hand, $2n+1=2^k-1-|T_l|$ and

$$Sq^{J(k)}-(T_l)=\{Sq^{J(k)_k}-(T_l)\}Sq^1$$

by definition, since $t'_{l-1}\geqq 2$ or $l-1=0$. Thus, we see the desired equality by Theorem 2.4. q.e.d.

(ii) and (iii) of the following corollary are due to Davis [2, Th. 2].

COROLLARY 2.16. (i) $\theta^{2k'}=Sq^{2k'}+Sq^{k'}\theta^{k'}$.

(ii) $\theta^{2k-l}=Sq^{J(k;l)}\theta^{l'-l}$ *for* $k\geqq l\geqq 1$,

where $J(k;l)=(k',(k-1)',\dots,l')$.

(iii) $\theta^{2k-k-1}=Sq^{k'}\theta^{k'-k-1}+Sq^{(k'-1,(k-1)'-1,\dots,1)}$ *for* $k\geqq 2$.

PROOF. By using (2.9) and (2.13), we see easily that

$$(2.17)\qquad \sum_I Sq^I=\sum_J Sq^J \quad\text{implies}\quad \sum_I(Sq^I-(t))=\sum_J(Sq^J-(t)).$$

(i) By Proposition 2.3, Theorem 2.4, (2.2)′ and (2.17), we see that

$$\begin{aligned}\theta^{2k'}&=\theta^{4k'-1}-(k)=Sq^{J(k+1)}-(k)\\&=Sq^{2k'-k'}(Sq^{J(k)}-(k-1))+Sq^{2k'}(Sq^{J(k)}-(k))=Sq^{k'}\theta^{k'}+Sq^{2k'}.\end{aligned}$$

(ii) We prove the equality by the induction on l. (ii) for $l=1$ is in Theorem 2.4(i). Assume (ii) for l. Then, by Proposition 2.3, (2.15) and (2.17), we see (ii) for $l+1(\leqq k)$ as follows:

$$\begin{aligned}\theta^{2k'-l-1}&=\theta^{2k'-l}-(1)=(Sq^{J(k;l+1)}Sq^{l'}\theta^{l'-l})-(1)\\&=Sq^{J(k;l+1)}(Sq^{l'}\theta^{l'-l}-(1))=Sq^{J(k;l+1)}(\theta^{2l'-l}-(1))=Sq^{J(k;l+1)}\theta^{2l'-l-1}.\end{aligned}$$

(iii) By (ii), $\theta^{2l'-l}=Sq^{l'}\theta^{l'-l}$ for any $l\geqq 1$. Thus,

$$\theta^{2l'-l-1} = \theta^{2l'-l} - (1) = Sq^{l'}\theta^{l'-l-1} + Sq^{l'-1}\theta^{l'-l}$$

for any $l\geqq 1$ in the same way. By using this equality for $l=k, k-1,\dots, 1$ and (2.15), we see immediately (iii). q.e.d.

The following Cartan formula for θ, which may be well-known, is used in the next section.

Proposition 2.18. *For any cohomology classes x and y,*

$$\theta^n(xy) = \sum_{i+j=n}(\theta^i x)(\theta^j y).$$

Proof. We can prove easily the formula by the induction on n, by using (2.1)' and the Cartan formula for Sq. q.e.d.

Remark 2.19. We remark that Proposition 2.6 (ii) can be proved by (2.8) and the Cartan formula

$$\theta^n(x\times y) = \sum_{i+j=n}(\theta^i x)\times(\theta^j y).$$

§ 3. Odd dimensional Wu classes

Let M^d be a closed d-manifold, and let

$$v_i\in H^i(M^d; Z_2)$$

be the ith Wu class of M^d, which is defined to be the element with

$$\langle v_i x, \mu\rangle = \langle Sq^i x, \mu\rangle \qquad \text{for every} \quad x\in H^{d-i}(M^d; Z_2).$$

Here $\mu\in H_d(M^d; Z_2)$ is the fundamental homology class and $\langle\ ,\ \rangle$ is the Kronecker index. Then, the kth Stiefel-Whitney class

$$w_k\in H^k(M^d; Z_2)$$

of M^d is represented by the Wu classes as the following Wu formula:

(3.1) ([6, Th. 11. 14]) $\qquad w_k = \sum_{i=0}^{k} Sq^i v_{k-i}.$

Conversely, the Wu class is represented by the Stiefel-Whitney classes as follows:

Proposition 3.2. $\qquad v_n = \sum_{i=1}^{n}\theta^{n-i}w_i,$

where $\theta^{n-i}\in\mathscr{A}(2)$ is the element given by (2.1).

Proof. By (3.1), $w_1=v_1+Sq^1v_0=v_1$. Suppose inductively that the equality holds for $n<k$. Then, by (3.1) and (2.1)′, we see that

$$\begin{aligned} v_k &= w_k + \textstyle\sum_{i=1}^{k-1} Sq^i v_{k-i} = w_k + \sum_{i=1}^{k-1} Sq^i(\sum_{j=1}^{k-i} \theta^{k-i-j} w_j) \\ &= w_k + \textstyle\sum_{j=1}^{k-1} (\sum_{i=1}^{k-j} Sq^i \theta^{k-j-i}) w_j = \sum_{j=1}^{k} \theta^{k-j} w_j, \end{aligned}$$

as desired. q.e.d.

To prove Theorems 3.9 and 3.10 which are the main results in this section, we prepare several lemmas, where we use the notations $t'=2^{t-1}$ for any positive integer t, and $l(I)=l$ and $\|I\|=i_1'+\cdots+i_l'$ for any sequence $I=(i_1,\dots,i_l)$ of positive integers.

LEMMA 3.3. (i) *If* $l=l_1'+\cdots+l_k'=\|L\|$ *for* $L=(l_1,\dots,l_k)$ *with* $l_1>\cdots>l_k\geqq 1$, *then*

$$\textstyle\sum_{l(I)=l} w_1^{2\|I\|}\theta^{m-2\|I\|}w_n = \sum_{l(J)=k} w_1^{\|J+L\|}\theta^{m-\|J+L\|}w_n,$$

where $J+L=(j_1+l_1,\dots,j_k+l_k)$ *for* $J=(j_1,\dots,j_k)$ *and* $\theta^j=0$ *if* $j<0$.

(ii) *If* $l=2^s\geqq 1$, *then*

$$\textstyle\sum_{l(I)=l} w_1^{2\|I\|}\theta^{m-2\|I\|}w_n = \sum_{i\geqq 2} w_1^{(i+s)'}\theta^{m-(i+s)'}w_n.$$

(iii) *If* $l=2^s-1\geqq 1$, *then*

$$\textstyle\sum_{l(I)=l} w_1^{2\|I\|}\theta^{m-2\|I\|}w_n = w_1^{2l}\theta^{m-2l}w_n + \sum_{k=1}^{s}\sum_{i\geqq 2} w_1^{\varphi(s;k,i)}\theta^{m-\varphi(s;k,i)}w_n,$$

where $\varphi(s;\,k,\,i)=(s+i)'+(s+2-k)'-2$.

PROOF. (i) In the left hand side of the equality, the sum of the terms for $I=(i_1,i_2,i_3,\dots,i_l)$ and $I'=(i_2,i_1,i_3,\dots,i_l)$ with $i_1\neq i_2$ is 0, and the term for $I=(i_1,i_1,i_3,\dots,i_l)$ is equal to

$$w_1^{2\|I''\|}\theta^{m-2\|I''\|}w_n \quad \text{with} \quad I''=(i_1+1,\,i_3,\dots,i_l).$$

Let $k=1$, i.e., $l=l_1'$. Then, by using these facts repeatedly, we see easily that the left hand side of the equality is equal to

$$\textstyle\sum_{l(J)=l/2} w_1^{4\|J\|}\theta^{m-4\|J\|}w_n,$$

and hence to $\sum_{i\geqq 1} w_1^{(i+l_1)'}\theta^{m-(i+l_1)'}w_n$, which is the right hand side of the equality. In the same way, we can prove (i) for $k>1$.

(ii) The equality is proved in the above proof.

(iii) Since $l=2^s-1=\|S\|$ where $S=(s,\,s-1,\dots,1)$, (i) implies that the left hand side of the equality in (iii) is equal to

$$(*) \qquad \textstyle\sum_{l(J)=s} w_1^{\|J+S\|}\theta^{m-\|J+S\|}w_n.$$

In this summation, let σ_k $(1\leqq k\leqq s)$ be the partial sum on

$$J=(j_1,\dots,j_s) \quad \text{with} \quad j_k \geqq 2 \quad \text{and} \quad j_{k+1}=\cdots=j_s=1.$$

Then, (*) is equal to

$$w_1^{2l}\theta^{m-2l}w_n + \sum_{k=1}^{s} \sigma_k,$$

since the term in (*) for $J=(1,\dots,1)$ is equal to the first term.

Now, by the same consideration as in the proof of (i), σ_s is equal to the partial sum on J with $j_{s-1}+1=j_s\geqq 2$, and hence to that on J with $j_{s-2}+2=j_{s-1}+2=j_s+1\geqq 3$, and so on. Hence, σ_s is equal to the partial sum on J with $j_1=j_2=\cdots=j_{s-1}=j_s-1\geqq 1$, which is clearly equal to $\sum_{i\geqq 2} w_1^{(i+s)'}\theta^{m-(i+s)'}w_n$.

Similarly, we see that

$$\sigma_k = \sum_{i\geqq 2} w_1^{\varphi(s;k,i)}\theta^{m-\varphi(s;k,i)}\, w_n.$$

Thus we have proved (iii). q.e.d.

LEMMA 3.4. (i) *For* $t'=2^{t-1}\geqq 2$,

$$\sum_{q=t'-1}^{2t'-2} \sum_{l(I)=q} w_1^{2\|I\|}\theta^{m-2\|I\|}\, w_n = w_1^{2t'-2}\theta^{m+2-2t'}\, w_n.$$

(ii) $\sum_{q\geqq 1} \sum_{l(I)=q} w_1^{2\|I\|}\theta^{m-2\|I\|}\, w_n = \sum_{t\geqq 3} w_1^{t'-2}\theta^{m+2-t'}\, w_n.$

PROOF. (i) For $t=2$, the above lemma implies the desired equality as follows:

$$\begin{aligned}&\sum_{q=1}^{2} \sum_{l(I)=q} w_1^{2\|I\|}\theta^{m-2\|I\|}\, w_n \\ &\quad = w_1^2\theta^{m-2}w_n + \sum_{i\geqq 2} w_1^{\varphi(1;1,i)}\theta^{m-\varphi(1;1,i)}\, w_n + \sum_{i\geqq 2} w_1^{(i+1)'}\theta^{m-(i+1)'}\, w_n \\ &\quad = w_1^2\theta^{m-2}w_n.\end{aligned}$$

We prove (i) by the induction on t. In the left hand side of the equality, we see easily by (i) of the above lemma that the sum on $q=t'+p$ with $1\leqq p\leqq t'-2$ is equal to

$$\sum_{i\geqq 2} w_1^{i't'} \{\sum_{p=1}^{t'-2} \sum_{l(J)=p} w_1^{2\|J\|}\theta^{m-i't'-2\|J\|}\, w_n\}.$$

By the inductive assumption, this is equal to

$$\begin{aligned}&\sum_{i\geqq 2} w_1^{i't'} \{\sum_{u=2}^{t-1} w_1^{2u'-2}\theta^{m-i't'+2-2u'}\, w_n\} \\ &\quad = \sum_{k=1}^{t-1} \sum_{i\geqq 2} w_1^{\varphi(t-1;k,i)}\theta^{m-\varphi(t-1;k,i)}\, w_n + \sum_{i\geqq 2} w_1^{i't'}\theta^{m-i't'}\, w_n.\end{aligned}$$

On the other hand, the terms for $q=t'-1$ and t' are given by (iii) and (ii) of the above lemma for $s=t-1$, respectively. Thus we see (i).

(ii) (ii) follows immediately from (i). q.e.d.

LEMMA 3.5.

(i) $\theta^{2l}(w_1w_{2m}) = \sum_{t\geqq 2} w_1^{t'-1}\theta^{2l+2-t'}\, w_{2m} + \sum_{t\geqq 3} w_1^{t'-2}\theta^{2l+2-t'}\, w_{2m+1},$

(ii) $\theta^{2l}(w_1 w_{2m+1}) = \sum_{t \geqq 2} w_1^{t'-1} \theta^{2l+2-t'} w_{2m+1}$.

PROOF. We notice that the equalities

$$Sq^1 w_{2m} = w_1 w_{2m} + w_{2m+1}, \quad Sq^1 w_{2m+1} = w_1 w_{2m+1} \tag{3.6}$$

hold as special cases of Wu's formula

$$\text{([10], [3])} \quad Sq^j w_i = \sum_{t=0}^{j} \binom{i-j+t-1}{t} w_{j-t} w_{i+t} \quad \textit{for} \quad j \leqq i. \tag{3.7}$$

By Proposition 2.18, (2.8), Corollary 2.14 and the first equality in (3.6), we see that

$$\begin{aligned}(3.8.t) \quad \theta^{2t}(w_1 w_{2m}) &= \textstyle\sum_{j \geqq 0} (\theta^j w_1)(\theta^{2t-j} w_{2m}) \\ &= w_1 \theta^{2t} w_{2m} + \textstyle\sum_{i \geqq 2} w_1^{i'} \theta^{2t-i'} Sq^1 w_{2m} \\ &= w_1 \theta^{2t} w_{2m} + \textstyle\sum_{i \geqq 2} w_1^{i'} \theta^{2t-i'} w_{2m+1} + \textstyle\sum_{i \geqq 2} w_1^{i'} \theta^{2t-i'}(w_1 w_{2m}).\end{aligned}$$

Consider the equality (3.8.l), and substitute (3.8.$l-i'/2$) for its last term $\theta^{2l-i'}(w_1 w_{2m})$ ($i'=2^{i-1} \geqq 2$) if $2l-i' \geqq 0$, and so on. Then, we see easily that

$$\begin{aligned}\theta^{2l}(w_1 w_{2m}) = w_1 \theta^{2l} w_{2m} &+ \textstyle\sum_{q \geqq 1} \sum_{l(I)=q} w_1^{1+2\|I\|} \theta^{2l-2\|I\|} w_{2m} \\ &+ \textstyle\sum_{q \geqq 1} \sum_{l(I)=q} w_1^{2\|I\|} \theta^{2l-2\|I\|} w_{2m+1}.\end{aligned}$$

Thus, (i) is seen by (ii) of the above lemma.

We can prove (ii) similarly by using the second equality in (3.6). q.e.d.

By the above lemmas, we have the following results.

THEOREM 3.9. *The equality in Proposition* 3.2 *can be rewritten as follows, where* $a \geqq 1$ *and* $t'=2^{t-1}$ *for any positive integer* t:

(i) $v_{2a} = \sum_{p \geqq 1} \theta^{2a-2p} w_{2p} + \sum_{p \geqq 0, t \geqq 2} w_1^{t'-1} \theta^{2a-2p-t'} w_{2p+1}$.

(ii) $v_{2a+1} = \sum_{p \geqq 1, t \geqq 2} w_1^{t'-1} \theta^{2a+2-2p-t'} w_{2p} + \sum_{p \geqq 1, t \geqq 3} w_1^{t'-2} \theta^{2a+2-2p-t'} w_{2p+1}$.

PROOF. By Proposition 3.2, Corollary 2.14 and the first equality in (3.6), we see that

$$\begin{aligned}v_{2a+1} &= \textstyle\sum_{p \geqq 1} (\theta^{2a+1-2p} w_{2p} + \theta^{2a-2p} w_{2p+1}) \\ &= \textstyle\sum_{p \geqq 1} \{\theta^{2a-2p}(w_1 w_{2p} + w_{2p+1}) + \theta^{2a-2p} w_{2p+1}\} \\ &= \textstyle\sum_{p \geqq 1} \theta^{2a-2p}(w_1 w_{2p}).\end{aligned}$$

Thus, we have (ii) by (i) of the above lemma.

(i) is shown in the same way. q.e.d.

THEOREM 3.10. *The odd dimensional Wu class v_{2a+1} of a closed manifold can be represented by the lower and even dimensional Wu classes and the first Stiefel-Whitney class w_1 by the equality*

$$v_{2a+1} = \sum_{i\geqq 2} w_1^{i'-1} v_{2a+2-i'}, \qquad (i'=2^{i-1}).$$

PROOF. The equality for $a=0$ is clear.

Let a be positive. Then, by the above theorem,

$$\begin{aligned}&\sum_{i\geqq 2} w_1^{i'-1} v_{2a+2-i'} \\ &= \sum_{i\geqq 2, p\geqq 1} w_1^{i'-1}\theta^{2a+2-2p-i'} w_{2p} + \sum_{i\geqq 2, p\geqq 0, t\geqq 2} w_1^{i'+t'-2}\theta^{2a+2-2p-i'-t'} w_{2p+1} \\ &\quad + \begin{cases} w_1^{2a+1} & (a=2^j-1) \\ 0 & (a\neq 2^j-1). \end{cases}\end{aligned}$$

Here, in the same way as in the proof of Lemma 3.3(ii), we see that the second term is equal to

$$\begin{aligned}&\sum_{i\geqq 2, p\geqq 0} w_1^{2i'-2}\theta^{2a+2-2p-2i'} w_{2p+1} \\ &\quad = \sum_{i\geqq 2} w_1^{2i'-2}\theta^{2a+2-2i'} w_1 + \sum_{i\geqq 2, p\geqq 1} w_1^{2i'-2}\theta^{2a+2-2p-2i'} w_{2p+1},\end{aligned}$$

whose first sum is equal to

$$w_1^{2a+1} \quad (a=2^j-1), \qquad 0 \quad (a\neq 2^j-1),$$

by (2.8). Thus we obtain the desired equality by (ii) of the above theorem.

q. e. d.

As an application of the above theorem, we obtain the following known result:

COROLLARY 3.11 ([5, Lemma 3]). *If a closed manifold M is orientable, then the odd-dimensional Wu classes of M vanish.*

PROOF. By [4, p. 244, Th. 12.1], the assumption is equivalent to $w_1=0$. Thus the corollary follows immediately from the above theorem. q. e. d.

§4. Wu classes of certain manifolds

In the rest of this paper, we only consider a closed manifold M whose ith Stiefel-Whitney class w_i satisfies

(4.1) $$w_i = 0 \quad \text{if } i \text{ is not a power of } 2;$$

i.e., we assume that the total Stiefel-Whitney class wM is given by

(4.1)′ $$wM = 1 + \sum_{b\geqq 1} w_{b'}, \quad w_{b'} \in H^{b'}(M; Z_2),$$

where we use at all times the notation

$$b' = 2^{b-1} \quad \text{for any positive integer } b.$$

Under the above assumption, we have the following

PROPOSITION 4.2. *If $c \geqq b+2$, then, $w_{b'}w_{c'}=0$.*

PROOF. $w_{c'-b'}=0$ by the assumption and (4.1). Therefore

$$0 = Sq^{2b'}w_{c'-b'} = \sum_{t=0}^{2b'}\binom{c'-b'-2b'+t-1}{t}w_{2b'-t}w_{c'-b'+t}$$

by (3.7), and the last sum is equal to $w_{b'}w_{c'}$ by (2.12) and (4.1). q. e. d.

By Proposition 3.2, (3.7) and this proposition, we see that the Wu class v_i can be written as a sum of cohomology classes $(w_{b'})^j(w_{2b'})^k$. More precisely, the purpose of this section is to prove the following

THEOREM 4.3. *The ith Wu class v_i of a closed manifold M satisfying the condition* (4.1) *can be represented by the Stiefel-Whitney classes $w_{b'}$ of M as follows, where*

$$i = a_1'+a_2'+\cdots+a_k' \quad \text{with} \quad a_1>a_2>\cdots>a_k\geqq 1:$$

(i) *If $k=1$, i.e., if $i=a'$ with $a\geqq 1$, then*

$$v_i = \sum_{b=1}^{a}(w_{b'})^{(a-b+1)'}.$$

(ii) *If $k=2$, i.e., if $i=a_1'+a_2'$ with $a_1>a_2\geqq 1$, then*

$$v_i = \sum_{b=1}^{a_2}\sum_{j=a_2+1}^{a_1}(w_{b'})^{(i-j')/b'}(w_{2b'})^{(j-b)'}.$$

(iii) *If $k\geqq 3$, then $v_i=0$.*

To prove this theorem, we prepare several lemmas.

LEMMA 4.4. *For any cohomology class y and $t'=2^{t-1}$,*

$$Sq^i y^{t'} = \begin{cases} (Sq^{i/t'}y)^{t'} & \text{if } i \text{ is a multiple of } t', \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We see easily by the Cartan formula that

$$Sq^{2a}z^2 = (Sq^a z)^2, \quad Sq^{2a+1}z^2 = 0.$$

These imply immediately the lemma. q. e. d.

LEMMA 4.5. (i) *For $b'=2^{b-1}\geqq 1$, $t'=2^{t-1}\geqq 1$ and $i\geqq 1$,*

$$Sq^i(w_{2b'})^{t'} = \begin{cases} (w_{b'})^{t'}(w_{2b'})^{t'} & \textit{if} \quad i=b't', \\ (w_{2b'})^{2t'} & \textit{if} \quad i=2b't', \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) *$Sq^I w_{2b'}=0$ if $|I|$ is not a multiple of b'.*

Proof. (i) By (3.7), (4.1) and Proposition 4.2, we see that

$$Sq^i w_{2b'} = w_i w_{2b'} = \begin{cases} w_{b'}w_{2b'} & \text{if} \quad i=b', \\ 0 & \text{otherwise,} \end{cases}$$

for $0<i<2b'$. Thus we see the equality for $t'=t=1$.

The lemma for $t>1$ follows immediately from that for $t=1$ and the above lemma.

(ii) (ii) is clear by (i), Proposition 4.2 and the Cartan formula. q.e.d.

Lemma 4.6. *For $q\geqq p+1\geqq 3$,*

$$Sq^{2b'(q'-p')}(w_{2b'})^{q'-p'+1} = \begin{cases} (w_{2b'})^{2q'-2p'+1} & \textit{if} \quad p>2, \\ (w_{2b'})^{2q'-3} + (w_{b'})^2(w_{2b'})^{2q'-4} & \textit{if} \quad p=2. \end{cases}$$

Proof. We prove the lemma by the induction on $q=p+1,\ p+2,\dots$. If $q=p+1$, then $q'-p'=p'$ and we see that

$$\begin{aligned} Sq^{2b'p'}(w_{2b'})^{p'+1} &= Sq^{2b'p'}\{(w_{2b'})^{p'}w_{2b'}\} \\ &=(w_{2b'})^{p'}Sq^{2b'p'}w_{2b'} + (w_{b'})^{p'}(w_{2b'})^{p'}Sq^{b'p'}w_{2b'} + (w_{2b'})^{2p'}w_{2b'} \end{aligned}$$

by the Cartan formula and the above lemma for $t=p$. Furthermore,

$$Sq^{2b'p'}w_{2b'} = 0, \quad Sq^{b'p'}w_{2b'} = \begin{cases} 0 & \text{if} \quad p>2 \\ (w_{2b'})^2 & \text{if} \quad p=2, \end{cases}$$

by the above lemma for $t=1$. Thus we see the equality for $q=p+1\geqq 3$.

By the Cartan formula, (i) of the above lemma and the dimensional reason that $Sq^i x=0$ for $i>\dim x$, we see easily that

$$\begin{aligned} Sq^{2b'(2q'-p')}(w_{2b'})^{2q'-p'+1} &= Sq^{2b'(2q'-p')}\{(w_{2b'})^{q'}(w_{2b'})^{q'-p'+1}\} \\ &= (w_{2b'})^{2q'}Sq^{2b'(q'-p')}(w_{2b'})^{q'-p'+1}. \end{aligned}$$

Thus, we see the equality by the induction on q. q.e.d.

Lemma 4.7. *For $q\geqq p+1\geqq 3$,*

$Sq^{2b'(q'-p'-1)}(w_{2b'})^{q'-p'+1}$

$$=\begin{cases} 0 & \text{if } p\geqq 4,\\ (w_{b'})^4(w_{2b'})^{2q'-10} & \text{if } p=3,\\ (w_{2b'})^{2q'-4}+(w_{b'})^2(w_{2b'})^{2q'-5}+\begin{cases} 0 \\ (w_{b'})^4(w_{2b'})^{2q'-6}\end{cases} & \begin{matrix}\text{if } p=2,\ q=3,\\ \text{if } p=2,\ q\geqq 4.\end{matrix}\end{cases}$$

PROOF. If $q=p+1$ or $p+2$, then the left hand side of the equality is equal to

$$Sq^{2b'p'-2b'}((w_{2b'})^{p'}(w_{2b'})) \quad \text{or} \quad Sq^{6b'p'-2b'}((w_{2b'})^{2p'}(w_{2b'})^{p'+1}),$$

respectively. Thus, we see the equality for $q=p+1$ or $p+2$ by the Cartan formula and Lemma 4.5 (i).

If $q\geqq p+2$, then we see easily that

$$Sq^{2b'(2q'-p'-1)}(w_{2b'})^{2q'-p'+1}=(w_{2b'})^{2q'}Sq^{2b'(q'-p'-1)}(w_{2b'})^{q'-p'+1}$$

by a way similar to the inductive proof of the above lemma. Thus, we see the equality by the induction on q. q.e.d.

LEMMA 4.8. *For $l\geqq 0$ and $b'=2^{b-1}\geqq 1$,*

(i) $Sq^{J(b+l;b)}w_{2b'}=\sum_{i=0}^{l}(w_{b'})^{4l'-4i'+1}(w_{2b'})^{2i'}$,

(ii) $Sq^{(b+l+1)'-b'}Sq^{J(b+l;b)}w_{2b'}=0$,

where $J(k;b)=(k',(k-1)',\dots,b')$.

PROOF. (i) The equality holds for $l=0$ by Lemma 4.5 (i).

Assume inductively the equality for l. Then

$$\begin{aligned} Sq^{J(b+l+1;b)}w_{2b'} &= Sq^{k'}Sq^{J(b+l;b)}w_{2b'} \quad (k=b+l+1)\\ &= Sq^{k'}\{\textstyle\sum_{i=0}^{l}(w_{b'})^{4l'-4i'+1}(w_{2b'})^{2i'}\}\\ &= \textstyle\sum_{i=0}^{l}\sum_{\varepsilon=0}^{2}\{Sq^{k'-2\varepsilon b'i'}(w_{b'})^{4l'-4i'+1}\}\{Sq^{2\varepsilon b'i'}(w_{2b'})^{2i'}\}\\ &= (w_{b'})^{8l'-1}w_{2b'}+\textstyle\sum_{i=0}^{l}\{Sq^{k'-4b'i'}(w_{b'})^{4l'-4i'+1}\}(w_{2b'})^{4i'}\\ &= \textstyle\sum_{i=0}^{l+1}(w_{b'})^{8l'-4i'+1}(w_{2b'})^{2i'}, \end{aligned}$$

as desired, by Lemmas 4.5 (i), 4.6 and Proposition 4.2.

(ii) The equality holds for $l=0$ by Lemma 4.5 (i). Assume $l\geqq 1$. By (i), it is sufficient to show that

$$\textstyle\sum_{i=0}^{l}Sq^{k'-b'}\{(w_{b'})^{4l'-4i'+1}(w_{2b'})^{2i'}\}=0 \qquad (k=b+l+1).$$

The left hand side is equal to

$$\begin{aligned} &\textstyle\sum_{i=0}^{l}\sum_{\varepsilon=0}^{2}\{Sq^{k'-b'-2\varepsilon b'i'}(w_{b'})^{4l'-4i'+1}\}\{Sq^{2\varepsilon b'i'}(w_{2b'})^{2i'}\}\\ &\quad=(w_{b'})^{8l'-2}w_{2b'}+\{Sq^{k'-2b'}(w_{b'})^{4l'-1}\}w_{b'}w_{2b'}+(w_{b'})^{8l'-6}(w_{b'})^2(w_{2b'})^2\\ &\qquad+\textstyle\sum_{i=0}^{l-1}\{Sq^{k'-b'-4b'i'}(w_{b'})^{4l'-4i'+1}\}(w_{2b'})^{4i'}=0, \end{aligned}$$

as desired, by Lemmas 4.5 (i), 4.6, Proposition 4.2 and Lemma 4.7. q.e.d.

Now, by using the above results and Theorem 2.4, we can prove the following lemma which implies (i) and (ii) of Theorem 4.3.

LEMMA 4.9. (i) *For* $a \geqq b \geqq 1$,

$$\theta^{a'-b'} w_{b'} = (w_{b'})^{(a-b+1)'}.$$

(ii) *If* $i = a_1' + a_2'$ *for* $a_1 > a_2 \geqq 1$ *and* $a_1 > b \geqq 1$, *then*

$$\theta^{i-2b'} w_{2b'} = \begin{cases} \sum_{j=a_2+1}^{a_1} (w_{b'})^{(i-j')/b'} (w_{2b'})^{(j-b)'} & \text{if } b \leqq a_2, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. (i) If $b=1$, then the equality is clear by (2.8). Also, the equality for $a=b$ is trivial.

Let $a > b > 1$. Then $a' - b' = a' - 1 - (b' - 1) = a' - 1 - |J(b-1)|$, where $J(b-1) = ((b-1)', (b-2)', \dots, 1)$, and Theorem 2.4 (ii) shows that

$$\theta^{a'-b'} w_{b'} = \{Sq^{J(a-1)} - (J(b-1))\} w_{b'} = \sum_J Sq^J w_{b'},$$

where $J = (j_1, \dots, j_{a-1})$ is given by

$$j_{p_s} = (a-p_s)' - (b-s)' \ (s=1, \dots, b-1), \quad j_p = (a-p)' \ (p \neq p_1, \dots, p_{b-1})$$

for $1 \leqq p_1 < \cdots < p_{b-1} \leqq a-1$. Since $Sq^j w_{b'} = 0$ for $0 < j < b'/2$ by Lemma 4.5 (i),

$$Sq^J w_{b'} = 0 \qquad \text{if} \quad (p_2, \dots, p_{b-1}) \neq (a-b+2, \dots, a-1),$$

and hence we see that

$$\theta^{a'-b'} w_{b'} = Sq^{J(a-1;b)} w_{b'} + \sum_{l=b}^{a-1} Sq^{J(a-1;l+1)} Sq^{l'-b'/2} Sq^{J(l-1;b-1)} w_{b'}.$$

This is equal to $(w_{b'})^{(a-b+1)'}$ by Lemmas 4.5 (i) and 4.8 (ii), and (i) is proved.

(ii) Let $a_2 > b \geqq 1$. Then, $i - 2b' = 2a_1' - 1 - (a_1' - a_2') - (2b' - 1)$ and

$$\theta^{i-2b'} = Sq^{(a_1-1)'} \theta^{(a_1-1)'+a_2'-2b'} + Sq^{a_1'} \theta^{a_2'-2b'}$$

by Corollary 2.5. Therefore, by the dimensional reason that $Sq^j x = 0$ if $j > \dim x$,

$$\theta^{i-2b'} w_{2b'} = Sq^{(a_1-1)'} \theta^{(a_1-1)'+a_2'-2b'} w_{2b'}.$$

By repeating this process, we see that

$$\theta^{i-2b'} w_{2b'} = Sq^{J(a_1-1;a_2)} \theta^{2a_2'-2b'} w_{2b'}.$$

By (i) and Lemma 4.4, the last is equal to

$$Sq^{J(a_1-1;a_2)}(w_{2b'})^{(a_2-b+1)'} = \{Sq^{J(a_1-a_2+b-1;b)}(w_{2b'})\}^{(a_2-b+1)'},$$

which is equal to the right hand side of the equality in (ii) by Lemma 4.8 (i).

Let $a_2=b$. Then, $i-2b'=a_1'-1-(b'-1)$ and we see that

$$\theta^{i-2b'}w_{2b'} = \{Sq^{J(a_1-1)} - (J(b-1))\}w_{2b'} = Sq^{J(a_1-1;b)}w_{2b'}$$

in the same way as in the proof of (i). Thus, we see (ii) for $a_2=b$ by Lemma 4.8 (i).

Let $b>a_2$. Then $i-2b'=a_1'+a_2'-2b'$ is not a multiple of b'. Thus $\theta^{i-2b'}w_{2b'}=0$ by Lemma 4.5 (ii) and (2.1). q.e.d.

PROOF OF (i) AND (ii) OF THEOREM 4.3. The desired results follow immediately from Proposition 3.2, the assumption (4.1), (2.8) and the above lemma. q.e.d.

To prove Theorem 4.3 (iii), we use the following two lemmas which are valid without assuming (4.1).

LEMMA 4.10. *Let $i=a_1'+\cdots+a_k'$ with $a_1>\cdots>a_k\geqq 1$ and $k\geqq 3$. If $b<a_k$, then*

$$\theta^{i-2b'}w_{2b'} = Sq^{A_2}\cdots Sq^{A_k}Sq^{a_k'(k'-1)}\theta^{(a_k+k-1)'-2b'}w_{2b'},$$

where $A_s=((a_{s-1}-1)'(s'-1), (a_{s-1}-2)'(s'-1),\ldots, (a_s+1)'(s'-1))$ and $Sq^{\phi}=1$.

PROOF. Set $i_s=a_{s+1}'+\cdots+a_k'$ for $1\leqq s\leqq k$. Then, we can prove that

$$\theta^{i-2b'}w_{2b'} = Sq^{A_2}\cdots Sq^{A_s}\theta^{\varphi_s-2b'}w_{2b'} \qquad (\varphi_s=(a_s+s)'-a_s'+i_s), \tag{4.11}$$

by the induction on s $(2\leqq s\leqq k)$ as follows.

If $a_1=a_2+1$, then (4.11) for $s=2$ is trivial. If $a_1>a_2+1$, then

$$i-2b' = a_1'+i_1-2b' = 2a_1'-1-(a_1'-i_1)-(2b'-1) \text{ with } a_1'-i_1\geqq(a_1-1)',$$

and we see in the same way as in the first part of the proof of Lemma 4.9 (ii) that

$$\theta^{i-2b'}w_{2b'} = Sq^{(a_1-1)'}\theta^{(a_1-1)'+i_1-2b'}w_{2b'}=\cdots= Sq^{J(a_1-1;a_2+1)}\theta^{2a_2'+i_1-2b'}w_{2b'},$$

by using Corollary 2.5. Thus we see (4.11) for $s=2$.

Assume inductively (4.11) for $s(<k)$. If $a_s=a_{s+1}+1$, then $\varphi_{s+1}=\varphi_s$ and (4.11) for $s+1$ is trivial. Let $a_s>a_{s+1}+1$. Then

$$\varphi_s-2b' = (a_s+s)'-a_s'+i_s-2b' = 2(a_s+s-1)'-1-(a_s'-i_s)-(2b'-1)$$

with $a_s'-i_s\geqq(a_s-1)'$, and in the same way, we see (4.11) for $t=s+1$ by

$$\begin{aligned}\theta^{\varphi_s-2b'}w_{2b'} &= Sq^{(a_s+s-1)'-(a_s-1)'}\theta^{(a_s+s-1)'-(a_s-1)'+i_s-2b'}w_{2b'}\\ &= \cdots = Sq^{A_t}\theta^{(a_t+t)'-(a_t+1)'+i_s-2b'}w_{2b'} = Sq^{A_t}\theta^{\varphi_t-2b'}w_{2b'}.\end{aligned}$$

Thus, we see (4.11). Furthermore, since $\varphi_k - 2b' = (a_k+k)' - a'_k - 2b' = 2(a_k+k-1)' - 1 - a'_k - (2b'-1)$, we see in the same way that

$$\theta^{\varphi_k - 2b'} w_{2b'} = Sq^{(a_k+k-1)' - a'_k} \theta^{(a_k+k-1)' - 2b'} w_{2b'}.$$

This equality and (4.11) for $s=k$ imply the lemma. q.e.d.

LEMMA 4.12. *For i in the above lemma and $b=a_k$,*

$$\theta^{i-2b'} w_{2b'} = Sq^{A_2} \cdots Sq^{A_l} Sq^{a'_l(l'-1)} \theta^{(a_l+l-1)'-b'} w_{2b'} \qquad (l=k-1).$$

PROOF. We see that (4.11) is also valid in the case $b=a_k$ for $2 \leqq s \leqq k-1=l$. Furthermore, since

$$\varphi_l - 2b' = (a_l+l)' - a'_l - b' = 2(a_l+l-1)' - 1 - a'_l - (b'-1),$$

we see in the same way as in the above proof that

$$\theta^{\varphi_l - 2b'} w_{2b'} = Sq^{(a_l+l-1)' - a'_l} \theta^{(a_l+l-1)' - b'} w_{2b'}.$$

Thus, we see the lemma. q.e.d.

Now, we use the assumption (4.1) in the following

LEMMA 4.13. *Let $q > p > b \geqq 1$. Then,*

$$Sq^{q'-p'} \{ \textstyle\sum_{i=1}^{q-b} (w_{b'})^{2(q-b)'-2i'+1} (w_{2b'})^{i'} \} = 0.$$

PROOF. Put

$$x_i = (w_{b'})^{2(q-b)'-2i'+1} \qquad \text{for} \quad 1 \leqq i \leqq q-b.$$

Then, by Lemma 4.5 (i) and the Cartan formula, we see that

$$(4.14) \qquad \begin{aligned} Sq^{q'-p'} \{ \textstyle\sum_{i=1}^{q-b} x_i (w_{2b'})^{i'} \} &= \textstyle\sum_1 + \sum_2 + \sum_3, \\ \textstyle\sum_1 &= \textstyle\sum_{i=1}^{q-b} (Sq^{q'-p'} x_i)(w_{2b'})^{i'}, \\ \textstyle\sum_2 &= \textstyle\sum_{i=1}^{q-b} (Sq^{q'-p'-b'i'} x_i)(w_{b'} w_{2b'})^{i'}, \\ \textstyle\sum_3 &= \textstyle\sum_{i=1}^{q-b} (Sq^{q'-p'-2b'i'} x_i)(w_{2b'})^{2i'}. \end{aligned}$$

Thus we can prove the lemma by showing

$$(4.15) \qquad \textstyle\sum_1 = \sum_{i=1}^{p-b} (w_{b'})^{4(q-b)'-2(p-b)'-2i'+1} (w_{2b'})^{i'},$$

$$(4.16) \qquad \begin{aligned} \textstyle\sum_2 = {} & (w_{b'})^{4(q-b)'-6(p-b)'+1} (w_{2b'})^{2(p-b)'} \\ & + (w_{b'})^{4(q-b)'-2(p-b)'-1} w_{2b'}, \end{aligned}$$

$$(4.17) \qquad \textstyle\sum_3 = \sum_{i=1}^{p-b} (w_{b'})^{4(q-b)'-2(p-b)'-4i'+1} (w_{2b'})^{2i'}.$$

Proof of (4.15). If $i \geqq p-b+1$, then $\dim x_i \leqq q'-2p'+b' < q'-p'$ and so $Sq^{q'-p'}x_i=0$. For $1 \leqq i \leqq p-b$, by the Cartan formula, Lemma 4.5 (i) and Proposition 4.2, we see that

$$\begin{aligned}(Sq^{q'-p'}x_i)(w_{2b'})^{i'} &= \{Sq^{q'-p'}((w_{b'})^{(q-b)'}(w_{b'})^{(q-b)'-2i'+1})\}(w_{2b'})^{i'}\\ &= (w_{b'})^{2(q-b)'}\{Sq^{(q-1)'-p'}(w_{b'})^{(q-b)'-2i'+1}\}(w_{2b'})^{i'}\\ &= \cdots = (w_{b'})^{2(q-b)'+\cdots+2(p-b)'-2i'+1}(w_{2b'})^{i'}\\ &= (w_{b'})^{4(q-b)'-2(p-b)'-2i'+1}(w_{2b'})^{i'}.\end{aligned}$$

Thus, we see (4.15).

Proof of (4.16). In the same way as in the above proof, we see (4.16) by the following

$$\begin{aligned}&(Sq^{q'-p'-b'i'}x_i)(w_{b'}w_{2b'})^{i'}\\ &\quad= (w_{b'})^{2(q-b)'}\{Sq^{(q-1)'-p'-b'i'}(w_{b'})^{(q-b)'-2i'+1}\}(w_{b'}w_{2b'})^{i'}\\ &\quad= \cdots = (w_{b'})^{4(q-b)'-8(p-b)'}\{Sq^{p'-b'i'}(w_{b'})^{4(p-b)'-2i'+1}\}(w_{b'}w_{2b'})^{i'};\end{aligned}$$

$$\begin{aligned}&\{Sq^{p'-b'i'}(w_{b'})^{4(p-b)'-2i'+1}\}(w_{b'}w_{2b'})^{i'}\\ &\qquad= \begin{cases}(w_{b'})^{2(p-b)'+1}(w_{2b'})^{2(p-b)'} & (\text{if } i=p-b+1)\\ (w_{b'})^{6(p-b)'-1}w_{2b'} & (\text{if } i=1)\\ 0 & (\text{otherwise}).\end{cases}\end{aligned}$$

Proof of (4.17). Let $i \leqq p-b-1$. Then, in a way similar to the above proof, we see that

$$\begin{aligned}&(Sq^{q'-p'-2b'i'}x_i)(w_{2b'})^{2i'}\\ &\quad= (w_{b'})^{2(q-b)'}\{Sq^{(q-1)'-p'-2b'i'}(w_{b'})^{(q-b)'-2i'+1}\}(w_{2b'})^{2i'}\\ &\quad= \cdots = (w_{b'})^{4(q-b)'-8(p-b)'}|\{Sq^{p'-2b'i'}(w_{b'})^{4(p-b)'-2i'+1}\}(w_{2b'})^{2i'}\\ &\quad= (w_{b'})^{4(q-b)'-6(p-b)'}\{Sq^{p'-2b'i'}(w_{b'})^{2(p-b)'-2i'+1}\}(w_{2b'})^{2i'}.\end{aligned}$$

Here, by using Lemma 4.6, we see that

$$\{Sq^{p'-2b'i'}(w_{b'})^{2(p-b)'-2i'+1}\}(w_{2b'})^{2i'} = (w_{b'})^{4(p-b)'-4i'+1}(w_{2b'})^{2i'}.$$

Let $i=p-b$. Then in the same way as above, we see that

$$(Sq^{q'-p'-2b'i'}x_i)(w_{2b'})^{2i'} = (w_{b'})^{4(q-b)'-6(p-b)'+1}(w_{2b'})^{2(p-b)'}.$$

Let $p-b < i \leqq q-b$. Then, in the same way,

$$\begin{aligned}&(Sq^{q'-p'-2b'i'}x_i)(w_{2b'})^{2i'}\\ &\quad= (w_{b'})^{2(q-b)'}\{Sq^{(q-1)'-p'-2b'i'}(w_{b'})^{(q-b)'-2i'+1}\}(w_{2b'})^{2i'}\\ &\quad= \cdots = (w_{b'})^{4(q-b)'-6i'}(Sq^{2b'i'-p'}w_{b'})(w_{2b'})^{2i'} = 0,\end{aligned}$$

because $Sq^{2b'i'-p'}w_{b'}=0$ by the dimensional reason. Thus, we see (4.17); and the proof of the lemma is complete. q.e.d.

LEMMA 4.18. *Let $i=a_1'+\cdots+a_k'$ with $a_1>\cdots>a_k\geqq 1$ and $k\geqq 3$. Then*

$$\theta^{i-b'}w_{b'} = 0 \qquad \text{for} \quad 1\leqq b\leqq a_1.$$

PROOF. If $b=1$, then the equality holds by (2.8) and the assumption.

Let $1\leqq b<a_k$. Then, by Lemmas 4.10 and 4.9 (i), we see that

$$\theta^{i-2b'}w_{2b'} = Sq^{A_2}\cdots Sq^{A_k}Sq^{\alpha}(w_{2b'})^{\beta},$$

where $\alpha=(a_k+k-1)'-a_k'$ and $\beta=(a_k+k-b-1)'$. Since $k\geqq 3$ by the assumption, α is not a multiple of $\beta b'=(a_k+k-2)'$. Therefore $Sq^{\alpha}(w_{2b'})^{\beta}=0$ by Lemmas 4.4 and 4.5 (i). Thus $\theta^{i-2b'}w_{2b'}=0$.

Let $b=a_k$. Then, by Lemma 4.12,

$$\theta^{i-2b'}w_{2b'} = Sq^{A_2}\cdots Sq^{A_{k-1}}Sq^{q'-p'}\theta^{q'-b'}w_{2b'} \qquad (q=a_{k-1}+k-2,\ p=a_{k-1}).$$

Furthermore, by Lemmas 4.9 (ii) and 4.13, we see that

$$Sq^{q'-p'}\theta^{q'-b'}w_{2b'} = Sq^{q'-p'}\{\textstyle\sum_{j=b+1}^{q}(w_{b'})^{(q'+b'-j')/b'}(w_{2b'})^{(j-b)'}\} = 0.$$

Thus $\theta^{i-2b'}w_{2b'}=0$.

Let $a_k<b$. Then, $i-2b'$ is not a multiple of b' by the assumption, and we see $\theta^{i-2b'}w_{2b'}=0$ by Lemma 4.5 (ii) and (2.1). q.e.d.

PROOF OF (iii) OF THEOREM 4.3. The desired result follows immediately from Proposition 3.2, the assumption (4.1) and the above lemma. q.e.d.

Thus, we have proved Theorem 4.3 completely. In the rest of this section, we consider some examples of closed manifolds which satisfy (4.1).

EXAMPLE 4.19. *Let RP^n be the real projective n-space. Then*

$$wRP^n = 1 + u^{b'} + u^{a'} \qquad \text{if} \quad n=a'+b'-1 \quad \text{with} \quad a>b\geqq 1,$$

where $u\in H^1(RP^n; Z_2)=Z_2$ is the generator.

PROOF. We see the desired result by the fact that

$$wRP^n = (1+u)^{n+1}$$

([6, Th. 4.5]) and (2.12). q.e.d.

For a (differentiable real) k-plane bundle $\zeta\to V$ over a closed d-manifold V, we denote by

$$p\colon RP(\zeta) \longrightarrow V$$

the associated projective space bundle with fiber RP^{k-1}. Then, $RP(\zeta)$ is a closed $(d+k-1)$-manifold.

Let ξ_n be the canonical line bundle over RP^n, and $m\xi_n$ be the m-fold Whitney sum of ξ_n. Consider the natural projection

$$p_i\colon RP^n \times RP^n \longrightarrow RP^n \qquad (i=1, 2)$$

of the product manifold $RP^n \times RP^n$ onto the ith factor, the induced bundle $p_i^* m\xi_n$ of $m\xi_n$ by p_i, and the Whitney sum

$$\xi(n, m) = p_1^* m\xi_n \oplus p_2^* m\xi_n,$$

which is a $2m$-plane bundle over $RP^n \times RP^n$. Then, we have the associated projective space bundle

$$p\colon RP(\xi(n, m)) \longrightarrow RP^n \times RP^n \quad \text{with fiber} \quad RP^{2m-1}.$$

EXAMPLE 4.20. *If*

$$n=a'+b'-1 \quad \textit{and} \quad m=a' \quad \textit{with} \quad a>b\geqq 1,$$

then the total Stiefel-Whitney class of the $(2n+2m-1(=4a'+2b'-3))$*-manifold* $RP(\xi(n, m))$ *is given by*

$$wRP(\xi(n, m)) = 1 + p^*\{(u_1^{b'}+u_2^{b'}) + (u_1u_2)^{b'} + (u_1u_2)^{a'}\},$$

where $u_i=p_i^*u \in H^1(RP^n \times RP^n; Z_2)$ *and* $u \in H^1(RP^n; Z_2)$ *is the generator.*

PROOF. For the projective space bundle $p\colon RP(\zeta)\to V$ of a k-plane bundle ζ over a closed manifold V, it is proved in [1, (23.3)] that

(4.21) $H^*(RP(\zeta); Z_2)$ *is the free* $H^*(V; Z_2)$*-module with basis* $1, c,\dots, c^{k-1}$, *with the relation*

$$c^k = \textstyle\sum_{i=1}^k p^*(w_i\zeta)c^{k-i},$$

where c *is the first Stiefel-Whitney class of the canonical line bundle over* $RP(\zeta)$ *and* $w_i\zeta$ *is the ith Stiefel-Whitney class of* ζ. *Furthermore, the total Stiefel-Whitney class of* $RP(\zeta)$ *is given by*

$$wRP(\zeta) = p^*(wV)\textstyle\sum_{i=0}^k p^*(w_i\zeta)(1+c)^{k-i}.$$

Consider the case that ζ is the $2m$-plane bundle $\xi(n, m)$ over the $2n$-manifold $RP^n \times RP^n$ in the example. Then,

$$w\xi(n, m) = \{p_1^*(1+u)^m\}\{p_2^*(1+u)^m\} = (1 + u_1^{a'})(1 + u_2^{a'})$$

$(m=a')$, and the first equality in (4.21) is

$$c^{2a'} = \{p^*(u_1^{a'} + u_2^{a'})\}c^{a'} + p^*(u_1u_2)^{a'}.$$

Therefore

$$\begin{aligned}&\textstyle\sum_{i=0}^{2m} p^*(w_i\xi(n, m))(1+c)^{2m-i}\\ &\quad= (1+c)^{2a'} + \{p^*(u_1^{a'} + u_2^{a'})\}(1+c)^{a'} + p^*(u_1u_2)^{a'} = 1 + p^*(u_1^{a'} + u_2^{a'}).\end{aligned}$$

Thus, by the last equality in (4.21) and Example 4.19, we see that

$$\begin{aligned}wRP(\xi(n, m)) &= p^*(w(RP^n\times RP^n)) \textstyle\sum_{i=0}^{2m} p^*(w_i\xi(n, m))(1+c)^{2m-i}\\ &= p^*\{(1 + u_1^{b'} + u_1^{a'})(1 + u_2^{b'} + u_2^{a'})(1 + u_1^{a'} + u_2^{a'})\}\\ &= 1 + p^*\{(u_1^{b'} + u_2^{b'}) + (u_1u_2)^{b'} + (u_1u_2)^{a'}\},\end{aligned}$$

as desired. q.e.d.

Similarly, we have the following

EXAMPLE 4.22. *If $n=b'-1$ and $m=a'$ with $b>a\geqq 1$, then*

$$wRP(\xi(n, m)) = 1 + p^*(u_1^{a'} + u_2^{a'}).$$

REMARK 4.23. In Proposition 4.2, the assumption is necessary. In fact,

$$(w_{b'})^2 \neq 0, \qquad w_{b'}w_{2b'} \neq 0$$

in Example 4.20, where $w_{b'}=p^*(u_1^{b'}+u_2^{b'})$ and $w_{2b'}=p^*(u_1u_2)^{b'}$.

Finally, in connection with the condition (4.1), we notice the following

REMARK 4.24. *Let M be a closed manifold.*

(i) *If $w_{b'}=0$ for some $b\geqq 1$, then $w_i=0$ for $b'\leqq i<2b'$.*

(ii) *If $wM=1+w_1+w_i$ $(i>1)$ or $wM=1+w_i$ $(i\geqq 1)$, and i is not a power of 2 in addition, then $w_i=0$.*

In fact, we can show (i) by using the equality

$$Sq^{i-b'}w_{b'} = w_i + \textstyle\sum_{j=b'}^{i-1}\binom{b'+j-i-1}{j-b'}w_{i-j}w_j \qquad (b'<i<2b')$$

of (3.7) and by the induction on i. (ii) is an immediate consequence of (i).

§5. Unoriented bordism classes of certain manifolds

The purpose in this section is to prove the following

THEOREM 5.1. *Assume that a closed manifold M satisfies* (4.1), *i.e., the* *total Stiefel-Whitney class wM is given by*

$$(5.2)\qquad wM = 1 + \textstyle\sum_{b\geqq 1} w_{b'}, \quad w_{b'} \in H^{b'}(M; Z_2) \qquad (b'=2^{b-1}),$$

and let

$$(5.3)\qquad \dim M = p_1'+\cdots+p_k' \quad \text{with} \quad p_1>\cdots>p_k\geqq 1 \qquad (p'=2^{p-1}).$$

(i) *If* $k\geqq 4$ *in* (5.3) *and*

$$(w_{b'})^{\dim M/b'} = 0 \qquad \text{for} \quad 2\leqq b\leqq p_k,$$

then the unoriented bordism class $[M]$ *of* M *is* 0.

(ii) *If* $\dim M$ *is odd and* $k\geqq 3$ *in* (5.3), *then* $[M]=0$.

THEOREM 5.4. (i) *If wM is given by*

$$wM = 1 + w_{b'} + w_{c'} \qquad \text{for some} \quad c>b\geqq 1,$$

and $k\geqq 2$ *in* (5.3), *then* $[M]=0$.

(ii) *If* $wM=1+w_1+w_i$ *where* $i>1$ *is not a power of* 2, *then* $w_i=0$ *and* $[M]=0$.

(iii) *If* $wM=1+w_i$ *for some* $i\geqq 1$, *then* $[M]=0$.

To prove these theorems, we study the Stiefel-Whitney numbers of M, which is assumed throughout this section to satisfy (5.2) and $k\geqq 2$ in (5.3), as follows.

By the assumption $k\geqq 2$ in (5.3), we put

$$(5.3)'\qquad \dim M = p'+q'+m \quad \text{with} \quad p>q \quad \text{and} \quad q'>m\geqq 0,$$

and consider the following cohomology classes in $H^*(M; Z_2)$:

$$(5.5)\qquad \begin{aligned} A_t(b) &= \textstyle\sum_{j=t+1}^{p-1} (w_{b'})^{((p-1)'+t'-j')/b'}(w_{2b'})^{(j-b)'} && (q\leqq t\leqq p-2,\ 1\leqq b\leqq t),\\ B(b) &= (w_{b'})^{(p-b+1)'} && (1\leqq b\leqq p),\\ B_s(b) &= \textstyle\sum_{j=s+1}^{p} (w_{b'})^{(p'+s'-j')/b'}(w_{2b'})^{(j-b)'} && (1\leqq s\leqq q,\ 1\leqq b\leqq s). \end{aligned}$$

Then, we have the following

LEMMA 5.6. $\sum_{b=1}^{t} A_t(b) = 0 \qquad (q\leqq t\leqq p-2)$,

$$\textstyle\sum_{b=1}^{p} B(b) = 0, \quad \sum_{b=1}^{s} B_s(b) = 0 \qquad (1\leqq s\leqq q).$$

PROOF. By Theorem 4.3 (i)–(ii), the ith Wu class v_i is equal to

$$\textstyle\sum_{b=1}^{t} A_t(b) \quad \text{if} \quad i=(p-1)'+t', \qquad \sum_{b=1}^{p} B(b) \quad \text{if} \quad i=p',$$

and $\sum_{b=1}^{s} B_s(b)$ if $i=p'+s'$, respectively. On the other hand, $v_i=0$ if $2i>\dim M$ by the definition of the Wu classes. Thus we see the lemma. q.e.d.

LEMMA 5.7. *For any b with $2 \leqq b \leqq q$,*

$$\tilde{A}_t(b) \equiv w_{b'}w_{2b'}A_t(b) = 0 \qquad (q \leqq t \leqq p-2),$$
$$\tilde{B}(b) \equiv w_{b'}w_{2b'}B(b) + w_{b'}w_{2b'}B(b+1) = 0,$$
$$\tilde{B}_s(b) \equiv w_{b'}w_{2b'}B_s(b) = 0 \qquad (b \leqq s \leqq q).$$

PROOF. Multiply the equalities in Lemma 5.6 by $w_{b'}w_{2b'}$. Then, we see the lemma by Proposition 4.2. q.e.d.

LEMMA 5.8. $\tilde{A}_t \equiv w_1 A_t(1) = 0 \qquad (q \leqq t \leqq p-2),$

$$\tilde{B} \equiv w_1 B(1) + w_1 B(2) = 0,$$
$$\tilde{B}_s \equiv w_1 B_s(1) = 0 \qquad (1 \leqq s \leqq q).$$

PROOF. In the same way, by multiplying the equalities in Lemma 5.6 by w_1, we see the lemma. q.e.d.

LEMMA 5.9. *For any b with $2 \leqq b \leqq q$, the equality*

$$(w_{b'})^{\alpha}(w_{2b'})^{\beta} = 0$$

holds for α and β given as follows:

(1) $\alpha = 1 + (p'+q')/b', \qquad \beta = 1.$

(2) $\alpha = 1 + (p'-q'+s')/b', \quad \beta = 2 + (q'-s')/b' \qquad (b < s \leqq q).$

(3) $\alpha = 1 + p'/2b', \qquad \beta = 1 + q'/b'.$

(4) $\alpha = 1 + t'/b', \qquad \beta = 1 + (p'-2t')/2b' \qquad (q \leqq t \leqq p-2).$

(5) $\alpha = 1, \qquad \beta = 2 + p'/2b'.$

PROOF. (3) By Lemma 5.7 and (5.5), we see that

$$\begin{aligned}(w_{b'})^{\alpha}(w_{2b'})^{\beta} &= (w_{b'})^{1+p'/2b'}(w_{2b'})^{1+q'/b'} + \tilde{A}_{q+1}(b) \\ &= (w_{b'})^{q'/b'}\tilde{A}_q(b) = 0 \qquad \text{if } p \geqq q+2;\end{aligned}$$
$$(w_{b'})^{\alpha}(w_{2b'})^{\beta} = (w_{b'}w_{2b'})^{1+p'/2b'} = \tilde{B}_q(b) = 0 \qquad \text{if } p = q+1.$$

(4)
$$\begin{aligned}(w_{b'})^{\alpha}(w_{2b'})^{\beta} &= (w_{b'})^{1+t'/b'}(w_{2b'})^{1+(p'-2t')/2b'} \\ &\quad + \tilde{A}_{p-2}(b)\textstyle\sum_{j=t+1}^{p-2}(w_{b'})^{((p-2)'+t'-j')/b'}(w_{2b'})^{(j'-2t')/2b'} \\ &= (w_{2b'})^{((p-2)'-t')/b'}\tilde{A}_t(b) = 0.\end{aligned}$$

(2) with $s=q$: $(w_{b'})^{1+p'/b'}(w_{2b'})^2$ is equal to

$$(w_{b'})^{1+p'/b'}(w_{2b'})^2 + \tilde{B}_{b+1}(b) = w_{b'}\tilde{B}_b(b) = 0.$$

(5) By the above result, we see that

$$w_{b'}(w_{2b'})^{2+p'/2b'} = w_{2b'}\tilde{B}(b) = 0.$$

(1) By (3) if $p=q+1$ and by (4) with $t=q$ if $p\geqq q+2$, we see that $(w_{b'})^{1+q'/b'}(w_{2b'})^{1+p'/2b'}=0$. Hence

$$(w_{b'})^{1+(p'+q')/b'}w_{2b'} = (w_{b'})^{q'/b'}\tilde{B}(b) = 0.$$

(2) with $b<s<q$: In the equality

$$\tilde{B}_s(b) = \sum_{j=s+1}^{p}(w_{b'})^{1+(p'+s'-j')/b'}(w_{2b'})^{1+j'/2b'},$$

$\sum_{j=q+1}^{p-1}$ is equal to

$$(w_{b'})^{(p'+2s'-2q')/2b'}\tilde{A}_q(b) = 0,$$

and the term for $j=p$ multiplied by $w_{2b'}$ is equal to $(w_{b'})^{1+s'/b'}(w_{2b'})^{2+p'/2b'}$, which is 0 by (5). Therefore

$$\sum_{j=s+1}^{q}(w_{b'})^{1+(p'+s'-j')/b'}(w_{2b'})^{2+j'/2b'} = w_{2b'}\tilde{B}_s(b) = 0.$$

By taking $s=q-1$ especially, we see that

$$(w_{b'})^{1+(2p'-q')/2b'}(w_{2b'})^{2+q'/2b'} = 0.$$

Thus, the desired equality is shown as follows:

$$\begin{aligned}&(w_{b'})^{1+(p'-q'+s')/b'}(w_{2b'})^{2+(q'-s')/b'}\\ &= \sum_{j=s+1}^{q}(w_{b'})^{1+(p'+s'-j')/b'}(w_{2b'})^{2+(j'+q'-2s')/2b'}\\ &= (w_{2b'})^{1+(q'-2s')/2b'}\tilde{B}_s(b) = 0.\end{aligned}$$

These complete the proof of Lemma 5.9. q.e.d.

We notice that the relations in Lemma 5.8 are obtained from those in Lemma 5.7 for $b=1$ by replacing $(w_1)^\alpha(w_2)^\beta$ by $(w_1)^\alpha(w_2)^{\beta-1}$. Thus, for $b=1$, Lemma 5.9 turns out the following

LEMMA 5.10. *The equality*

$$(w_1)^\alpha(w_2)^{\beta-1} = 0$$

holds for α and β which are given by the equalities obtained from (1)–(5) *of Lemma* 5.9 *by setting $b=1$.*

To study the Stiefel-Whitney numbers of M, we consider cohomology classes

$$(w_{b'})^{k(b,l)}(w_{2b'})^l \in H^{\dim M}(M; Z_2) \qquad (b\geqq 1,\ l\geqq 0),$$

where the integer $k(b,l)$ is given by

(5.11) $\quad k(b,l)b'+2lb' = \dim M = p'+q'+m \qquad (p'>q'>m\geqq 0).$

LEMMA 5.12. *If* $(w_{b'})^{\alpha}(w_{2b'})^{\beta}=0$ *for some* α *and* β, *then*

$$(w_{b'})^{k(b,l)}(w_{2b'})^{l} = 0 \qquad \textit{for} \quad \beta \leqq l \leqq n(\alpha) = (\dim M - \alpha b')/2b'.$$

PROOF. The lemma is clear, since $k(b, l) \geqq \alpha$ for the above l by (5.11).

q.e.d.

By using Lemmas 5.9 and 5.12, we see the following

LEMMA 5.13. *In* (5.11), *assume that*

$$(5.14) \qquad m = ar' \quad \textit{for} \quad r \geqq 1 \quad \textit{and an odd integer} \quad a \geqq 3.$$

Then, for any b *with* $2 \leqq b \leqq r$,

$$(w_{b'})^{k(b,l)}(w_{2b'})^{l} = 0 \quad \textit{if} \quad 1 \leqq l \leqq (\dim M - b')/2b'.$$

PROOF. For α and β given in Lemma 5.9, we see easily that β and $n(\alpha)$ in the above lemma are given as follows, where $n_0=(m-b')/2b'$:

(1) $\beta = 1,$ $\quad n(\alpha) = n_0.$

(2) $\beta = 2 + (q'-s')/b',$ $\quad n(\alpha) = n_0 + (q'-(s-1)')/b' \quad (b < s \leqq q).$

(3) $\beta = 1 + q'/b',$ $\quad n(\alpha) = n_0 + ((p-1)'+q')/2b'.$

(4) $\beta = 1 + (p'-2t')/2b',$ $\quad n(\alpha) = n_0 + (p'+q'-t')/2b' \quad (q \leqq t \leqq p-2).$

(5) $\beta = 2 + p'/2b',$ $\quad n(\alpha) = (\dim M - b')/2b'.$

Thus, for these β and $n(\alpha)$,

$$(5.15) \qquad (w_{b'})^{k(b,l)}(w_{2b'})^{l} = 0 \qquad (\beta \leqq l \leqq n(\alpha)).$$

Here, we notice that $n_0=(ar'-b')/2b' \geqq 1$ by the assumptions (5.14) and $b \leqq r$. Therefore, we see immediately that $n(\alpha)$ in (1) (resp. (2) for $s=u \geqq b+2$, (2) for $s=b+1$, (3), (4) for $t=v>q$ or (4) for $t=q$) is not smaller than $\beta-1$ of β in (2) for $s=q$ (resp. (2) for $s=u-1$, (3), (4) for $t=p-2$, (4) for $t=v-1$ or (5)). Thus, we have the lemma by (5.15). q.e.d.

LEMMA 5.16. *In* (5.11), *assume that*

$$(5.17) \qquad m = ar' \quad \textit{for} \quad r \geqq 1 \quad \textit{and an odd integer} \quad a \geqq 1.$$

Then

$$(w_1)^{k(1,l)}(w_2)^{l} = 0 \quad \textit{if} \quad 0 \leqq l \leqq (\dim M - 1)/2.$$

PROOF. By using Lemma 5.10 instead of Lemma 5.9, we see the lemma in the same way as in the above proof, since we have

$$(w_1)^{k(1,l)}(w_2)^{l} = 0 \qquad (\beta - 1 \leqq l \leqq n(\alpha)),$$

instead of (5.15), for β and $n(\alpha)$ obtained from the above (1)−(5) by setting $b=1$, where $n_0=(ar'-1)/2\geqq 0$. q.e.d.

Now, we are ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. (i) By the assumption that $k\geqq 4$ in (5.3), we see (5.14) where $r=p_k$. Therefore, by the above two lemmas and Proposition 4.2, we see immediately that all the Stiefel-Whitney numbers of M are 0 except for

$$\langle (w_{b'})^{k(b,0)}, \mu\rangle \qquad (2\leqq b\leqq r=p_k).$$

Thus the desired result is an immediate consequence of the theorem of R. Thom (cf. [8, p. 95, Th.]) that

(5.18) $[M]=0$ *if all the Stiefel-Whitney numbers of M are* 0.

(ii) By the assumption that $\dim M$ is odd and $k\geqq 3$, we see (5.17) with $r=1$. Thus we see that all the Stiefel-Whitney numbers of M are 0 by the above lemma and Proposition 4.2, and that $[M]=0$ by (5.18). q.e.d.

To prove Theorem 5.4, we notice the following

LEMMA 5.19. *Assume that*

$$(*) \qquad wM = 1 + w_{b'} + w_{2b'} \qquad \textit{for some} \quad b\geqq 1,$$

and let $k(b, l)$ be the integer given by (5.11). *Then*

$$(w_{b'})^{k(b,l)}(w_{2b'})^l = 0 \qquad \textit{for} \quad 0\leqq l\leqq \dim M/2b'.$$

PROOF. By the assumption (*), Lemma 5.7 for b in (*) holds without multiplying $w_{b'}w_{2b'}$. Thus, we see by the same proof as in Lemma 5.9 that

$$(w_{b'})^{\alpha-1}(w_{2b'})^{\beta-1} = 0$$

for α and β given by (1)–(5) in Lemma 5.9, and hence we have

$$(w_{b'})^{k(b,l)}(w_{2b'})^l = 0 \qquad (\beta-1\leqq l\leqq n(\alpha-1))$$

instead of (5.15) by Lemma 5.12. Here, $n(\alpha-1)=n(\alpha)+1/2$ and so $n(\alpha-1)$ is given by the equalities obtained from those of $n(\alpha)$ in (1)–(5) in the proof of Lemma 5.13 by replacing n_0 with $n_0+1/2=m/2b'\geqq 0$ and $(\dim M-b')/2b'$ with $\dim M/2b'$. Therefore, we have the lemma in the same way as in the proof of Lemma 5.13. q.e.d.

PROOF OF THEOREM 5.4. (i) Let $c=b+1$. Then, the desired result follows immediately from the above lemma and (5.18).

Let $c>b+1$. Then, by the second equality in Lemma 5.6,

$$(w_{b'})^{(p-b+1)'} + (w_{c'})^{(p-c+1)'} = 0.$$

By Proposition 4.2, this equality implies that

$$(w_{b'})^{(p-b+1)'+1} = 0 \quad \text{and} \quad (w_{c'})^{(p-c+1)'+1} = 0.$$

Hence $(w_{b'})^{k(b,0)}=0=(w_{c'})^{k(c,0)}$ and all the Stiefel-Whitney numbers of M are 0. Thus, the desired result for $c>b+1$ follows immediately from (5.18).

(ii), (iii) By Remark 4.24(ii), it is sufficient to show that

(*) *if* $wM = 1 + w_{b'}$ *for some* $b \geqq 1$, *then* $[M] = 0$.

If $k \geqq 2$ in (5.3), then (*) is a special case of (i).

Let $k=1$ in (5.3), i.e., $\dim M=p'$ for some $p \geqq 1$. Then, by the assumption of (*), Theorem 4.3(i) and the dimensional reason, we see that

$$(w_{b'})^{(p-b+1)'} = v_{p'} = 0.$$

Thus $[M]=0$ by (5.18). q.e.d.

EXAMPLE 5.20. *The unoriented bordism classes of the* $(4a'+2b'-3)$-*manifold* $RP(\xi(a'+b'-1, a'))$ *given in Example* 4.20 *and the* $(2b'+2a'-3)$-*manifold* $RP(\xi(b'-1, a'))$ *given in Example* 4.22 *are all* 0.

Finally, we notice that Theorem 5.4(i) does not hold if $k=1$ in (5.3) (i.e., dim M is a power of 2), as is seen by the following two examples.

EXAMPLE 5.21. *Consider the closed* $(2n+2(=2^t))$-*manifold* $RP(n, n, 0)$ $=RP(p_1^*\xi_n \oplus p_2^*\xi_n \oplus p_3^*\xi_0)$ $(n=t'-1,\ t=2, 3, 4)$, *given in* [9, Lemma 3.4], *where* p_i *is the projection of* $RP^n \times RP^n \times RP^0$ *onto the ith factor and* ξ_i *is the canonical line bundle over* RP^i. *Then,*

$$[RP(n, n, 0)] \neq 0, \quad w_i RP(n, n, 0) = 0 \qquad \text{for} \quad i \geqq 3.$$

PROOF. The first assertion is valid, because $[RP(n, n, 0)]$ is indecomposable by [9, Lemma 3.4]. The second assertion is shown by using [11, Lemma 2.9] and [6, p. 39, Prop. 4]. q.e.d.

EXAMPLE 5.22. *For* $RP^{p'}$ *with* $p>1$, *it holds that*

$$[RP^{p'}] \neq 0 \quad \text{and} \quad wRP^{p'} = 1 + w_1 + w_{p'}.$$

PROOF. This is clear by Example 4.19 and $(w_1)^{p'} \neq 0$. q.e.d.

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