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Nonlinear differential systems with monotone solutions

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Consider the vector equation

$$\mathbf{x}' = -\mathbf{f}(t, \mathbf{x}) \tag{1}$$

where $\mathbf{f}$: $[a, \infty) \times R^n \to R^n$ is continuous and sufficiently regular so that

(i) solutions of (1) vary continuously with initial data assigned at $t = a$,

and

(ii) solutions of (1) can be continued until some components become unbounded.

We seek to show the existence of a "monotone solution" $\mathbf{x}(t)$ of (1) whose components satisfy $x_i(t) > 0$ and $x_i'(t) < 0$ in $[a, \infty)$ for $1 \leq i \leq n$.

The existence of such monotone solutions was first established by Hartman and Wintner [5] under explicit bounds on $\mathbf{f}(t, \mathbf{x})$ which assured that all solutions of (1) can be continued to $t = \infty$. This requirement was removed in [6] for the case where n is even by means of a corollary of Sperner's lemma and for general n by means of Ważewski retracts in [2] (see also [3] and [4], Chapter XIV, Problems 2.8 and 2.9, for alternate techniques). The purpose of this note is to show how a different form of Sperner's lemma leads to the more general results obtained by the theory of retracts and also to note the nonlinear form of criteria which assure that the "monotone solution" of the scalar equation $y^{(n)} - (-1)^n \cdot f(t, y) = 0$ tends to zero.

Writing $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) in case all components of a vector $\mathbf{x}$ satisfy $x_i \geq 0$ ($x_i > 0$), we formulate the following hypotheses

(A) $\mathbf{f}(t, \mathbf{x}) > \mathbf{0}$ whenever $\mathbf{x} > \mathbf{0}$ and $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$;

(B) $f_i(t, \mathbf{x}) > 0$ whenever $\mathbf{x} \geq \mathbf{0}$ and $x_i = 0$ and some $x_j > 0$, for $1 \leq i \leq n$ and $j \neq i$.

These hypotheses are essentially satisfied when a scalar n-th order equation

$$y^{(n)} - (-1)^n f(t, y) = 0 \tag{2}$$

satisfying

(C) $f(t, y) > 0$ whenever $y > 0$ and $f(t, 0) \equiv 0$,

is represented as a first order system (1) by the transformation $x_i = (-1)^{i-1} y^{(i-1)}$, $1 \leq i \leq n$.

Our basic result is the following:

THEOREM. *If* $\mathbf{f}(t, \mathbf{x})$ *satisfies* (A) *and* (B), *then* (1) *has a monotone solution* $\mathbf{x}(t)$ *satisfying* $\mathbf{x}(t)>\mathbf{0}$ *and* $\mathbf{x}'(t)<\mathbf{0}$ *in* $[a, \infty)$.

The proof will be based on the following corollary of Sperner's Lemma [1; p. 377].

LEMMA. *Let* T^n *be an* $n-1$ *dimensional simplex with vertices* $\mathbf{e}_1,\dots, \mathbf{e}_n$. *If there exists a closed covering* $\{A_1,\dots, A_n\}$ *of* T^n *with the property that for every subsequence* $\{i_1, i_2,\dots, i_r\}$ *of* $\{1, 2,\dots, n\}$ *with* $1\le r<n$

$$\overline{\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_r}} \subset A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_r}, \tag{2}$$

then $\bigcap_{i=1}^n A_i$ *is not empty.*

PROOF OF THEOREM. Consider the $n-1$ dimensional simplex T^n defined in the positive n-tant of R^n by $x_1+\cdots+x_n=1$. Calling the vertices of T^n $\mathbf{e}_1,\dots, \mathbf{e}_n$ (where $\mathbf{e}_i$ is the vector $\mathbf{x}$ with $x_j=\delta_{ji}$) we denote by $\mathbf{x}(t; \mathbf{x}_0)$ the solution of (1) satisfying $\mathbf{x}(a)=\mathbf{x}_0$ for $\mathbf{x}_0\in T^n$. It follows from (A) that solutions $\mathbf{x}(t; \mathbf{x}_0)$ of (1) will remain in the n dimensional simplex T^{n+1} (with vertices $\mathbf{0}, \mathbf{e}_1,\dots, \mathbf{e}_n$) unless they exist across one of the closed faces corresponding to $x_i=0$ for some $i\in\{1,\dots, n\}$. Identifying $i=0$ with $i=n$, we define for $1\le i\le n$

$$A_i = \{\mathbf{x}_0\in T^n \mid \mathbf{x}(t; \mathbf{x}_0) \text{ does not exit } T^{n+1} \text{ with } x_{i-1} > 0\}$$

and note that each A_i is closed. Furthermore if $\mathbf{x}(a; \mathbf{x}_0)=\mathbf{e}_i$ for some i, then $x_{i-1}(a; \mathbf{e}_i)=0$ while $x'_{i-1}(a; \mathbf{e}_i)=-f_{i-1}(t, \mathbf{e}_i)<0$. It follows that $\mathbf{x}(t; \mathbf{e}_i)$ exits T^{n+1} at $\mathbf{e}_i$, and therefore that $\mathbf{e}_i\in A_i$ for $1\le i\le n$.

To show that the hypotheses of the Lemma are satisfied, we proceed by induction to consider a face $\overline{\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_r}}$ with $r<n$. Since $\overline{\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_r}}$ is bounded by faces of lower dimension, it suffices to show that the interior of $\overline{\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_r}}$ is contained in $A_{i_1}\cup\cdots\cup A_{i_r}$. To that end we note that since $r<n$, there must exist an index $i_q\in\{i_1,\dots, i_r\}$ such that i_q-1 is not an element of $\{i_1,\dots, i_r\}$. Then for $\mathbf{x}_0\in(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_r})^\circ$ the i_q coordinate of $\mathbf{x}_0$ is positive while the i_q-1 is zero, so that $x_{i_q-1}(a; \mathbf{x}_0)=0$ while $x'_{i_q-1}(a; \mathbf{x}_0)=-f_{i_q-1}(t; \mathbf{x}_0)<0$ by (B). Therefore $\mathbf{x}(t; \mathbf{x}_0)$ exits T^{n+1} with $x_{i_q-1}=0$, showing that $\overline{\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_r}}\subset A_{i_q}\subset A_{i_1}\cup\cdots\cup A_{i_r}$.

Applying the Lemma, we consider a solution $\mathbf{x}(t; \mathbf{x}_0)$ with $\mathbf{x}_0\in\bigcap_{i=1}^n A_i$. From the definition of A_i it follows that such $\mathbf{x}(t)$ can exit T^{n+1} only at $\mathbf{x}=\mathbf{0}$, but this possibility is precluded by the assumption $\mathbf{f}(t, \mathbf{0})\equiv\mathbf{0}$. Therefore $\mathbf{x}(t; \mathbf{x}_0)>\mathbf{0}$ for $a\le t<\infty$, and by (A) we must have $\mathbf{x}'(t; \mathbf{x}_0)<\mathbf{0}$ for $a\le t<\infty$.

REMARKS

1. Condition (B) only imposes conditions on the *faces* of T^{n+1} corresponding to $x_i=0$ while condition (A) imposes conditions in the interior of T^{n+1}. However (A) can clearly be weakened to

(A)′ $\sum_{i=1}^{n} f_i(t, \mathbf{x}_0) > 0$ for $\mathbf{x}_0 \in T^n$ and $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$

to establish the weaker conclusion that some solution $\mathbf{x}(t; \mathbf{x}_0)$ remains in T^{n+1} for $a \leq t < \infty$. Thus our methods yield the generalizations obtained in [2] by the theory of retracts. Also, if (A) is weakened to

(A)″ $\mathbf{f}(t, \mathbf{x}) \geq \mathbf{0}$ whenever $\mathbf{x} > \mathbf{0}$ and $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$,

then by continuity we can still conclude the existence of a solution satisfying $\mathbf{x} > \mathbf{0}$ and $\mathbf{x}' \leq \mathbf{0}$ for $a \leq t < \infty$.

2. In case (1) represents the scalar equation

(2) $$y^{(n)} - (-1)^n f(t, y) = 0$$

satisfying condition (C) above, our theorem establishes the existence of a monotone solution $y(t)$ satisfying $(-1)^{i-1} y^{(i-1)}(t) > 0$ for $a \leq t < \infty$. It is clear that such solutions satisfy $\lim_{t\to\infty} y^{(i)}(t) = 0$ for $i \geq 1$, but it does not necessarily follow that $\lim_{t\to\infty} y(t) = 0$. However, noting that the monotone solution of (2) satisfies the integral equation

$$y(t) = y(\infty) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s)) ds,$$

one can apply well known techniques (see for example [7], [8]) to establish the following: *in case $f(t, y)$ is monotone* (increasing or decreasing) *in y, some monotone solution of* (2) *will tend to a positive limit if and only if* $\int^\infty t^{n-1} f(t, c) dt < \infty$ *for some $c > 0$.*

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