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Localization of differential operators and holomorphic continuation of the solutions

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1. Introduction

The holomorphic continuation of solutions of linear partial differential equations across the multiple characteristic surfaces is the subject of this paper. In the preceding note [4] we attack this problem with the aid of the Goursat problem. Since the existence-domain of solutions of the Goursat problem is determined not only by the principal parts of the equations but also by their lower order terms, the results in [4] depend on the "*weighted principal parts*" of the operators which are not contained in the principal parts.

The purpose of this paper is to improve the results in [4] so that the theorems are also valid under the similar assumptions only on the principal parts of the operators and the properties of the boundary surfaces.

Let $P(z, \partial_z)$ be a linear partial differential operator with holomorphic coefficients defined near a point p in $\mathbf{C}^n$ and Ω be an open set with the C^2 boundary $\partial\Omega$ which contains p. Though the property of the holomorphic continuation is free from the choice of the local coordinates, we here employ the weighted local coordinates at p such that the normal direction z_1 of $\partial\Omega$ at p is assigned the weight 2, while the tangential directions $z_2, \ldots, z_n$ are each assigned the weight 1. The motivation of this employment is that the boundary $\partial\Omega$ can be approximated by the quadratic hypersurfaces of the form

$$\mathrm{Re}\, z_1 = \mathrm{Re} \sum a_{ij} z_i z_j + \sum b_{ij} z_i \bar{z}_j.$$

To make this paper self-contained, some properties related to the weighted coordinates are restated in the next section which is the summary of the section 2 in [4]. In the third section the basic theorem is proved under some fixed local coordinates. The idea of the proof is due to Hörmander [1] and used also by Treves and Zachmanoglou to show the uniqueness of the Cauchy problem (see the references of [3]) and in [3] to obtain the holomorphic continuation theorem. The key point of this idea is to construct the family of surfaces which are non-characteristic with respect to $P(z, \partial_z)$ and cover a neighborhood of p. This basic theorem is the generalization of the theorem of the simple characteristic case. In the last section, § 4, we study the geometric conditions on $P(z, \partial_z)$ and $\partial\Omega$ to insure the existence of the local coordinates in the third section. The assumptions on $P(z, \partial_z)$ and $\partial\Omega$ are made in relation to the localization of $P(z, \partial_z)$ at (p, N), where N is the complex normal direction of $\partial\Omega$ at p, and the bicharacteristic space of its localization. The localization of an operator is due to Hörmander [2] to research the location of the singularities of the distribution solutions of $P(D)u=0$. Our method in this paper is also applicable to show the uniqueness of the Cauchy problem near the multiple characteristic surfaces. The details in this area will be published elsewhere [5].

2. Weighted coordinates

Since we deal with the local problem, the local coordinates have always the same origin. The local coordinates $(z_1,\dots, z_n)$ are the weighted coordinates with the weight $(2, 1,\dots, 1)$ if z_1 has the weight 2 and z_j $(j=2,\dots, n)$ has the weight 1. The weight of a monomial z^α is equal to $2\alpha_1+\alpha_2+\cdots+\alpha_n$. A holomorphic function $f(z)$ at 0 has the weight l if l is the lowest weight among the monomials in the Taylor expansion of $f(z)$ at 0. We remark that if $f=0$ then the weight of f is $+\infty$. For a differential monomial $(\partial/\partial z)^\alpha$, its weight is defined by $-2\alpha_1-\alpha_2-\cdots-\alpha_n$. Similarly the weight of $a(z)(\partial/\partial z)^\alpha$ is equal to weight $(a(z))$ + weight $((\partial/\partial z)^\alpha)$. The weight of a linear partial differential operator $P(z, \partial_z)=\Sigma\, a_\alpha(z)(\partial/\partial z)^\alpha$ is determined by min. weight $(a_\alpha(z)(\partial/\partial z)^\alpha)$.

Let $(z_1,\dots, z_n)$ and $(w_1,\dots, w_n)$ be local coordinates with the same origin. We say that these coordinates are equivalent as the weighted coordinates if w_j has the same weight as z_j as a holomorphic function of z, and the converse is also true. Then the weights of functions or differential operators are invariant under the equivalent weighted coordinates. We also remark that if the weights of $(\xi_1,\dots, \xi_n)$ are each assigned by $(-2, -1,\dots, -1)$, then the weight of $P_m(z, \xi)$, the principal part of P, is invariant.

3. The basic theorem

The differential operator studied in this section is the following one:

$$P(z, \partial_z) = \left(\frac{\partial}{\partial z_1}\right)^{m-l}\left(\frac{\partial}{\partial z_2}\right)^{l} + \sum a_\alpha(z)\left(\frac{\partial}{\partial z}\right)^{\alpha} \tag{3.1}$$

where $a_\alpha(z)$ are holomorphic in some neighborhood U of O and the summation is taken over the multi-indices α such that $|\alpha|\leq m$. The domain Ω is given by

$$\Omega = \{z\in U \mid \rho(z) < 0\} \tag{3.2}$$

where ρ is a real-valued C^2 function such that

$$\rho(O) = 0, \quad \frac{\partial\rho}{\partial z_1}(O) = 1, \quad \frac{\partial\rho}{\partial z_j}(O) = 0, \quad j = 2,\dots, n. \tag{Ω.1}$$

We consider this local coordinates as the weighted coordinates with the weights (2, 1,..., 1). Then we make the following conditions on the principal part $P_m(z, \partial_z)$ of the operator (3.1).

(P.1) *Every weight of* $a_\alpha(z)(\partial/\partial z)^\alpha$ *in* $P_m(z, \partial_z)$ *is larger than or equal to* $l-2m$ $=$ *the weight of* $(\partial/\partial z_1)^{m-l}(\partial/\partial z_2)^l$.

(P.2) *For the term in* P_m *with the weight* $l-2m$, *its coefficient does not vanish at* O, *that is*

$$\text{weight}\,[(a_\alpha(z)-a_\alpha(O))(\partial/\partial z)^\alpha] \geqq l-2m+1,$$

and especially $a_\alpha(O)=0$ *when* $\alpha=(m-l, l, 0,\dots, 0)$ *in the second terms of the right hand side of* (3.1).

(P.3) *There exists an integer* μ $(2\leqq\mu\leqq n)$ *such that the term in* P_m *with the weight* $l-2m$ *is generated only by* $\partial/\partial z_1,\dots, \partial/\partial z_\mu$.

REMARK 3.1. If P is simple characteristic at (O, N) with $N=(1, 0,\dots, 0)$, then it is possible to choose the local coordinates such that P is in the form (3.1) with $l=1$ and $\alpha_1<m-1$ in the sum of the second terms. In this case all conditions (P.1$\sim$3) with $\mu=2$ are automatically fulfilled.

REMARK 3.2. In P_m the condition (P.1) is only restrictive on the terms of the order larger than $m-l$ with respect to $\partial/\partial z_1$. In fact, by (P.1),

$$\text{weight}\,(a_\alpha(z)) \geqq \max\{0, l-2m+2\alpha_1+\alpha_2+\cdots+\alpha_n = l-m+\alpha_1\}.$$

REMARK 3.3. The conditions (P.1) and (P.2) imply that the term of the weight $l-2m$ in P_m is essentially of the form $a_\alpha(O)(\partial/\partial z)^\alpha$ with $\alpha_1=m-l$.

Concerning the boundary function $\rho(z)$ of $\partial\Omega$, the following conditions are imposed in addition to (Ω.1).

(Ω.2) $$\frac{\partial^2\rho}{\partial x_2^2}(O)<0, \qquad \textit{where} \quad x_2=\operatorname{Re} z_2.$$

(Ω.3) $$\frac{\partial^2\rho}{\partial z_i\partial z_j}(O)=0, \quad \textit{if} \quad 3\leqq i \textit{ or } j\leqq\mu, \quad 2\leqq i, j\leqq n.$$

(Ω.4) $$\frac{\partial^2\rho}{\partial z_i\partial \bar z_j}(O)=0, \quad \textit{if} \quad 3\leqq i \textit{ or } j\leqq\mu, \quad 2\leqq i, j\leqq n.$$

We remark that if $\mu=2$ in (P.3), then (Ω.3) and (Ω.4) become empty. Such a case is happened when P is simple characteristic at (O, N).

REMARK 3.4. It is easy to show that these conditions (Ω.2$\sim$4) are independent of the choice of the defining function $\rho(z)$.

Now the basic theorem is as follows:

THEOREM 3.1. *Let $P(z, \partial_z)$ be a differential operator of the form* (3.1) *which satisfies the conditions* (P.1~3), *and Ω be an open set given by* (3.2) *with the conditions* (Ω.1~4). *If $u(z)$ is a holomorphic solution of $Pu=f$, where f is holomorphic near* 0, *then $u(z)$ can be holomorphically prolonged across $\partial\Omega$ at* 0.

For the rest of this section we devote ourselves to prove this theorem.

LEMMA 3.1. *The conditions* (P.1~3) *are invariant under the transformation of the coordinates of the following form*:

$$(3.3)\quad \begin{cases} w_1 = z_1 + \text{a holomorphic function of the weight} \geqq 3, \\ w_j = z_j + \text{a holomorphic function of the weight} \geqq 2, \quad j = 2,\dots, n. \end{cases}$$

PROOF. By (3.3) we have

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial w_1} + \text{terms of the weight} > -2,$$

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial w_j} + \text{terms of the weight} > -1, \quad j = 2,\dots, n.$$

Then the invariance of (P.1~3) is trivial.

LEMMA 3.2. *Let $\rho(z)$ be a defining function of Ω with the condition* (Ω.1). *Then by changing the defining function $\rho(z)$ if necessarily we may assume that*

$$(\Omega.5)\qquad \frac{\partial^2 \rho}{\partial z_1 \partial \bar{z}_j}(O) = 0, \qquad j = 1,\dots, n.$$

PROOF. By (Ω.1) $\rho(z)$ can be expressed as follows:

$$\rho(z) = z_1 + \bar{z}_1 + \sum a_{ij} z_i z_j + \sum \bar{a}_{ij} \bar{z}_i \bar{z}_j + (\textstyle\sum_{j=1}^n b_j z_j)\bar{z}_1 + (\textstyle\sum_{j=1}^n \bar{b}_j \bar{z}_j) z_1 + \textstyle\sum_{i,j \geqq 2} b_{ij} z_i \bar{z}_j + o(|z|^2).$$

If we set $H(z) = \sum_{j=1}^n (b_j z_j + \bar{b}_j \bar{z}_j)$, then $r(z) = \rho(z) \exp[-H(z)]$ becomes also the defining function of Ω. It now easy to see that this $r(z)$ satisfies (Ω.5).

REMARK 3.5. (1) Since ρ is real-valued, (Ω.5) implies that

$$(\Omega.5)^{bis}\qquad \frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_j}(O) = 0, \qquad j = 1,\dots, n.$$

(2) By Remark 3.4 we may assume that $\rho(z)$ satisfies (Ω.1~5) under the conditions (Ω.1~4).

LEMMA 3.3. *Let $\rho(z)$ be a defining function of Ω which satisfies the con-*

ditions (Ω.1$\sim$5). *Then by the suitable transformation of the coordinates of the form* (3.3), *we may assume that*

$$(\Omega.6) \qquad \begin{cases} \dfrac{\partial^2 \rho}{\partial z_1 \partial z_j}(O) = 0, \\ \dfrac{\partial^2 \rho}{\partial \bar{z}_1 \partial \bar{z}_j}(O) = 0, \end{cases} \qquad j = 1,\dots, n,$$

in addition to the conditions (Ω.1$\sim$5).

PROOF. By (Ω.1$\sim$5), $\rho(z)$ can be written as

$$\rho(z) = z_1 + \bar{z}_1 + \textstyle\sum_{j=1}^n (a_j z_j) z_1 + \sum_{j=1}^n (\bar{a}_j \bar{z}_j) \bar{z}_1$$
$$+ \textstyle\sum^* (a_{ij} z_i z_j + \bar{a}_{ij} \bar{z}_i \bar{z}_j + b_{ij} z_i \bar{z}_j) + o(|z|^2),$$

where $\sum^*$ means that the summation is taken over the set $\{(i, j) \mid i, j \geqq \mu+1$ or $i=2$ and $j \geqq \mu+1$ or $j=2$ and $i \geqq \mu+1\}$. Here we introduce the new coordinates as

$$w_1 = z_1 + (\textstyle\sum_{j=1}^n a_j z_j) z_1,$$
$$w_j = z_j, \qquad j = 2,\dots, n.$$

Then it is easy to see that ρ satisfies the conditions (Ω.1$\sim$6) under the coordinates $(w_1,\dots, w_n)$. This proves the lemma.

LEMMA 3.4. *If a real-valued C^2 function ρ satisfies the conditions* (Ω.1$\sim$6), *then there exist positive constants α and M such that for any $\varepsilon>0$ the following inequality holds in a sufficiently small neighborhood V of* 0.

$$(3.4) \qquad \rho(z) \leqq z_1 + \bar{z}_1 - \alpha x_2^2 + \varepsilon(|z_1|^2 + |z_3|^2 + \cdots + |z_\mu|^2)$$
$$+ M(y_2^2 + |z_{\mu+1}|^2 + \cdots + |z_n|^2),$$

where $y_2 = \mathrm{Im}\, z_2$.

PROOF. We expand ρ in the Taylor series up to the order 2. The first order part is equal to $z_1 + \bar{z}_1$. For the terms of the second order we divide these into three groups: (i) the terms containing only z_2 and $\bar{z}_2$, (ii) the terms consisted by the product of z_2 or $\bar{z}_2$ and $z_{\mu+1},\dots, z_n$ or $\bar{z}_{\mu+1},\dots, \bar{z}_n$, (iii) the quadratic terms of $z_{\mu+1},\dots, z_n$ and their complex conjugates. For the sums of the terms in each groups, we estimate these respectively by

$$-\alpha' x_2^2 + \beta y_2^2, \quad \text{with} \quad \alpha', \beta > 0 \quad \text{by} \quad (\Omega.2)$$
$$\varepsilon |z_2|^2 + (C/\varepsilon)(|z_{\mu+1}|^2 + \cdots + |z_n|^2), \quad \varepsilon > 0,$$
$$C'(|z_{\mu+1}|^2 + \cdots + |z_n|^2).$$

Then the inequality (3.4) is easily derived.

REMARK 3.6. The function of the right hand side of (3.4) is pluri-subharmonic at 0 if and only if $M \geqq \alpha$.

Set $\psi(z)$ as

$$\psi(z) = z_1 + \bar{z}_1 - \alpha x_2^2 + \varepsilon(|z_1|^{1+\delta} + |z_3|^2 + \cdots + |z_\mu|^2) + M(y_2^2 + |z_{\mu+1}|^2 + \cdots + |z_n|^2), \tag{3.5}$$

where δ is a parameter such that $0<\delta<1/2$. Then Lemma 3.4 shows that in some neighborhood V of 0, the open set $\{\psi(z)<0\}$ is contained in Ω. We here remark that V can be chosen independently on the parameter δ.

Nextly we construct the family of surfaces. Define $\phi(z)$ as

$$\phi(z) = z_1 + \bar{z}_1 - (\alpha/2)rx_2 + 2\varepsilon(|z_1|^{1+\delta} + |z_3|^2 + \cdots + |z_\mu|^2) + 2M(y_2^2 + |z_{\mu+1}|^2 + \cdots + |z_n|^2), \tag{3.6}$$

where $r>0$ is a parameter and determined later.

LEMMA 3.5. *If $s \leqq \alpha r^2$, then the set $\{\psi(z)\geqq 0\} \cap \{\phi(z)\leqq s\}$ is compact and contained in $U(r)$, where*

$$U(r) = \{z \mid |x_1| \leqq 2\alpha r^2,\ |z_1|^{1+\delta} \leqq 3(\alpha/\varepsilon)r^2,\ |z_2| \leqq (2 + (3\alpha/M)^{1/2})r,$$
$$|z_j| \leqq (3\alpha/\varepsilon)^{1/2}r\ (j = 3,\dots,\mu),\ |z_k| \leqq (3\alpha/M)^{1/2}r\ (k = \mu + 1,\dots, n)\}.$$

PROOF. Set $R_\mu^2 = |z_1|^{1+\delta} + |z_3|^2 + \cdots + |z_\mu|^2$ and $R_n^2 = |z_{\mu+1}|^2 + \cdots + |z_n|^2$. For any $z \in \{\psi(z)\geqq 0\} \cap \{\phi(z)\leqq s\}$ we have

$$2\alpha x_2^2 - 4x_1 \leqq 2My_2^2 + 2\varepsilon R^2 + 2MR_n^2 \leqq s + (\alpha/2)rx_2 - 2x_1,$$

which implies the next two inequalities:

$$2x_1 \geqq 2\alpha x_2^2 - (\alpha/2)rx_2 - s,$$
$$s + (\alpha/2)rx_2 - 2x_1 \geqq 0.$$

Then it is easily derived that $|x_1|\leqq 2\alpha r^2$ and $|x_2|\leqq 2r$ provided that $s\leqq\alpha r^2$. Using these estimates we have

$$0 \leqq 2My_2^2 + 2\alpha R_\mu^2 + 2MR_n^2 \leqq 6\alpha r^2.$$

Thus the lemma is proved.

Now we determine the parameters ε and r so that the surface $\phi(z)=s$ is non-characteristic with respect to $P(z, \partial_z)$ in some neighborhood of O. Let $Q(\partial_z)$

be the sum of the terms in P_m with the weight exactly $l-2m$. By Remark 3.3 and the condition (P.3), $Q(\partial_z)$ is expressed as follows:

$$Q(\partial_z)=\left(\frac{\partial}{\partial z_1}\right)^{m-l}\left(\frac{\partial}{\partial z_2}\right)^{l}+\sum a_\alpha\left(\frac{\partial}{\partial z}\right)^{\alpha} \tag{3.7}$$

where the summation is taken over the multi-indices such that $\alpha_1=m-l$, $\alpha_2+\cdots+\alpha_\mu=l$, $\alpha_2<l$ and $\alpha_{\mu+1}=\cdots=\alpha_n=0$. In (3.7), every a_α is a constant.

Let $V(r)$ be the set defined by

$$\begin{aligned} V(r)=\{z\,|\,&|z_1|\leqq 6(\alpha/\varepsilon)r^2,\ |z_2|\leqq(2+(3\alpha/M)^{1/2})r,\\ &|z_j|\leqq(3\alpha/\varepsilon)^{1/2}r \quad (j=3,\dots,\mu),\\ &|z_k|\leqq(3\alpha/M)^{1/2}r \quad (k=\mu+1,\dots,n)\}. \end{aligned} \tag{3.8}$$

If we set $\xi_j=\partial\phi/\partial z_j$, then we have the next estimates on $V(r)$ for $0<\varepsilon<1/4$ and $0<\delta<1/2$:

$$\begin{cases} 1/4\leqq|\xi_1|\leqq 2 & \text{if}\quad 6\alpha r^2\leqq 1,\\ \alpha r/4\leqq|\xi_2|\leqq C(\alpha,M)r, & \\ |\xi_j|\leqq 2(3\alpha)^{1/2}\varepsilon^{1/2}r & (j=3,\dots,\mu),\\ |\xi_k|\leqq 2(3\alpha M)^{1/2}r & (k=\mu+1,\dots,n), \end{cases} \tag{3.9}$$

where $C(\alpha, M)$ is a constant depending on α and M.

LEMMA 3.6. *If we take ε $(0<\varepsilon<1/4)$ sufficiently small, then $Q(\xi)$ does not vanish on $V(r)$.*

PROOF. We use the notation $C(\alpha, M)$ which is a different constant in each position depending on α and M. By (3.9),

$$|\xi_1^{m-l}\xi_2^{l}|\geqq C(\alpha,M)r^l$$

$$|a_\alpha\xi^\alpha|\leqq C(\alpha,M)\varepsilon^{(\alpha_3+\cdots+\alpha_\mu)/2}r^l,\quad \text{if}\quad |\alpha|=m\quad\text{and}\quad \alpha_1=m-l.$$

If we put these estimates into the corresponding terms in (3.7), we have that

$$|Q(\xi)|\geqq\{C(\alpha,M)-C'(\alpha,M)\varepsilon^{(\alpha_3+\cdots+\alpha_\mu)/2}\}r^l.$$

Since $\alpha_3+\cdots+\alpha_\mu\neq 0$, $Q(\xi)$ does not vanish for a sufficiently small ε. This proves the lemma.

From now on, the constant ε is taken as in this lemma and always fixed. For the determination of the parameter r, we have the next lemma.

LEMMA 3.7. *If we take r sufficiently small, then $P_m(z, \xi)$ does not vanish on $V(r)$.*

PROOF. If the weight of a holomorphic function $a(z)$ is equal to k, then the inequality

$$\sup_{V(r)} |a(z)| \leqq \text{Const.}\, r^k$$

holds for a sufficiently small r. Thus for a term $a(z)(\partial/\partial z)^\alpha$ in P_m with the weight larger than $l-2m$, the inequality

$$\begin{aligned}\text{weight}\, a(z) &\geqq l - 2m + 1 + 2\alpha_1 + \alpha_2 + \cdots + \alpha_n \\ &= l - m + \alpha_1 + 1\end{aligned}$$

implies

$$|a(z)\xi^\alpha| \leqq \text{Const.}\, r^{l-m+\alpha_1+1}\, \text{Const.}\, r^{\alpha_2+\cdots+\alpha_n} = \text{Const.}\, r^{l+1}.$$

While on $V(r)$, $|Q(\xi)| \geqq \text{Const.}\, r^l$. Since P_m is the sum of Q and the terms of the weight larger than $l-2m$, we can choose r so that P_m does not vanish on $V(r)$. This proves the lemma.

Lastly we determine the parameter δ such that $U(r)$ in Lemma 3.5 is contained in $V(r)$. Here ε and r are already fixed numbers.

LEMMA 3.8. *If we take δ $(0<\delta<1/2)$ sufficiently small, then $U(r)$ is contained in $V(r)$.*

PROOF. It is sufficient to choose δ so that

$$3(\alpha/\varepsilon)r^2 \leqq \{6(\alpha/\varepsilon)r^2\}^{1+\delta}.$$

This proves the lemma.

Under the preparation of Lemmas 3.1~8, we now make the proof of the basic theorem. The key lemma of this proof is the following one:

LEMMA 3.9 ([3, Lemma 1]). *Suppose that there exist a real-valued C^1 function $\phi(z)$ and constants s_0, s_1 such that*

(i) $P_m(z, \text{grad}_z\, \phi(z)) \neq 0$, *in some neighborhood V of 0,*
(ii) $s_0 < \phi(0) < s_1$,
(iii) $\{z \in V \mid \phi(z) \leqq s_1\} \cap \overline{\Omega^c}$ *is compact,*
(iv) $\{z \in V \mid \phi(z) \leqq s_0\} \cap \overline{\Omega^c}$ *is empty,*
(v) $\{z \in V \mid \phi(z) \leqq s_0\}$ *is not empty,*
(vi) $\{z \in V \mid \phi(z) < s\}$ *is simply connected for $s_0 < s < s_1$.*

Then every holomorphic solution $u(z)$ in Ω of the equation $Pu=f$, where f is *holomorphic in V, can be prolonged holomorphically in the set* $\{z \in V \mid \phi(z) < s_1\}$.

The proof of this lemma is based on Zerner's theorem which states that the holomorphic continuation theorem holds across the non-characteristic surface. The detail is presented in [3].

PROOF OF THEOREM 3.1. By Lemma 3.4 we may take Ω as the set $\{z \in V \mid \psi(z) < 0\}$. Now take $V(r)$ in Lemma 3.7 as the neighborhood V of O in Lemma 3.9. Then the condition (i) is fulfilled by Lemma 3.7. Set $s_0 = -\alpha r^2$ and $s_1 = \alpha r^2$. Then the condition (ii) becomes trivial and the condition (iii) is derived from Lemmas 3.5 and 3.8. The other conditions (iv), (v) and (vi) are easily proved from the expression (3.6) of $\phi(z)$, so we omit their proofs. This ends the proof of Theorem 3.1.

4. Choice of the local coordinates in the basic theorem

The contents of this section is similar to that of section 4 in [4]. Let $(z_1,\dots, z_n)$ be the local coordinates such that the surface $z_1 = 0$ is tangent to $\partial\Omega$ at $z = O$. We consider this coordinates as the weighted coordinates with the weights $(2, 1,\dots, 1)$. The other local coordinates with the same property become equivalent to this coordinates as the weighted coordinates.

Let $P(z, \partial_z)$ be a linear differential operator of order m with holomorphic coefficients which is characteristic at 0 in the cotangential direction $N = (1, 0,\dots, 0)$. We set l the multiplicity of P at (O, N). That is

$$P_m(O, N + t\zeta) = L(\zeta)t^l + \text{higher order terms of } t \tag{4.1}$$

where P_m is the principal part of P and $L(\zeta)$ is a non-zero polynomial of ζ. This polynomial $L(\zeta)$ is called the localization of P_m at (O, N), which is originally introduced by Hörmander [2] to analyze the location of the singularities of the solution u of the equation $Pu = 0$. When $N = (1, 0,\dots, 0)$, (4.1) means that in $P_m(O, \partial_z)$ there is none of the terms of order larger than $m - l$ with respect to $\partial/\partial z_1$ and the sum of the coefficients of $(\partial/\partial z_1)^{m-l}$ is equal to $L(\partial/\partial z_2,\dots, \partial/\partial z_n)$. Therefore $L(\zeta)$ is a homogeneous polynomial of degree l in the variables $(\zeta_2,\dots, \zeta_n)$. Since the weight of $L(\partial/\partial z_2,\dots, \partial/\partial z_n)(\partial/\partial z_1)^{m-l}$ is equal to $l - 2m$, we make the assumption:

(P.I) *The weight of* $P_m(z, \xi)$ *is equal to* $l - 2m$, *if the weight of* ξ *are assigned by* $(-2, -1,\dots, -1)$.

Relating to the localization $L(\zeta)$ of P_m, we introduce some complex linear spaces in the holomorphic tangent space T_O and the cotangent space T_O^* of the surface $\partial\Omega$ at O. For the polynomial $L(\zeta)$, we set

(4.2) $$\Lambda^*(L) = \{\eta \in T_O^* \mid L(\xi + t\xi) = L(\xi) \text{ for all } t \text{ and } \xi\},$$

which is a linear subspace, and we introduce the annihilator

(4.3) $$\Lambda(L) = \{v \in T_O \mid \langle v, \eta\rangle = 0 \text{ for any } \eta \in \Lambda^*(L)\},$$

where $\langle\ ,\ \rangle$ denotes the contraction between cotangent vectors and tangent vectors. $\Lambda(L)$ is the smallest subspace along which $L(\partial/\partial z)$ operates and is called the bicharacteristic space of P at (O, N). These subspaces are introduced by Hörmander [2].

DEFINITION 4.1. A holomorphic function $\phi(z)$ with $\mathrm{grad}_z\, \phi(O)=N$ is said to be *a weighted characteristic function of* $P(z, \partial_z)$ if it satisfies the following condition:

(4.4) $$\text{weight}\, P_m(z, t\, \mathrm{grad}_z\, \phi(z)) \geqq l - 2m + 1,$$

where the complex parameter t is assigned the weight -2.

We remark here that the above definition is weaker than that of [4]. To find such a weighted characteristic function $\phi(z)$, it is sufficient that ϕ has the form

$$\phi(z) = z_1 + \textstyle\sum_{i,j\geqq 2} a_{ij} z_i z_j.$$

Assume that

(P.II) *there exists a weighted characteristic function* $\phi(z)$.

By the suitable equivalent change of the weighted coordinates we can assume that $\phi(z)=z_1$. Then the following proposition is easy to prove.

PROPOSITION 4.1. *If* $\phi(z)=z_1$, *then* (4.4) *is equivalent to that there is none of the differential monomials of the weight* $l-2m$ *in* $P_m(z, \partial_z)$ *which is generated only by* $\partial/\partial z_1$.

We now fix some weighted characteristic function $\phi(z)$ and consider the local coordinates $(z_1,\dots, z_n)$ as $\phi(z)=z_1$ (mod weight 3) and each z_j $(j=2,\dots, n)$ has the weight 1. This means that the coordinates transformation considered from now on has the following form:

(4.5) $$\begin{cases} w_1 = z_1 + \text{a holomorphic function of the weight} \geqq 3, \\ w_j = \sum_{k=2}^n c_{jk} z_k + \text{a holomorphic function of the weight} \geqq 2. \end{cases}$$

Then we make the last assumption on P_m such that

(P.III) weight $[P_m(z, \xi) - P_m(O, \xi)] \geqq l - 2m + 1$.

PROPOSITION 4.2. *This assumption* (P.III) *is invariant under the change of variables of the form* (4.5).

PROOF. If we remark that by (4.5),

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial w_1} + \text{terms of the weight larger than } -2,$$

$$\frac{\partial}{\partial z_j} = \sum_{k=2}^{n} c_{kj} \frac{\partial}{\partial w_k} + \text{terms of the weight larger than } -1,$$

the invariance of (P.III) is easy to prove.

Assumption (P.III) means that the terms with the lowest weight in P_m do not degenerate at O.

Now we proceed to examine the conditions on Ω under the assumptions (P.I$\sim$III).

(Ω.I) *There exists a holomorphic curve $\zeta(t)$ in the weighted characteristic surface $S=\{z_1=0\}$ with the following conditions*:

(i) $\zeta(O)=0$, $d\zeta(O)\in \Lambda(L)$,

(ii) *there exists a covector $\xi_0 \in T_O^*$ such that*

$$L(\xi_0) \neq 0 \quad \text{and} \quad \langle d\zeta(O), \xi_0 \rangle \neq 0,$$

(iii) *there exists a non-zero complex number t_0 such that*

$$\frac{d^2}{d\tau^2}(\rho(t_0\tau))|_{\tau=0} < 0$$

for a real parameter τ.

The geometric meaning of these conditions is that the tangent vector $d\zeta(O)$ belongs to the bicharacteristic space of P and the normal curvature of $\partial\Omega$ at O in some real direction of $d\zeta(O)$ is negative and, at the same time, $L(\xi)$ is non-characteristic at ξ_0.

REMARK 4.1. The covector ξ_0 in the above condition (ii) does not belong to $\Lambda^*(L)$. Indeed this follows from the relations $L(\xi_0)\neq 0 = L(\xi_0+(-1)\xi_0)$.

We introduce the linear spaces

$$\Lambda(\xi_0) = \{v \in T_O \mid \langle v, \xi_0 \rangle = 0\},$$

$$\Lambda_0(L) = \Lambda(\xi_0) \cap \Lambda(L).$$

Since $\xi_0 \notin \Lambda^*(L)$, $\Lambda(\xi_0)$ is not contained in $\Lambda(L)$. If the dimension of $\Lambda(L)$ is $\mu-1$, where $\mu \geqq 2$ because $L\neq 0$, it follows that $\dim \Lambda_0(L)=\mu-2$.

Concerning the subspace $\Lambda_0(L)$ we assume that

(Ω.II) *the Levi form of $\partial\Omega$ is degenerate on $\Lambda_0(L)$,*

(Ω.III) *$\partial\Omega$ is tangent to $S=\{z_1=0\}$ at 0 of the second order holomorphically in $\Lambda_0(L)$.*

These conditions are the same as in [4]. If Ω is given by $\{\rho(z)<0\}$, then the complex Hessian form of ρ at O is defined by

$$H_{\rho(O)}(t, s) = \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(O) dz_i(t) d\bar{z}_j(s) \tag{4.6}$$

for two holomorphic tangent vectors t and s in T_O. Using (4.6), (Ω.II) means that if s belongs to $\Lambda_0(L)$, the linear form $H_{\rho(O)}(\cdot, s)$ vanishes on T_O. The condition (Ω.III) means by definition (see [4]) that for all holomorphic vector fields X and Y which are tangent to S, $(XY\rho)(O)=0$ if $X(O)$ or $Y(O)$ belongs to $\Lambda_0(L)$.

Now we construct the local coordinates $(z_1,\dots, z_n)$ so that the operator $P(z, \partial_z)$ is reduced to the form (3.1) and all assumptions in the basic theorem are satisfied. The method of this construction is the same as that in [4].

First we fix the weighted characteristic function $\phi(z)$ in (P.II) and set $\phi(z) = z_1$.

Secondly we choose the tangential coordinates $(z_2,\dots, z_n)$ so that

$$\begin{cases} \text{(i)} & \partial/\partial z_2 = d\zeta(O), \\ \text{(ii)} & \Lambda_0(L) \text{ is generated by } \partial/\partial z_3,\dots, \partial/\partial z_\mu. \end{cases} \tag{4.7}$$

PROPOSITION 4.3. *In these coordinates, L is non-characteristic at the covector dz_2.*

PROOF. If we write ξ_0 as

$$\xi_0 = c_2 dz_2 + \cdots + c_n dz_n,$$

then by (4.7),

$$\langle \xi_0, \partial/\partial z_j \rangle = 0, \quad j = 3,\dots, \mu.$$

Thus $\xi_0 = c_2 dz_2 + \xi'$ where $\xi' \in \Lambda^*(L)$. Since $\Lambda^*(L)$ is generated by $dz_{\mu+1},\dots, dz_n$, we have

$$L(c_2 dz_2) = L(c_2 dz_2 + \xi') = L(\xi_0) \neq 0.$$

which proves the proposition.

From this proposition, $L(\partial/\partial z)$ is written as

$$L(\partial_z) = a\left(\frac{\partial}{\partial z_2}\right)^l + \sum a_\alpha \left(\frac{\partial}{\partial z}\right)^\alpha$$

where the summation is taken over the multi-indices $|\alpha|=l$, $\alpha_1=0$ and $\alpha_2<l$. Thus the operator $P(z, \partial_z)$ is expressed in the form (3.1) under this coordinates. Since the equivalences between (P.1) and (P.I), (P.2) and (P.III), (Ω.2) and (Ω.I), (Ω.3) and (Ω.III), (Ω.4) and (Ω.II) are trivial, and (P.3) follows from (4.7), (Ω.1) follows from the form of the weighted characteristic function $\phi(z)$ of (P.II), all conditions in Theorem 3.1 are satisfied under this coordinates. Summing up these results we have the final theorem:

THEOREM 4.1. *Let $P(z, \partial_z)$ be a differential operator of order m with holomorphic coefficients in a neighborhood of p and Ω be an open set with C^2 boundary $\partial\Omega \ni p$. We suppose that $P(z, \partial_z)$ and Ω satisfy the conditions* (P.I$\sim$III) *and* (Ω.I$\sim$III). *Then if $u(z)$ is a holomorphic solution of $Pu=f$ in Ω, where f is holomorphic near p, then $u(z)$ can be holomorphically prolonged across $\partial\Omega$ at p.*

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