HIROSHIMA MATH. J.
10 (1980), 495–516

Basic results on oscillation for differential equations with deviating arguments

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(Received December 6, 1979)
(Revised February 12, 1980)

1. Preliminaries

This article deals with oscillatory and asymptotic properties of the solutions of differential equations with deviating arguments. The study is concentrated to differential equations of "fundamental" forms, since, in view of a comparison principle introduced by Staikos and Sficas [20, 21, 22], the results obtained can be extended for differential equations of more general forms in such a manner that is rather technical. More precisely, it is enough to treat here the n-th order differential equation with deviating arguments

$$(1) \qquad x^{(n)}(t) + a(t)\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) = 0, \quad t \geqq t_0,$$

where the functions a, φ, σ_j $(j=1,\dots, m)$ are continuous and

$$\lim_{t\to\infty} \sigma_j(t) = \infty \quad (j = 1,\dots, m).$$

Most of the results here are concerned with the case where the function a is of constant sign. Then the equation (1) can be written in the form

$$(1)' \qquad x^{(n)}(t)I(a) + |a(t)|\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) = 0, \quad t \geqq t_0,$$

where

$$I(a) = \begin{cases} +1, & \text{if } a \geqq 0, \\ -1, & \text{if } a \leqq 0, \end{cases}$$

is the so called *sign index* of the function a. For technical reasons, it is then more convenient to work with the differential inequalities

$$(1)'_{\leqq} \qquad x^{(n)}(t)I(a) + |a(t)|\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) \leqq 0, \quad t \geqq t_0,$$

and

$$(1)'_{\geqq} \qquad x^{(n)}(t)I(a) + |a(t)|\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) \geqq 0, \quad t \geqq t_0,$$

associated to the equation (1)′.

The function φ, or equivalently, the differential equation (1) and the differential inequalities $(1)'_{\leqq}$ and $(1)'_{\geqq}$ are said to be:

(A_1) *strongly sublinear to the right* if φ is positive and increasing on $\mathbf{R}_+^m$, $\mathbf{R}_+ = (0, \infty)$, and such that

$$\int_{+0} \frac{dy}{\varphi(y,\dots, y)} < \infty;$$

(A_2) *strongly sublinear to the left* if φ is negative and increasing on $\mathbf{R}_-^m$, $\mathbf{R}_- = (-\infty, 0)$, and such that

$$\int_{-0} \frac{dy}{\varphi(y,\dots, y)} < \infty;$$

(A_3) *strongly sublinear* if it is strongly sublinear to the both right and left, and similarly,

(B_1) *strongly superlinear to the right* if φ is positive and increasing on $\mathbf{R}_+^m$ and such that

$$\int^{\infty} \frac{dy}{\varphi(y,\dots, y)} < \infty;$$

(B_2) *strongly superlinear to the left* if φ is negative and increasing on $\mathbf{R}_-^m$ and such that

$$\int^{-\infty} \frac{dy}{\varphi(y,\dots, y)} < \infty;$$

(B_3) *strongly superlinear* if it is strongly superlinear to the both right and left.

The pattern for the definitions above is the simple ordinary differential equation

$$x^{(n)} + a(t)x^{\alpha} = 0,$$

where the exponent α is the ratio of odd natural numbers. Obviously, it is strongly sublinear for $0<\alpha<1$ and strongly superlinear for $\alpha>1$. This ordinary equation has been treated by Ličko and Švec [12] and its oscillatory and asymptotic behavior is well established. As far as we know, the terms sublinear and superlinear differential equations have first been introduced by Coffman and Wong [1]. Our definitions, though not identical to theirs, essentially follow the same spirit and justification.

It is noteworthy to list here some consequences of the above definitions:

(a) *If the function φ is strongly sublinear to the right, then*

$$\lim_{y\to+0} y^{-1}\varphi(y,\dots, y) = \infty.$$

Indeed, for any $y>0$ we have

$$(2) \qquad 2\int_{y/2}^{y} \frac{dz}{\varphi(z,\dots,z)} \geqq 2\,(y/2)\frac{1}{\varphi(y,\dots,y)} = \frac{y}{\varphi(y,\dots,y)}$$

where obviously

$$\lim_{y\to+0}\int_{y/2}^{y} \frac{dz}{\varphi(z,\dots,z)} = 0.$$

(b) *If the function φ is strongly sublinear to the left, then*

$$\lim_{y\to-0} y^{-1}\varphi(y,\dots,y) = \infty.$$

Indeed, for any $y<0$ we have

$$(3) \qquad -2\int_{y}^{y/2} \frac{dz}{\varphi(z,\dots,z)} \geqq (-2)\,(-y/2)\frac{1}{\varphi(y,\dots,y)} = \frac{y}{\varphi(y,\dots,y)}$$

where obviously

$$\lim_{y\to-0}\int_{y}^{y/2} \frac{dz}{\varphi(z,\dots,z)} = 0.$$

(c) *If the function φ is strongly superlinear to the right, then*

$$\lim_{y\to\infty} y^{-1}\varphi(y,\dots,y) = \infty.$$

This follows from (2), since now we have

$$\lim_{y\to\infty}\int_{y/2}^{y} \frac{dz}{\varphi(z,\dots,z)} = 0.$$

(d) *If the function φ is strongly superlinear to the left, then*

$$\lim_{y\to-\infty} y^{-1}\varphi(y,\dots,y) = \infty.$$

This follows from (3), since we have

$$\lim_{y\to-\infty}\int_{y}^{y/2} \frac{dz}{\varphi(z,\dots,z)} = 0.$$

THE KIGURADZE FIRST LEMMA. *Let u be a positive and k-times differentiable function on an interval $[A, \infty)$ with its k-th derivative $u^{(k)}$ nonpositive and not identically zero on any interval of the form $[B, \infty)$.*

Then there exist a $t_u \geqq A$ and an integer ℓ, $0 \leqq \ell \leqq k-1$, with $k+\ell$ odd and such that

$$u^{(j)} > 0 \quad on \quad [t_u, \infty) \quad (j = 0,\dots,\ell)$$

and, provided that $\ell < k-1$,

$$(-1)^{\ell+j}u^{(j)} > 0 \quad on \quad [t_u, \infty) \quad (j = \ell + 1, \dots, k-1).$$

THE KIGURADZE SECOND LEMMA. *Let u be a positive and k-times differentiable function on an interval $[A, \infty)$ with its k-th derivative $u^{(k)}$ nonnegative and not identically zero on any interval of the form $[B, \infty)$.*

Then for some $t_u \geqq A$ either (i) *or* (ii) *holds:*

(i) *For every $j=0,\dots,k-1$*

$$u^{(j)} > 0 \quad on \quad [t_u, \infty).$$

(ii) *There exists an integer ℓ, $0 \leqq \ell \leqq k-2$, with $k+\ell$ even and such that*

$$u^{(j)} > 0 \quad on \quad [t_u, \infty) \quad (j = 0, \dots, \ell)$$

and

$$(-1)^{\ell+j}u^{(j)} > 0 \quad on \quad [t_u, \infty) \quad (j = \ell + 1, \dots, k-1).$$

THE KIGURADZE THIRD LEMMA. *Let u be a positive and k-times differentiable function on an interval $[A, \infty)$ with its k-th derivative $u^{(k)}$ of constant sign and not identically zero on any interval of the form $[B, \infty)$.*

We set

$$M_s = [1/2 - |s - 1/2|]^{k-1}/2^{k-1}(k-1)!.$$

If

$$u^{(k-1)}(t)u^{(k)}(t) \leqq 0 \quad for\ all\ large\ t,$$

then for every θ, $0<\theta<1$, we have

$$u(\theta t) \geqq M_\theta t^{k-1}|u^{(k-1)}(t)| \quad for\ all\ large\ t \tag{4}$$

and when, in addition $\lim_{t\to\infty} u(t) \neq 0$, then

$$u(t) \geqq \theta M_{1/2} t^{k-1}|u^{(k-1)}(t)| \quad for\ all\ large\ t. \tag{5}$$

Moreover, if u is increasing, the last inequality holds also for $\theta=1$.

REMARK 1.1. The Kiguradze first and second lemmas are versions of results that appeared in Kiguradze [5, 6]. They are stated here in a form suitable for our purposes. The Kiguradze third lemma has also partly appeared in Kiguradze [5, 6] where the inequality (4) is proved for a certain value of θ. Inequality (5) appeared in Sficas [16]. The version presented here is due to Grammatikopoulos, Sficas and Staikos [3].

In the next sections we make use of Theorems 1.1 and 1.2 below which have been given by Staikos in [19].

THEOREM 1.1. *Let the function a be of constant sign and let the differential* *inequality* $(1)'_{\leqq}$ *[respectively the inequality* $(1)'_{\geqq}$*] subject to the conditions*:

(C_1) *The function* φ *is defined at least on the set*

$$\{y \in \mathbf{R}^m: (\forall i) y_i > 0 \quad or \quad (\forall i) y_i < 0\}$$

and has the following sign property:

$$(\forall i = 1, \dots, m) y_i > 0 \Longrightarrow \varphi(y_1, \dots, y_m) > 0$$

and

$$(\forall i = 1, \dots, m) y_i < 0 \Longrightarrow \varphi(y_1, \dots, y_m) < 0.$$

(C_2) $$\int^{\infty} t^{n-1}|a(t)|dt = \infty.$$

Then for all eventually nonnegative [respectively nonpositive] and bounded solutions x of the inequality $(1)'_{\leqq}$ *[respectively of* $(1)'_{\geqq}$*] we have*

(6) $$\lim_{t\to\infty} x^{(j)} = 0 \quad monotonically \quad (j = 0, \dots, n-1).$$

Moreover, nonoscillatory such solutions occur only when a is nonnegative and n is odd, or a nonpositive and n even.

THEOREM 1.2. *Consider the differential equation* (1) *subject to the condition* (C_1).

If the function a is of constant sign, then the condition

(C_3) $$\int^{\infty} t^{n-1}|a(t)|dt < \infty$$

is a necessary and sufficient condition in order that the equation (1) *have a solution x so that the* $\lim_{t\to\infty} x(t)$ *exists in* $\mathbf{R} - \{0\}$.

We need further the following basic asymptotic result. For other related to it as well as for various generalizations in this line we refer to Philos, Sficas and Staikos [13].

THEOREM 1.3. *Consider the differential equation* (1) *subject to the conditions* (C_1) *and*:

(C_4) *For some constant* $\mu \neq 0$,

$$\int^{\infty} |a(t)|\,|\varphi(\mu\sigma_1^{n-1}(t), \dots, \mu\sigma_m^{n-1}(t))|dt < \infty.$$

If the function φ *is increasing, then there exists a solution x of the equation* (1) *so that the* $\lim_{t\to\infty} x^{(n-1)}(t)$ *exists in* $\mathbf{R} - \{0\}$.

2. The case of sublinearity

First of all, we are interested in some results concerning the differential inequalities $(1)'_{\leqq}$ and $(1)'_{\geqq}$. The oscillatory and asymptotic behavior of the equation (1) follows then easily. For our purposes, we introduce here the functions τ_j $(j=1,\ldots, m)$ defined by

$$\tau_j(t) = \min\{t, \sigma_j(t)\}. \tag{7}$$

THEOREM 2.1. *Let the function a be of constant sign and let* φ *have on* $\mathbf{R}_+^m$ *the exponential property*

$$\varphi(y_1 z_1,\ldots, y_m z_m) \geqq K\varphi(y_1,\ldots, y_m)\varphi(z_1,\ldots, z_m), \tag{8}$$

where K is a positive constant.

If the differential inequality $(1)'_{\leqq}$ *is strongly sublinear to the right and*

$$\int^{\infty} |a(t)|\varphi(\tau_1^{n-1}(t),\ldots, \tau_m^{n-1}(t))dt = \infty, \tag{C$_5$}$$

then:

(i) *For nonnegative a, every eventually nonnegative solution x of the inequality* $(1)'_{\leqq}$ *satisfies* (6).

(ii) *For nonpositive a, every eventually nonnegative solution x of the inequality* $(1)'_{\leqq}$ *satisfies* (6) *or*

$$\lim_{t\to\infty} x^{(j)}(t) = \infty \quad \textit{monotonically} \quad (j = 0,\ldots, n-1). \tag{9}$$

Moreover, for such nonoscillatory solutions x, (6) *occurs only when n is odd in case* (i), *while in case* (ii) *it occurs only when n is even.*

PROOF. By (8), for $y>0$ we have

$$K\varphi(y,\ldots, y)\varphi(y^{-1},\ldots, y^{-1}) \leqq \varphi(1,\ldots, 1).$$

But, because of the sublinearity of the function φ,

$$\lim_{y\to\infty} y\varphi(y^{-1},\ldots, y^{-1}) = \lim_{z\to+0} z^{-1}\varphi(z,\ldots, z) = \infty$$

and consequently for all large y,

$$K\varphi(y,\ldots, y) \leqq K\varphi(y,\ldots, y)y\varphi(y^{-1},\ldots, y^{-1}) \leqq \varphi(1,\ldots, 1)y.$$

Therefore, since $\lim_{t\to\infty} \tau_j(t) = \infty$ $(j=1,\ldots, m)$ and φ is increasing on $\mathbf{R}_+^m$, by (7), we have that for all large t

$$K\varphi(\tau_1^{n-1}(t),\ldots, \tau_m^{n-1}(t)) \leqq K\varphi(t^{n-1},\ldots, t^{n-1}) \leqq \varphi(1,\ldots, 1)t^{n-1}.$$

Thus, condition (C_5) implies (C_2).

Now, we consider the differential inequality

$$z^{(n)}(t)I(a) + |a(t)|\psi(z[\sigma_1(t)],\dots, z[\sigma_m(t)]) \leqq 0, \tag{10}$$

where

$$\psi(y_1,\dots, y_m) = \begin{cases} \varphi(y_1,\dots, y_m), & \text{if } (y_1,\dots, y_m) \in \operatorname{dom}\varphi \text{ and } (\forall j) y_j \geqq 0 \\ -\varphi(-y_1,\dots, -y_m), & \text{if } (-y_1,\dots, -y_m) \in \operatorname{dom}\varphi \text{ and } (\forall j) y_j \leqq 0. \end{cases}$$

Since, under the assumptions of the theorem, condition (C_2) is satisfied, we can apply Theorem 1.1 for the inequality (10) to obtain that every eventually nonnegative and bounded solution x of the differential inequality $(1)'_{\leqq}$ satisfies the conclusion of the theorem.

It remains to prove the theorem for the unbounded solutions. To this end, let us consider an unbounded solution x of the inequality $(1)'_{\leqq}$, which without loss of generality is supposed nonnegative on the whole interval $[t_0, \infty)$. Moreover, because of $\lim_{t\to\infty} \tau_j(t) = \infty$, let T, $T \geqq t_0$, be chosen sufficiently large so that for $j = 1,\dots, m$

$$\tau_j(t) \geqq \max\{t_0, 0\} \quad \text{for every} \quad t \geqq T.$$

From $(1)'_{\leqq}$ and the positivity of φ on $\mathbf{R}^m_+$ we obtain

$$x^{(n)}(t)I(a) \leqq 0 \quad \text{for every} \quad t \geqq T \tag{11}$$

and therefore all derivatives $x^{(j)}$ $(j = 0,\dots, n-1)$ are eventually monotone. Thus, because of the unboundedness of x, $\lim_{t\to\infty} x(t) = \infty$ and hence, without loss of generality, we suppose that $n > 1$ and

$$x(t) > 0 \quad \text{and} \quad x'(t) \geqq 0 \quad \text{for every} \quad t \geqq T. \tag{12}$$

Moreover, $x^{(n-1)}(t)$ is not identically zero for all large t. Indeed, we obviously have

$$\varphi(x\langle\sigma(t)\rangle) \equiv \varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) > 0 \quad \text{for all large } t$$

and consequently, in the opposite case, from $(1)'_{\leqq}$ we get

$$|a(t)| = 0 \quad \text{for all large } t,$$

a contradiction to the condition (C_5).

Now, let t_x, $t_x \geqq T$, be assigned to the function x as in the Kiguradze first or second lemma applied for $k = n$ and let us suppose that $\lim_{t\to\infty} x^{(n-1)}(t) = 0$. Then, by (11), we must have

$$x^{(n-1)}(t)I(a) > 0 \quad \text{for every} \quad t \geqq t_x. \tag{13}$$

Therefore, we can apply the Kiguradze third lemma for $u=x$ and $k=n$ to obtain

$$x(t) \geqq M_{1/2}t^{n-1}|x^{(n-1)}(t)| \quad \text{for all large } t.$$

From (11) and (13), we get that $|x^{(n-1)}|$ is decreasing and hence, by $\lim_{t\to\infty} \tau_j(t) = \infty$, (7) and (13),

$$x[\tau_j(t)] \geqq M_{1/2}\tau_j^{n-1}(t)x^{(n-1)}(t)I(a) \quad \text{for every} \quad t \geqq t_1,$$

where $t_1 \geqq t_x$ is sufficiently large and $j=1,\dots, m$. Thus, by using $(1)'_{\leqq}$, (7), (12), the increasing character of φ and its exponential property (8), for $t \geqq t_1$ we have

$$\begin{aligned} x^{(n)}(t)I(a) &\leqq -|a(t)|\varphi(x\langle\sigma(t)\rangle) \leqq -|a(t)|\varphi(x\langle\tau(t)\rangle) \\ &\leqq -|a(t)|\varphi(M_{1/2}\tau_1^{n-1}(t)x^{(n-1)}(t)I(a),\dots, M_{1/2}\tau_m^{n-1}(t)x^{(n-1)}(t)I(a)) \\ &\leqq -K|a(t)|\varphi(\tau_1^{n-1}(t),\dots, \tau_m^{n-1}(t))\varphi(M_{1/2}x^{(n-1)}(t)I(a),\dots, M_{1/2}x^{(n-1)}(t)I(a)), \end{aligned}$$

i.e.

$$|a(t)|\varphi(\tau_1^{n-1}(t),\dots, \tau_m^{n-1}(t)) \leqq -\frac{1}{K}\frac{x^{(n)}(t)I(a)}{\varphi(M_{1/2}x^{(n-1)}(t)I(a),\dots, M_{1/2}x^{(n-1)}(t)I(a))}.$$

Hence, by integration,

$$\int_{t_1}^{\infty} |a(t)|\varphi(\tau_1^{n-1}(t),\dots, \tau_m^{n-1}(t))\, dt \leqq \frac{1}{KM_{1/2}}\int_{+0}^{M_{1/2}x^{(n-1)}(t_1)I(a)} \frac{dy}{\varphi(y,\dots, y)} < \infty,$$

a contradiction to condition (C_5).

We have thus proved that the $\lim_{t\to\infty} x^{(n-1)}(t)$, which obviously exists in the extended real line $\mathbf{R}^*$, must be nonzero. Therefore and since x is positive

$$(14) \qquad x^{(n-1)}(t) > 0 \quad \text{for every} \quad t \geqq t_x.$$

For nonpositive a, by Taylor's formula and (11), we get

$$x(t) \geqq x(t_x) + \frac{x'(t_x)}{1!}(t-t_x) + \cdots + \frac{x^{(n-1)}(t_x)}{(n-1)!}(t-t_x)^{n-1}$$

and hence there exists a positive constant μ such that

$$(15) \qquad x(t) \geqq \mu t^{n-1} \quad \text{for all large } t.$$

For nonnegative a, by (11) and (14), we can apply the Kiguradze third lemma for $u=x$ and $k=n$ to obtain

$$x(t) \geqq M_{1/2}t^{n-1}|x^{(n-1)}(t)| \quad \text{for all large } t$$

and consequently (15), since $\lim_{t\to\infty} |x^{(n-1)}(t)| > 0$.

Since $\lim_{t\to\infty} \tau_j(t) = \infty$, (15) gives

$$x[\tau_j(t)] \geqq \mu\tau_j^{n-1}(t) \quad \text{for every} \quad t \geqq t_2,$$

where $t_2 \geqq t_x$ is sufficiently large and $j=1,\dots, m$. Thus by (7), (12), the increasing character of φ and (8), from the inequality $(1)'_{\leqq}$ we obtain that for every $t \geqq t_2$

$$\begin{aligned} x^{(n)}(t)I(a) &\leqq -|a(t)|\varphi(x\langle\sigma(t)\rangle) \leqq -|a(t)|\varphi(x\langle\tau(t)\rangle) \\ &\leqq -|a(t)|\varphi(\mu\tau_1^{n-1}(t),\dots, \mu\tau_m^{n-1}(t)) \\ &\leqq -K\varphi(\mu,\dots, \mu)\,|a(t)|\varphi(\tau_1^{n-1}(t),\dots, \tau_m^{n-1}(t)). \end{aligned}$$

Therefore, by integration,

$$\begin{aligned} &I(a)\lim_{t\to\infty} x^{(n-1)}(t) \\ &\leqq I(a)x^{(n-1)}(t_2) - K\varphi(\mu,\dots, \mu)\int_{t_2}^{\infty} |a(t)|\varphi(\tau_1^{n-1}(t),\dots, \tau_m^{n-1}(t))dt \end{aligned}$$

and hence, by condition (C_5),

$$\lim_{t\to\infty} x^{(n-1)}(t) = -I(a)\infty. \tag{16}$$

But, because of (14), this is the case where a is nonpositive. Hence, (9) is easily derived from (16).

THEOREM 2.2. *Let the function a be of constant sign and let φ have on $\mathbf{R}_-^m$ the exponential property*

$$-\varphi(-y_1z_1,\dots, -y_mz_m) \geqq K\varphi(y_1,\dots, y_m)\varphi(z_1,\dots, z_m), \tag{17}$$

where K is a positive constant.

If the differential inequality $(1)'_{\geqq}$ is strongly sublinear to the left and

$$\int^{\infty} |a(t)|\varphi(-\tau_1^{n-1}(t),\dots, -\tau_m^{n-1}(t))dt = -\infty, \tag{C_6}$$

then:

(i) *For nonnegative a, every eventually nonpositive solution x of the inequality $(1)'_{\geqq}$ satisfies (6).*

(ii) *For nonpositive a, every eventually nonpositive solution x of the inequality $(1)'_{\geqq}$ satisfies (6) or*

$$\lim_{t\to\infty} x^{(j)}(t) = -\infty \quad \textit{monotonically} \quad (j = 0,\dots, n-1). \tag{18}$$

Moreover, for such nonoscillatory solutions x, (6) occurs only when n is odd in case (i), *while in case* (ii) *it occurs only when n is even.*

PROOF. The substitution $z=-x$ transforms the inequality $(1)'_{\geqq}$ into an inequality of the form $(1)'_{\leqq}$ satisfying the assumptions of Theorem 2.1. Indeed, the transformed inequality is

$$z^{(n)}(t)I(a) + |a(t)|\hat{\varphi}(z[\sigma_1(t)],\dots, z[\sigma_m(t)]) \leqq 0,$$

where $\hat{\varphi}(y_1,\dots, y_m) = -\varphi(-y_1,\dots, -y_m)$. Thus, since $\hat{\varphi}$ is strongly sublinear to the right, the theorem follows immediately from Theorem 2.1.

Since the solutions of the equation (1) are the solutions common to both differential inequalities $(1)'_{\leqq}$ and $(1)'_{\geqq}$, we get immediately the following corollary.

COROLLARY 2.1. *Let the function a be of constant sign and let* φ *have on* $\mathbf{R}^m_{\mp}$ *and* $\mathbf{R}^m_{=}$ *the exponential properties* (8) *and* (17) *respectively.*

If the equation (1) *is strongly sublinear, then under conditions* (C_5) *and* (C_6) *we have*:

(i) *For a nonnegative and n even, every solution of* (1) *is oscillatory.*

(ii) *For a nonnegative and n odd, every solution x of* (1) *is oscillatory or satisfies* (6).

(iii) *For a nonpositive and n even, every solution x of* (1) *is oscillatory or satisfies one of* (6), (9) *and* (18).

(iv) *For a nonpositive and n odd, every solution x of* (1) *is oscillatory or satisfies one of* (9) *and* (18).

We now turn our attention to a particular class of differential equations of the form (1), which includes the ordinary, retarded equations and some other of advanced or mixed type. This class is characterized by the condition

(C_7) *For every* $j = 1,\dots, m$,

$$\limsup_{t\to\infty} t^{-1}\sigma_j(t) < \infty.$$

Under this condition and for any strongly sublinear function φ having on $\mathbf{R}^m_{\mp}$ and $\mathbf{R}^m_{=}$ the exponential properties (8) and (17) respectively, one can prove that conditions (C_5) and (C_6) are respectively equivalent to the following ones:

$$(C_8) \qquad \int^{\infty} |a(t)|\varphi(\sigma_1^{n-1}(t),\dots, \sigma_m^{n-1}(t))dt = \infty$$

and

$$(C_9) \qquad \int^{\infty} |a(t)|\varphi(-\sigma_1^{n-1}(t),\dots, -\sigma_m^{n-1}(t))dt = -\infty.$$

Actually, we have the following "if and only if" corollary.

COROLLARY 2.2. *Let the functions a and* φ *be as in Corollary* 2.1 *and let condition* (C_7) *be satisfied.*

If the equation (1) *is strongly sublinear, then both conditions* (C_8) *and* (C_9) *constitute a necessary and sufficient condition in order to have the conclusion of Corollary* 2.1.

PROOF. The necessity follows immediately from Theorem 1.3. For the sufficiency, as stated above, we have to prove the equivalence of conditions (C_5) and (C_6) to (C_8) and (C_9) respectively. But, because of the exponential properties (8), (17) and the increasing character of φ, it is enough to prove that there exists a positive constant M such that for $j=1,\dots, m$

$$M\sigma_j(t) \leqq \tau_j(t) \leqq \sigma_j(t) \quad \text{for all large } t. \tag{19}$$

To this end, we observe that the right part of (19) is obvious by (7). The left one follows from condition (C_7), since, by (7) again, for all large t with $\tau_j(t)\neq 0$ we have

$$\frac{\sigma_j(t)}{\tau_j(t)} = \begin{cases} 1, & \text{if} \quad \sigma_j(t) \leqq t \\ t^{-1}\sigma_j(t)\,, & \text{if} \quad \sigma_j(t) < t\,. \end{cases}$$

The equivalence of conditions (C_5) and (C_6) to (C_8) and (C_9), in general, ceases to hold when condition (C_7) fails. So, condition (C_7) can not be removed from the above corollary. This is illustrated by the following four examples of advanced differential equations. These equations fail to satisfy condition (C_7). However, they satisfy the rest of the assumptions of Corollary 2.2 including conditions (C_8) and (C_9).

EXAMPLE 2.1. The equation

$$x''(t) + (1/4)\, t^{-2}x^{1/3}(t^3) = 0, \quad t \geqq 1$$

has the nonoscillatory solution $x(t)=t^{1/2}$, a contradiction to conclusion (i) of Corollary 2.1.

EXAMPLE 2.2. The equation

$$x'''(t) + 6(t^{12} + t^6)^{-1/3}x^{1/3}(t^6) = 0, \quad t \geqq 1$$

has the solution $x(t)=1+t^{-1}$ for which $\lim_{t\to\infty} x(t)=1$, a contradiction to conclusion (ii) of Corollary 2.1.

EXAMPLE 2.3. The equation

$$x''''(t) - (9/16)\, t^{-4}x^{1/3}(t^3) = 0, \quad t \geqq 1$$

has the solution $x(t)=t^{3/2}$ for which we have

$$\lim_{t\to\infty} x^{(j)}(t) = \begin{cases} \infty & \text{for} \quad j = 0, 1 \\ 0 & \text{for} \quad j = 2, 3, \end{cases}$$

a contradiction to conclusion (iii) of Corollary 2.1.

EXAMPLE 2.4. The equation

$$x'''(t) - (3/8)\, t^{-3}x^{1/3}(t^3) = 0, \quad t \geqq 1$$

has the solution $x(t) = t^{1/2}$ for which we have

$$\lim_{t\to\infty} x^{(j)}(t) = \begin{cases} \infty & \text{for } j = 0 \\ 0 & \text{for } j = 1, 2, \end{cases}$$

a contradiction to conclusion (iv) of Corollary 2.1.

REMARK 2.1. It would be desirable to study the sublinear case without the condition of the exponential properties (8) and (17). From the arguments presented in this section though, it is apparent that the role of these conditions is essential to our method. Perhaps another method would turn out to be more successful and the ensuing results would be of significant importance. This is said in view of the fact that in the superlinear case which follows in the next section no such condition is imposed. We hasten to add that as far as we know the sublinear equations that appeared in the bibliography satisfy the above mentioned exponential properties. So, one usually encounters sublinear differential equations of the form (1) with the continuous function φ defined by

$$\varphi(y_1,\dots, y_m) = |y_1|^{\alpha_1}\cdots|y_m|^{\alpha_m}\operatorname{sgn} y_1$$

at least on the set $\{y \in \mathbf{R}^m\colon (\forall i)y_i > 0 \text{ or } (\forall i)y_i < 0\}$. The simplest case when $m = 1$, i.e. $\varphi(y) = |y|^\alpha \operatorname{sgn} y$ drew much attention in the bibliography.

REMARK 2.2. The results of this section concerning the sublinear case are presented in a form which is essentially new. Parts of Theorems 2.1 and 2.2 are covered by Kusano [8, Theorem 3.2]. Also the results of Kusano and Onose [9, 10] concerning special cases of higher order differential equations are very close to ours.

3. The case of superlinearity

We treat this case by starting again with the differential inequalities $(1)'_{\leqq}$ and $(1)'_{\geqq}$.

THEOREM 3.1. *Let the function a be of constant sign and let τ be a continuously differentiable function on $[t_0, \infty)$ with nonnegative derivative, $\lim_{t\to\infty} \tau(t) = \infty$ and such that*

$$\tau(t) \leqq \min\{t, \sigma_1(t),\dots, \sigma_m(t)\} \quad \textit{for every} \quad t \geqq t_0. \tag{20}$$

If the differential inequality $(1)'_{\leqq}$ is strongly superlinear to the right and

$$(C_{10}) \qquad \int^{\infty} \tau^{n-1}(t)\,|a(t)|dt = \infty,$$

then the conclusion of Theorem 2.1 *holds.*

PROOF. Obviously, by (20), condition (C_{10}) implies (C_2) and hence, as in the proof of Theorem 2.1, it is enough to prove the theorem for the unbounded solutions only. So, let us consider an unbounded solution x of the inequality $(1)'_{\leqq}$, which without loss of generality is supposed nonnegative on the whole interval $[t_0, \infty)$. Also, due to $\lim_{t\to\infty} \tau(t)=\infty$, we choose a sufficiently large T, $T\geqq t_0$, so that

$$\tau(t) \geqq \max\{t_0, 0\} \quad \text{for every} \quad t \geqq T.$$

As in the proof of Theorem 2.1, we also derive (11) and (12), i.e.

$$x^{(n)}(t)I(a) \leqq 0 \quad \text{for every } t \geqq T$$

and

$$x(t) > 0 \quad \text{and} \quad x'(t) \geqq 0 \quad \text{for every} \quad t \geqq T.$$

Finally, $x^{(n-1)}(t)$ is not identically zero for all large t, since, because of condition (C_{10}), the same holds true for $a(t)$.

Now, let t_x, $t_x\geqq T$, be assigned to the function x as in the Kiguradze first or second lemma applied for $k=n$ and let us suppose

$$(21) \qquad x^{(n-1)}(t)I(a) > 0 \quad \text{for every} \quad t \geqq t_x.$$

By setting

$$(22) \qquad z(t) = -x^{(n-1)}(t)I(a)\int_{t_x}^{t} \frac{\tau^{n-2}(s)\tau'(s)}{\varphi(x[\tau(s)],\dots, x[\tau(s)])}ds,$$

we obtain from inequality $(1)'_{\leqq}$

$$(23) \qquad z'(t) \geqq |a(t)|\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)])\int_{t_x}^{t} \frac{\tau^{n-2}(s)\tau'(s)}{\varphi(x[\tau(s)],\dots, x[\tau(s)])}ds$$

$$- x^{(n-1)}(t)I(a)\frac{\tau^{n-2}(t)\tau'(t)}{\varphi(x[\tau(t)],\dots, x[\tau(t)])}.$$

But, because of (20), the increasing character of φ and the fact that $\lim_{t\to\infty} \tau(t) =\infty$, we have

$$(24) \qquad |a(t)|\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)])\int_{t_x}^{t} \frac{\tau^{n-2}(s)\tau'(s)}{\varphi(x[\tau(s)],\dots, x[\tau(s)])}ds$$

$$\geqq |a(t)|\frac{\varphi(x[\sigma_1(t)],\dots, x[\sigma_m(t)])}{\varphi(x[\tau(t)],\dots, x[\tau(t)])}\int_{t_x}^{t} \tau^{n-2}(s)\tau'(s)ds$$

$$\geqq |a(t)| \int_{t_x}^{t} \tau^{n-2}(s)\tau'(s)ds = \frac{1}{n-1}\tau^{n-1}(t)|a(t)|\left(1-\frac{\tau^{n-1}(t_x)}{\tau^{n-1}(t)}\right)$$
$$\geqq K\tau^{n-1}(t)\,|a(t)|$$

for some constant $K>0$ and for all large t. On the other hand, since $\lim_{t\to\infty}\tau(t)=\infty$, from the Kiguradze third lemma applied for $u=x'$, $\theta=1/2$ and $k=n-1$, it follows that

$$x'[\tau(t)/2] \geqq M_{1/2}\tau^{n-2}(t)\,|x^{(n-1)}[\tau(t)]| \quad \text{for all large } t.$$

Since, by (11) and (21), $|x^{(n-1)}|$ is decreasing on $[t_x, \infty)$ and because of (20) we have

$$x'[\tau(t)/2] \geqq M_{1/2}\tau^{n-2}(t)\,|x^{(n-1)}(t)| \quad \text{for all large } t.$$

Using this inequality, by (21) and the increasing character of φ, we get

$$\begin{aligned}(25)\qquad & x^{(n-1)}(t)I(a)\,\tau^{n-2}(t)\tau'(t)/\varphi(x[\tau(t)],\dots, x[\tau(t)]) \\ &= \tau^{n-2}(t)|x^{(n-1)}(t)|\,\tau'(t)/\varphi(x[\tau(t)],\dots, x[\tau(t)]) \\ &\leqq 2x'[\tau(t)/2][\tau(t)/2]'/M_{1/2}\varphi(x[\tau(t)/2],\dots, x[\tau(t)/2])\end{aligned}$$

for all large t. From (23), (24) and (25), it follows that for some $t_1 \geqq t_x$ and every $t \geqq t_1$

$$z'(t) \geqq K\tau^{n-1}(t)\,|a(t)| - 2x'[\tau(t)/2][\tau(t)/2]'/M_{1/2}\varphi(x[\tau(t)/2],\dots, x[\tau(t)/2])$$

Thus, by integration, we have

$$z(t) \geqq z(t_1) + K\int_{t_1}^{t} \tau^{n-1}(s)|a(s)|ds - \frac{2}{M_{1/2}}\int_{x[\tau(t_1)/2]}^{x[\tau(t)/2]} \frac{dy}{\varphi(y,\dots, y)},$$

where $\lim_{t\to\infty} x[\tau(t)/2]=\infty$. Since

$$\int^{\infty} \frac{dy}{\varphi(y,\dots, y)} < \infty,$$

by condition (C_{10}), we obtain $\lim_{t\to\infty} z(t)=\infty$, which contradicts the nonpositivity of z derived from (12), (21) and (22). This means that (21) fails.

We have thus proved that

$$(26)\qquad x^{(n-1)}(t)I(a) < 0 \quad \text{for every} \quad t \geqq t_x.$$

By virtue of the Kiguradze first lemma, this is the case where a is nonpositive and consequently, by (11) and (26) we have

$$x^{(n)}(t) \geqq 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for every} \quad t \geqq t_x.$$

Thus, by Taylor's formula, we obtain

$$x(t) \geqq x(t_x) + \frac{x'(t_x)}{1!}(t-t_x) + \cdots + \frac{x^{(n-1)}(t_x)}{(n-1)!}(t-t_x)^{n-1}$$

for any $t \geqq t_x$ and consequently that there exists a constant $L>0$ with

$$x(t) \geqq Lt^{n-1} \quad \text{for all large } t.$$

Hence, because of $\lim_{t\to\infty} \tau(t) = \infty$,

$$x[\tau(t)] \geqq L\tau^{n-1}(t) \quad \text{for all large } t.$$

But φ is increasing on $\mathbf{R}^m_+$ and $\lim_{t\to\infty} y^{-1}\varphi(y,\dots,y) = \infty$. Therefore, for some $t_2 \geqq t_x$ and every $t \geqq t_2$

$$\varphi(x[\tau(t)],\dots,x[\tau(t)]) \geqq \varphi(L\tau^{n-1}(t),\dots,L\tau^{n-1}(t)) \geqq L\tau^{n-1}(t).$$

From inequality $(1)'_{\leqq}$ and because of the increasing character of φ for $t \geqq t_2$ we obtain

$$\begin{aligned} x^{(n)}(t) &\geqq |a(t)|\varphi(x[\sigma_1(t)],\dots,x[\sigma_m(t)]) \geqq |a(t)|\varphi(x[\tau(t)],\dots,x[\tau(t)]) \\ &\geqq L\tau^{n-1}(t)\,|a(t)|. \end{aligned}$$

Finally, by integration,

$$x^{(n-1)}(t) \geqq x^{(n-1)}(t_2) + L\int_{t_2}^{t} \tau^{n-1}(s)\,|a(s)|ds$$

and hence, by condition (C_{10}), $\lim_{t\to\infty} x^{(n-1)}(t) = \infty$. This easily implies that the solution x satisfies (9).

THEOREM 3.2. *Let the functions a and τ be as in Theorem* 3.1.

If the differential inequality $(1)'_{\geqq}$ is strongly superlinear to the left and satisfies condition (C_{10}), *i.e.*

$$\int^{\infty} \tau^{n-1}(t)\,|a(t)|dt = \infty,$$

then the conclusion of Theorem 2.2 *holds.*

PROOF. It is enough to verify all assumptions of Theorem 3.1 for the differential inequality below, obtained from the inequality $(1)'_{\geqq}$ by the transformation $z=-x$,

$$z^{(n)}(t)I(a) + |a(t)|\hat{\varphi}(z[\sigma_1(t)],\dots,z[\sigma_m(t)]) \leqq 0,$$

where the function $\hat{\varphi}$ is defined by the formula

$$\hat{\varphi}(y_1,\dots, y_m) = -\varphi(-y_1,\dots, -y_m).$$

The following corollary concerning the equation (1), is immediately obtained from the above Theorems 3.1 and 3.2.

COROLLARY 3.1. *Let the functions a and τ be as in Theorem 3.1.*
If the equation (1) *is strongly superlinear and satisfies condition* (C_{10}), *i.e.*

$$\int^{\infty} \tau^{n-1}(t)\,|a(t)|dt = \infty,$$

then:
(i) *For a nonnegative and n even, every solution of* (1) *is oscillatory.*
(ii) *For a nonnegative and n odd, every solution x of* (1) *is oscillatory or satisfies* (6).
(iii) *For a nonpositive and n even, every solution x of* (1) *is oscillatory or satisfies one of* (6), (9) *and* (18).
(iv) *For a nonpositive and n odd, every solution x of* (1) *is oscillatory or satisfies one of* (9) *and* (18).

Differential equations of the form (1) subject to the condition
(C_{11}) *For every* $j=1,\dots, m$

$$\liminf_{t\to\infty} t^{-1}\sigma_j(t) > 0$$

include obviously the ordinary, advanced equations and some other ones of retarded or mixed type. For such differential equations we can take $\tau(t)=ct$, where

$$c = \min\{1,\ 2^{-1}\liminf_{t\to\infty} t^{-1}\sigma_1(t),\dots,\ 2^{-1}\liminf_{t\to\infty} t^{-1}\sigma_m(t)\}$$

and therefore, condition (C_{10}) specializes then to the condition (C_2), i.e.

$$\int^{\infty} t^{n-1}|a(t)|dt = \infty.$$

COROLLARY 3.2. *Let the function a be of constant sign and let condition* (C_{11}) *be satisfied.*
If the equation (1) *is strongly superlinear, then the condition* (C_2) *is a necessary and sufficient condition in order to have the conclusion of Corollary* 3.1.

PROOF. The sufficiency of the condition (C_2) is obvious by Corollary 3.1. Its necessity follows from Theorem 1.2.

Finally, we remark that condition (C_2), though necessary to have the conclusion of Corollary 3.1, it is not always sufficient when (C_{11}) fails. This follows immediately by the examples 3.1–3.4 below, where for each case the conclusion of Corollary 3.1 fails.

EXAMPLE 3.1. For the retarded differential equation

$$x''''(t) + (15/16)\, t^{-4} x^3(t^{1/3}) = 0, \quad t \geqq 1$$

condition (C_2) is satisfied while (C_{11}) fails. This equation has the nonoscillatory solution $x(t) = t^{1/2}$, a contradiction to conclusion (i) of Corollary 3.1.

EXAMPLE 3.2. For the retarded differential equation

$$x'''(t) + (10/27)\, t^{-5/2} x^{7/5}(t^{1/2}) = 0, \quad t \geqq 1$$

condition (C_2) is satisfied while (C_{11}) fails. This equation has the solution $x(t) = t^{5/3}$ for which $\lim_{t\to\infty} x(t) = \infty$, a contradiction to conclusion (ii) of Corollary 3.1.

EXAMPLE 3.3. For the retarded differential equation

$$x''''(t) - (40/81)\, t^{-7/2} x^{7/5}(t^{1/2}) = 0, \quad t \geqq 1$$

condition (C_2) is satisfied while (C_{11}) fails. This equation has the solution $x(t) = t^{5/3}$ for which we have

$$\lim_{t\to\infty} x^{(j)}(t) = \begin{cases} \infty & \text{for } j = 0, 1 \\ 0 & \text{for } j = 2, 3, \end{cases}$$

a contradiction to conclusion (iii) of Corollary 3.1.

EXAMPLE 3.4. For the retarded differential equation

$$x'''(t) - (3/8)\, t^{-3} x^3(t^{1/3}) = 0, \quad t \geqq 1$$

condition (C_2) is also satisfied while (C_{11}) fails. This equation has the solution $x(t) = t^{1/2}$ for which we have

$$\lim_{t\to\infty} x^{(j)}(t) = \begin{cases} \infty & \text{for } j = 0 \\ 0 & \text{for } j = 1, 2, \end{cases}$$

a contradiction to conclusion (iv) of Corollary 3.1.

REMARK 3.1. The results of this section concerning the superlinear case are of high degree of generality. The version presented here is essentially based on Grammatikopoulos, Sficas and Staikos [3]. Theorems 3.1, 3.2 and their Corollary 3.1 are originated in the papers of Kartsatos [4], Ryder and Wend [14] and Sficas and Staikos [17]. For closely related results we refer to Kusano [8], Kusano and Onose [11] and Sficas and Staikos [18].

4. Some applications-The mixed case

The oscillatory and asymptotic results on differential inequalities obtained previously in sections 2 and 3 permit a further generalization which unify Corollaries 2.1 and 3.1, i.e. sublinear and superlinear cases. More precisely, the differential equation under consideration is of the form

$$(27)\quad x^{(n)}(t) + a_1(t)\varphi_1(x[\sigma_{11}(t)],\dots, x[\sigma_{1m_1}(t)]) + a_2(t)\varphi_2(x[\sigma_{21}(t)],\dots, x[\sigma_{2m_2}(t)])$$
$$= h(t;\, x\langle g_0(t)\rangle,\dots, x^{(N-1)}\langle g_{N-1}(t)\rangle), \qquad t \geqq t_0,$$

where all functions involved are continuous and moreover φ_1 has on $\mathbf{R}_+^{m_1}$ and $\mathbf{R}_-^{m_1}$ respectively the exponential properties

$$\varphi_1(x_1y_1,\dots, x_{m_1}y_{m_1}) \geqq K\varphi_1(x_1,\dots, x_{m_1})\varphi_1(y_1,\dots, y_{m_1})$$

and

$$-\varphi_1(-x_1y_1,\dots, -x_{m_1}y_{m_1}) \geqq K\varphi_1(x_1,\dots, x_{m_1})\varphi_1(y_1,\dots, y_{m_1})$$

for some positive constant K.

COROLLARY 4.1. *Let the functions a_1, a_2 be of constant sign with $I(a_1) = I(a_2)$ and let h have the following sign property:*

(C_{12}) *For every t, $y_0,\dots, y_{N-1}$,*

$$(\forall i) y_{0i} > 0 \Longrightarrow I(a_1)h(t;\, y_0,\dots, y_{N-1}) \leqq 0,$$

$$(\forall i) y_{0i} < 0 \Longrightarrow I(a_1)h(t;\, y_0,\dots, y_{N-1}) \geqq 0.$$

Moreover, let the functions τ_j ($j=1,\dots, m_1$),

$$\tau_j(t) = \min\{t, \sigma_{1j}(t)\}$$

and let τ be a continuously differentiable function on $[t_0, \infty)$ with nonnegative derivative, $\lim_{t\to\infty}\tau(t)=\infty$ and such that

$$\tau(t) \leqq \min\{t, \sigma_{21}(t),\dots, \sigma_{2m_2}(t)\} \quad \textit{for every} \quad t \geqq t_0.$$

If φ_1 is strongly sublinear and φ_2 strongly superlinear, then, under condition

$$(C_{13})\quad \int^{\infty} [|a_1(t)|\,|\varphi_1(\delta\tau_1^{n-1}(t),\dots, \delta\tau_{m_1}^{n-1}(t))| + \tau^{n-1}(t)|a_2(t)|]dt = \infty,\ \delta = \pm 1,$$

the equation (27) satisfies the following:

(i) *For a_1, a_2 nonnegative and n even, every solution is oscillatory.*

(ii) *For a_1, a_2 nonnegative and n odd, every solution x is oscillatory or* *satisfies* (6).

(iii) *For a_1, a_2 nonpositive and n even, every solution x is oscillatory or satisfies one of* (6), (9) *and* (18).

(iv) *For a_1, a_2 nonpositive and n odd, every solution x is oscillatory or satisfies one of* (9) *and* (18).

PROOF. Because of condition (C_{12}) and the sign properties of the sublinear function φ_1 and the superlinear one φ_2, the nonnegative solutions of the equation (27) are also solutions of both inequalities

(28) $$x^{(n)}(t)I(a_1) + |a_1(t)|\varphi_1(x[\sigma_{11}(t)],\dots, x[\sigma_{1m_1}(t)]) \leqq 0$$

and

(29) $$x^{(n)}(t)I(a_2) + |a_2(t)|\varphi_2(x[\sigma_{21}(t)],\dots, x[\sigma_{2m_2}(t)]) \leqq 0.$$

Similarly, the nonpositive solutions of the equation (27) are also solutions of both inequalities

(30) $$x^{(n)}(t)I(a_1) + |a_1(t)|\varphi_1(x[\sigma_{11}(t)],\dots, x[\sigma_{1m_1}(t)]) \geqq 0$$

and

(31) $$x^{(n)}(t)I(a_2) + |a_2(t)|\varphi_2(x[\sigma_{21}(t)],\dots, x[\sigma_{2m_2}(t)]) \geqq 0.$$

Since condition (C_{13}) implies

$$\int^{\infty} |a_1(t)|\varphi_1(\delta\tau_1^{n-1}(t),\dots, \delta\tau_{m_1}^{n-1}(t))dt = \delta\infty, \quad \delta = \pm 1$$

or

$$\int^{\infty} \tau^{n-1}(t)\,|a_2(t)|dt = \infty$$

we can apply respectively Theorems 2.1 and 2.2 for the sublinear inequalities (28) and (30), or Theorems 3.1 and 3.2 for the superlinear inequalities (29) and (31) in order to obtain the conclusion of the corollary.

Now, we confine ourselves to dealing with a typical example of a differential equation of the form (27), namely the equation

(32) $$x^{(n)}(t) + a_1(t)\,|x|^{\alpha_1}[\sigma_1(t)] \operatorname{sgn} x[\sigma_1(t)] + a_2(t)\,|x|^{\alpha_2}[\sigma_2(t)] \operatorname{sgn} x[\sigma_2(t)] = 0, \quad t \geqq t_0,$$

where $0<\alpha_1<1<\alpha_2$.

COROLLARY 4.2. *Let the functions a_1, a_2 be of constant sign with $I(a_1) = I(a_2)$ and let σ_1, σ_2 be such that*

(C_{14}) $\limsup_{t\to\infty} t^{-1}\sigma_1(t) < \infty$ *and* $\liminf_{t\to\infty} t^{-1}\sigma_2(t) > 0$.

Then, the condition

(C_{15}) $$\int^\infty [|a_1(t)|\sigma_1^{(n-1)\alpha_1}(t) + t^{n-1}|a_2(t)|]dt = \infty$$

is sufficient in order that the equation (32) *satisfy the conclusion of Corollary* 4.1. *Moreover, condition* (C_{15}) *is also necessary in the extreme cases where either* $a_1=0$ *or* $a_2=0$.

PROOF. Because of the second part of condition (C_{14}), we can apply Corollary 4.1 for the equation (32) by taking $\tau(t)=ct$, where c is an appropriate positive constant. So, condition (C_{13}) specializes then to

$$\int^\infty [|a_1(t)|(\min\{t, \sigma_1(t)\})^{(n-1)\alpha_1} + t^{n-1}|a_2(t)|]dt = \infty,$$

which in turn, by virtue of the first part of condition (C_{14}), is equivalent to the condition (C_{15}). Thus the sufficiency of (C_{15}) follows immediately from Corollary 4.1.

In the extreme cases where either $a_1=0$ or $a_2=0$, condition (C_{15}) takes respectively the form

$$\int^\infty t^{n-1}|a_2(t)|dt = \infty \qquad \text{or} \qquad \int^\infty \sigma_1^{(n-1)\alpha_1}(t)\,|a_1(t)|dt = \infty.$$

Hence, by applying Corollary 3.2 or 2.2, we get also the necessity of condition (C_{15}) for the two considered cases.

REMARK 4.1. As far as we know, it is an open question whether condition (C_{15}) is also necessary for the strict mixed case. For the particular case where $n=2$ and a_1, a_2 are nonnegative, Gollwitzer [2] claims that the necessity always holds true. His claim is based on the assertion that, when condition (C_{15}) fails, then there exists a solution x with $\lim_{t\to\infty} x(t)=1$. But we can illustrate that this assertion fails to be true even in the case of ordinary differential equations. Indeed, for the sublinear differential equation

(33) $$x''(t) + t^{-2}x^{3/5}(t) = 0, \quad t \geqq 1$$

condition (C_{15}) fails. On the other hand, we have

$$\int^\infty tt^{-2}dt = \int^\infty t^{-1}dt = \infty$$

and hence, by Theorem 1.1, equation (33) has no bounded nonoscillatory solution.

REMARK 4.2. The results of this section are indicative of the significance of the approach used in the previous sections 2 and 3. The strictly mixed sublinear-superlinear case is not encountered in the bibliography. The special case of the equation (32) has been recently studied by Koplatadze and Chanturia [7]. Also, for the extreme cases of (32) and in connection with higher order differential equations we mention the work of Ševelo and Vareh [15]. Similarly, a considerable amount of work has been done for second order equations to which the so called Emden-Fowler equation can be reduced. For a survey and an extensive bibliography we refer to the article of Wong [23].

ACKNOWLEDGMENT. The author wishes to thank the referee for his helpful suggestions.

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