HIROSHIMA MATH. J.
11 (1981), 143–172

Fourier transforms and elementary solutions of invariant differential operators on homogeneous vector bundles over compact homogeneous spaces

Masaaki EGUCHI and Keisaku KUMAHARA

(Received August 25, 1980)

§ 1. Introduction

Let G be a connected compact Lie group and K be a closed subgroup of G. Let τ and σ be finite-dimensional unitary representations of K. We denote by E_τ the homogeneous hermitian vector bundle associated with τ. Let $\mathscr{D}(E_\tau)$ be the space of all C^∞ sections of E_τ with usual topology and $\mathscr{D}'(E_\tau)$ be its dual space. A homogeneous differential operator of $\mathscr{D}(E_\sigma)$ to $\mathscr{D}(E_\tau)$ is a left invariant differential operator on $\mathscr{D}(G/K)$ when $\tau=\sigma=$ the identity representation of K. Let $\mathscr{D}$ be $\mathscr{D}(G)$ or $\mathscr{D}(G/K)$. In [1] Cerezo and Rouvière have determined when an invariant differential operator on $\mathscr{D}$ has an elementary solution, by using the Fourier transforms of $\mathscr{D}$ and $\mathscr{D}'$. On the other hand, N. R. Wallach has defined the Fourier transform on E_τ and determined the images of $\mathscr{D}(E_\tau)$ and $\mathscr{D}'(E_\tau)$ in [2].

The main purpose of the present paper is to generalize the notion of elementary solutions to vector bundle case and to characterize homogeneous differential operators which have elementary solutions. For this purpose we adopt a different definition from [2] of the Fourier transform as a direct generalization of [1].

Let V_τ be the representation space of τ. Sections of E_τ can be identified with V_τ-valued functions f which satisfy $f(xk)=\tau(k^{-1})f(x)$ for all $x\in G$ and $k\in K$. We first define the Fourier transforms of vector valued functions in § 2. In § 4 we study the images of $\mathscr{D}(E_\tau)$ and $\mathscr{D}'(E_\tau)$ by the Fourier transform which is the restriction of the above Fourier transform. In § 3 and § 5 we characterize homogeneous differential operators which have elementary solutions.

In [2] Wallach has determined which homogeneous differential operator is globally hypoelliptic. In § 6 we show when a globally hypoelliptic operator has an elementary solution.

§ 2. Fourier transforms of vector valued functions

2.1. Let G be a connected compact Lie group and let dx be the normalized Haar measure on G so that the total measure is one. For a finite-dimensional complex Hilbert space W we denote by $(\, , \,)_W$ and $\| \; \|_W$ the inner product and the norm, respectively. Let $C(G; W)$ be the set of all continuous W-valued functions on G and $C^\infty(G; W)$ be the set of all infinitely differentiable W-valued functions on G. And we denote by $L^2(G; W)$ the set of all measurable W-valued functions f which satisfy

$$\|f\|_2 = \left(\int_G \|f(x)\|_W^2 dx\right)^{1/2} < +\infty.$$

We put

$$(f, g) = \int_G (f(x), g(x))_W dx.$$

Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G. Let π_γ be a fixed representative of $\gamma \in \hat{G}$ and V_γ the representation space of π_γ. We put $d(\gamma) = \dim_{\boldsymbol{C}} V_\gamma$. Let $W(\gamma) = \mathrm{End}\,(V_\gamma) \otimes W$ and $W(\hat{G}) = \bigcup_{\gamma \in \hat{G}} W(\gamma)$ (disjoint union).

DEFINITION. For any $f \in C(G; W)$ we define a $W(\hat{G})$-valued function $\hat{f}$ on $\hat{G}$ by

$$\hat{f}(\gamma) = \int_G \pi_\gamma(x)^{-1} \otimes f(x) dx.$$

We call $\hat{f}$ the *Fourier transform* of f.

Let $e_1, \ldots, e_m$ $(m = \dim_{\boldsymbol{C}} W)$ be an orthonormal basis of W. For any two elements $A = \sum_{j=1}^m L_j \otimes e_j$ and $B = \sum_{j=1}^m L'_j \otimes e_j$ of $W(\gamma)$ we put

$$(A, B)_{\gamma, W} = \sum_{j=1}^m \mathrm{tr}\,(L'^*_j L_j),$$

where the asterisk denotes the adjoint of the matrix. Then it is not difficult to see that $(\, , \,)_{\gamma, W}$ is independent of the choice of the orthonormal basis of W and it defines an inner product of $W(\gamma)$. We put $\|A\|_{\gamma, W} = (A, A)_{\gamma, W}^{1/2}$ and we put $(A, B)_\gamma = (A, B)_{\gamma, W}$ and $\|A\|_\gamma = \|A\|_{\gamma, W}$ in the case when $W = \boldsymbol{C}$. For any $W(\hat{G})$-valued function a on $\hat{G}$ such that $a(\gamma) \in W(\gamma)$, we put

$$\|a\|^2 = \sum_{\gamma \in \hat{G}} d(\gamma) \|a(\gamma)\|_{\gamma, W}^2.$$

Let $L^2(\hat{G}; W)$ be the set of all $W(\gamma)$-valued functions a on $\hat{G}$ such that $a(\gamma) \in W(\gamma)$ for all $\gamma \in \hat{G}$ and $\|a\| < +\infty$. Then $L^2(\hat{G}; W)$ is a Hilbert space with the norm $\| \; \|$ and the inner product

$$(a, b) = \sum_{\gamma \in \hat{G}} d(\gamma) (a(\gamma), b(\gamma))_{\gamma, W}.$$

The following lemma is well known.

LEMMA 1. *When* $\dim_{\boldsymbol{C}} W=1$, *i.e.*, $W=\boldsymbol{C}$, *we have the following assertions*:
(1) (*Parseval's equality*) *If* $f\in C(G;\boldsymbol{C})$, *then*

$$\|f\|_2 = \|\hat{f}\|;$$

(2) (*Inversion formula*) *If* $f\in C^\infty(G;\boldsymbol{C})$, *then*

$$f(x) = \sum_{\gamma\in\hat{G}} d(\gamma)\,\mathrm{tr}\,(\pi_\gamma(x)\hat{f}(\gamma));$$

(3) *The mapping* $f\mapsto\hat{f}$ *can be extended to an isometry of* $L^2(G;\boldsymbol{C})$ *onto* $L^2(\hat{G};\boldsymbol{C})$.

For any element $A=\sum_{j=1}^m L_j\otimes e_j\in W(\gamma)$ we put

$$\mathrm{tr}\,(A) = \sum_{j=1}^m \mathrm{tr}\,(L_j)e_j.$$

Then we can easily see that tr is independent of the choice of the basis of W and that it is a linear mapping of $W(\gamma)$ to W.

Let I_W be the identity operator on W. If we apply the above lemma to each coordinate, we have immediately the following lemma.

LEMMA 2. (1) *If* $f\in C(G;W)$, *then*

$$\|f\|_2 = \|\hat{f}\|;$$

(2) *If* $f\in C^\infty(G;W)$, *then*

$$f(x) = \sum_{\gamma\in\hat{G}} d(\gamma)\,\mathrm{tr}\,((\pi_\gamma(x)\otimes I_W)\hat{f}(\gamma));$$

(3) *The mapping* $f\mapsto\hat{f}$ *can be extended to an isometry of* $L^2(G;W)$ *onto* $L^2(\hat{G};W)$.

2.2. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{G}$ the universal enveloping algebra of the complexification of $\mathfrak{g}$. We identify $\mathfrak{G}$ with the algebra of all left invariant differential operators on G. Let V be another finite-dimensional complex Hilbert space. Then any element $D=\sum_j \xi_j\otimes L_j$ of $\mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V,W)$ defines a linear mapping of $C^\infty(G;V)$ to $C^\infty(G;W)$ by

$$(Df)(x) = \sum_j((\xi_j\otimes L_j)f)(x) = \sum_j L_j((\xi_j f)(x)).$$

Let $d\pi_\gamma$ be the differential representation of π_γ. For any $\gamma\in\hat{G}$ and $\xi\in\mathfrak{G}$ we put $\hat{\xi}(\gamma)=d\pi_\gamma(\xi)$. Let $X_1,\dots,X_n$ ($n=\dim G$) be a basis of $\mathfrak{g}$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_p)$, $\alpha_j\in\boldsymbol{N}$, $1\le\alpha_j\le n$, of length p we put $X^\alpha=X_{\alpha_1}\cdots X_{\alpha_p}$ and $(X^\alpha)^*=(-1)^p X_{\alpha_p}\cdots X_{\alpha_1}$. If $\xi=\sum_\alpha c_\alpha X^\alpha$ ($c_\alpha\in\boldsymbol{C}$), we put $\xi^*=\sum_\alpha \bar{c}_\alpha(X^\alpha)^*$. Then ξ^* is the L^2-adjoint of ξ, i.e., for any $f,g\in C^\infty(G;\boldsymbol{C})$, $(\xi f,g)=(f,\xi^* g)$. It is easy to see that

$$(\xi^*)\hat{}(\gamma) = (\hat{\xi}(\gamma))^*.$$

For any $\gamma \in \hat{G}$ and $D = \sum_j \xi_j \otimes L_j \in \mathfrak{G} \otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$ we put

$$\hat{D}(\gamma) = \sum_j \hat{\xi}_j(\gamma) \otimes L_j.$$

Then $\hat{D}(\gamma)$ is an element of $\mathrm{End}(V_\gamma) \otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$. We put

$$D^* = \sum_j \xi_j^* \otimes L_j^*.$$

Then we have $(D^*f, g) = (f, Dg)$ for all $f \in C^\infty(G; W)$ and $g \in C^\infty(G; V)$. As is easily seen, we have

$$(D^*)\hat{}(\gamma) = (\hat{D}(\gamma))^*.$$

If $X \in \mathfrak{g}$ and $L \in \mathrm{Hom}_{\boldsymbol{C}}(V, W)$, then we have

$$((X \otimes L)f)\hat{}(\gamma) = (\hat{X}(\gamma) \otimes L)\hat{f}(\gamma)$$

for any $f \in C^\infty(G; V)$. Hence we have the following lemma.

LEMMA 3. *Let $D \in \mathfrak{G} \otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$. Then*

$$(Df)\hat{}(\gamma) = \hat{D}(\gamma)\hat{f}(\gamma), \qquad (f \in C^\infty(G; V),\ \gamma \in \hat{G}).$$

The following two lemmas are analogies of Proposition 4 of [1].

LEMMA 4. *Let $D \in \mathfrak{G} \otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$. Then the following statements* (1) *and* (2) *are equivalent*:

(1) *The mapping D of $C^\infty(G; V)$ to $C^\infty(G; W)$ is injective*;

(2) *For all $\gamma \in \hat{G}$ there exists a left inverse of $\hat{D}(\gamma)$.*

LEMMA 5. *Let $D \in \mathfrak{G} \otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$. Then the following statements* (1) *and* (2) *are equivalent*:

(1) *The image of $C^\infty(G; V)$ by D is dense in $C^\infty(G; W)$*;

(2) *For all $\gamma \in \hat{G}$ there exists a right inverse of $\hat{D}(\gamma)$.*

PROOF OF LEMMA 4. (2)⇒(1). Let $f \in C^\infty(G; V)$ and assume that $Df = 0$. Then, for all $\gamma \in \hat{G}$, $\hat{D}(\gamma)\hat{f}(\gamma) = (Df)\hat{}(\gamma) = 0$ by Lemma 3. Let $\hat{D}(\gamma)_{\mathrm{L}}^{-1}$ be a left inverse of $\hat{D}(\gamma)$, i.e. $\hat{D}(\gamma)_{\mathrm{L}}^{-1} \in \mathrm{End}(V_\gamma) \otimes \mathrm{Hom}_{\boldsymbol{C}}(W, V) = \mathrm{Hom}_{\boldsymbol{C}}(W, V)(\gamma)$ and $\hat{D}(\gamma)_{\mathrm{L}}^{-1}\hat{D}(\gamma) = I_{V_\gamma} \otimes I_V$. Then $\hat{f}(\gamma) = \hat{D}(\gamma)_{\mathrm{L}}^{-1}\hat{D}(\gamma)\hat{f}(\gamma) = 0$ for all $\gamma \in \hat{G}$. Therefore, we have $f = 0$ by Lemma 2.

(1)⇒(2). Let us assume that $\hat{D}(\gamma)A = 0$ for $A \in V(\gamma)$. We define a function $f \in C^\infty(G; V)$ by

$$f(x) = d(\gamma)\,\mathrm{tr}\,((\pi_\gamma(x) \otimes I_V)A).$$

Then $\hat{f}(\gamma) = A$ and $\hat{f}(\gamma') = 0$ for $\gamma' \neq \gamma$, $\gamma' \in \hat{G}$. Hence we have $(Df)\hat{}(\gamma') = 0$ for all

$\gamma' \in \hat{G}$. Therefore, $Df=0$ and hence $f=0$ by the assumption. By the irreducibility of π_γ we have $A=0$. Thus we see that $\hat{D}(\gamma)$ is injective. Q.E.D.

PROOF OF LEMMA 5. (2)⇒(1). We assume $A \in W(\gamma)$ and put $f(x) = \mathrm{tr}((\pi_\gamma(x) \otimes I_V)\hat{D}(\gamma)_R^{-1}A) \in C^\infty(G; V)$, where $\hat{D}(\gamma)_R^{-1}$ is a right inverse of $\hat{D}(\gamma)$. Then $\hat{f}(\gamma) = \hat{D}(\gamma)_R^{-1}A/d(\gamma)$ and $\hat{f}(\gamma')=0$ for $\gamma' \neq \gamma$, $\gamma' \in \hat{G}$. Hence we have $(Df)^\wedge(\gamma) = A/d(\gamma)$ and $(Df)^\wedge(\gamma')=0$ for $\gamma' \neq \gamma$. Therefore, we have $(Df)(x) = \mathrm{tr}((\pi_\gamma(x) \otimes I_W)A)$. Since the set of linear combinations of $\mathrm{tr}((\pi_\gamma(x) \otimes I_W)A)$, $A \in W(\gamma)$ and $\gamma \in \hat{G}$, is dense in $C^\infty(G; W)$, the image of D is dense in $C^\infty(G; \mathrm{W})$.

(1)⇒(2). By the denseness of the image of D in $C^\infty(G; W)$ we can see that D^* is injective. Hence there exists a left inverse $(D^*)^\wedge(\gamma)_L^{-1}$ of $(D^*)^\wedge(\gamma)$ for all $\gamma \in \hat{G}$ by Lemma 4. As $\hat{D}(\gamma) = ((D^*)^\wedge(\gamma))^*$, $((D^*)^\wedge(\gamma)_L^{-1})^*$ is a right inverse of $\hat{D}(\gamma)$. Q.E.D.

Let us fix a positive definite inner product on $\mathfrak{g}$ which is invariant under $\mathrm{Ad}(G)$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to this inner product. Let p be a positive integer. For a multi-index α of length p we put $|\alpha| = p$. For any $\ell \in \boldsymbol{N}$ we define a differential operator D_ℓ by

$$D_\ell = \sum_{0 \le |\alpha| \le \ell} (X^\alpha)^* X^\alpha,$$

where $X^\alpha = X_{\alpha_1} \cdots X_{\alpha_p}$. The following two lemmas are due to Cerezo and Rouvière [1, p. 564].

LEMMA 6. *Let $\Delta = -\sum_{i=1}^n X_i^2$ be the Laplacian of G. Then $D_\ell = \sum_{j=0}^{\ell} \Delta^j$.*

Hence D_ℓ is an element of the center of $\mathfrak{G}$. Therefore, $\hat{D}_\ell(\gamma)$ is a scalar operator of V_γ for all $\gamma \in \hat{G}$. We put $\hat{D}_\ell(\gamma) = d_\ell(\gamma) I_{V_\gamma}$.

LEMMA 7. *For any $\gamma \in \hat{G}$,*

$$1 \le d_0(\gamma) \le d_1(\gamma) \le \cdots \le d_\ell(\gamma) \le \cdots.$$

And for any $\ell \in \boldsymbol{N}$ and α such that $|\alpha| \le \ell$,

$$\|(X^\alpha)^\wedge(\gamma)v\|_{V_\gamma} \le d_\ell(\gamma)^{1/2} \|v\|_{V_\gamma} \quad \text{for all} \quad v \in V_\gamma.$$

2.3. If we identify $(X^\alpha)^\wedge(\gamma)$ with $(X^\alpha)^\wedge(\gamma) \otimes I_W$, we have

$$\begin{aligned} \|X^\alpha f\|_2^2 &= \sum_{\gamma \in \hat{G}} d(\gamma) \|(X^\alpha)^\wedge(\gamma)\hat{f}(\gamma)\|_{\gamma,W}^2 \\ &\le \sum_{\gamma \in \hat{G}} d(\gamma) d_\ell(\gamma) \|\hat{f}(\gamma)\|_{\gamma,W}^2 \qquad (|\alpha| \le \ell) \\ &\le \sum_{\gamma \in \hat{G}} d(\gamma) \|\hat{D}_\ell(\gamma)\hat{f}(\gamma)\|_{\gamma,W}^2 = \|D_\ell f\|_2^2. \end{aligned}$$

For $f \in C^\infty(G; W)$ we set $\mu_\alpha^1(f) = \|X^\alpha f\|_2$ and $\mu_\ell^2(f) = \|D_\ell f\|_2$. Then we get the following lemma.

LEMMA 8. *The topology on* $C^\infty(G; W)$ *defined by the system of seminorms* $\{\mu_\alpha^1\}_\alpha$ *coincides with the topology on it defined by the system of seminorms* $\{\mu_\ell^2\}_\ell$.

We topologize $C^\infty(G; W)$ by the above seminorms and denote it by $\mathscr{D}(G; W)$. Let $f \in L^2(G; W)$. Then, by Sobolev's lemma, f is infinitely differentiable if and only if $X^\alpha f \in L^2(G; W)$ for all multi-indices α. Hence we have the following.

LEMMA 9. *Let* $f \in L^2(G; W)$. *Then* $f \in \mathscr{D}(G; W)$ *if and only if*

$$\sum_{\gamma\in\hat{G}} d(\gamma) \|(X^\alpha)^\wedge(\gamma)\hat{f}(\gamma)\|_{\gamma,W}^2 < +\infty \qquad \text{for all multi-indices} \quad \alpha.$$

Let $T: f \mapsto \langle T, f\rangle$ be a linear mapping of $\mathscr{D}(G; W)$ to $\boldsymbol{C}$. Then T is continuous if and only if there exist a constant $C>0$ and an integer $\ell \geq 0$ such that $|\langle T, f\rangle| \leq C\|D_\ell f\|_2$ for all $f \in \mathscr{D}(G; W)$. We denote by $\mathscr{D}'(G; W)$ the set of all continuous linear functionals on $\mathscr{D}(G; W)$. Let $\hat{W}$ be the dual space of W and we denote by $\langle \phi, w\rangle_W$ the value of $\phi \in \hat{W}$ at $w \in W$. Let $e_1, \dots, e_m$ be an orthonormal basis of W and $\phi_1, \dots, \phi_m$ be its dual basis of $\hat{W}$. Let $a_{pq}^\gamma(x)$, $p, q = 1, \dots, d(\gamma)$, be the matrix entries of $\pi_\gamma(x)$ with respect to a fixed orthonormal basis in V_γ. If E_{pq}^γ are matrix units, then $\pi_\gamma(x) \otimes w = \sum_{p,q=1}^{d(\gamma)} E_{pq}^\gamma \otimes a_{pq}^\gamma(x) w$, $w \in W$. The functions $x \mapsto a_{pq}^\gamma(x) w$ are members of $\mathscr{D}(G; W)$. For $T \in \mathscr{D}'(G; W)$ we put

$$\hat{T}(\gamma)_j = \sum_{p,q=1}^{d(\gamma)} \langle T, a_{pq}^\gamma(x^{-1})e_j\rangle E_{pq}^\gamma, \quad \hat{T}(\gamma^*)_j = \sum_{p,q=1}^{d(\gamma)} \langle T, a_{pq}^\gamma(x)e_j\rangle E_{pq}^\gamma$$

and

$$\hat{T}(\gamma) = \sum_{j=1}^m \hat{T}(\gamma)_j \otimes \phi_j, \quad \hat{T}(\gamma^*) = \sum_{j=1}^m \hat{T}(\gamma^*)_j \otimes \phi_j.$$

Then $\hat{T}(\gamma)$ and $\hat{T}(\gamma^*)$ are elements of $\hat{W}(\gamma) = \mathrm{End}\,(V_\gamma) \otimes \hat{W}$.

With any $g \in \mathscr{D}(G; \hat{W})$ we associate an element $\theta(g)$ of $\mathscr{D}'(G; W)$ by

$$\langle \theta(g), f\rangle = \int_G \langle g(x), f(x)\rangle_W dx, \qquad f \in \mathscr{D}(G; W). \tag{2.1}$$

Then we have the following immediately.

LEMMA 10. *Let* $g \in \mathscr{D}(G; W)$. *Then* $\theta(g)^\wedge(\gamma) = \hat{g}(\gamma)$ *for all* $\gamma \in \hat{G}$.

For $A = \sum_{j=1}^m L_j \otimes e_j \in W(\gamma)$ and $B = \sum_{j=1}^m L'_j \otimes \phi_j \in \hat{W}(\gamma)$ we put

$$\langle B, A\rangle_{\gamma,W} = \sum_{j=1}^m \mathrm{tr}\,(L'_j L_j).$$

If $f \in \mathscr{D}(G; W)$ and $f(x) = \sum_{j=1}^m f_j(x) e_j$, then

$$f(x) = \sum_{\gamma\in\hat{G}} d(\gamma) \sum_{j=1}^m \mathrm{tr}\,(\pi_\gamma(x)\hat{f}_j(\gamma)) e_j,$$

which converges in the sense of the topology of $\mathscr{D}(G; W)$. Hence we have, for $T \in \mathscr{D}'(G; W)$,

$$\langle T, f\rangle = \sum_{\gamma\in\hat{G}} d(\gamma) \sum_{j=1}^m \sum_{p,q=1}^{d(\gamma)} \langle T, a_{pq}^\gamma(x)e_j\rangle \hat{f}_j(\gamma)_{qp}$$

$$= \sum_{\gamma\in\hat{G}} d(\gamma) \sum_{j=1}^{m} \operatorname{tr}(\hat{T}(\gamma^*)_j \hat{f}_j(\gamma)) = \sum_{\gamma\in\hat{G}} d(\gamma) \langle \hat{T}(\gamma^*), \hat{f}(\gamma)\rangle_{\gamma,W}.$$

Then there exist a constant $C>0$ and an $\ell\in\boldsymbol{N}$ such that

$$C\|D_\ell f\|_2 \geq |\sum_{\gamma\in\hat{G}} d(\gamma)\langle \hat{T}(\gamma^*), \hat{f}(\gamma)\rangle_{\gamma,W}|$$
$$= |\sum_{\gamma\in\hat{G}} d(\gamma) d_\ell(\gamma)^{-1}\langle \hat{T}(\gamma^*), (D_\ell f)^{\wedge}(\gamma)\rangle_{\gamma,W}|.$$

By Lemmas 7 and 5, $D_\ell \mathscr{D}(G; W)$ is dense in $\mathscr{D}(G; W)$ and hence so is in $L^2(G; W)$. Therefore, the mapping $h\mapsto\Phi(h)$ defined by

$$\Phi(h) = \sum_{\gamma\in\hat{G}} d(\gamma) d_\ell(\gamma)^{-1}\langle \hat{T}(\gamma^*), \hat{h}(\gamma)\rangle_{\gamma,W}$$

is a continuous linear functional on $L^2(G; W)$. Then by Riesz's theorem and Lemma 2 there exists a function $a\in L^2(\hat{G}; W)$ such that

$$(\hat{h}, a) = \Phi(h) \quad \text{and} \quad \|a\| \leq C.$$

If we put $h(x)=D_\ell(\operatorname{tr}(\pi_\gamma(x)L)w)$ for $L\in\operatorname{End}(V_\gamma)$ and $w\in W$, then we see that $\hat{h}(\gamma) = d_\ell(\gamma)d(\gamma)^{-1}L\otimes w$ and $\hat{h}(\gamma')=0$ for $\gamma'\neq\gamma$, $\gamma'\in\hat{G}$. Hence

$$d_\ell(\gamma)^{-1}\langle \hat{T}(\gamma^*), L\otimes w\rangle_{\gamma,W} = (L\otimes w, a(\gamma))_{\gamma,W} = \langle a(\gamma)^*, L\otimes w\rangle_{\gamma,W},$$

where A^* for $A=\sum_{j=1}^{m} L_j\otimes e_j\in W(\gamma)$ is defined by $A^*=\sum_{j=1}^{m} L_j^*\otimes\phi_j$. Then we have $a(\gamma)^*=d_\ell(\gamma)^{-1}\hat{T}(\gamma^*)$. Then

$$\sum_{\gamma\in\hat{G}} d(\gamma)d_\ell(\gamma)^{-2}\|\hat{T}(\gamma^*)\|^2_{\gamma,\hat{W}} = \sum_{\gamma\in\hat{G}} d(\gamma)\,\|a(\gamma)^*\|^2_{\gamma,\hat{W}}$$
$$= \sum_{\gamma\in\hat{G}} d(\gamma)\,\|a(\gamma)\|^2_{\gamma,W} = \|a\|^2 \leq C^2. \tag{2.2}$$

Let $W_o=\sum_{j=1}^{m}\boldsymbol{R}e_j$ be the real form of W generated by $e_1,\dots, e_m$. Then $W=W_o+(-1)^{1/2}W_o$. We denote by conj the conjugation of W with respect to W_o : $\operatorname{conj}(w_1+(-1)^{1/2}w_2)=w_1-(-1)^{1/2}w_2$ $(w_1, w_2\in W_o)$. The conjugation $\bar{f}$ of $f\in\mathscr{D}(G; W)$ is defined by $\bar{f}(x)=\operatorname{conj}(f(x))$. Then the conjugation $\bar{T}$ of $T\in\mathscr{D}'(G; W)$ is defined by $\langle\bar{T}, f\rangle=\overline{\langle T, \bar{f}\rangle}$. Then we can easily see that $\hat{T}(\gamma^*)_j = \hat{\bar{T}}(\gamma)_j^*$ for all $j=1,\dots, m$. Hence by (2.2) we have

$$\sum_{\gamma\in\hat{G}} d(\gamma)d_\ell(\gamma)^{-2}\|\hat{\bar{T}}(\gamma)\|^2_{\gamma,\hat{W}} \leq C^2. \tag{2.3}$$

Since T and $\bar{T}$ are simultaneously members of $\mathscr{D}'(G; W)$, T satisfies the same type of inequality as (2.3). That is, there are a $C>0$ and an $\ell\in N$ such that

$$\sum_{\gamma\in\hat{G}} d(\gamma)d_\ell(\gamma)^{-2}\|\hat{T}(\gamma)\|^2_{\gamma,\hat{W}} \leq C^2.$$

Conversely, we assume that a $\hat{W}(\hat{G})$-valued function b on $\hat{G}$ satisfies; (1) $b(\gamma)\in\hat{W}(\gamma)$ for all $\gamma\in\hat{G}$ and (2) the sum

$$\sum_{\gamma\in\hat{G}} d(\gamma)d_\ell(\gamma)^{-2}\|b(\gamma)\|^2_{\gamma,\hat{W}}$$

has a finite value, say C^2 ($C>0$), for some $\ell \in \boldsymbol{N}$. Let $b(\gamma)=\sum_{j=1}^{m} b(\gamma)_j \otimes \phi_j$. We put

$$\langle T_b, f\rangle = \sum_{\gamma\in\hat{G}} d(\gamma) \sum_{j=1}^{m} \{\mathrm{tr}\,(b(\gamma)_j^* \hat{f}_j(\gamma))\}^- \tag{2.4}$$

for $f\in \mathscr{D}(G; W)$. By taking $f(x)=\{a_{qp}^{\gamma}(x)\}^- \cdot e_j$, we can see that $\hat{T}_b(\gamma)=b(\gamma)$.

DEFINITION. We call the $\hat{W}(\hat{G})$-valued function $\hat{T}$ on $\hat{G}$ the *Fourier transform of* $T\in \mathscr{D}'(G; W)$.

Then we have already proved the following.

PROPOSITION 1. *A $\hat{W}(\hat{G})$-valued function b on $\hat{G}$ is the Fourier transform of some $T\in\mathscr{D}'(G; W)$ if and only if it satisfies the following conditions* (1) *and* (2):

(1) *$b(\gamma)\in \hat{W}(\gamma)$ for all $\gamma\in\hat{G}$;*

(2) *There exists an $\ell\in\boldsymbol{N}$ such that*

$$\sum_{\gamma\in\hat{G}} d(\gamma) d_\ell(\gamma)^{-2} \|b(\gamma)\|_{\gamma,\hat{W}}^2 < +\infty.$$

In particular, if $W=\boldsymbol{C}$ and $T=\delta$ (Dirac's delta), then $\hat{\delta}(\gamma)=I_{V_\gamma}$ and $\|\hat{\delta}(\gamma)\|_\gamma^2 = d(\gamma)$. Hence we have the following corollary.

COROLLARY (Cerezo and Rouvière [1, p. 567]). *There is an integer $\ell_o\in\boldsymbol{N}$ such that*

$$\sum_{\gamma\in\hat{G}} (d(\gamma)/d_{\ell_o}(\gamma))^2 < +\infty.$$

For each $f\in C^\infty(G; W)$ and α we set $\mu_\alpha^3(f)=\sup_{x\in G} \|X^\alpha f(x)\|_W$.

LEMMA 11. *The topology of $C^\infty(G; W)$ defined by the system of seminorms $\{\mu_\alpha^3\}_\alpha$ coincides with that of $\mathscr{D}(G; W)$.*

PROOF. Let $f\in C^\infty(G; W)$. Then for any multi-index α

$$\begin{aligned}
\|X^\alpha f(x)\|_W^2 &= \|\textstyle\sum_{\gamma\in\hat{G}} d(\gamma)\,\mathrm{tr}\,(\pi_\gamma(x)\otimes I_W)(X^\alpha f)^\wedge(\gamma)\|_W^2 \\
&\le (\textstyle\sum_{\gamma\in\hat{G}} d(\gamma) d_\ell(\gamma)^{-1} \|\mathrm{tr}\,(\pi_\gamma(x)\otimes I_W)(D_\ell X^\alpha f)^\wedge(\gamma)\|_W)^2 \\
&\le (\textstyle\sum_{\gamma\in\hat{G}} d(\gamma)^2 d_\ell(\gamma)^{-2})(\sum_{\gamma\in\hat{G}} \|\mathrm{tr}\,(\pi_\gamma(x)\otimes I_W)(D_\ell X^\alpha f)^\wedge(\gamma)\|_W^2) \\
&\le (\textstyle\sum_{\gamma\in\hat{G}} d(\gamma)^2 d_\ell(\gamma)^{-2})(\sum_{\gamma\in\hat{G}} d(\gamma)\, \|(D_\ell X^\alpha f)^\wedge(\gamma)\|_{\gamma,W}^2) \\
&= (\textstyle\sum_{\gamma\in\hat{G}} d(\gamma)^2 d_\ell(\gamma)^{-2})\, \|X^\alpha D_\ell f\|_2^2.
\end{aligned}$$

Hence if $\ell>2\ell_o+|\alpha|$, then there exists a constant $C>0$ such that

$$\|X^\alpha f(x)\|_W \le C\|D_\ell f\|_2$$

for all $f\in C^\infty(G; W)$ and for all $x\in G$ by the corollary of Proposition 1. Conversely,

$$\|D_\ell f\|_2^2 = \int_G \|D_\ell f(x)\|_W^2 dx \leq \sup_{x\in G} \|D_\ell f(x)\|_W^2. \qquad \text{Q.E.D.}$$

Hence we can call any element of $\mathscr{D}'(G; W)$ a distribution on G. The following lemma is due to Cerezo and Rourière [1, Lemma 3].

LEMMA 12. *For any $j_1,\dots,j_p \in \boldsymbol{N}$ there exists a constant $C>0$ such that*

$$d_{j_1}(\gamma)\cdots d_{j_p}(\gamma) \leq C d_\ell(\gamma)$$

for all $\ell \geq 2(j_1+\cdots+j_p)$ and for all $\gamma \in \hat{G}$.

From Proposition 1 and Lemma 12 we have the following.

LEMMA 13. *A $\hat{W}(\hat{G})$-valued function b on $\hat{G}$ such that $b(\gamma)\in\hat{W}(\gamma)$ for all $\gamma\in\hat{G}$ is the Fourier transform of a distribution on G if and only if there exist a $C>0$ and an $\ell\in\boldsymbol{N}$ such that*

$$\|b(\gamma)\|_{\gamma,\hat{W}} \leq C d_\ell(\gamma)/d(\gamma)^{1/2} \qquad \textit{for all} \quad \gamma\in\hat{G}.$$

PROOF. The necessity can be seen easily. So we prove the sufficiency. By Lemma 12 there is a $C'>0$ such that $d_\ell(\gamma)d_{\ell_o}(\gamma) \leq C' d_{2(\ell+\ell_o)}(\gamma)$ for every $\gamma\in\hat{G}$. Then

$$\begin{aligned} &\|b(\gamma)\|_{\gamma,\hat{W}}/d_{2(\ell+\ell_o)}(\gamma) \\ &\quad = (\|b(\gamma)\|_{\gamma,\hat{W}}/d_\ell(\gamma))(d_\ell(\gamma)d_{\ell_o}(\gamma)/d_{2(\ell+\ell_o)}(\gamma))/d_{\ell_o}(\gamma) \leq CC' d_{\ell_o}(\gamma)^{-1}. \end{aligned} \tag{2.5}$$

Hence

$$\textstyle\sum_{\gamma\in\hat{G}} d(\gamma)d_{2(\ell+\ell_o)}(\gamma)^{-2}\|b(\gamma)\|_{\gamma,\hat{W}}^2 \leq C^2C'^2 \sum_{\gamma\in\hat{G}} d(\gamma)^2 d_{\ell_o}(\gamma)^{-2} < +\infty. \tag{2.6}$$

Therefore, by Proposition 1 there exists a distribution T such that $\hat{T}=b$.

Q.E.D.

To obtain (2.6) from (2.5) it is sufficient to use the inequality $\|b(\gamma)\|_{\gamma,\hat{W}} \leq C d_\ell(\gamma)d(\gamma)^{1/2}$. Hence we have the following corollary.

COROLLARY. *The following statements* (1), (2) *and* (3) *are equivalent*:

(1) *There exist an $\ell\in\boldsymbol{N}$ and a $C>0$ such that $\|b(\gamma)\|_{\gamma,\hat{W}} \leq C d_\ell(\gamma)/d(\gamma)^{1/2}$ for all $\gamma\in\hat{G}$;*

(2) *There exist an $\ell\in\boldsymbol{N}$ and a $C>0$ such that $\|b(\gamma)\|_{\gamma,\hat{W}} \leq C d_\ell(\gamma)$ for all $\gamma\in\hat{G}$;*

(3) *There exist an $\ell\in\boldsymbol{N}$ and a $C>0$ such that $\|b(\gamma)\|_{\gamma,\hat{W}} \leq C d_\ell(\gamma)d(\gamma)^{1/2}$ for all $\gamma\in\hat{G}$.*

LEMMA 14. *Let a be a $W(\hat{G})$-valued function on $\hat{G}$ such that $a(\gamma)\in W(\gamma)$ for all $\gamma\in\hat{G}$. Then a is the Fourier transform of a function of $\mathscr{D}(G; W)$ if and only if for any $\ell\in\boldsymbol{N}$ there is a constant $C_\ell>0$ such that $\|a(\gamma)\|_{\gamma,W} \leq C_\ell d(\gamma)^{1/2}/d_\ell(\gamma)$*

for all $\gamma \in \hat{G}$.

PROOF. The necessity is easy to see by Lemma 9. To prove the sufficiency we choose a constant C'_ℓ so that $d_{\ell_o}(\gamma)d_\ell(\gamma) \leq C'_\ell d_{2(\ell+\ell_o)}(\gamma)$. If there exists a constant C_ℓ which satisfies the inequality in the lemma, then for any $\ell \in \boldsymbol{N}$

$$\begin{aligned} &\textstyle\sum_{\gamma\in\hat{G}} d(\gamma)d_\ell(\gamma)^2 \|a(\gamma)\|^2_{\gamma,W} \\ &\leq C'^2_\ell \textstyle\sum_{\gamma\in\hat{G}} (d(\gamma)/d_{\ell_o}(\gamma)^2) d_{2(\ell+\ell_o)}(\gamma)^2 \|a(\gamma)\|^2_{\gamma,W} \\ &\leq (C'_\ell C_{2(\ell+\ell_o)})^2 \textstyle\sum_{\gamma\in\hat{G}} (d(\gamma)/d_{\ell_o}(\gamma))^2 < +\infty. \end{aligned}$$

In particular, if we put $\ell=0$, we then have $a \in L^2(\hat{G}; W)$. Hence we obtain the sufficiency from Lemmas 7 and 9. Q.E.D.

By Lemma 9, if $f \in \mathscr{D}(G; W)$, then $\|\hat{f}(\gamma)\|_{\gamma,W} \leq C(d_\ell(\gamma)d(\gamma)^{1/2})^{-1}$.

COROLLARY. *The following statements* (1), (2) *and* (3) *are equivalent*:

(1) *For any* $\ell \in \boldsymbol{N}$ *there exists a* $C>0$ *such that* $\|a(\gamma)\|_{\gamma,W} \leq Cd(\gamma)^{-1/2}d_\ell(\gamma)^{-1}$ *for all* $\gamma \in \hat{G}$;

(2) *For any* $\ell \in \boldsymbol{N}$ *there exists a* $C>0$ *such that* $\|a(\gamma)\|_{\gamma,W} \leq Cd_\ell(\gamma)^{-1}$ *for all* $\gamma \in \hat{G}$;

(3) *For any* $\ell \in \boldsymbol{N}$ *there exists a* $C>0$ *such that* $\|a(\gamma)\|_{\gamma,W} \leq Cd(\gamma)^{1/2}d_\ell(\gamma)^{-1}$ *for all* $\gamma \in \hat{G}$.

Let $\mathscr{D}(\hat{G}; W)$ be the set of all $W(\hat{G})$-valued functions a on $\hat{G}$ such that $a(\gamma)$ belong to $W(\gamma)$ and satisfy a condition in the above corollary. For $a \in \mathscr{D}(\hat{G}; W)$ we set

$$\hat{\mu}_\ell(a) = \sup_{\gamma\in\hat{G}} (d_\ell(\gamma)/d(\gamma)^{1/2}) \|a(\gamma)\|_{\gamma,W}.$$

We topologize $\mathscr{D}(\hat{G}; W)$ by the system of seminorms $\{\hat{\mu}_\ell\}_{\ell\in\boldsymbol{N}}$. Let $\mathscr{D}'(\hat{G}; W)$ be the set of all $\hat{W}(\hat{G})$-valued functions b on $\hat{G}$ such that $b(\gamma) \in \hat{W}(\gamma)$ and that there exist an $\ell \in \boldsymbol{N}$ and a $C>0$ satisfying $\|b(\gamma)\|_{\gamma,\hat{W}} \leq Cd_\ell(\gamma)d(\gamma)^{-1}$ for all $\gamma \in \hat{G}$.

Let $f \in \mathscr{D}(G; W)$. For any $\ell \in \boldsymbol{N}$ we choose $\ell' \in \boldsymbol{N}$ so that $\ell' \geq 2(\ell+\ell_o)$. Then by Lemma 12 there is a constant $C_1>0$ such that $d_\ell(\gamma)d_{\ell_o}(\gamma) \leq C_1 d_{\ell'}(\gamma)$ for all $\gamma \in \hat{G}$. Then we have

$$\begin{aligned} \|D_\ell f\|_2^2 &= \textstyle\sum_{\gamma\in\hat{G}} d(\gamma)d_\ell(\gamma)^2 \|\hat{f}(\gamma)\|^2_{\gamma,W} \\ &= \textstyle\sum_{\gamma\in\hat{G}} (d(\gamma)^2 d_\ell(\gamma)^2/d_{\ell'}(\gamma)^2)(d_{\ell'}(\gamma)^2/d(\gamma)) \|\hat{f}(\gamma)\|^2_{\gamma,W} \\ &\leq C_1^2 \textstyle\sum_{\gamma\in\hat{G}} (d(\gamma)/d_{\ell_o}(\gamma))^2 \{\sup_{\gamma\in\hat{G}} (d_{\ell'}(\gamma)/d(\gamma)^{1/2}) \|\hat{f}(\gamma)\|_{\gamma,W}\}^2. \end{aligned}$$

If we put $C=C_1(\sum_{\gamma\in\hat{G}} (d(\gamma)/d_{\ell_o}(\gamma))^2)^{1/2}$, then we have

$$\|D_\ell f\|_2 \leq C \sup_{\gamma\in\hat{G}} (d_{\ell'}(\gamma)/d(\gamma)^{1/2}) \|\hat{f}(\gamma)\|_{\gamma,W}.$$

Conversely, we can easily see that

$$\sup_{\gamma\in\hat{G}} (d_\ell(\gamma)/d(\gamma)^{1/2}) \|\hat{f}(\gamma)\|_{\gamma,W} \leq \|D_\ell f\|_2.$$

Because of this bijectiveness of the Fourier transform we have thus proved that the Fourier transform gives a topological isomorphism between $\mathscr{D}(G; W)$ and $\mathscr{D}(\hat{G}; W)$.

For any $b\in\mathscr{D}'(\hat{G}; W)$ we defined $T_b\in\mathscr{D}'(G; W)$ in (2.4), which is the inverse Fourier transform of b. Let $f_a\in\mathscr{D}(G; W)$ be the inverse Fourier transform of $a\in\mathscr{D}(\hat{G}; W)$. We put $\langle b, a\rangle=\langle T_b, f_a\rangle$. Then the mapping $a\mapsto\langle b, a\rangle$ is a continuous linear functional on $\mathscr{D}(\hat{G}; W)$. Conversely, if $a\mapsto\Phi(a)$ is a continuous linear functional on $\mathscr{D}(\hat{G}; W)$, then the mapping $f\mapsto\Phi(\hat{f})$ is a continuous linear functional on $\mathscr{D}(G; W)$. Therefore, there is a distribution $T\in\mathscr{D}'(G; W)$ such that $\langle T, f\rangle=\Phi(\hat{f})$. Then, for any $a\in\mathscr{D}(\hat{G}; W)$, $\Phi(a)=\langle T, f_a\rangle=\langle\hat{T}, a\rangle$. Hence $\Phi=\hat{T}\in\mathscr{D}'(\hat{G}; W)$. Therefore, $\mathscr{D}'(\hat{G}; W)$ is the space of all continuous linear functionals on $\mathscr{D}(\hat{G}; W)$. We endow $\mathscr{D}'(G; W)$ and $\mathscr{D}'(\hat{G}; W)$ with the weak topologies as the conjugate spaces of $\mathscr{D}(G; W)$ and $\mathscr{D}(\hat{G}; W)$, respectively. We have now obtained the following theorem.

THEOREM 1. *The Fourier transform gives topological isomorphisms of $\mathscr{D}(G; W)$ onto $\mathscr{D}(\hat{G}; W)$ and also of $\mathscr{D}'(G; W)$ onto $\mathscr{D}'(\hat{G}; W)$.*

§3. Differential equations on G

3.1. We use the following identifications of linear spaces. Let W, W_1, W_2 and W_3 be finite-dimensional complex Hilbert spaces.

(a) $W_1 \otimes W_2 = \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, W_2)$.

For $w_1\in W_1$, $w_2\in W_2$ and $\phi\in\hat{W}_1$, $(w_1\otimes w_2)(\phi)=\langle\phi, w_1\rangle_{W_1}w_2$.

(b) $\mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_2) = (\mathrm{Hom}_{\boldsymbol{C}}(W_1, W_2))^\wedge$.

For $L\in\mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_2)$ and $M\in\mathrm{Hom}_{\boldsymbol{C}}(W_1, W_2)$, $\langle L, M\rangle_{\mathrm{Hom}_{\boldsymbol{C}}(W_1,W_2)}=\mathrm{tr}\,({}^tLM)$.

(c) $\mathscr{D}(G; W_1)\otimes W_2 = \mathscr{D}(G; W_1\otimes W_2)$.

Let $e_1^{(1)},\dots, e_{m_1}^{(1)}$ and $e_1^{(2)},\dots, e_{m_2}^{(2)}$ be orthonormal bases of W_1 and W_2, respectively. If $f\in\mathscr{D}(G; W_1\otimes W_2)$ and $f(x)=\sum_{i,j} f_{ij}(x)e_i^{(1)}\otimes e_j^{(2)}$ $(f_{ij}(x)\in\boldsymbol{C})$, then $f_j=\sum_i f_{ij}e_i^{(1)}\in\mathscr{D}(G; W_1)$ and $f=\sum_j f_j\otimes e_j^{(2)}\in\mathscr{D}(G; W_1)\otimes W_2$.

(d) $\mathscr{D}'(G; W_1)\otimes W_2 = \mathscr{D}'(G; W_1\otimes\hat{W}_2)$.

Let $\phi_1^{(2)},\dots, \phi_{m_2}^{(2)}$ be the dual basis of $e_1^{(2)},\dots, e_{m_2}^{(2)}$. Let $T=\sum_j T_j\otimes e_j^{(2)}\in\mathscr{D}'(G; W_1)\otimes W_2$ $(T_j\in\mathscr{D}'(G; W_1))$ and $f=\sum_j f_j\otimes\phi_j^{(2)}\in\mathscr{D}(G; W_1)\otimes\hat{W}_2=\mathscr{D}(G; W_1\otimes\hat{W}_2)$. Then $\langle T, f\rangle=\sum_j\langle T_j, f_j\rangle$.

Let $\mathscr{L}(\mathscr{D}(G; W_1), W_2)$ be the set of all continuous linear mappings of $\mathscr{D}(G; W_1)$ to W_2.

(e) $\mathscr{L}(\mathscr{D}(G; W_1), W_2) = \mathscr{D}'(G; W_1) \otimes W_2$.

Let $T \in \mathscr{L}(\mathscr{D}(G; W_1), W_2)$ and $f \in \mathscr{D}(G; W_1)$. Then

$$T(f) = \sum_j \langle \phi_j^{(2)}, T(f)\rangle_{W_2} e_j^{(2)}.$$

We put $\langle T_j, f\rangle = \langle \phi_j^{(2)}, T(f)\rangle_{W_2}$. Then $T_j \in \mathscr{D}'(G; W_1)$ and $T = \sum_j T_j \otimes e_j^{(2)} \in \mathscr{D}'(G; W_1) \otimes W_2$.

(f) $\mathscr{D}(G; \hat{W}) \subset \mathscr{D}'(G; W)$.

Here we identify $f \in \mathscr{D}(G; \hat{W})$ with $\theta(f) \in \mathscr{D}'(G; W)$ (see (2.1)).

(g) $\mathrm{Hom}_{\boldsymbol{C}}(W_2, W_3) \subset \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Hom}_{\boldsymbol{C}}(W_1, W_2), \mathrm{Hom}_{\boldsymbol{C}}(W_1, W_3))$.

Let $L \in \mathrm{Hom}_{\boldsymbol{C}}(W_2, W_3)$. We identify L with the mapping $M \mapsto LM$ of $\mathrm{Hom}_{\boldsymbol{C}}(W_1, W_2)$ to $\mathrm{Hom}_{\boldsymbol{C}}(W_1, W_3)$.

The convolutions $S*T$ of two distributions S and T of $\mathscr{D}'(G; \boldsymbol{C})$ and $T*f$ of $T \in \mathscr{D}'(G; \boldsymbol{C})$ and a function $f \in \mathscr{D}(G; \boldsymbol{C})$ are defined as follows. For any $F \in \mathscr{D}(G; \boldsymbol{C})$ we put $({}_xF)(y) = F(xy)$, $x, y \in G$, and $\check{F}(x) = F(x^{-1})$. The function $x \mapsto F^S(x) = \langle S, {}_xF\rangle$ is an element of $\mathscr{D}(G; \boldsymbol{C})$. Then $\langle S*T, F\rangle = \langle T, F^S\rangle$. On the other hand $(T*f)(x) = \langle T, {}_x\check{f}\rangle$. Then we have $(S*T)\hat{}(\gamma) = \hat{S}(\gamma)\hat{T}(\gamma)$ and $(T*f)\hat{}(\gamma) = \hat{T}(\gamma)\hat{f}(\gamma)$ for all $\gamma \in \hat{G}$.

Now let us define the convolutions of distributions on vector valued functions. Let

$$S = \sum_i S_i \otimes B_i \in \mathscr{D}'(G; \boldsymbol{C}) \otimes \mathrm{Hom}_{\boldsymbol{C}}(W_2, W_3) = \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_2, \hat{W}_3)),$$

$$T = \sum_j T_j \otimes C_j \in \mathscr{D}'(G; \boldsymbol{C}) \otimes \mathrm{Hom}_{\boldsymbol{C}}(W_1, W_2) = \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_2)),$$

$$F = \sum_s F_s \otimes A_s \in \mathscr{D}(G; \boldsymbol{C}) \otimes \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_3) = \mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_3)).$$

We put

$$\langle S*T, F\rangle = \sum_{i,j,s} \langle S_i * T_j, F_s\rangle \operatorname{tr}({}^tC_j{}^tB_iA_s).$$

If $f = \sum_t f_t \otimes w_t \in \mathscr{D}(G; \boldsymbol{C}) \otimes W_1 = \mathscr{D}(G; W_1)$, we put

$$(T*f)(x) = \sum_{j,t} (T_j * f_t)(x) \otimes C_j w_t.$$

Then it is not difficult to see that $S*T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_3))$ and $T*f \in \mathscr{D}(G; W_2)$ and that

$$(S*T)\hat{}(\gamma) = \hat{S}(\gamma)\hat{T}(\gamma) \quad \text{and} \quad (T*f)\hat{}(\gamma) = \hat{T}(\gamma)\hat{f}(\gamma) \qquad \text{for all} \quad \gamma \in \hat{G}.$$

3.2. For any $\xi = \sum_\alpha c_\alpha X^\alpha \in \mathfrak{G}$ we put ${}^t\xi = \sum_\alpha c_\alpha (X^\alpha)^*$ (here the notation is the same as in §2). Let $D=\sum_j \xi_j \otimes L_j \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(W_1, W_2)$. We define ${}^tD \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_2, \hat{W}_1)$ by

$${}^tD = \sum_j {}^t\xi_j \otimes {}^tL_j.$$

For any $T \in \mathscr{D}'(G; \hat{W}_1)$ and $S \in \mathscr{D}'(G; W_2)$ we define $DT \in \mathscr{D}'(G; \hat{W}_2)$ and ${}^tDS \in \mathscr{D}'(G; W_1)$ by

$$\langle DT, g\rangle = \langle T, {}^tDg\rangle, \qquad g \in \mathscr{D}(G; \hat{W}_2),$$
$$\langle {}^tDS, f\rangle = \langle S, Df\rangle, \qquad f \in \mathscr{D}(G; W_1).$$

Clearly, the mapping D: $\mathscr{D}'(G; \hat{W}_1) \to \mathscr{D}'(G; \hat{W}_2)$ and ${}^tD$: $\mathscr{D}'(G; W_2) \to \mathscr{D}'(G; W_1)$ are continuous.

Let $D \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(W_3, W)$. By the identification (g), D can be considered as an element of $\mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Hom}_{\boldsymbol{C}}(W_1, W_3), \mathrm{Hom}_{\boldsymbol{C}}(W_1, W))$ and hence as a mapping of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_3))$ to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}))$. On the other hand, if we regard $D \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Hom}_{\boldsymbol{C}}(W_2, W_3), \mathrm{Hom}_{\boldsymbol{C}}(W_2, W))$, then D is a mapping of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_2, \hat{W}_3))$ to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_2, \hat{W}))$. Let

$$T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_2)) \quad \text{and} \quad S \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_2, \hat{W}_3)).$$

Then we have

$$DS \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_2, \hat{W})), \qquad D(S*T) \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W})),$$
$$(DS)^\wedge(\gamma)\hat{T}(\gamma) = \hat{D}(\gamma)(S*T)^\wedge(\gamma) \qquad \text{for all} \quad \gamma \in \hat{G}.$$

Therefore, we have

$$D(S*T) = (DS)*T.$$

Let $f \in \mathscr{D}(G; W_1)$ and $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}_2))$. If $D \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(W_2, W)$, then $D(T*f) \in \mathscr{D}(G; W)$, $DT \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}_1, \hat{W}))$ and $(DT)*f \in \mathscr{D}(G; W)$. Since $(DT)^\wedge(\gamma)\hat{f}(\gamma) = \hat{D}(\gamma)(T*f)^\wedge(\gamma)$ for all $\gamma \in \hat{G}$, we have

$$D(T*f) = (DT)*f.$$

3.3. Let δ_W be a continuous linear mapping of $\mathscr{D}(G; W)$ to W so that $\delta_W(f) = f(1)$, $f \in \mathscr{D}(G; W)$. Regarding that δ_W is an element of $\mathscr{D}'(G; W\otimes\hat{W})$ by the identifications (d) and (e), let us calculate the Fourier transform $\hat{\delta}_W$ of δ_W.

LEMMA 15. *As an element of* $\mathrm{End}(V_\gamma \otimes W)$,

$$\hat{\delta}_W(\gamma) = I_{V_\gamma \otimes W} \qquad \textit{for all} \quad \gamma \in \hat{G}.$$

PROOF.

$$\hat{\delta}_W(\gamma) = \sum_{i,j} \sum_{p,q} \langle \delta_W, a_{pq}^\gamma(x^{-1})e_i \otimes \phi_j\rangle E_{pq}^\gamma \otimes \phi_i \otimes e_j.$$

We put $f(x)=\sum_s f_s(x)\otimes\phi_s$ and $f_s(x)=\delta_{sj}a^{\gamma}_{pq}(x^{-1})e_i$, where δ_{sj} is Kronecker's delta. Then

$$\langle\delta_W, f\rangle = \sum_s \langle(\delta_W)_s, f_s\rangle = \sum_s \langle\phi_s, \delta_W(f_s)\rangle_W = \sum_s\langle\phi_s, f_s(1)\rangle_W = \delta_{ij}\delta_{pq}.$$

Hence we have

$$\hat{\delta}_W(\gamma) = \sum_{p=1}^{d(\gamma)} E^{\gamma}_{pp} \otimes \sum_{j=1}^{m} \phi_j \otimes e_j.$$

Therefore,

$$\hat{\delta}_W(\gamma) = I_{V_\gamma} \otimes I_W = I_{V_\gamma \otimes W}. \qquad \text{Q. E. D.}$$

3.4. Let $D \in \mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V, W)$. Let us consider the differential equation

$$Df = u, \tag{3.1}$$

where $u \in \mathscr{D}(G; W)$.

PROPOSITION 2. *If for any $\gamma \in \hat{G}$ there exists a right inverse $\hat{D}(\gamma)^{-1}_{\mathrm{R}}$ of $\hat{D}(\gamma)$ and if for any $\ell \in \boldsymbol{N}$ there is a constant $C_\ell > 0$ such that*

$$\|\hat{D}(\gamma)^{-1}_{\mathrm{R}}\hat{u}(\gamma)\|_{\gamma,V} \leq C_\ell d(\gamma)^{1/2} d_\ell(\gamma)^{-1}$$

for avery $\gamma \in \hat{G}$, then the function $f(x)$ defined by

$$f(x) = \sum_{\gamma\in\hat{G}} d(\gamma)\,\mathrm{tr}\,(\pi_\gamma(x)\hat{D}(\gamma)^{-1}_{\mathrm{R}}\hat{u}(\gamma)) \tag{3.2}$$

is a function of $\mathscr{D}(G; V)$ and is a solution of (3.1).

PROOF. By Lemma 14 we know that $f \in \mathscr{D}(G; V)$. And $\hat{f}(\gamma)=\hat{D}(\gamma)^{-1}_{\mathrm{R}}\hat{u}(\gamma)$. Hence $\hat{D}(\gamma)\hat{f}(\gamma)=\hat{u}(\gamma)$ for all $\gamma \in \hat{G}$. Therefore, $Df=u$. Q. E. D.

3.5. Let $D \in \mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V, W)$.

DEFINITION. If a distribution $E \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ satisfies $DE=\delta_W$, we call it an *elementary solution* of D.

LEMMA 16. *If D has an elementary solution, then D is a surjective mapping of $\mathscr{D}(G; V)$ to $\mathscr{D}(G; W)$.*

PROOF. Let E be an elementary solution of D. For any $u \in \mathscr{D}(G; W)$ we put $f=E*u$. Then we have $f \in \mathscr{D}(G; V)$ and $Df=DE*u=\delta_W*u=u$ from §§3.1, 3.2 and 3.3. Q. E. D.

THEOREM 2. *Let $D \in \mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V, W)$. Then the following conditions are equivalent*:

(1) *D has an elementary solution*;

(2) *The mapping D of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{W}))$ is*

surjective;

(3) *For any* $\gamma \in \hat{G}$, $\hat{D}(\gamma)$ *has a right inverse* $\hat{D}(\gamma)_{\mathrm{R}}^{-1}$ *which satisfies that there exist a constant* $C>0$ *and an* $\ell \in \boldsymbol{N}$ *such that*

$$\|\hat{D}(\gamma)^{-1}\|_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(W,V)} \leq C d_{\ell}(\gamma)/d(\gamma)^{1/2} \qquad \text{for all} \quad \gamma \in \hat{G}.$$

PROOF. Let E be an elementary solution of D. For any distribution $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{W}))$, we put $S=E*T$. Then $S \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ and $DS=T$. Implication from (2) to (1) is trivial.

On the other hand, $\hat{E}(\gamma)$ is a right inverse of $\hat{D}(\gamma)$ for all $\gamma \in \hat{G}$. As $E \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$, we have the inequalities in (3). Conversely, we assume that $\hat{D}(\gamma)$ has a right inverse as in (3). Then the mapping $\gamma \mapsto \hat{D}(\gamma)_{\mathrm{R}}^{-1}$ is clearly in $\mathscr{D}'(\hat{G}; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ and hence it is the Fourier transform of a distribution $E \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$. E is obviously an elementary solution of D. Q.E.D.

The operation of D on $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ is as follows. By our identifications, $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))=\mathscr{D}'(G; \boldsymbol{C})\otimes \hat{W} \otimes V$ and $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{W})) =\mathscr{D}'(G; \boldsymbol{C})\otimes \hat{W} \otimes W$. From § 3.2 D maps $\mathscr{D}'(G; \hat{V})=\mathscr{D}'(G; \boldsymbol{C})\otimes V$ to $\mathscr{D}'(G; \hat{W}) =\mathscr{D}'(G; \boldsymbol{C})\otimes W$. If $D=\sum \xi_j \otimes L_j \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$ and $T\otimes v \in \mathscr{D}'(G; \boldsymbol{C})\otimes V$, Then $D(T\otimes v)=\sum \xi_j T \otimes L_j v$ and, as a mapping of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{W}))$, $D(T\otimes \phi \otimes v)=\sum \xi_j T\otimes \phi \otimes L_j v$. Thus we have the following lemma.

LEMMA 17. *Let* $D \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$. *If* D *maps* $\mathscr{D}'(G; \hat{V})$ *onto* $\mathscr{D}'(G; \hat{W})$, *then* D *has an elementary solution. Similarly, if the transpose* ${}^tD$ *of* D *maps* $\mathscr{D}'(G; W)$ *onto* $\mathscr{D}'(G; V)$, *then* ${}^tD$ *has an elementary solution.*

3.6. In this number we assume that $\dim_{\boldsymbol{C}} V=\dim_{\boldsymbol{C}} W$.

LEMMA 18. *Let* $D \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$. *Then* D *has an elementary solution if and only if so does* ${}^tD$.

PROOF. Let E be an elementary solution of D. Then $\hat{E}(\gamma)$ is a right inverse of $\hat{D}(\gamma)$ and is also a left inverse of $\hat{D}(\gamma)$, i.e. $\hat{E}(\gamma)\hat{D}(\gamma)=I_{V_{\gamma}\otimes V}$. By Lemma 4, D is an injective mapping of $\mathscr{D}(G; V)$ to $\mathscr{D}(G; W)$. Hence D is a continuous bijective mapping between two Fréchet spaces. Thus D is a topological isomorphism between $\mathscr{D}(G; V)$ and $\mathscr{D}(G; W)$. For any $S \in \mathscr{D}'(G; V)$ the linear functional $g \mapsto \langle S, D^{-1}g\rangle$ on $\mathscr{D}(G; W)$ is continuous. Then there exists $T \in \mathscr{D}'(G; W)$ such that $\langle T, g\rangle=\langle S, D^{-1}g\rangle$. Hence $\langle T, Df\rangle=\langle S, f\rangle$ for all $f \in \mathscr{D}(G; V)$, therefore, $S={}^tDT$. Then by Lemma 17, ${}^tD$ has an elementary solution. Conversely, if ${}^tD$ has an elementary solution, D has clearly an elementary solution since ${}^t({}^tD) =D$. Q.E.D.

THEOREM 3. *We assume that* $\dim_{\boldsymbol{C}} V=\dim_{\boldsymbol{C}} W$. *Let* $D \in \mathfrak{G}\otimes \mathrm{Hom}_{\boldsymbol{C}}(V, W)$

be an invariant differential operator of $\mathscr{D}(G; V)$ *to* $\mathscr{D}(G; W)$. *Then the following conditions are equivalent*:

(1) *D has an elementary solution*;

(2) ${}^tD$ *has an elementary solution*;

(3) $D(\mathscr{D}(G; V))=\mathscr{D}(G; W)$;

(4) *D is a topological isomorphism of* $\mathscr{D}(G; V)$ *onto* $\mathscr{D}(G; W)$;

(5) *D maps* $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ *onto* $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{W}))$;

(6) $D(\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V})))\supset\mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(W, W))$;

(7) *For any* $\gamma\in\hat{G}$, $\hat{D}(\gamma)$ *is invertible. And there exist a constant* $C>0$ *and an* $\ell\in\boldsymbol{N}$ *such that*

$$\|\hat{D}(\gamma)^{-1}\|_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(W,V)}\leq Cd_\ell(\gamma)/d(\gamma)^{1/2}\qquad \textit{for all}\quad \gamma\in\hat{G}.$$

PROOF. The equivalence of (1) and (2) is given in Lemma 18. The equivalences between (1), (5) and (7) are in Theorem 2. Lemma 16 gives the implication from (1) to (3).

If (3) holds, the surjectivity implies the injectivity from Lemmas 4 and 5. Then as in the proof of Lemma 18, D is a topological isomorphism of $\mathscr{D}(G; V)$ onto $\mathscr{D}(G; W)$. By the duality, ${}^tD$ is a topological isomorphism of $\mathscr{D}'(G; W)$ onto $\mathscr{D}'(G; V)$. Hence by Lemma 17, ${}^tD$ has an elementary solution.

The implication from (5) to (6) is a matter of course. Conversely, let us assume that $D(\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))\supset\mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(W, W))$. We put $g_\gamma(x)=d(\gamma)\,\mathrm{tr}\,(\pi_\gamma(x))I_W$, $\gamma\in\hat{G}$. Then there is $T^\gamma\in\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ with $DT^\gamma=g_\gamma$. Hence we have $\hat{D}(\gamma)(T^\gamma)\hat{\ }(\gamma)=\hat{g}_\gamma(\gamma)=I_{V_\gamma}\otimes I_W$. Thus $\hat{D}(\gamma)$ is invertible. Let us consider the bilinear form $\langle\!\langle h,f\rangle\!\rangle=\langle\theta(f), h\rangle$ defined on the product space of $\mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{W}))$ with $\mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(W, W))$, where the former is a metrizable space with seminorms $h\mapsto\|D_\ell{}^tDh\|_2$ $(\ell\in\boldsymbol{N})$ and the latter is a Fréchet space. If $\langle h,f\rangle$ is separately continuous, it is continuous. It is trivial that it is continuous with respect to f for any fixed h. By our assumption, there is an $S\in\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{W}, \hat{V}))$ such that $DS=f$. Hence $\langle\!\langle h,f\rangle\!\rangle=\langle DS, h\rangle=\langle S, {}^tDh\rangle$. Thus this is continuous with respect to h for any fixed f. Hence there exist a constant C and ℓ, $\ell'\in\boldsymbol{N}$ such that

$$\left|\int_G\langle h(x), f(x)\rangle_{\mathrm{Hom}_{\boldsymbol{C}}(W,W)}\,dx\right|\leqq C\|D_\ell f\|_2\|D_{\ell'}{}^tDh\|_2.$$

If $f(x)=\sum_{i,j}f_{ij}(x)\phi_i\otimes e_j$, we put $f^*(x)=\sum_{i,j}\overline{f_{ij}(x)}e_i\otimes\phi_j$. Let A be an element of $\mathrm{End}\,(V_\gamma)\otimes\mathrm{Hom}_{\boldsymbol{C}}(W, W)$ and put $f(x)=d(\gamma)\,\mathrm{tr}\,((\pi_\gamma(x)\otimes I_W)A)$ and $h=f^*$. Then we have

$$\|f\|_2^2=d(\gamma)\,\|A\|^2_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(W,W)},$$

$$\|D_\ell f\|_2^2=d_\ell(\gamma)^2d(\gamma)\,\|A\|^2_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(W,W)}$$

and

$$\|D_{\ell'}\, {}^tDf^*\|_2^2 = \|D_{\ell'}\overline{D^*f}\|_2^2 = d_{\ell'}(\gamma)^2 d(\gamma)\, \|\hat{D}(\gamma)^* A\|^2_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(W,W)}.$$

Let $A=\hat{D}(\gamma)^{*-1}$. Then

$$\|\hat{D}(\gamma)^{-1}\|_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(W,V)} \le C d_{\ell}(\gamma) d_{\ell'}(\gamma) d(\gamma)^{1/2}.$$

By Lemma 12 there is a constant C' such that

$$d_{\ell}(\gamma) d_{\ell'}(\gamma) \le C' d_{2(\ell+\ell')}(\gamma)$$

for all $\gamma \in \hat{G}$. Thus we have completed the proof. Q.E.D.

§4. Fourier transforms on homogeneous vector bundles

4.1. Let K be a closed subgroup of G. Let τ be a finite-dimensional unitary representation of K. Let $E_\tau = G \times_\tau V_\tau$ be the homogeneous vector bundle associated with τ. We identify the spaces of (continuous) sections, smooth sections and L^2-sections with $C(G;\tau)$, $C^\infty(G;\tau)$ and $L^2(G;\tau)$, respectively, which are the set of all functions f of $C(G; V_\tau)$, $C^\infty(G; V_\tau)$ and $L^2(G; V_\tau)$ such that $f(xk) = \tau(k)^{-1} f(x)$ for $x \in G$ and $k \in K$, respectively. Since $C^\infty(G;\tau)$ is a closed subspace of $\mathscr{D}(G; V_\tau)$, it is also a Fréchet space and we denote it by $\mathscr{D}(G;\tau)$.

We define the *Fourier transform* of sections on E_τ by the restriction of the Fourier transform of $C(G; V_\tau)$ to $C(G;\tau)$.

Let δ_τ be the continuous linear mapping of $\mathscr{D}(G; V_\tau)$ to V_τ defined by

$$\delta_\tau(f) = \int_K \tau(k)^{-1} f(k)\,dk, \qquad (f \in \mathscr{D}(G; V_\tau)).$$

Regarding δ_τ as an element of $\mathscr{D}'(G; V_\tau \otimes \hat{V}_\tau) = \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))$, let us calculate $\hat{\delta}_\tau$.

LEMMA 19. $$\hat{\delta}_\tau(\gamma) = \int_K \pi_\gamma(k) \otimes \tau(k)\,dk.$$

PROOF.

$$\begin{aligned}\hat{\delta}_\tau(\gamma) &= \textstyle\sum_{i,j=1}^{d(\tau)} \sum_{p,q=1}^{d(\gamma)} \langle \delta_\tau, a^\gamma_{pq}(x^{-1}) e_i \otimes \phi_j \rangle E^\gamma_{pq} \otimes \phi_i \otimes e_j \\ &= \textstyle\sum_{i,j} \sum_{p,q} \sum_{s=1}^{d(\tau)} \langle \phi_s, \delta_{sj} \sum_{t=1}^{d(\tau)} \int_K a^\gamma_{pq}(k) \tau_{ti}(k) e_t\, dk \rangle_{V_\tau} E^\gamma_{pq} \otimes \phi_i \otimes e_j \\ &= \textstyle\sum_{i,j} \sum_{p,q} \int_K a^\gamma_{pq}(k) E^\gamma_{pq} \otimes \tau_{ji}(k) \phi_i \otimes e_j = \int_K \pi_\gamma(k) \otimes \tau(k)\,dk. \end{aligned}$$

Q.E.D.

By the invariance and the normalization of the measure we have

$$(\pi_\gamma(k) \otimes \tau(k))\hat{\delta}_\tau(\gamma) = \hat{\delta}_\tau(\gamma)(\pi_\gamma(k) \otimes \tau(k)) = \hat{\delta}_\tau(\gamma) \tag{4.1}$$

for every $k \in K$ and hence

$$\hat{\delta}_\tau(\gamma)^2 = \hat{\delta}_\tau(\gamma).$$

Hence $\hat{\delta}_\tau(\gamma)$ is a projection. Moreover, we can see that $\hat{\delta}_\tau(\gamma)$ is self-adjoint with respect to the inner product $(\ ,\ )_{\gamma,V_\tau}$. Hence $\hat{\delta}_\tau(\gamma)$ is an orthogonal projection of $V_\tau(\gamma)=\mathrm{End}\,(V_\gamma)\otimes V_\tau$ onto $I(\gamma, \tau)=\mathrm{Im}\,\hat{\delta}_\tau(\gamma)$, the image of $\hat{\delta}_\tau(\gamma)$, with kernel $N(\gamma, \tau) =\ker \hat{\delta}_\tau(\gamma)$. The space $V_\tau(\gamma)$ is the orthogonal direct sum:

$$V_\tau(\gamma) = I(\gamma, \tau) \oplus N(\gamma, \tau).$$

Let $\mathscr{D}(\hat{G};\tau)=\{a \in \mathscr{D}(\hat{G}; V_\tau) \mid a(\gamma) \in I(\gamma, \tau) \textit{ for all } \gamma \in \hat{G}\}$. Then $\mathscr{D}(\hat{G};\tau)$ is a closed subspace of $\mathscr{D}(\hat{G}; V_\tau)$.

THEOREM 4. *The Fourier transform gives a topological isomorphism of* $\mathscr{D}(G;\tau)$ *onto* $\mathscr{D}(\hat{G};\tau)$.

PROOF. Let $f \in \mathscr{D}(G;\tau)$. Then

$$\hat{f}(\gamma) = \int_G \pi_\gamma(x)^{-1} \otimes f(x)dx = \int_G \pi_\gamma(xk^{-1})^{-1} \otimes f(xk^{-1})dx = (\pi_\gamma(k) \otimes \tau(k))\hat{f}(\gamma)$$

for every $\gamma \in \hat{G}$. Therefore, $\hat{f}(\gamma) \in I(\gamma, \tau)$ and hence $\hat{f} \in \mathscr{D}(\hat{G};\tau)$.

Conversely, let $a \in \mathscr{D}(\hat{G};\tau)$. Then by (4.1) we have

$$a(\gamma) = \hat{\delta}_\tau(\gamma)a(\gamma) = (\pi_\gamma(k) \otimes \tau(k))\hat{\delta}_\tau(\gamma)a(\gamma) = (\pi_\gamma(k) \otimes \tau(k))a(\gamma).$$

Hence

$$(\pi_\gamma(k) \otimes I_{V_\tau})a(\gamma) = (I_{V_\gamma} \otimes \tau(k)^{-1})a(\gamma).$$

If we put

$$f(x) = \sum_{\gamma\in\hat{G}} d(\gamma)\,\mathrm{tr}\,((\pi_\gamma(x) \otimes I_{V_\tau})a(\gamma)),$$

then $f \in \mathscr{D}(G; V_\tau)$, and by the above relation we have

$$f(xk) = \tau(k)^{-1}f(x).$$

Hence $f \in \mathscr{D}(G;\tau)$. Q.E.D.

Let $L^2(\hat{G};\tau)$ be the set of all $a \in L^2(\hat{G}; V_\tau)$ such that $a(\gamma) \in I(\gamma, \tau)$ for all $\gamma \in \hat{G}$. Then $L^2(\hat{G};\tau)$ is a closed subspace of $L^2(\hat{G}; V_\tau)$ and is the completion of $\mathscr{D}(\hat{G};\tau)$ in $L^2(\hat{G}; V_\tau)$. Hence we have the following.

COROLLARY. *The Fourier transform extended to* $L^2(G; V_\tau)$ *gives an isometry of* $L^2(G;\tau)$ *onto* $L^2(\hat{G};\tau)$.

From the definition of the convolution in § 3,

$$(\delta_\tau * f)(x) = \sum_{\gamma\in\hat{G}} d(\gamma)\,\mathrm{tr}\,((\pi_\gamma(x) \otimes I_{V_\tau})\hat{\delta}_\tau(\gamma)\hat{f}(\gamma)).$$

Because of the uniform convergence of the series, we can show that

$$(\delta_\tau * f)(x) = \int_K \tau(k) f(xk) dk.$$

Thus we have the following.

LEMMA 20. *A V_τ-valued function $f \in \mathscr{D}(G; V_\tau)$ is a section which belongs to $\mathscr{D}(G; \tau)$ if and only if $\delta_\tau * f = f$.*

4.2. We denote by $\mathscr{D}'(G; \tau)$ the set of all continuous linear functionals on $\mathscr{D}(G; \tau)$. For $T \in \mathscr{D}'(G; V_\tau)$ we denote by $\Phi(T)$ the restriction of T to $\mathscr{D}(G; \tau)$. Then it is clear that $\Phi(T) \in \mathscr{D}'(G; \tau)$. Conversely, if $S \in \mathscr{D}'(G; \tau)$, we can define a distribution on $\mathscr{D}(G; V_\tau)$ by $f \mapsto \langle S, \delta_\tau * f \rangle$. For any $T \in \mathscr{D}'(G; V_\tau)$ let us define $T_\tau \in \mathscr{D}'(G; V_\tau)$ by

$$\langle T_\tau, f \rangle = \langle T, \delta_\tau * f \rangle, \qquad (f \in \mathscr{D}(G; V_\tau)).$$

We put $\mathscr{D}'_\tau = \{T \in \mathscr{D}'(G; V_\tau) \mid T_\tau = T\}$.

LEMMA 21. *The mapping Φ gives a linear isomorphism of $\mathscr{D}'_\tau$ onto $\mathscr{D}'(G; \tau)$.*

PROOF. For any $S \in \mathscr{D}'(G; \tau)$ we put $T = S_\tau \in \mathscr{D}'(G; V_\tau)$. Then $T_\tau = (S_\tau)_\tau = S_\tau = T$. Hence $T_\tau \in \mathscr{D}'_\tau$ and from Lemma 20 we have $\Phi(T) = S$. We next assume that $\Phi(T) = \Phi(T')$, $T, T' \in \mathscr{D}'_\tau$. Then for any $f \in \mathscr{D}(G; V_\tau)$,

$$\langle T, f \rangle = \langle T_\tau, f \rangle = \langle T, \delta_\tau * f \rangle = \langle T', \delta_\tau * f \rangle = \langle T'_\tau, f \rangle = \langle T', f \rangle.$$

Thus we have proved that Φ is an isomorphism. Q.E.D.

Remark that if we endow $\mathscr{D}'(G; \tau)$ with the weak topology, then Φ is a topological isomorphism of $\mathscr{D}'_\tau$, which is a closed subspace of $\mathscr{D}'(G; V_\tau)$, onto $\mathscr{D}'(G; \tau)$. We thus identify $\mathscr{D}'(G; \tau)$ with $\mathscr{D}'_\tau$ hereafter.

Let $\tilde{\tau}$ be the contragredient representation of τ on $\hat{V}_\tau$.

LEMMA 22. *For any $T \in \mathscr{D}'(G; V_\tau)$ we have*

$$\hat{T}_\tau(\gamma) = \hat{\delta}_{\tilde{\tau}}(\gamma) \hat{T}(\gamma) \quad \text{and} \quad T_\tau = \delta_{\tilde{\tau}} * T.$$

PROOF. Let $f(x) = a^\gamma_{pq}(x^{-1}) e_j \in \mathscr{D}(G; V_\tau)$. Then

$$\delta_\tau * f(x) = \int_K a^\gamma_{pq}(k^{-1}x^{-1}) \tau(k) e_j dk = \sum_{r=1}^{d(\gamma)} \sum_{i=1}^{d(\tau)} \int_K a^\gamma_{pr}(k) \tau_{ij}(k^{-1}) a^\gamma_{rq}(x^{-1}) e_i dk.$$

Then

$$\hat{T}_\tau(\gamma) = \int_K \sum_{r,q} \sum_i \pi_\gamma(k) \langle T, a^\gamma_{rq}(x^{-1}) e_i \rangle E^\gamma_{rq} \otimes \tilde{\tau}(k) \phi_i dk = \hat{\delta}_{\tilde{\tau}}(\gamma) \hat{T}(\gamma).$$

Q.E.D.

Thus we can characterize the Fourier image of $\mathscr{D}'(G;\tau)$. Let

$$\mathscr{D}'(\hat{G};\tau)=\{b\in\mathscr{D}'(\hat{G};V_\tau)\,|\,\hat{\delta}_{\tilde{\tau}}(\gamma)b(\gamma)=b(\gamma)\quad \textit{for all}\quad \gamma\in\hat{G}\}.$$

THEOREM 5. *The Fourier transform gives a topological isomorphism of* $\mathscr{D}'(G;\tau)$ *onto* $\mathscr{D}'(\hat{G};\tau)$.

4.3. Let θ be the injection of $\mathscr{D}(G;V_\tau)$ into $\mathscr{D}'(G;\hat{V}_\tau)$ defined by (2.1).

LEMMA 23. $\theta(\mathscr{D}(G;\tau))\subset\mathscr{D}'(G;\tilde{\tau})$.

PROOF. Let $f\in\mathscr{D}(G;\tau)$. For any $g\in\mathscr{D}(G;\hat{V}_\tau)$ we have

$$\begin{aligned}\langle\theta(f)_{\tilde{\tau}},g\rangle &= \langle\theta(f),\delta_{\tilde{\tau}}*g\rangle=\int_G\langle f(x),\int_K\tilde{\tau}(k)g(xk)dk\rangle_{V_\tau}dx\\ &=\int_G\langle\int_K\tau(k)f(xk)dk,\,g(x)\rangle_{V_\tau}dx=\int_G\langle(\delta_\tau*f)(x),\,g(x)\rangle_{V_\tau}dx\\ &=\int_G\langle f(x),g(x)\rangle_{V_\tau}dx=\langle\theta(f),g\rangle.\end{aligned}$$

Therefore, $\theta(f)_{\tilde{\tau}}=\theta(f)$ and hence $\theta(f)\in\mathscr{D}'(G;\tilde{\tau})$. Q.E.D.

§5. Differential equations on homogeneous vector bundles

5.1. Let σ and τ be finite-dimensional unitary representations of K. The adjoint action of K on $\mathfrak{g}$ can be lifted to an action on $\mathfrak{G}$. We denote it by $k\cdot\xi$, $\xi\in\mathfrak{G}$ and $k\in K$. Then K acts on $\mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V_\sigma,V_\tau)$ by

$$k\cdot D=\textstyle\sum_j k\cdot\xi_j\otimes\tau(k)L_j\sigma(k)^{-1}$$

for $D=\sum_j\xi_j\otimes L_j$, $\xi_j\in\mathfrak{G}$ and $L_j\in\mathrm{Hom}_{\boldsymbol{C}}(V_\sigma,V_\tau)$. We put

$$HD_K(\sigma,\tau)=\{D\in\mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V_\sigma,V_\tau)\,|\,k\cdot D=D\quad \textit{for all}\ k\in K\}.$$

We know that any $D\in HD_K(\sigma,\tau)$ maps $\mathscr{D}(G;\sigma)$ to $\mathscr{D}(G;\tau)$ (see N. R. Wallach [2, §5.4.7]). And we know that any homogeneous differential operator of E_σ to E_τ corresponds to an element of $HD_K(\sigma,\tau)$ (see N. R. Wallach [2, §5.4.11]). Since

$$d\pi_\gamma(\mathrm{Ad}\,(k)X)=\pi_\gamma(k)\hat{X}(\gamma)\pi_\gamma(k)^{-1}$$

for any $k\in K$ and for any $X\in\mathfrak{g}$, we have

$$(k\cdot D)\hat{}\,(\gamma)=(\pi_\gamma(k)\otimes\tau(k))\hat{D}(\gamma)(\pi_\gamma(k)^{-1}\otimes\sigma(k)^{-1})$$

for each $D\in\mathfrak{G}\otimes\mathrm{Hom}_{\boldsymbol{C}}(V_\sigma,V_\tau)$. Hence if $D\in HD_K(\sigma,\tau)$, then we have

$$\hat{D}(\gamma)(\pi_\gamma(k) \otimes \sigma(k)) = (\pi_\gamma(k) \otimes \tau(k))\hat{D}(\gamma) \tag{5.1}$$

for all $k \in K$. Therefore, we have

$$\hat{D}(\gamma)\hat{\delta}_\sigma(\gamma) = \hat{\delta}_\tau(\gamma)\hat{D}(\gamma) \tag{5.1$'$}$$

for all $\gamma \in \hat{G}$. Hence we have $\hat{D}(\gamma)I(\gamma, \sigma) \subset I(\gamma, \tau)$ and $\hat{D}(\gamma)N(\gamma, \sigma) \subset N(\gamma, \tau)$.

5.2. We denote by $\tilde{\sigma}$ and $\tilde{\tau}$ the contragredient representations of σ and τ on $\hat{V}_\sigma$ and $\hat{V}_\tau$, respectively, as before. Then K acts on $\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma)$ by $L \mapsto \tilde{\sigma}(k)L\tilde{\tau}(k)^{-1}$, and we denote it by $\tilde{\lambda}(k)L$. For $F \in \mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ we put

$$F^k(x) = F(k^{-1}xk), \qquad (k \in K).$$

Let $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$. We define $\lambda(k)T$ and T^k, which are members of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$, by

$$\langle \lambda(k)T, F\rangle = \langle T, \tilde{\lambda}(k^{-1})F\rangle \quad \text{and} \quad \langle T^k, F\rangle = \langle T, F^{k^{-1}}\rangle,$$

where $\tilde{\lambda}(k^{-1})F$ is a function defined by $(\tilde{\lambda}(k^{-1})F)(x) = \tilde{\lambda}(k^{-1})(F(x))$. Let $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o$ be the set of all $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ such that

$$T^k = \lambda(k^{-1})T \qquad \text{for all} \quad k \in K.$$

LEMMA 24. *Let $b \in \mathscr{D}'(\hat{G}; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$. Then b is the Fourier transform of some $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o$ if and only if*

$$(\pi_\gamma(k) \otimes \sigma(k))b(\gamma) = b(\gamma)(\pi_\gamma(k) \otimes \tau(k)) \qquad \textit{for all} \quad k \in K.$$

PROOF. Let $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o$. Let $e_1, \dots, e_{d(\tau)}$ and $e'_1, \dots, e'_{d(\sigma)}$ be orthonormal bases of V_τ and V_σ, respectively. And let $\phi_1, \dots, \phi_{d(\tau)}$ and $\phi'_1, \dots, \phi'_{d(\sigma)}$ be their dual bases. Then

$$\begin{aligned}(T^k)^\wedge(\gamma) &= \textstyle\sum_{i=1}^{d(\tau)} \sum_{j=1}^{d(\sigma)} \sum_{p,q=1}^{d(\gamma)} \langle T, a^\gamma_{pq}(kx^{-1}k^{-1})e_i \otimes \phi'_j\rangle\, E^\gamma_{pq} \otimes \phi_i \otimes e'_j \\ &= (\pi_\gamma(k) \otimes I_{V_\sigma})\hat{T}(\gamma)(\pi_\gamma(k)^{-1} \otimes I_{V_\tau}).\end{aligned}$$

On the other hand, by the identification $\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma)$ with $V_\tau \otimes \hat{V}_\sigma$ we have

$$\tilde{\lambda}(k)(e_i \otimes \phi'_j)(\psi) = (\tau(k)e_i \otimes \tilde{\sigma}(k)\phi'_j)(\psi)$$

for $\psi \in \hat{V}_\tau$. Hence

$$\begin{aligned}(\lambda(k^{-1})T)^\wedge(\gamma) &= \textstyle\sum_i \sum_j \sum_{p,q} \langle \lambda(k^{-1})T, a^\gamma_{pq}(x^{-1})e_i \otimes \phi'_j\rangle E^\gamma_{pq} \otimes \phi_i \otimes e'_j \\ &= \textstyle\sum_s \sum_t \sum_{p,q} \langle T, a^\gamma_{pq}(x^{-1})e_s \otimes \phi'_t\rangle E^\gamma_{pq} \otimes \tilde{\tau}(k^{-1})\phi_s \otimes \sigma(k^{-1})e'_t \\ &= (I_{V_\gamma} \otimes \sigma(k^{-1}))\hat{T}(\gamma)(I_{V_\gamma} \otimes \tau(k)).\end{aligned}$$

Thus we have

$$(\pi_\gamma(k) \otimes \sigma(k))\hat{T}(\gamma) = \hat{T}(\gamma)(\pi_\gamma(k) \otimes \tau(k)) \qquad \text{for all} \quad k \in K.$$

Conversely, let b be an element of $\mathscr{D}'(\hat{G}; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ such that

$$(\pi_\gamma(k) \otimes \sigma(k))b(\gamma) = b(\gamma)(\pi_\gamma(k) \otimes \tau(k)) \qquad \text{for all} \quad k \in K.$$

Then by Theorem 1 there is a unique $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ such that $\hat{T}=b$. Hence for any $\gamma \in \hat{G}$

$$\begin{aligned}(T^k)\hat{}(\gamma) &= (\pi_\gamma(k) \otimes I_{V_\sigma})b(\gamma)(\pi_\gamma(k)^{-1} \otimes I_{V_\tau})\\ &= (I_{V_\gamma} \otimes \sigma(k)^{-1})(\pi_\gamma(k) \otimes \sigma(k))b(\gamma)(\pi_\gamma(k)^{-1} \otimes I_{V_\tau})\\ &= (I_{V_\gamma} \otimes \sigma(k)^{-1})b(\gamma)(\pi_\gamma(k) \otimes \tau(k))(\pi_\gamma(k)^{-1} \otimes I_{V_\tau})\\ &= (I_{V_\gamma} \otimes \sigma(k)^{-1})b(\gamma)(I_{V_\gamma} \otimes \tau(k)) = (\lambda(k^{-1})T)\hat{}(\gamma).\end{aligned}$$

Therefore, $T^k = \lambda(k^{-1})T$. Hence $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o$. Q.E.D.

COROLLARY. *If* $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o$ *and* $u \in \mathscr{D}(G; \tau)$, *then* $T*u \in \mathscr{D}(G; \sigma)$.

PROOF. We know that $T*u \in \mathscr{D}(G; V_\sigma)$. From Lemma 24

$$(\pi_\gamma(k) \otimes \sigma(k))\hat{T}(\gamma)\hat{u}(\gamma) = \hat{T}(\gamma)(\pi_\gamma(k) \otimes \tau(k))\hat{u}(\gamma) = \hat{T}(\gamma)\hat{u}(\gamma).$$

Hence $T*u \in \mathscr{D}(G; \sigma)$ by Theorem 4. Q.E.D.

5.3. DEFINITION. Let $D \in HD_K(\sigma, \tau)$. When there exists a distribution $E \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o$ such that

$$DE = \delta_\tau,$$

we call E an *elementary solution* of D.

We assume that D has an elementary solution E. Let $u \in \mathscr{D}(G; \tau)$. Then from the corollary of Lemma 24, $f = E*u$ is in $\mathscr{D}(G; \sigma)$. And

$$\hat{D}(\gamma)\hat{f}(\gamma) = \hat{D}(\gamma)\hat{E}(\gamma)\hat{u}(\gamma) = \hat{\delta}_\tau(\gamma)\hat{u}(\gamma) = \hat{u}(\gamma) \qquad \text{for all} \quad \gamma \in \hat{G}.$$

Therefore, f is a solution of the differential equation $Df = u$. Thus we have the following lemma.

LEMMA 25. *Let* $D \in HD_K(\sigma, \tau)$. *If* D *has an elementary solution, then* D *is a surjective mapping of* $\mathscr{D}(G; \sigma)$ *to* $\mathscr{D}(G; \tau)$.

We assume again that E is an elementary solution of D. By Lemma 24 we have

$$\hat{E}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{\delta}_\sigma(\gamma)\hat{E}(\gamma).$$

Hence $\hat{E}(\gamma)I(\gamma, \tau) \subset I(\gamma, \sigma)$ and $\hat{E}(\gamma)N(\gamma, \tau) \subset N(\gamma, \sigma)$. We put $\hat{D}(\gamma, \sigma) = \hat{D}(\gamma)|_{I(\gamma,\sigma)}$ and $\hat{E}(\gamma, \tau) = \hat{E}(\gamma)|_{I(\gamma,\tau)}$. Then

$$\hat{D}(\gamma, \sigma)\hat{E}(\gamma, \tau) = I_{I(\gamma,\tau)},$$

where $I_{I(\gamma,\tau)}$ is the identity operator on $I(\gamma, \tau)$. Hence $\hat{E}(\gamma, \tau)$ is a right inverse of $\hat{D}(\gamma, \sigma)$.

Taking $\sigma = \tau$ we define $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_o$ as before. Then by Lemma 24, $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau)_o$ is the subspace of $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))$ such that

$$(\pi_\gamma(k) \otimes \tau(k))\hat{T}(\gamma) = \hat{T}(\gamma)(\pi_\gamma(k) \otimes \tau(k)) \qquad \text{for all} \quad k \in K.$$

We put

$$\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_* = \{T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_o \mid \delta_\tau * T = T\},$$

$$\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))^* = \{T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_o \mid \delta_\sigma * T = T\}.$$

Then $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))$ belongs to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$ if and only if

$$\hat{\delta}_\tau(\gamma)\hat{T}(\gamma) = \hat{T}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{T}(\gamma) \qquad \text{for all} \quad \gamma \in \hat{G}.$$

And $T \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ belongs to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ if and only if

$$\hat{\delta}_\sigma(\gamma)\hat{T}(\gamma) = \hat{T}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{T}(\gamma) \qquad \text{for all} \quad \gamma \in \hat{G}.$$

Clearly, $\delta_\tau \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$.

Let E be an elementary solution of D. Then if we put $E_1 = \delta_\sigma * E$, we have

$$\hat{D}(\gamma)\hat{E}_1(\gamma) = \hat{D}(\gamma)\hat{\delta}_\sigma(\gamma)\hat{E}(\gamma) = \hat{D}(\gamma)\hat{E}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{\delta}_\tau(\gamma)^2 = \hat{\delta}_\tau(\gamma)$$

for all $\gamma \in \hat{G}$. Hence E_1 is also an elementary solution of D. And E_1 is a member of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$.

Let $L \in \mathrm{Hom}_{\boldsymbol{C}}(I(\gamma, \tau), I(\gamma, \sigma))$. If we denote by $\|L\|_{\gamma,\tau,\sigma}$ the Hilbert-Schmidt norm of L, then

$$\|L \oplus O\|_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(V_\tau,V_\sigma)} = \|L\|_{\gamma,\tau,\sigma},$$

where O is the null operator of $N(\gamma, \tau)$ to $N(\gamma, \sigma)$.

THEOREM 6. *Let $D \in HD_K(\sigma, \tau)$. Then the following conditions are equivalent:*

(1) *D has an elementary solution;*

(2) *The mapping D of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$ is surjective;*

(3) *For any $\gamma \in \hat{G}$, $\hat{D}(\gamma, \sigma)$ has a right inverse $\hat{D}(\gamma, \sigma)_{\mathrm{R}}^{-1}$ which satisfies that there exist a constant $C > 0$ and an $\ell \in \boldsymbol{N}$ such that*

$$\|\hat{D}(\gamma, \sigma)_{\mathrm{R}}^{-1}\|_{\gamma,\tau,\sigma} \leq C d_\ell(\gamma)/d(\gamma)^{1/2} \qquad \textit{for all} \quad \gamma \in \hat{G}.$$

PROOF. We have shown the implication (1)⇒(3).

(1)⇒(2). Since for $S\in\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ we have

$$\hat{\delta}_\tau(\gamma)\hat{D}(\gamma)\hat{S}(\gamma) = \hat{D}(\gamma)\hat{\delta}_\sigma(\gamma)\hat{S}(\gamma) = \hat{D}(\gamma)\hat{S}(\gamma)$$

by (5.1)′ and

$$\hat{D}(\gamma)\hat{S}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{D}(\gamma)\hat{S}(\gamma),$$

we see that D is a mapping of $\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ to $\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$. Let E be an elementary solution of D. For $T\in\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$ we put $S=E_1*T$. Then $S\in\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$. And we have

$$\hat{\delta}_\sigma(\gamma)\hat{S}(\gamma) = \hat{\delta}_\sigma(\gamma)\hat{E}_1(\gamma)\hat{T}(\gamma) = \hat{E}_1(\gamma)\hat{T}(\gamma) = \hat{S}(\gamma)$$

and

$$\hat{S}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{E}_1(\gamma)\hat{T}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{E}_1(\gamma)\hat{T}(\gamma) = \hat{S}(\gamma).$$

Hence $S\in\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$. Moreover,

$$\hat{D}(\gamma)\hat{S}(\gamma) = \hat{D}(\gamma)\hat{E}_1(\gamma)\hat{T}(\gamma) = \hat{\delta}_\tau(\gamma)\hat{T}(\gamma) = \hat{T}(\gamma).$$

Hence $DS=T$. Thus D is surjective.

The implication (2)⇒(1) is trivial.

(3)⇒(1). We put $b(\gamma)=\hat{D}(\gamma,\sigma)_{\mathrm{R}}^{-1}\oplus O$, where O is the null operator of $N(\gamma,\tau)$ to $N(\gamma,\sigma)$. Then $b(\gamma)\in\mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\sigma)(\gamma)=\mathrm{End}(V_\gamma)\otimes\mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\sigma)$ for all $\gamma\in\hat{G}$ and there are a $C>0$ and an $\ell\in\boldsymbol{N}$ such that

$$\|b(\gamma)\|_{\gamma,\mathrm{Hom}_{\boldsymbol{C}}(V_\tau,V_\sigma)} \le Cd_\ell(\gamma)/d(\gamma)^{1/2} \qquad \text{for every} \quad \gamma\in\hat{G}.$$

Hence by Lemma 13 $b\in\mathscr{D}'(\hat{G};\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ and we then have that there is $T\in\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))$ such that $\hat{T}=b$. We put $E=\delta_\sigma*T*\delta_\tau$. Then $E\in\mathscr{D}'(G;\mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ and

$$\begin{aligned}\hat{D}(\gamma)\hat{E}(\gamma) &= \hat{D}(\gamma)\hat{\delta}_\sigma(\gamma)\hat{T}(\gamma)\hat{\delta}_\tau(\gamma)\\ &= \hat{\delta}_\tau(\gamma)(\hat{D}(\gamma,\sigma)\oplus\hat{D}(\gamma)|_{N(\gamma,\sigma)})(\hat{D}(\gamma,\sigma)_{\mathrm{R}}^{-1}\oplus O)\hat{\delta}_\tau(\gamma)\\ &= \hat{\delta}_\tau(\gamma)\hat{D}(\gamma,\sigma)\hat{D}(\gamma,\sigma)_{\mathrm{R}}^{-1}\hat{\delta}_\tau(\gamma) = \hat{\delta}_\tau(\gamma)^3 = \hat{\delta}_\tau(\gamma).\end{aligned}$$

Hence $DE=\delta_\tau$. Q.E.D.

LEMMA 26. *If $D\in HD_K(\sigma,\tau)$, then ${}^tD\in HD_K(\tilde{\tau},\tilde{\sigma})$.*

PROOF. Let $k\in K$ and $X, Y\in\mathfrak{g}$. Then

$$\begin{aligned}{}^t(k\cdot XY) &= {}^t((\mathrm{Ad}(k)X)(\mathrm{Ad}(k)Y) = (\mathrm{Ad}(k)Y)(\mathrm{Ad}(k)X)\\ &= k\cdot(YX) = k\cdot{}^t(XY).\end{aligned}$$

Hence for any $\xi \in \mathfrak{G}$

$$^t(k \cdot \xi) = k \cdot {}^t\xi.$$

Thus we have, for $D = \sum_j \xi_j \otimes L_j \in \mathfrak{G} \otimes \mathrm{Hom}_{\boldsymbol{C}}(V_\sigma, V_\tau)$,

$$^t(k \cdot D) = \sum_j k \cdot {}^t\xi_j \otimes {}^t\sigma(k^{-1})^t L_j{}^t\tau(k^{-1})^{-1} = k \cdot {}^tD.$$

Therefore, $D \in HD_K(\sigma, \tau)$ if and only if ${}^tD \in HD_K(\tilde{\tau}, \tilde{\sigma})$. Q.E.D.

LEMMA 27. *Let $D \in HD_K(\sigma, \tau)$. If D maps $\mathscr{D}'(G; \tilde{\sigma})$ onto $\mathscr{D}'(G; \tilde{\tau})$, then D has an elementary solution. Similarly, if ${}^tD$ maps $\mathscr{D}'(G; \tau)$ onto $\mathscr{D}'(G; \sigma)$, then ${}^tD$ has an elementary solution.*

PROOF. By Lemma 26, D maps $\mathscr{D}'(G; \tilde{\sigma})$ into $\mathscr{D}'(G; \tilde{\tau})$. By Theorem 6 it is enough to prove the surjectivity of the map D of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ to $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$. We choose bases as in the proof of Lemma 24. Let $S \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$. Under the identification $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau)) = \mathscr{D}'(G; \boldsymbol{C}) \otimes \hat{V}_\tau \otimes V_\tau$, we can write

$$S = \sum_{i,j} S_{ij} \otimes \phi_i \otimes e_j, \qquad S_{ij} \in \mathscr{D}'(G; \boldsymbol{C}).$$

Then

$$\hat{S}(\gamma) = \sum_{i,j} \hat{S}_{ij}(\gamma) \otimes \phi_i \otimes e_j,$$

where $\hat{S}_{ij}(\gamma) \in \boldsymbol{C}(\gamma) = \mathrm{End}(V_\gamma)$. We put

$$S_i = \sum_j S_{ij} \otimes e_j \qquad (1 \leqq i \leqq d(\tau)).$$

Then $S_i \in \mathscr{D}'(G; \boldsymbol{C}) \otimes V_\tau = \mathscr{D}'(G; \hat{V}_\tau)$. Let $\hat{\delta}_\tau(\gamma) = \sum_s L_s \otimes M_s$, $L_s \in \mathrm{End}(V_\gamma)$ and $M_s \in \mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\tau)$. We have

$$\hat{\delta}_\tau(\gamma)\hat{S}(\gamma) = \sum_{i,j} \sum_s L_s \hat{S}_{ij}(\gamma) \otimes \phi_i \otimes M_s e_j = \hat{S}(\gamma).$$

On the other hand,

$$\hat{\delta}_\tau(\gamma)\hat{S}_i(\gamma) = \sum_j \sum_s L_s \hat{S}_{ij}(\gamma) \otimes M_s e_j.$$

Hence $\hat{\delta}_\tau(\gamma)\hat{S}_i(\gamma) = \hat{S}_i(\gamma)$ for all $\gamma \in \hat{G}$. This shows that $S_i \in \mathscr{D}'(G; \tilde{\tau})$. From our assumption it follows that there are $T_i \in \mathscr{D}'(G; \tilde{\sigma})$ such that $DT_i = S_i$. If $D = \sum_t \xi_t \otimes A_t \in \mathfrak{G} \otimes \mathrm{Hom}_{\boldsymbol{C}}(V_\sigma, V_\tau)$ and $T_i = \sum_p T_{ip} \otimes e'_p \in \mathscr{D}'(G; \boldsymbol{C}) \otimes V_\sigma$, then

$$DT_i = \sum_p \sum_t \xi_t T_{ip} \otimes A_t e'_p = \sum_j S_{ij} \otimes e_j.$$

We put

$$T = \sum_{i,p} T_{ip} \otimes \phi_i \otimes e'_p \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma)).$$

Then

$$DT = \sum \sum \xi_t T_{ip} \otimes \phi_i \otimes A_t e'_p = \sum S_{ij} \otimes \phi_i \otimes e_j = S.$$

We put $T' = T*\delta_\tau$. Since $T_i \in \mathscr{D}'(G; \tilde{\sigma})$, we can see that $\hat{\delta}_\sigma(\gamma)\hat{T}_i(\gamma) = \hat{T}_i(\gamma)$. Hence

$$\hat{\delta}_\sigma(\gamma)\hat{T}'(\gamma) = \hat{T}'(\gamma)\hat{\delta}_\tau(\gamma) = \hat{T}'(\gamma),$$

i.e. $T' \in \mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$. And

$$DT' = (DT)*\delta_\tau = S*\delta_\tau = S. \qquad \text{Q.E.D.}$$

By the injection θ, $\mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\tau))$ can be regarded to be a subset of $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))$. We define $\mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\tau))_*$ as the subspace of all $f \in \mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\tau))$ which satisfy $\hat{\delta}_\tau(\gamma)\hat{f}(\gamma) = \hat{f}(\gamma)\hat{\delta}_\tau(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \hat{G}$, that is, $f = \delta_\tau * f = f*\delta_\tau$.

5.4. THEOREM 7. *We suppose that* $\mathrm{rank}\,\hat{\delta}_\sigma(\gamma) = \mathrm{rank}\,\hat{\delta}_\tau(\gamma)$ *for all* $\gamma \in \hat{G}$. *Let* $D \in HD_K(\sigma, \tau)$. *Then the following conditions are equivalent*:

(1) *D has an elementary solution*;

(2) ${}^tD$ *has an elementary solution*;

(3) $D(\mathscr{D}(G; \sigma)) = \mathscr{D}(G; \tau)$;

(4) *D is a topological isomorphism of* $\mathscr{D}(G; \sigma)$ *onto* $\mathscr{D}(G; \tau)$;

(5) *D maps* $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ *onto* $\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$;

(6) $D(\mathscr{D}'(G; \mathrm{Hom}_{\boldsymbol{C}}(\hat{V}_\tau, \hat{V}_\sigma))_* \subset \mathscr{D}(G; \mathrm{Hom}_{\boldsymbol{C}}(V_\tau, V_\tau))_*$;

(7) *For any* $\gamma \in \hat{G}$, $\hat{D}(\gamma, \sigma)$ *is invertible. And there exist a constant* $C > 0$ *and an* $\ell \in \boldsymbol{N}$ *such that*

$$\|\hat{D}(\gamma, \sigma)^{-1}\|_{\gamma,\tau,\sigma} \leq C d_\ell(\gamma)/d(\gamma)^{1/2} \qquad \text{for all} \quad \gamma \in \hat{G}.$$

PROOF. The implication from (1) to (3) is given in Lemma 25.

(3)⇒(4). If D is a surjection from $\mathscr{D}(G; \sigma)$ to $\mathscr{D}(G; \tau)$, then $\hat{D}(\gamma, \sigma)$ is a surjection from $I(\gamma, \sigma)$ to $I(\gamma, \tau)$ for all $\gamma \in \hat{G}$. Then $\hat{D}(\gamma, \sigma)$ is regular, for $\dim_{\boldsymbol{C}} I(\gamma, \sigma) = \dim_{\boldsymbol{C}} I(\gamma, \tau)$. If $Df = 0$ for $f \in \mathscr{D}(G; \sigma)$, then $\hat{D}(\gamma, \sigma)\hat{f}(\gamma) = 0$. Hence $\hat{f}(\gamma) = 0$ for all $\gamma \in \hat{G}$. Therefore, D is a continuous bijective mapping of $\mathscr{D}(G; \sigma)$ to $\mathscr{D}(G; \tau)$ and is a topological isomorphism.

(4)⇒(2). We put $D_1 = D|_{\mathscr{D}(G;\sigma)}$. Then the inverse mapping D_1^{-1} of D_1 is a continuous mapping of $\mathscr{D}(G; \tau)$ onto $\mathscr{D}(G; \sigma)$. For any $S \in \mathscr{D}'(G; \sigma)$ the linear functional $T: g \mapsto \langle S, D_1^{-1}g \rangle$ is continuous on $\mathscr{D}(G; \tau)$, i.e. $T \in \mathscr{D}'(G; \tau)$. Regarding S and T as elements of $\mathscr{D}'(G; V_\sigma)$ and $\mathscr{D}'(G; V_\tau)$, respectively, we get $S_\sigma = S$ and $T_\tau = T$. Then for any $f \in \mathscr{D}(G; V_\sigma)$

$$\begin{aligned} \langle S, f \rangle &= \langle S_\sigma, f \rangle = \langle S, \delta_\sigma * f \rangle = \langle T, D(\delta_\sigma * f) \rangle \\ &= \langle T, \delta_\tau * (Df) \rangle = \langle T_\tau, Df \rangle = \langle T, Df \rangle = \langle {}^tDT, f \rangle. \end{aligned}$$

Hence $S = {}^tDT$. Thus we have proved that ${}^tD$ is a surjective mapping of

$\mathscr{D}'(G; \tau)$ to $\mathscr{D}'(G; \sigma)$. Then ${}^tD$ has an elementary solution from Lemma 27.

If ${}^tD$ has an elementary solution, then combining the fact $D={}^t({}^tD)$ with the implications proved above, we have (1).

(2)⇒(5). If ${}^tD$ has an elementary solution, ${}^tD$ is a topological isomorphism of $\mathscr{D}(G; \tilde{\tau})$ onto $\mathscr{D}(G; \tilde{\sigma})$ as mentioned above. Then by the duality D is a topological isomorphism of $\mathscr{D}'(G; \tilde{\sigma})$ onto $\mathscr{D}'(G; \tilde{\tau})$ in the sense of the weak topology. Then, as we have proved in the proof of Lemma 27, D is a surjection from $\mathscr{D}'(G; \mathrm{Hom}_{\mathbf{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ to $\mathscr{D}'(G; \mathrm{Hom}_{\mathbf{C}}(\hat{V}_\tau, \hat{V}_\tau))_*$.

The implication from (5) to (6) is trivial.

By Theorem 6, (1) is equivalent to (7).

Now let us assume (6). We put

$$h_\gamma(x) = d(\gamma)\int_K \mathrm{tr}\,(\pi_\gamma(xk))\tau(k)dk.$$

Then $h_\gamma \in \mathscr{D}(G; \mathrm{Hom}_{\mathbf{C}}(V_\tau, V_\tau))$. We set

$$g_\gamma(x) = d(\gamma)\,\mathrm{tr}\,(\pi_\gamma(x))I_{V_\tau}.$$

Then $h_\gamma = \delta_\tau * g$. As we have seen in the proof of Theorem 3,

$$\hat{g}_\gamma(\gamma) = I_{V_\gamma} \otimes I_{V_\tau}.$$

Hence

$$\hat{h}_\gamma(\gamma) = \hat{\delta}_\tau(\gamma)\hat{g}_\gamma(\gamma) = \hat{\delta}_\tau(\gamma).$$

Therefore, $h_\gamma \in \mathscr{D}(G; \mathrm{Hom}_{\mathbf{C}}(V_\tau, V_\tau))_*$ and by our assumption there is a $T^\gamma \in \mathscr{D}'(G; \mathrm{Hom}_{\mathbf{C}}(\hat{V}_\tau, \hat{V}_\sigma))_*$ such that $DT^\gamma = h_\gamma$. Then $(T^\gamma)\hat{\ }(\gamma)|_{I(\gamma,\tau)}$ is the inverse of $\hat{D}(\gamma, \sigma)$. If we replace A by $\hat{D}(\gamma, \sigma)^{*-1} \oplus O$, where O is the null operator of $N(\gamma, \tau)$ to $N(\gamma, \sigma)$, the rest of the proof is the same as that of Theorem 3. Then we can prove that $\hat{D}(\gamma, \sigma)^{-1}$ satisfies the condition in (7). Q.E.D.

§6. Global hypoellipticity and elementary solutions

Let σ and τ be finite-dimensional unitary representations of a closed subgroup K of G. By Lemma 23 we can consider that $\mathscr{D}(G; \sigma)$ is a subspace of $\mathscr{D}'(G; \tilde{\sigma})$.

DEFINITION. Let $D \in HD_K(\sigma, \tau)$. Then, D is called *globally hypoelliptic* if whenever $DT \in \mathscr{D}(G; \tau)$ for $T \in \mathscr{D}'(G; \tilde{\sigma})$ then $T \in \mathscr{D}(G; \sigma)$.

We set $\hat{G}(\sigma) = \{\gamma \in \hat{G} \mid \hat{\delta}_\sigma(\gamma) \neq 0\}$. For $A \in \mathrm{Hom}_{\mathbf{C}}(I(\gamma, \sigma), I(\gamma, \tau))$ we put

$$m_\gamma^\sigma(A) = \inf\{\|Av\|_{\gamma, V_\tau} \mid \|v\|_{\gamma, V_\sigma} = 1,\ v \in I(\gamma, \sigma)\}.$$

For $D \in HD_K(\sigma, \tau)$ we put $m_\gamma^\sigma(D) = m_\gamma^\sigma(\hat{D}(\gamma, \sigma))$. The following theorem, which characterizes globally hypoelliptic operators, is due to N. R. Wallach [2, Theorem 5.8.3].

THEOREM 8 (Wallach). *Let* $D\in HD_K(\sigma, \tau)$. *Then the following conditions* (1) *and* (2) *are equivalent*:

(1) *D is globally hypoelliptic*;

(2) *There are an* $\ell\in \boldsymbol{N}$, *a* $C>0$ *and a finite subset F of* $\hat{G}(\sigma)$ *such that*

$$m_\gamma^\sigma(D)\geq Cd_\ell(\gamma)^{-1} \qquad \textit{for all} \quad \gamma\in\hat{G}(\sigma)\backslash F.$$

For $B\in \mathrm{Hom}_{\boldsymbol{C}}(I(\gamma, \tau), I(\gamma, \sigma))$ we put

$$M_\gamma^\tau(B)=\sup\{\|Bw\|_{\gamma,V_\sigma} \mid \|w\|_{\gamma,V_\tau}=1,\ w\in I(\gamma, \tau)\}.$$

For a homogeneous differential operator $D\in HD_K(\sigma, \tau)$ we put $M_\gamma^\tau(D^{-1})=M_\gamma^\tau(\hat{D}(\gamma, \sigma)^{-1})$ if $\hat{D}(\gamma, \sigma)$ is invertible.

COROLLARY. *Let* $D\in HD_K(\sigma, \tau)$. *If* $\mathrm{rank}\,\hat{\delta}_\sigma(\gamma)=\mathrm{rank}\,\hat{\delta}_\tau(\gamma)$ *for all* $\gamma\in\hat{G}$, *then the following conditions are equivalent*:

(1) *D is globally hypoelliptic*;

(2) *There exist an* $\ell\in\boldsymbol{N}$, *a* $C>0$ *and a finite subset F of* $\hat{G}(\sigma)$ *such that, for* $\gamma\in\hat{G}(\sigma)\backslash F$, $\hat{D}(\gamma, \sigma)$ *is invertible and*

$$M_\gamma^\tau(D^{-1})\leq Cd_\ell(\gamma).$$

PROOF. (1)⇒(2). By Theorem 8 there are an $\ell\in\boldsymbol{N}$, a $C'>0$ and a finite subset F of $\hat{G}(\sigma)$ such that $m_\gamma^\sigma(D)\geq C'd_\ell(\gamma)^{-1}$ for $\gamma\in\hat{G}(\sigma)\backslash F$. Hence, for $\gamma\in\hat{G}(\sigma)\backslash F$, $\hat{D}(\gamma, \sigma)$ is invertible. Remark that if $A\in\mathrm{Hom}_{\boldsymbol{C}}(I(\gamma, \sigma), I(\gamma, \tau))$ is invertible, then $M_\gamma^\tau(A^{-1})=m_\gamma^\sigma(A)^{-1}$. Therefore, if we put $C=C'^{-1}$, we have (2).

(2)⇒(1). Let $\gamma\in\hat{G}(\sigma)\backslash F$. Let $v\in I(\gamma, \sigma)$ be $\|v\|_{\gamma,V_\sigma}=1$. Then $\hat{D}(\gamma, \sigma)v\neq 0$ and $\|\hat{D}(\gamma, \sigma)v\|_{\gamma,V_\tau}\neq 0$. Hence

$$\begin{aligned}1&=\|v\|_{\gamma,V_\sigma}=\|\hat{D}(\gamma, \sigma)^{-1}(\hat{D}(\gamma, \sigma)v/\|\hat{D}(\gamma, \sigma)v\|_{\gamma,V_\tau})\|_{\gamma,V_\sigma}\|\hat{D}(\gamma, \sigma)v\|_{\gamma,V_\tau}\\&\leq M_\gamma^\tau(D^{-1})\,\|\hat{D}(\gamma, \sigma)v\|_{\gamma,V_\tau}\leq Cd_\ell(\gamma)\,\|\hat{D}(\gamma, \sigma)v\|_{\gamma,V_\tau}.\end{aligned}$$

Therefore,

$$m_\gamma^\sigma(D)\geq C^{-1}d_\ell(\gamma)^{-1} \qquad \text{for all} \quad \gamma\in\hat{G}(\sigma)\backslash F.$$

Hence by Theorem 8, D is globally hypoelliptic. Q.E.D.

THEOREM 9. *We assume that* $\mathrm{rank}\,\hat{\delta}_\sigma(\gamma)=\mathrm{rank}\,\hat{\delta}_\tau(\gamma)$ *for all* $\gamma\in\hat{G}$. *Let* $D\in HD_K(\sigma, \tau)$. *Then D has an elementary solution if and only if it is globally hypoelliptic and is an injective mapping of* $\mathscr{D}(G; \sigma)$ *to* $\mathscr{D}(G; \tau)$.

PROOF. Suppose that D has an elementary solution. Then by Theorem 7, $\hat{D}(\gamma, \sigma)$ is invertible for every $\gamma \in \hat{G}$ and there exist a constant $C>0$ and an $\ell \in \boldsymbol{N}$ such that

$$\|\hat{D}(\gamma, \sigma)^{-1}\|_{\gamma,\tau,\sigma} \leq C d_{\ell}(\gamma) \qquad \text{for every} \quad \gamma \in \hat{G}.$$

Hence if $w \in I(\gamma, \tau)$ and $\|w\|_{\gamma,V_\tau}=1$, then

$$\|\hat{D}(\gamma, \sigma)^{-1} w\|_{\gamma,V_\sigma} \leq C d_{\ell}(\gamma)\, \|w\|_{\gamma,V_\tau} = C d_{\ell}(\gamma) \qquad \text{for every} \quad \gamma \in \hat{G}.$$

Therefore,

$$M_{\gamma}^{\tau}(D^{-1}) \leq C d_{\ell}(\gamma)$$

and then D is globally hypoelliptic by the corollary of Theorem 8.

Conversely, suppose that D is globally hypoelliptic and injective. We first show that $\hat{D}(\gamma, \sigma)$ is invertible for all $\gamma \in \hat{G}$. Let us assume that $\hat{D}(\gamma, \sigma)v=0$ for $v \in I(\gamma, \sigma)$. We set $f(x)=d(\gamma)\operatorname{tr}((\pi_\gamma(x)\otimes I_{V_\tau})v)$. Then $\hat{f}(\gamma)=v$ and $\hat{f}(\gamma')=0$ for $\gamma' \neq \gamma$, $\gamma' \in \hat{G}$ and hence $\hat{\delta}_\sigma(\gamma')\hat{f}(\gamma')=\hat{f}(\gamma')$ for all $\gamma' \in \hat{G}$. Then $f \in \mathscr{D}(G; \sigma)$ and $Df=0$. Hence $f=0$. By the irreducibility of π_γ, $v=0$. Thus we have proved that $\hat{D}(\gamma, \sigma)$ is injective and hence invertible. By the corollary of Theorem 8 there are an $\ell \in \boldsymbol{N}$, a $C>0$ and a finite subset F of $\hat{G}(\sigma)$ such that

$$M_{\gamma}^{\tau}(D^{-1}) \leq C d_{\ell}(\gamma) \qquad \text{for every} \quad \gamma \in \hat{G}(\sigma)\backslash F.$$

If $w \in I(\gamma, \tau)$, we then have

$$\|\hat{D}(\gamma, \sigma)^{-1} w\|_{\gamma,V_\sigma} \leq C d_{\ell}(\gamma)\, \|w\|_{\gamma,V_\tau} \qquad \text{for any} \quad \gamma \in \hat{G}(\sigma)\backslash F.$$

We put $C'=\max\{C, d_\ell(\gamma)^{-1}\|\hat{D}(\gamma, \sigma)^{-1}\|_{\gamma,\tau,\sigma}\ (\gamma \in F)\}$. Let g be any function in $\mathscr{D}(G; \tau)$. Then for any $j \in \boldsymbol{N}$ we can find $C_j>0$ so that

$$\|\hat{g}(\gamma)\|_{\gamma,V_\tau} \leq C_j d_j(\gamma)^{-1}$$

(Lemma 14 and its corollary). For any $t \in \boldsymbol{N}$ we choose j so that $j \geq 2(t+\ell)$. Then we know that there is a constant $C''>0$ such that $d_\ell(\gamma)d_t(\gamma) \leq C'' d_j(\gamma)$ for any $\gamma \in \hat{G}$. We define a $V_\sigma(\hat{G})$-valued function a on $\hat{G}$ by

$$a(\gamma) = \hat{D}(\gamma, \sigma)^{-1}\hat{g}(\gamma).$$

Then it is easy to see that $a(\gamma) \in I(\gamma, \sigma)$ for all $\gamma \in \hat{G}$. Since we have

$$\|a(\gamma)\|_{\gamma,V_\sigma} \leq C' C_j C'' d_j(\gamma)^{-1} \qquad \text{for any} \quad \gamma \in \hat{G},$$

a is the Fourier transform of a function $f \in \mathscr{D}(G; \sigma)$. Since

$$(Df)\hat{\ }(\gamma) = \hat{D}(\gamma)\hat{D}(\gamma, \sigma)^{-1}\hat{g}(\gamma) = \hat{g}(\gamma),$$

we have $Df=g$. Therefore, $D(\mathscr{D}(G; \sigma))=\mathscr{D}(G; \tau)$. Hence by Theorem 7, D has an elementary solution. Q. E. D.

References

[1] A. Cerezo and F. Rouvière, *Solution élémentaire d'un opérateur differentiel linéaire invariant à gauche sur un groupe de Lie réel compact et sur un espace homogène réductif compact,* Ann. Sci. École. Norm. Sup., 4e série, **2** (1969), 561–581.

[2] N. R. Wallach, *Harmonic Analysis on Homogeneous Spaces,* Marcel Dekker, New York, 1973.

Department of Mathematics,
Faculty of Integrated Arts and Sciences,
Hiroshima University
and
Department of Mathematics,
Faculty of General Education,
Tottori University