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Bernstein's theorem and translation invariant operators

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We propose to give non-homogeneous versions of some results on Fourier transforms and translation invariant operators of homogeneous Besov and Hardy spaces. Our first aim is to derive an analogue of Herz's version of Bernstein's theorem ([8]) for the non-homogeneous Besov spaces of Taibleson ([15]). Two proofs of this theorem will be presented. The first proof is quite elementary; the main tool is an inequality due to Flett ([5]). Our second proof is based on a relation linking non-homogeneous and homogeneous spaces, which allows us to pass from non-homogeneous to homogeneous spaces and then to use the theorem of Herz. As it turns out, the spaces describing integrability of Fourier transforms of distributions in non-homogeneous Besov spaces arise naturally as intermediate spaces between weighted L^p spaces ([6]), and they also generalize some algebras of Beurling ([2]).

Our second group of results concerns translation invariant operators. It is a known fact that Besov spaces can be used to measure smoothness of translation invariant operators on Lebesgue or Besov spaces ([15], [14]). The results of Johnson ([10], [11]) give necessary and/or sufficient conditions, in terms of homogeneous Besov spaces, for operators on Hardy spaces to be bounded and translation invariant. We generalize these results to the local Hardy spaces defined in [7], and improve or supplement some results of Taibleson and Stein-Zygmund ([15], [14]).

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NOTATION. Our notation is standard. We use R^n to denote the n-dimensional euclidean space and R^{n+1}_+ to denote the cartesian product $R^n \times]0, \infty[$. An element of R^{n+1}_+ is denoted by (x, t), where $x \in R^n$ and $0<t<\infty$. The Fourier transform is defined by

$$\mathscr{F} f(x) = \hat{f}(x) = \int e^{-2\pi i x \cdot y} f(y) dy,$$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$, and the integral is extended over all of R^n unless otherwise indicated. If u is an infinitely differentiable function on R^{n+1}_+, then $D^k_{n+1} u$ stands for $(\partial/\partial t)^k u$. For measurable functions u on R^{n+1}_+, we write

$$M_p(u;\, t) = \|u(\cdot,\, t)\|_p, \quad 0 < p \leq \infty.$$

The Gauss-Weierstrass kernel on R_+^{n+1} is denoted by W, i.e.,

$$W(x,\, t) = W_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t).$$

If $f \in \mathscr{S}'$, $1 \leq p \leq \infty$ and $u(\cdot,\, t) = W_t * f$, then $M_p(u;\, t)$ is non-increasing in t; we call u the Gauss-Weierstrass integral of f.

We adopt the current notation $B_{p,q}^\alpha$ and $\dot{B}_{p,q}^\alpha$ to denote the non-homogeneous and homogeneous Besov spaces considered by Taibleson ([15]) and Herz ([8]), respectively. We shall mainly use the characterizations via traces of temperatures on R_+^{n+1} given by Flett ([4]) and Johnson ([9]). Let $-\infty < \alpha < \infty$, $1 \leq p,\, q \leq \infty$ and k be a non-negative integer greater than $\alpha/2$. Then

$$B_{p,q}^\alpha = \Big\{f \in \mathscr{S}';\, B_{p,q}^\alpha(f) = \sup_{t \geq 1/2} M_p(u;\, t) + \Big(\int_0^1 [t^{k-\alpha/2} M_p(D_{n+1}^k u;\, t)]^q t^{-1} dt\Big)^{1/q} < \infty\Big\}$$

and

$$\dot{B}_{q,q}^\alpha = \Big\{f \in \mathscr{S}';\, \dot{B}_{p,q}^\alpha(f) = \Big(\int_0^\infty [t^{k-\alpha/2} M_p(D_{n+1}^k u;\, t)]^q t^{-1} dt\Big)^{1/q} < \infty\Big\},$$

where $u(\cdot,\, t) = W_t * f$, and in general $\Big(\int_X |g(x)|^p dv(x)\Big)^{1/p}$ means v-ess $\sup_X |g|$ if $p = \infty$ and v is a measure on X.

All immaterial constants are denoted by C, c,... hereafter; they are not necessarily the same on any two consecutive occurrences.

§1. The spaces $\dot{K}_{p,q}^\alpha$ and $K_{p,q}^\alpha$

In the paper [8], Herz introduced a space $\dot{K}_{p,q}^\alpha$ which is very useful in the study of Fourier transforms of distributions in $\dot{B}_{p,q}^\alpha$. (We use a dot in the notation for Herz's spaces in accordance with the current usage of notation to indicate that they are "homogeneous" spaces.) We shall adopt a characterization of $\dot{K}_{p,q}^\alpha$ given by Johnson [10] as its definition (cf. also [5]). Let $-\infty < \alpha < \infty$, $1 \leq p,\, q \leq \infty$ and $0 < \gamma < \delta < \infty$. A measurable function f is in the space $\dot{K}_{p,q}^\alpha$ if

$$\dot{k}_{p,q}^\alpha(f) = \dot{k}_{p,q,\gamma,\delta}^\alpha(f) = \Big(\int_0^\infty \Big(t^{\alpha p} \int_{\gamma t \leq |x| \leq \delta t} |f(x)|^p\Big)^{q/p} t^{-1} dt\Big)^{1/q} < \infty.$$

It was proved in [10] that if $\{\lambda_j\}_{j=-\infty}^\infty$ is a sequence of positive numbers such that $1 < \rho \leq \lambda_{j+1}/\lambda_j \leq \sigma < \infty$ for $j = 0,\, \pm 1,\, \pm 2, \ldots$, then

$$\dot{K}_{p,q}^\alpha(f) = \Big(\sum_{j=-\infty}^\infty \Big(\int_{\lambda_j \leq |x| \leq \lambda_{j+1}} (|x|^\alpha |f(x)|)^p dx\Big)^{q/p}\Big)^{1/q} \approx \dot{k}_{p,q}^\alpha(f),$$

where $F(h)\approx G(h)$, for h in some class K, means that there exist positive constants C and c such that

$$cG(h) \le F(h) \le CG(h).$$

Various other families of equivalent norms for $\dot{K}^{\alpha}_{p,q}$ were given in [8], [5] and [10].

Now we turn to the non-homogeneous version of $\dot{K}^{\alpha}_{p,q}$. Let α, p, q, γ and δ be as above. We define $K^{\alpha}_{p,q}$ as the space of those measurable functions f with

$$k^{\alpha}_{p,q}(f) = k^{\alpha}_{p,q,\gamma,\delta}(f) = \Big(\int_{|x|\le\delta} |f(x)|^p dx\Big)^{1/p} + \Big(\int_1^{\infty}\Big(t^{\alpha p}\int_{\gamma t\le|x|\le\delta t} |f(x)|^p dx\Big)^{q/p} t^{-1}dt\Big)^{1/q} < \infty.$$

A repeated application of change of variables shows that if $0<\gamma<\delta<\infty$ and $0<\eta<\mu<\infty$, then $k^{\alpha}_{p,q,\gamma,\delta}(f)\approx k^{\alpha}_{p,q,\eta,\mu}(f)$. An argument similar to the proof of Theorem 2 of [10] gives another family of equivalent norms for $K^{\alpha}_{p,q}$.

LEMMA 1. *Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $1<\rho\le\lambda_{j+1}/\lambda_j\le\sigma<\infty$ for $j=1, 2,\dots$. Then*

$$K^{\alpha}_{p,q}(f) = \Big(\int_{|x|\le\lambda_1} |f(x)|^p dx\Big)^{1/p} + \Big(\sum_{j=1}^{\infty}\Big(\int_{\lambda_j\le|x|\le\lambda_{j+1}} (|x|^{\alpha}|f(x)|)^p dx\Big)^{q/p}\Big)^{1/q} \approx k^{\alpha}_{p,q}(f).$$

We also need an inequality due to Flett.

LEMMA 2 ([5; Theorem 4]). *Let a, μ be positive numbers, and $0<p\le\infty$. Then for every measurable function H: $]0, \infty[\to[0, \infty[$ we have*

$$\Big(\int_0^{\infty}\Big(t^{\mu}\int_0^{\infty} e^{-ar^2t}H(r)dr\Big)^p t^{-1}dt\Big)^{1/p} \approx \Big(\int_0^{\infty}\Big(t^{-2\mu}\int_0^{t} H(r)dr\Big)^p t^{-1}dt\Big)^{1/p} \approx \Big(\int_0^{\infty}\Big(t^{-2\mu}\int_t^{t\sqrt{2}} H(r)dr\Big)^p t^{-1}dt\Big)^{1/p}.$$

The same result holds if each inner integral in the above is replaced by the essential supremum of its integrand over the same range.

LEMMA 3. *Let $\beta>\alpha$, $a>0$, $1\le p, q\le\infty$, $I=\{|x|\le\sqrt{2}\}$, $J=\{|x|\ge\sqrt{2}\}$ and $I_t=\{\sqrt{2}\le|x|\le t\}$ for $t\ge\sqrt{2}$. Then*

$$k^{\alpha}_{p,q}(f) \approx \Big(\int_I |f(x)|^p dx\Big)^{1/p} + \Big(\int_{\sqrt{2}}^{\infty}\Big(t^{-p(\beta-\alpha)}\int_{I_t} |f(x)|^p|x|^{p\beta}dx\Big)^{q/p} t^{-1}dt\Big)^{1/q}$$
$$\approx \Big(\int_I |f(x)|^p dx\Big)^{1/p} + \Big(\int_0^{\infty}\Big(t^{p(\beta-\alpha)/2}\int_J e^{-a|x|^2t}|f(x)|^p|x|^{p\beta}dx\Big)^{q/p} t^{-1}dt\Big)^{1/q}.$$

PROOF (cf. [5; p. 544]). The proof is carried out only in the case $p<\infty$ and $q<\infty$ since the other cases are similar, in fact simpler. Let $\gamma=1$ and $\delta=\sqrt{2}$ in the definition of $k^{\alpha}_{p,q}(f)$. For simplicity put $G(r)=\int_{|\sigma|=1}|f(r\sigma)|^p d\sigma$. Then

$$k^{\alpha}_{p,q}(f)=\Big(\int_I |f(x)|^p dx\Big)^{1/p}+\Big(\int_1^{\infty}\Big(t^{\alpha p}\int_t^{t\sqrt{2}}G(r)r^{n-1}dr\Big)^{q/p}t^{-1}dt\Big)^{1/q}.$$

Denoting the second term of the right hand side of the above by A, we see that

$$A\approx\Big(\int_1^{\infty}\Big(t^{-p(\beta-\alpha)}\int_t^{t\sqrt{2}}G(r)r^{n+p\beta-1}dr\Big)^{q/p}t^{-1}dt\Big)^{1/q}.$$

Letting

$$H(r)=\begin{cases} G(r)r^{n+p\beta-1} & \text{if } r\geq\sqrt{2}\\ 0 & \text{if } r<\sqrt{2}\end{cases}$$

in Lemma 2 and $P=\{|x|\leq 2\}$, we have

$$\begin{aligned}k^{\alpha}_{p,q}(f)&\approx\Big(\int_P |f(x)|^p dx\Big)^{1/p}+\Big(\int_{\sqrt{2}}^{\infty}\Big(t^{-p(\beta-\alpha)}\int_t^{t\sqrt{2}}H(r)dr\Big)^{q/p}t^{-1}dt\Big)^{1/q}\\ &\approx\Big(\int_I |f(x)|^p dx\Big)^{1/p}+\Big(\int_{\sqrt{2}}^{\infty}\Big(t^{-p(\beta-\alpha)}\int_{\sqrt{2}}^{t}H(r)dr\Big)^{q/p}t^{-1}dt\Big)^{1/q}\\ &\approx\Big(\int_I |f(x)|^p dx\Big)^{1/q}+\Big(\int_0^{\infty}\Big(t^{p(\beta-\alpha)/2}\int_{\sqrt{2}}^{\infty}H(r)e^{-ar^2t}dr\Big)^{q/p}t^{-1}dt\Big)^{1/q},\end{aligned}$$

which implies the conclusion of the lemma.

We shall list elementary properties of the spaces $K^{\alpha}_{p,q}$. They either are easily proved or can be derived from the corresponding properties of the spaces $\dot{K}^{\alpha}_{p,q}$ in [8] (cf. also [5]). Let $1\leq p,\ q,\ r\leq\infty$ and $-\infty<\alpha,\ \beta<\infty$.

(a) $K^{\alpha}_{p,q}$ is a Banach space.

(b) $K^{0}_{p,p}=L^p$.

(c) $K^{\alpha}_{p,q}\subset L^p$ if $\alpha>0$.

(d) The dual space of $K^{\alpha}_{p,q}$ is $K^{-\alpha}_{p',q'}$ if p and q are finite $(1/p+1/p'=1/q+1/q'=1)$.

(e) $K^{\alpha}_{p,q}\subseteq K^{\alpha}_{p,r}$ if $q\leq r$.

(f) $K^{\alpha-n/r}_{r,q}\subseteq K^{\alpha-n/p}_{p,q}$ if $p\leq r$.

(g) $K^{\alpha}_{p,q}\subset\dot{K}^{\alpha}_{p,q}$ if $\alpha>0$.

(h) $\dot{K}^{\alpha}_{p,q}\subset K^{\alpha}_{p,q}$ if $\alpha<0$.

(i) $K^{0}_{p,q}\subseteq\dot{K}^{0}_{p,q}$ if $p\leq q$.

(j) $\dot{K}^{0}_{p,q}\subseteq K^{0}_{p,q}$ if $q\leq p$.

(k) $f\in K^{\alpha}_{p,q}$ if and only if $(1+|x|)^{\beta}f\in K^{\alpha-\beta}_{p,q}$.

§ 2. Bernstein's theorem

LEMMA 4. *Let $\alpha\leq 0$, $1\leq p\leq 2$, $1\leq q\leq\infty$ and f be a distribution in $B^{\alpha}_{p,q}$. Assume that u is the Gauss-Weierstrass integral of f and $0<\delta<\infty$. Then $\hat{f}(\xi)=\hat{u}(\xi,\delta)\exp(4\pi^2|\xi|^2\delta)$ and it is a function of temperate growth. Actually, for any $\beta<\alpha$ we have*

$$\left(\int[|\hat{f}(\xi)|(1+|\xi|)^{\beta}]^{p'}d\xi\right)^{1/p'}\leq CB^{\alpha}_{p,q}(f).$$

PROOF. Since $M_p(u;t)$ is non-increasing in t, it follows from the characterization for $B^{\alpha}_{p,q}$ given before Section 1 that for $t\leq 1/2$

$$M_p(u;t)\leq CB^{\alpha}_{p,q}(f)t^{\alpha/2}\qquad(\alpha<0),\tag{1}$$

$$M_p(u;t)\leq CB^{0}_{p,q}(f)\log(1/t)\qquad(\alpha=0).\tag{2}$$

Put $g(\xi)=\hat{u}(\xi,\delta)\exp(4\pi^2|\xi|^2\delta)$. Since $u(\cdot,t+\delta)=W_t*u(\cdot,\delta)=W_\delta*u(\cdot,t)$, $t>0$, we see that

$$g(\xi)\exp(-4\pi^2|\xi|^2t)=\hat{u}(\xi,t),\quad t>0.$$

Therefore, it follows from Hausdorff-Young's theorem that

$$\left(\int_{\{|\xi|\leq 2\}}|g(\xi)|^{p'}d\xi\right)^{1/p'}\leq CM_p(u;1)\leq CB^{\alpha}_{p,q}(f).\tag{3}$$

On the other hand, for every $t\leq 1/2$, Hausdorff-Young's theorem, (1) and (2) imply that

$$\left(\int_{I_j}[|g(\xi)|\exp(-4\pi^2|\xi|^2t)]^{p'}d\xi\right)^{1/p'}\leq CM_p(u;t)$$
$$\leq\begin{cases}CB^{\alpha}_{p,q}(f)t^{\alpha/2} & (\alpha<0)\\ CB^{0}_{p,q}(f)\log(1/t) & (\alpha=0),\end{cases}$$

where $I_j=\{2^j\leq|\xi|^2\leq 2^{j+1}\}$ and $j=1,2,\ldots$. Let $\alpha<0$. Taking $t=2^{-j}$ in the above, we have

$$\left(\int_{I_j}[|g(\xi)|\,|\xi|^{\alpha}]^{p'}d\xi\right)^{1/p'}\leq CB^{\alpha}_{p,q}(f).$$

Now

$$\begin{aligned}\left(\sum_{j=1}^{\infty}\int_{I_j}[|g(\xi)|\,|\xi|^{\beta}]^{p'}d\xi\right)^{1/p'}&\leq C\left(\sum_{j=1}^{\infty}2^{p'j(\beta-\alpha)/2}\int_{I_j}[|g(\xi)|\,|\xi|^{\alpha}]^{p'}d\xi\right)^{1/p'}\\&\leq C\left(\sum_{j=1}^{\infty}2^{j(\beta-\alpha)/2}\right)B^{\alpha}_{p,q}(f)\leq CB^{\alpha}_{p,q}(f).\end{aligned}\tag{4}$$

Similarly, we obtain (4) for the case $\alpha=0$. These estimates, combined with (3), complete the proof of the lemma if we can show that $\hat{f}=g$. In fact, the last assertion follows easily by noting that for any $\psi\in\mathscr{S}$,

$$\begin{aligned}\int u(x,t)\psi(x)dx &= \int \hat{u}(\xi,t)\mathscr{F}^{-1}\psi(\xi)d\xi\\ &= \int g(\xi)\exp(-4\pi^2|\xi|^2t)\mathscr{F}^{-1}\psi(\xi)d\xi \to \mathscr{F}^{-1}g(\psi) \quad \text{as} \quad t\to 0\end{aligned}$$

by Lebesgue's dominated convergence theorem.

Now we are ready to prove our main result of this section.

THEOREM 1. *If* $1\le p\le 2$, $1\le q\le\infty$ *and* $-\infty<\alpha<\infty$, *then the Fourier transform is a continuous map of* $B^\alpha_{p,q}$ *into* $K^\alpha_{p',q}$ *and of* $K^\alpha_{p,q}$ *into* $B^\alpha_{p',q}$.

PROOF. Let f be a distribution in $B^\alpha_{p,q}$ and k be a non-negative integer greater than $\alpha/2$. Set $u=W_t*f$ and $v=D^k_{n+1}u$. Noting that $\hat{v}(\xi,t)=(-4\pi^2|\xi|^2)^k$ $\exp(-4\pi^2|\xi|^2t)\hat{f}(\xi)$, we derive from Hausdorff-Young's theorem that

$$(5)\qquad \begin{aligned}M_p(v;t) &\ge C\Big(\int_{I_j}[|\xi|^{2k-\alpha}\exp(-4\pi^2|\xi|^2t)\,|\xi|^\alpha|\hat{f}(\xi)|]^{p'}d\xi\Big)^{1/p'}\\ &\ge C2^{j(k-\alpha/2)}\exp(-4\pi^2 2^{j+1}t)\Big(\int_{I_j}[|\xi|^\alpha|\hat{f}(\xi)|]^{p'}d\xi\Big)^{1/p'}\end{aligned}$$

for any $j=1,2,\ldots$, where I_j is as in the proof of Lemma 4. Letting $t=2^{-j}$, $j=1,2,\ldots$, in (5) and summing up, we obtain

$$\begin{aligned}\Big(\sum_{j=1}^\infty\Big(\int_{I_j}[|\xi|^\alpha|\hat{f}(\xi)|]^{p'}d\xi\Big)^{q/p'}\Big)^{1/q} &\le C\Big(\sum_{j=1}^\infty[2^{-j(k-\alpha/2)}M_p(v;2^{-j})]^q\Big)^{1/q}\\ &\le CB^\alpha_{p,q}(f).\end{aligned}$$

This, combined with Lemma 4 and Lemma 1, proves the first part of the theorem.

Conversely, assume that $g\in K^\alpha_{p,q}$ and k is a non-negative integer greater than $\alpha/2$. Let $w=W_t*\hat{g}$ and $s=D^k_{n+1}w$. Since $\mathscr{F}[(-4\pi^2|\xi|^2)^k\exp(-4\pi^2|\xi|^2t)g]$ $=s$, Hausdorff-Young's theorem gives

$$M_{p'}(s;t)\le C\Big\{\Big(\int_I|g(\xi)|^pd\xi\Big)^{1/p}+\Big(\int_J[|\xi|^{2k}\exp(-4\pi^2|\xi|^2t)\,|g(\xi)|]^pd\xi\Big)^{1/p}\Big\},$$

where $I=\{|\xi|\le\sqrt{2}\}$ and $J=\{|\xi|>\sqrt{2}\}$. Thus, it follows from Lemma 3 that

$$\begin{aligned}\Big(\int_0^1[t^{k-\alpha/2}M_{p'}(s;t)]^qt^{-1}dt\Big)^{1/q} &\le C\Big\{\Big(\int_I|g(\xi)|^pd\xi\Big)^{1/p}\\ &+\Big(\int_0^\infty\Big(t^{(k-\alpha/2)p}\int_J\exp(-4\pi^2|\xi|^2t)[|\xi|^{2k}|g(\xi)|]^pd\xi\Big)^{q/p}t^{-1}dt\Big)^{1/q}\Big\}\le Ck^\alpha_{p,q}(g).\end{aligned}$$

Similarly, $\mathscr{F}[\exp(-4\pi^2|\xi|^2 t)g]=w(\cdot, t)$ and

$$M_{p'}(w; t) \leq C k^{\alpha}_{p,q}(g), \quad t \geq 1/2$$

by Lemma 3. The proof of the theorem is hence complete.

COROLLARY 1. *Let* $-\infty<\alpha<\infty$, $1\leq p\leq 2$ *and* $1\leq q\leq\infty$.

(i) *The Fourier transform is an isomorphism between* $B^{\alpha}_{2,q}$ *and* $K^{\alpha}_{2,q}$.

(ii) *The Fourier transform is a continuous map of* L^p *into* $B^0_{p',p}$ *and of* $B^0_{p,p'}$ *into* $L^{p'}$.

(iii) *The Fourier transform is a continuous map of* $B^{\alpha+n/p-n/2}_{2,q}$ *into* $K^{\alpha}_{p,q}$ *and of* $K^{\alpha}_{p',q}$ *into* $B^{\alpha+n/p'-n/2}_{2,q}$.

PROOF. Part (iii) follows from Theorem 1 and (f) in Section 1.

REMARK 1. In the above corollary, part (ii) and part (iii) in some particular cases were obtained earlier by Taibleson [15; III, Theorem 1]. We note that (ii) gives our Theorem 1 for $-\infty<\alpha<\infty$ and $q=p'$ by using (h) in Section 1 and the fact that Bessel potentials generate Besov spaces. A similar remark can be made for (iii).

REMARK 2. (i) Since $B^{\alpha}_{p,q}=L^p\cap\dot{B}^{\alpha}_{p,q}$ and $\dot{B}^{-\alpha}_{p,q}\subset B^{-\alpha}_{p,q}$ for $\alpha>0$, the result of Herz [8] implies that the Fourier transform is a continuous map of $B^{\alpha}_{p,q}$ into $K^{\alpha}_{p',q}$ and of $\dot{K}^{-\alpha}_{p,q}$ into $B^{-\alpha}_{p',q}$ if $1\leq p\leq 2$ and $\alpha>0$. We refer to [10] for further applications of Herz's results as well as their relations to results of Peetre and others.

(ii) In the paper [12], Mizuhara stated that the Fourier transform is a continuous map of $B^{\alpha}_{p,q}$ into $\dot{K}^{\alpha}_{p',q}$ and of $\dot{K}^{-\alpha}_{p,q}$ into $B^{-\alpha}_{p',q}$ for $1\leq p\leq 2$ and $-\infty<\alpha<\infty$. However, as also noted by the author (see the Autorreferat in [Zbl. Math. 381(1979), #42012]), his proof is only valid for $\alpha>0$; in fact, it is not difficult to give counterexamples in case $\alpha<0$. In view of (i), the above mentioned results of Mizuhara are corollaries of those of Herz. The author recently learned that substantially the same results as those in Theorem 1 were also obtained by Mizuhara by a somewhat different method ([18]).

REMARK 3. If we extend the definitions of $B^{\alpha}_{p,q}$ and $K^{\alpha}_{p,q}$ to $0<q<1$, then Theorem 1 is still valid.

§3. Other proofs and some applications of Theorem 1

Our second proof of Theorem 1 is derived by using a special case of the next proposition, which is very useful in passing from non-homogeneous to homogeneous spaces.

PROPOSITION 1. *Let* $-\infty<\alpha<\infty$ *and* $1\leq p, q\leq\infty$. *Let* $\Phi\in\mathcal{S}$ *be such that* $\int\Phi(x)dx=1$ *and* $\int x^{\kappa}\Phi(x)dx=0$ *for all multi-indexes* κ *with* $0\neq|\kappa|\leq N$, *where* N *is the greatest non-negative integer not exceeding* $-\alpha$. *Then*

$$\dot{B}^{\alpha}_{p,q}(f-\Phi*f)\leq CB^{\alpha}_{p,q}(f) \qquad \textit{for all} \quad f\in B^{\alpha}_{p,q}.$$

PROOF. Let $u=W_t*f$ and $w=W_t*(f-\Phi*f)$. First, observe that for any non-negative integer r we have

$$M_p(D^r_{n+1}w;\, t)\leq(1+\|\Phi\|_1)M_p(D^r_{n+1}u;\, t).$$

Hence, it follows that

$$\tag{6} \begin{gathered} \sup_{t\geq1/2}M_p(w;\, t)\leq CB^{\alpha}_{p,q}(f), \\ \left(\int_0^1[t^{k-\alpha/2}M_p(D^k_{n+1}w;\, t)]^q t^{-1}dt\right)^{1/q}\leq CB^{\alpha}_{p,q}(f), \end{gathered}$$

where k is a non-negative integer greater than $\alpha/2$. It is easily seen that

$$\tag{7} M_p(D^k_{n+1}w;\, t)\leq CB^{\alpha}_{p,q}(f)\,\|D^k_{n+1}(W(\cdot,\, t/2)-W(\cdot,\, t/2)*\Phi)\|_1 \quad \text{for all} \quad t\geq1.$$

Now let $x\in R^n$ and $s\geq1/2$ be fixed, and $\psi(y)=(4\pi)^{-n/2}\exp(-|y|^2/4)$. Then it follows from the assumptions on Φ that

$$\begin{aligned} &D^k_{n+1}(W(x,\, s)-W(\cdot,\, s)*\Phi(x)) \\ &\quad=(-1)^k s^{-k-n/2}\int\Phi(y)[\Delta^k\psi(x/\sqrt{s})-\Delta^k\psi((x-y)/\sqrt{s})]dy \\ &\quad=s^{-k-n/2}\textstyle\sum_{|\kappa|=N+1}c_\kappa\displaystyle\int\Phi(y)(y/\sqrt{s})^{\kappa}D^{\kappa}(\Delta^k\psi)(z/\sqrt{s})dy \\ &\quad=s^{-k-n/2}\textstyle\sum_{|\kappa|=N+1}\displaystyle\left(\int_{\{|y|\geq|x|/2\}}+\int_{\{|y|<|x|/2\}}\right)=I_1(x)+I_2(x), \end{aligned}$$

where z is a point on the segment joining x and $x-y$. Taking $0<\gamma<1$ such that $N+1>\gamma-\alpha$ and noting that $\psi\in\mathcal{S}$, we obtain

$$\begin{aligned} &|I_1(x)|\leq C(1+|x|)^{-n-\gamma}s^{-k-(n+N+1)/2}, \\ &|I_2(x)|\leq\begin{cases} C(1+|x|)^{-n-\gamma}s^{-k-(N+1-\gamma)/2} & \text{if} \quad |x|\geq\sqrt{s} \\ Cs^{-k-(n+N+1)/2} & \text{if} \quad |x|<\sqrt{s}. \end{cases} \end{aligned}$$

Consequently,

$$\|D^k_{n+1}(W(\cdot,\, s)-W(\cdot,\, s)*\Phi)\|_1\leq Cs^{-k-(N+1-\gamma)/2},$$

which, together with (6) and (7), implies the conclusion of the proposition.

REMARK 4. (i) The author partially owes the idea of the proof to [7].

(ii) Since $B^{\alpha}_{p,q}$ is contained in $\dot{B}^{\alpha}_{p,q}$ if $\alpha>0$, the proposition is of interest only in the case $\alpha\leq 0$.

(iii) The integer N in the proposition is in some sense best possible. For if $\int x^{\gamma}\Phi(x)dx\neq 0$ for some $|\gamma|\leq N\neq 0$, then it is not difficult to find a function $f\in\mathscr{S}$ (and hence $\in B^{\alpha}_{1,q}$) such that $f-\Phi*f\notin\dot{B}^{\alpha}_{1,q}$ ($\alpha<0$, $q<\infty$). Our proposition may be also considered as a generalization of a known result of Besov spaces (cf. [1; Theorem 6.3.2]).

SECOND PROOF OF THEOREM 1. Let $\Phi\in\mathscr{S}$ be such that $\hat{\Phi}=1$ on $\{|\xi|\leq 1/2\}$, $\hat{\Phi}=0$ on $\{|\xi|\geq 1\}$. Let $f\in B^{\alpha}_{p,q}$. Proposition 1 then implies that $f-\Phi*f\in\dot{B}^{\alpha}_{p,q}$. Therefore, it follows from a result of Herz that $\hat{g}=(1-\hat{\Phi})\hat{f}\in\dot{K}^{\alpha}_{p',q}$ (cf. [8; Proposition 3.1]). This fact and Lemma 4 give the first part of the theorem.

Conversely, let $h\in K^{\alpha}_{p,q}$. Let ϕ be a bounded function such that $\phi=0$ on $\{|x|\leq 1/2\}$ and $\phi=1$ on $\{|x|\geq 1\}$. Then $\phi h\in\dot{K}^{\alpha}_{p,q}$. Therefore, Proposition 3.1′ of [8] implies that $(\phi h)^{\wedge}\in\dot{B}^{\alpha}_{p',q}$. Let k be a non-negative integer greater than $\alpha/2$. Set $u=W_t*\hat{h}$ and $v=W_t*(\phi h)^{\wedge}$. Since

$$D^k_{n+1}u - D^k_{n+1}v = \mathscr{F}[(-4\pi^2|\xi|^2)^k\exp(-4\pi^2|\xi|^2t)(1-\phi)h],$$

we obtain

$$M_{p'}(D^k_{n+1}u - D^k_{n+1}v;\, t)\leq C\Big(\int_{|x|\leq 1}|h(x)|^p dx\Big)^{1/p}.$$

Consequently,

$$\Big(\int_0^1[t^{k-\alpha/2}M_{p'}(D^k_{n+1}u;\, t)]^q t^{-1}dt\Big)^{1/q}\leq CK^{\alpha}_{p,q}(h).$$

Next, let $t\geq 1/2$ and $I_j=\{2^j\leq|x|^2\leq 2^{j+1}\}$, $j=1, 2,\dots$. Observing that $u(\cdot, t)=\mathscr{F}[\exp(-4\pi^2|x|^2t)h]$, we derive that

$$\begin{aligned} &cM_{p'}(u;\, t)\\ &\leq\Big(\int_{\{|x|\leq 2\}}|h(x)|^p dx\Big)^{1/p}+\Big(\textstyle\sum_{j=1}^{\infty}\displaystyle\int_{I_j}[\exp(-4\pi^2|x|^2t)\,|h(x)|]^p dx\Big)^{1/p}\\ &\leq C\Big(\textstyle\sum_{j=1}^{\infty}[2^{-j\alpha/2}\exp(-2\pi^2 2^j)]^{q'}\Big)^{1/q'}K^{\alpha}_{p,q}(h).\end{aligned}$$

The proof of the converse part is thus complete.

Let ω be a measurable function such that $\omega(x)>0$ for almost every x in R^n. Define

$$L^p_{\omega}=\{f;\ \|f\|_{p,\omega}=\|f\omega\|_p<\infty\}.$$

For quasi-Banach spaces A_0 and A_1, let $(A_0, A_1)_{\theta,q}$ and $[A_0, A_1]_{\theta}$, $0<\theta<1$ and

$0<q\le\infty$, denote the intermediate spaces between A_0 and A_1 by the real method and the complex method, respectively.

LEMMA 5 ([6; Theorem (3.7)]). *Let* $1\le p,\ q\le\infty$ *and* $0<\theta<1$. *Then each of the following quantities gives an equivalent norm for* $(L^p, L^p_\omega)_{\theta,q}$:

$$\text{(i)}\quad \left(\sum_{j=-\infty}^{\infty}\left(r^{j\theta p}\int_{r^{j-1}<\omega(x)\le r^j}|f(x)|^p dx\right)^{q/p}\right)^{1/q},\quad r>1;$$

$$\text{(ii)}\quad \left(\int_0^\infty\left(t^{(1-\theta)p}\int_{t\omega(x)\le 1}[|f(x)|\omega(x)]^p dx\right)^{q/p}t^{-1}dt\right)^{1/q};$$

$$\text{(iii)}\quad \left(\int_0^\infty\left(t^{-\theta p}\int_{t\omega(x)>1}|f(x)|^p dx\right)^{q/p}t^{-1}dt\right)^{1/q}.$$

The connection with interpolation theory is given by

PROPOSITION 2. (i) *If* $\omega(x)=(1+|x|)^\beta$ (*or* $(1+|x|^2)^{\beta/2}$), $\beta\ne 0$, $0<\theta<1$ *and* $1\le p,\ q\le\infty$, *then* $K^{\beta\theta}_{p,q}=(L^p, L^p_\omega)_{\theta,q}$.

(ii) *If* $1<p<\infty$, *then* $\mathscr{A}^p=K^{n(1-1/p)}_{p,1}$ *and* $\mathscr{B}^p=K^{-n/p}_{p,\infty}$, *where* $\mathscr{A}^p$ *and* $\mathscr{B}^p$ *are the algebras defined by Beurling* [2; pp. 9–10].

PROOF. The assertion (i) follows from Lemmas 1 and 5.

To prove (ii) let $\omega(x)=(1+|x|)^{-n}$. Then, by (i) and Lemma 5(iii), an equivalent norm for $K^{-n/p}_{p,\infty}=(L^p, L^p_\omega)_{1/p,\infty}$ is given by

$$\sup_{t>0}\left(t^{-1}\int_{(1+|x|)^n\le t}|f(x)|^p dx\right)^{1/p}\approx\sup_{t>0}\left((1+t^n)^{-1}\int_{|x|\le t}|f(x)|^p dx\right)^{1/p}.$$

Thus, we conclude from a result of Beurling [2; p. 10] that $K^{-n/p}_{p,\infty}=\mathscr{B}^p$. The fact that $K^{n(1-1/p)}_{p,1}=\mathscr{A}^p$ follows by duality.

REMARK 5. In view of (ii) of the above proposition, Corollary 1(i) for $\alpha=1/2$ and $q=1$ in the 1-dimensional case was obtained earlier by Beurling [2; Theorem IX].

REMARK 6. Another proof of Theorem 1, communicated to the author by Professor H. Triebel, can be obtained by using Proposition 2(i) and interpolation theorems for Besov spaces (cf. [1; Theorem 6.2.4] or [16; Theorem 2.2.10]). One can equally obtain as much information by using the characterization of Besov spaces via the spectral decomposition of Peetre (cf. [16]).

A tempered distribution f is in the local Hardy space h^p, $0<p<\infty$, if $u^+(x)=\sup_{0<t<1}|\Phi_t * f(x)|\in L^p$, where $\Phi\in\mathscr{S}$ with $\int\Phi(x)dx=1$ and $\Phi_t(y)=t^{-n}\Phi(y/t)$ (cf. [7]). The space h^p is equipped with the quasi-norm $\|f\|_{h^p}=\|u^+\|_p$.

LEMMA 6. (i) $h^p\subset B^{n(1-1/p)}_{1,1}$ *for* $0<p<1$, *and* $B^0_{1,1}\subset h^1\subset B^0_{1,2}$.

(ii) $(h^p, h^q)_{\theta,r}=h^r$, $0<p, q<\infty$ *and* $1/r=(1-\theta)/p+\theta/q$.

PROOF. It was proved in [3] that h^p is identical with the Triebel-Lizorkin space $F^0_{p,2}$ defined in [16] for $0<p<\infty$. The assertion (i) then follows from this and the inclusion relations between Besov and Triebel-Lizorkin spaces (cf. [16; p. 103]), whereas (ii) was already observed in [3].

Our next corollary gives non-homogeneous version of results of Fefferman, Peetre and Johnson (cf. [13], [11]).

COROLLARY 2. (i) *If* $f\in h^p$, $0<p\leq 2$, *then* $(1+|\xi|)^{n(1-2/p)}\hat{f}\in L^p$.

(ii) *If* $f\in h^p$, $0<p<1$, *then the least decreasing radial majorant of* $(1+|\xi|)^{-n/p}\hat{f}$ *is in* L^1.

(iii) *If* $f\in h^1$, *then the least decreasing radial majorant of* $(1+|\xi|)^{-n/2}\hat{f}$ *is in* L^2.

PROOF. We begin with the proof of (i) (cf. [13]). If $f\in h^2=L^2$, then $\hat{f}\in L^2$. If $f\in h^p$, $0<p<1$, then $|\hat{f}(\xi)|\leq C(1+|\xi|)^{n(1/p-1)}$ since $h^p\subset B^{n(1-1/p)}_{1,1}$ by Lemma 6. It follows that $(1+|\xi|)^n\hat{f}\in L^{p,\infty}(\nu)$, where $d\nu(\xi)=(1+|\xi|)^{-2n}d\xi$. (Here $L^{p,\infty}(\nu)$ stands for the Marcinkiewicz space (=weak $L^p(\nu)$).) Hence, the result follows by interpolation on account of Lemma 6.

Before proceeding on with the proofs of (ii) and (iii), we make an observation. Let $\alpha>0$. Then for any $\beta>\alpha$, it follows from Proposition 2 that

$$\begin{aligned} K^{-\alpha}_{\infty,q}(g) &\approx \left(\int_0^\infty (t^{\beta-\alpha}\operatorname{ess\,sup}_{1+|x|\geq t}[|g(x)|(1+|x|)^{-\beta}])^q t^{-1}dt\right)^{1/q} \\ &\geq C\left(\int_0^\infty ((1+t)^{\beta-\alpha}\operatorname{ess\,sup}_{|x|\geq t}[|g(x)|(1+|x|)^{-\beta}])^q(1+t)^{-1}dt\right)^{1/q}. \end{aligned} \tag{8}$$

Now let $f\in h^p$, $0<p<1$. Lemma 6 and Theorem 1 imply that $\hat{f}\in K^{n(1-1/p)}_{\infty,1}$. Noting that $\hat{f}$ is continuous and by taking $\beta>\alpha=n(1/p-1)$ with $n+\alpha-\beta>0$ in (8), we obtain

$$\|f\|_{h^p}\geq c\int_0^\infty (1+t)^{n-1}(\sup_{|x|\geq t}[|\hat{f}(x)|(1+|x|)^{-n/p}])dt,$$

which implies (ii).

Finally, let $f\in h^1$. Since $h^1\subset B^0_{1,2}$, we derive that $\hat{f}\in K^0_{\infty,2}$. Therefore, $(1+|\xi|)^\alpha\hat{f}\in K^{-\alpha}_{\infty,2}$ for any $\alpha>0$. The rest follows from (8) in a way similar to the proof of (ii).

§4. Translation invariant operators

Let X and Y be Banach or quasi-Banach spaces continuously embedded in

$\mathscr{S}'$ such that $\mathscr{S}$ is contained in X. Adopting the notation of Johnson ([10]), we let

$$Cv(X, Y) = \{T \in \mathscr{S}';\ \|T*\psi\|_Y \leq C\|\psi\|_X \text{ for all } \psi \in \mathscr{S}\},$$
$$M(X, Y) = \{\hat{T}\ ;\ T \in Cv(X, Y)\}.$$

Most of the results in this section are non-homogeneous versions of those of Johnson ([10], [11]). A common technique used in the proofs of many statements on $Cv(X, Y)$ is to apply to the operators in $Cv(X, Y)$ two sets of test functions in X: $\{D_{n+1}^k W(\cdot, s)\}_{0<s\leq 1}$ for some appropriate k and $\{W(\cdot, s)\}_{s\geq 1/2}$. Another useful tool is the relation between local Hardy spaces and non-homogeneous Besov spaces given in Lemma 6. We shall be rather brief and leave details to the interested reader. We should remark that the idea of using the Gauss-Weierstrass kernel to study translation invariant operators is Johnson's. First, we prepare some lemmas.

LEMMA 7. (i) *Let* $1\leq p, q, r, s\leq \infty$, $-\infty<\alpha, \beta<\infty$, $0\leq 1/u=1/p+1/r-1\leq 1$, $0\leq 1/v=1/q+1/s\leq 1$, $f\in B_{p,q}^\alpha$ *and* $g\in B_{r,s}^\beta$. *Then* $f*g\in B_{u,v}^{\alpha+\beta}$ *and*

$$B_{u,v}^{\alpha+\beta}(f*g) \leq CB_{p,q}^\alpha(f)B_{r,s}^\beta(g)$$

(cf. [15; II, Lemma 1]).

(ii) *If* $1<r\leq 2\leq s\leq\infty$ *and* $1/s=1/r+1/p-1$, *then*

$$B_{p,\infty}^0 \subseteq Cv(L^r, L^s)$$

(cf. [15; III, Theorem 2]).

(iii) $[h^p, h^q]_\theta=h^r$, *where* $0<\theta<1$, $0<p, q<\infty$ *and* $1/r=(1-\theta)/p+\theta/q$ (cf. [17; p. 1158], [3]).

LEMMA 8. *Let* $-\infty<\alpha<\infty$ *and* $1\leq p, q\leq\infty$. *If* k *is a non-negative integer and* $s>0$, *then*

$$c(s+1)^{-k-n/2p'} \leq B_{p,q}^\alpha(D_{n+1}^k W(\cdot, s)) \leq C[(s+1)^{-k-n/2p'} + s^{-k-\alpha/2-n/2p'}].$$

PROOF. Simple computations.

LEMMA 9. *Let* k *be a non-negative integer and* $s>0$. *Then*

$$\|D_{n+1}^k W(\cdot, s)\|_{h^p} \approx \begin{cases} s^{-k-n/2p'} & \text{if } \ p > n/(n+2k) \\ (1+s)^{-k-n/2p'} & \text{if } \ p < n/(n+2k), \end{cases}$$
$$\|D_{n+1}^k W(\cdot, s)\|_{h^p} \leq C[1 + \log((s+1)/s)] \qquad \text{if } \ p = n/(n+2k).$$

PROOF. First, we note that $\|D_{n+1}^k W(\cdot, s)\|_{h^p} \approx \|u^+\|_p$, where $u(x, t)=D_{n+1}^k W(x, s+t)=(s+t)^{-k}W(x, s+t)P(|x|^2/4(s+t))$, P is a polynomial of degree

k and $u^+(x)=\sup_{0<t<1}|u(x,t)|$. It can be proved that there exist two positive numbers (possibly equal) $0<\lambda_1\le\lambda_2<\infty$ such that

$$u^+(x)\approx(s+1)^{-k-n/2}|P(|x|^2/4(s+1))\exp(-|x|^2/4(s+1))| \quad \text{if} \quad |x|^2>\lambda_2(s+1),$$
$$u^+(x)\approx s^{-k-n/2}|P(|x|^2/4s)\exp(-|x|^2/4s)| \quad \text{if} \quad |x|^2<\lambda_1 s,$$
$$u^+(x)\le C|x|^{-2k-n} \quad \text{if} \quad \lambda_1 s\le|x|^2\le\lambda_2(s+1).$$

The conclusion of the lemma follows from these estimates.

Let bmo denote the space of locally integrable functions b with

$$\|b\|_{bmo}=\max\left(\sup_{|Q|<1}|Q|^{-1}\int_Q|b(x)-b_Q|dx,\ \sup_{|Q|\ge1}|Q|^{-1}\int_Q|b(x)|dx\right)<\infty,$$

where the supremum is taken over the family of cubes Q with sides parallel to coordinate axes, $b_Q=|Q|^{-1}\int_Q b(x)dx$ and $|Q|$ denotes the Lebesgue measure of Q. Let $L^{p,\infty}$ denote the Marcinkiewicz space with norm $\|\cdot\|_{p,\infty}$.

THEOREM 2. (i) *If $0<\delta<n$, $p=n/\delta$, $f\in B^\delta_{1,\infty}$, $g\in L^{p,\infty}$ and $h\in bmo$, then*

$$\|f*g\|_{bmo}\le CB^\delta_{1,\infty}(f)\,\|g\|_{p,\infty},$$
$$B^\delta_{\infty,\infty}(f*h)\le CB^\delta_{1,\infty}(f)\,\|h\|_{bmo}.$$

(ii) *If $1\le a,b,d,p,q\le\infty$, $-\infty<\alpha,\beta<\infty$ and $0\le1/d=1/a+1/b-1$, then*

$$Cv(B^\alpha_{a,p},B^\beta_{b,q})\subseteq B^{\beta-\alpha-2n/a'}_{d,\infty}.$$

In particular, if $a=1$ and $p\le q$, then

$$Cv(B^\alpha_{1,p},B^\beta_{b,q})=B^{\beta-\alpha}_{b,\infty}.$$

(iii) *If $-\infty<\alpha,\beta<\infty$, $1\le a,p<\infty$, $1<q\le\infty$ and $p\le q$, then*

$$Cv(B^\alpha_{a,p},B^\beta_{\infty,q})=B^{\beta-\alpha}_{a',\infty}.$$

(iv) *If $1\le p\le q\le\infty$, $-\infty<\alpha<\infty$ and $1\le s\le\infty$, then*

$$Cv(L^p,L^q)\subseteq Cv(B^\alpha_{p,s},B^\alpha_{q,s}).$$

Consequently, if $1<p\le2\le q<\infty$, then

$$Cv(L^p,L^q)=Cv(B^0_{p,2},B^0_{q,2})=Cv(\dot B^0_{p,2},\dot B^0_{q,2}).$$

PROOF. The proof of the first assertion of (i) can be done by modifying the argument in the proof of Theorem 2(i) of [14]. For the second assertion, observe that $\mathscr{S}\subset B^0_{1,1}\subset h^1$ by Lemma 6. Since $\mathscr{S}$ is dense in both $B^0_{1,1}$ and h^1, by considering dual spaces (cf. [4; Theorem 26] and [7; Lemma 4, Corollary 1]), we see that $bmo \subset B^0_{\infty,\infty}$. The second assertion of (i) then follows from this inclusion and Lemma 7(i).

To prove (ii) let $T \in Cv(B^\alpha_{a,p}, B^\beta_{b,q})$ and $u = W_t * T$. Take a non-negative integer k greater than $-\alpha/2$, and let $f = D^k_{n+1} W(\cdot, s)$, $s > 0$. Let r be a non-negative integer greater than $\beta/2$. Then Young's inequality gives

$$M_d(D^{k+r}_{n+1} u;\, s+t) \le C t^{-n/2a'} M_b(D^{k+r}_{n+1} u;\, s+t/2).$$

Since $W_t * T * f = D^k_{n+1} u(\cdot, s+t)$ and $T \in Cv(B^\alpha_{a,p}, B^\beta_{b,q})$, we derive that

$$CB^\alpha_{a,p}(f) \ge B^\beta_{b,q}(T*f) \ge c\left(\int_0^{1/2} [t^{r-\beta/2+n/2a'} M_d(D^{k+r}_{n+1} u;\, s+t)]^q t^{-1} dt\right)^{1/q},$$

which, together with Lemma 8, implies that

$$\sup_{0<s\le 1} s^{k+r-(\beta-\alpha-2n/a')/2} M_d(D^{k+r}_{n+1} u;\, s) \le C.$$

Similarly, taking $f_1 = W(\cdot, s)$ and arguing as above, we see that

$$\sup_{s \ge 1/2} M_d(u;\, s) \le C.$$

The remaining part of (ii) is then derived by using Lemma 7(i).

The assertion (iii) follows from (ii) and duality results of Besov spaces.

Finally, let $T \in Cv(L^p, L^q)$, $\psi \in \mathscr{S}$, $u = W_t * \psi$ and $w = W_t * T * \psi$. Since $M_q(w;\, t) \le CM_p(u;\, t)$, it follows that

$$\sup_{t\ge 1/2} M_q(w;\, t) + \left(\int_0^1 [t^{-\alpha/2} M_q(w;\, t)]^s t^{-1} dt\right)^{1/s}$$
$$\le C\left\{\sup_{t\ge 1/2} M_p(u;\, t) + \left(\int_0^1 [t^{-\alpha/2} M_p(u;\, t)]^s t^{-1} dt\right)^{1/s}\right\},$$

and hence $T \in Cv(B^\alpha_{p,s}, B^\alpha_{q,s})$ for any $\alpha < 0$. Since $Cv(B^\alpha_{p,s}, B^\alpha_{q,s})$ is invariant with respect to α, we obtain the inclusion relation in (iv). Now, if $1 < p \le 2 \le q < \infty$, then $L^p \subseteq B^0_{p,2}$ and $B^0_{q,2} \subseteq L^q$ (cf. [15; I, Theorem 15]). Hence the converse inclusion $Cv(B^0_{p,2}, B^0_{q,2}) \subseteq Cv(L^p, L^q)$ follows. The identity $Cv(B^0_{p,2}, B^0_{q,2}) = Cv(\dot{B}^0_{p,2}, \dot{B}^0_{q,2})$ is finally deduced from this and the fact that $Cv(L^p, L^q) = Cv(\dot{B}^0_{p,2}, \dot{B}^0_{q,2})$ (cf. [11; Theorem 12]).

THEOREM 3. (i) $Cv(h^1, h^p) = Cv(h^1, L^p) \subseteq B^0_{p,\infty}$, $\quad 1 \le p < \infty$.

(ii) $Cv(h^1, L^p) = B^0_{p,\infty}$, $\quad 2 \le p < \infty$.

(iii) $B^0_{p,r} \subseteq Cv(h^1, h^p)$, $\quad 1 \le p \le 2$ *and* $1/r = 1/p - 1/2$.

(iv) $Cv(h^1, h^p) \subseteq Cv(L^r, L^s)$, $\quad 1 \le p < 2$, $1 < r \le 2$ *and* $1/s = 1/r + 1/p - 1$.

PROOF. The equality $Cv(h^1, h^p) = Cv(h^1, L^p)$ for $1 < p < \infty$ is obvious since

$h^p = L^p$ in this case. For $p=1$, this equality follows from the characterization of h^1 by the n "*Riesz transforms*" $\{r_1,\dots, r_n\}$ and the fact that r_j, $j=1,\dots, n$, are bounded on h^1 (cf. [7; Theorems 2 and 4]). The remaining part of (i) as well as (ii) and (iii) is verified by applying to the operator in question two sets of test functions $\{D_{n+1}W(\cdot, s)\}$ and $\{W(\cdot, s)\}$, and by using the inclusion relations in Lemma 6.

To prove (iv), let $T \in Cv(h^1, h^p)$. Then we derive from (i) and Lemma 7(ii) that $T \in Cv(L^2, L^\lambda)$, $1/\lambda = 1/p - 1/2$. Therefore, the desired result follows by interpolation (Lemma 7(iii)).

THEOREM 4. *If either* $0<p<1\le q\le\infty$ *or* $p=1$ *and* $2\le q\le\infty$, *then*

(i) $Cv(h^p, \mathscr{F}L^q) = \{T;\ \hat{T} \in K_{q,\infty}^{n(1/p-1)}\}$,

(ii) $Cv(h^p, B_{a,q}^{\alpha}) = B_{a,\infty}^{\alpha-n(1-1/p)}$, $\quad 1\le a\le\infty$ *and* $-\infty<\alpha<\infty$,

(iii) $Cv(h^p, L^q) = B_{q,\infty}^{n(1/p-1)}$.

Here $\mathscr{F}L^q = \{\hat{f};\ f\in L^q\}$.

PROOF. Assume that $T\in Cv(h^p, \mathscr{F}L^q)$, $0<p\le 1\le q\le\infty$. Let k be a non-negative integer so that $p>n/(n+2k)$ and $f=D_{n+1}^k W(\cdot, s)$. Then an argument similar to the proof of Lemma 4 and the assumption on T show that $\hat{T}$ is a function of temperate growth and $(T*f)\hat{}(\xi) = (-4\pi^2|\xi|^2)^k \exp(-4\pi^2|\xi|^2 s)\hat{T}(\xi)$. Since $T\in Cv(h^p, \mathscr{F}L^q)$, we obtain

$$\sup_{0<s<\infty} s^{k-n(1/p-1)/2}\left(\int_{|\xi|\ge 1}[\exp(-4\pi^2 s|\xi|^2)\,|\hat{T}(\xi)|\,|\xi|^{2k}]^q d\xi\right)^{1/q} \le C$$

by Lemma 9. On the other hand, by taking $f_1 = W(\cdot, 1)$, we see that

$$\left(\int_{|\xi|\le 2}|\hat{T}(\xi)|^q d\xi\right)^{1/q} \le C.$$

Hence $\hat{T}\in K_{q,\infty}^{n(1/p-1)}$ by Lemma 3.

Conversely, let $\hat{T}\in K_{q,\infty}^{n(1/p-1)}$ and $g\in h^p$. If $0<p<1\le q\le\infty$, then Lemma 6 implies that $\hat{g}\in K_{\infty,1}^{n(1-1/p)}$. Therefore $\hat{T}\hat{g}\in L^q$ by an easy computation. For $p=1$ and $2\le q\le\infty$, by using the inclusion $h^1\subset B_{1,2}^0$, we obtain the same conclusion.

The proofs of (ii) and (iii) can be carried out in the same spirit by exploiting the inclusion relations between local Hardy spaces and Besov spaces given in Lemma 6 and by using Lemma 9. These arguments are familiar by now.

REMARK 7. Theorems 2, 3 and 4, combined with Theorem 1, give useful criteria for multipliers in $M(X, Y)$, where X and Y are either local Hardy or Besov spaces considered in the theorems.

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