HIROSHIMA MATH. J.
11 (1981), 9–20

On the initial value problem for the Navier-Stokes equations in L^p spaces

Tetsuro MIYAKAWA

(Received June 25, 1980)

1. Introduction

On a bounded domain D in R^n $(n \geq 3)$ with smooth boundary S we consider the initial value problem for the Navier-Stokes equation

$$\text{(N.S)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u, \operatorname{grad})u + \operatorname{grad} q = f & \text{in} \quad D \times (0, T), \\ \operatorname{div} u = 0 & \text{in} \quad D \times (0, T), \\ u = 0 & \text{on} \quad S \times (0, T), \\ u(x, 0) = a(x) & \text{in} \quad D. \end{cases}$$

Here $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$, $q = q(x, t)$ and $f = f(x, t) = (f_1(x, t), \dots, f_n(x, t))$ are the velocity, the pressure and the given external force respectively, and $(u, \operatorname{grad}) = \sum_j u_j \partial/\partial x_j$. Our main concern is in the existence and uniqueness problem of strong solutions of (N.S) in the Banach space $(L^p(D))^n$, $n < p < \infty$. In treating this problem we employ the method of Kato and Fujita [2], [7] and transform the equation (N.S) to the following evolution equation in the Banach space X_p:

$$\text{(I)} \qquad \frac{du}{dt} + Au + P(u, \operatorname{grad})u = Pf, \quad t > 0, \quad u(0) = a \in X_p.$$

Here X_p is the closed subspace of $(L^p(D))^n$ consisting of all solenoidal vector fields on D whose normal components vanish on S, and $A = -P\Delta$ is the Stokes operator with P denoting the projection onto X_p. See [4] for the details. Kato and Fujita [2], [7] considered the equation (I) in X_2, $n = 3$, and proved the existence and uniqueness, generally local in time, of strong solutions for initial data in $D(A^{1/4})$ under a certain assumption on Pf. In this paper we shall show that the above restriction on the initial data can be removed by considering (I) in X_p, $n < p < \infty$. Further, we show that, as is done in [2], [7], the solution exists globally if the data are sufficiently small. What is basic for our discussion is the estimation of the nonlinear term $P(u, \operatorname{grad})u$ by the fractional powers of the Stokes operator, the existence of which is assured by the fact that the Stokes operator generates a bounded holomorphic semigroup in X_p, $1<p<\infty$ (see Theorem 2.1 below). It is to be noted that our estimates for the nonlinear term are weaker than those given in [11] for the Navier-Stokes equation with the Neumann condition. We also note that in the case of the Neumann condition we can solve (I) even in X_n. (This fact is not mentioned explicitly in [11].) In order to include the case $p=n$ for the problem with the no-slip boundary condition, it seems necessary to determine explicitly the domains of fractional powers of the Stokes operator.

In the final section we apply our results to the uniqueness and regularity problem for weak solutions. In particular we improve a result of von Wahl [15] concerning the regularity of weak solutions in four dimensions.

Recently von Wahl [16] has announced results similar to ours. In addition, he studies regularity in Hölder spaces and improves the regularity result of Serrin [12]. (See Remark 3.8.)

2. Solvability and uniqueness in X_p

It is proved in [4] that the following direct sum decomposition holds for each p, $1<p<\infty$:

$$(1) \qquad (L^p(D))^n = X_p \oplus G_p, \quad X_p^* = X_{p'}, \quad X_p^\perp = G_{p'}, \qquad p' = p/(p-1),$$

where X_p is the closure in $(L^p(D))^n$ of the space of all $u \in (C_0^\infty(D))^n$ such that $\operatorname{div} u=0$ in D, and G_p is the closed subspace of $(L^p(D))^n$ consisting of gradients of scalar functions in $W^{1,p}(D)$. (Here and hereafter we use the standard notation of Sobolev spaces.) Let $P=P_p$ be the projection onto X_p along G_p. Then applying this to both sides of the Navier-Stokes equation, one can derive the following evolution equation in X_p:

$$(\mathrm{I}) \qquad \frac{du}{dt} + Au = Fu + Pf, \qquad t > 0, \quad u(0) = a \in X_p.$$

Here $Fu=-P(u, \operatorname{grad})u$, and $A=A_p=-P\Delta$ denotes the Stokes operator defined on $D(A)=X_p \cap (W_0^{1,p}(D))^n \cap (W^{2,p}(D))^n$. It is shown in [4] that A is a closed operator in X_p and that $A_p^*=A_{p'}$. The regularity theorem for the stationary Stokes equation tells us that the spectrum of $A=A_p$ does not depend on p and is contained to $(0, +\infty)$.

THEOREM 2.1. *$-A$ generates a bounded holomorphic semigroup, e^{-tA}, in X_p.*

To prove this, we make use of the following

LEMMA (Solonnikov [14], von Wahl [15]). *For each $f \in L^p(0, T; X_p)$ and* *each* $a \in X_p \cap (W_0^{1,p}(D))^n \cap (W^{2,p}(D))^n$, $p \neq 3/2$, *there exists a unique function* $u \in L^p(0, T; D(A))$ *with* $u' \in L^p(0, T; X_p)$ *such that* $du/dt + Au = f$ *on* $(0, T)$, $u(0) = a$. *Moreover, there is a constant* $C = C(p, D) > 0$ *such that*

$$(2) \qquad \int_0^T \|u'(t)\|_{0,p}^p dt + \int_0^T \|u(t)\|_{2,p}^p dt \le C \left\{ \|a\|_{2,p}^p + \int_0^T \|f(t)\|_{0,p}^p dt \right\}.$$

This lemma is proved in Solonnikov [14] in the case $n = 3$. Necessary modifications in higher dimensions are given in von Wahl [15].

Proof of Theorem 2.1. We fix λ, $\mathrm{Re}\,\lambda > 0$, and $g \in X_p$, $p \neq 3/2$. Put $v = (\lambda + A)^{-1} g$, and choose $h(t) \in C^1([0, \infty); R)$ such that $0 \le h \le 1$, $h(t) = 1$ if $t \ge 1$, and $h(t) = 0$ if $0 \le t \le 1/2$. Then, the function $u(t) = v e^{\lambda t} h(\varepsilon t)$, $\varepsilon > 0$, satisfies

$$u' + Au = g e^{\lambda t} h(\varepsilon t) + \varepsilon v e^{\lambda t} h'(\varepsilon t), \qquad u(0) = 0.$$

Since $u'(t) = \lambda v e^{\lambda t} h(\varepsilon t) + \varepsilon v e^{\lambda t} h'(\varepsilon t)$ it follows from (2) that

$$\begin{aligned}(3) \qquad & (1 + |\lambda|^p) \|v\|_{0,p}^p \int_0^T e^{p(\mathrm{Re}\lambda)t} |h(\varepsilon t)|^p dt \\ & \le C_p \left\{ \int_0^T \|u'(t)\|_{0,p}^p dt + \int_0^T \|u(t)\|_{0,p}^p dt + \varepsilon^p \|v\|_{0,p}^p \int_0^T e^{p(\mathrm{Re}\lambda)t} |h'(\varepsilon t)|^p dt \right\} \\ & \le C_p \left\{ \|g\|_{0,p}^p \int_0^T e^{p(\mathrm{Re}\lambda)t} |h(\varepsilon t)|^p dt + 2\varepsilon^p \|v\|_{0,p}^p \int_0^T e^{p(\mathrm{Re}\lambda)t} |h'(\varepsilon t)|^p dt \right\}.\end{aligned}$$

Here we apply the following

Lemma 2.2. *Let* $0 < \delta < 1$ *be fixed. Then there exist for each* λ, $\mathrm{Re}\,\lambda > 0$, *constants* $T_\lambda > 0$ *and* $\varepsilon_\lambda > 0$ *such that*

$$(4) \qquad C_p \varepsilon_\lambda^p \int_0^{T_\lambda} e^{p(\mathrm{Re}\lambda)t} |h'(\varepsilon_\lambda t)|^p dt \le \delta (1 + |\lambda|^p) \int_0^{T_\lambda} e^{p(\mathrm{Re}\lambda)t} |h(\varepsilon_\lambda t)|^p dt.$$

Admitting this lemma for a moment we continue the proof of Theorem 2.1. Applying (4) to (3) we see

$$\begin{aligned} & (1 - \delta)(1 + |\lambda|^p) \|v\|_{0,p}^p \int_0^{T_\lambda} e^{p(\mathrm{Re}\lambda)t} |h(\varepsilon_\lambda t)|^p dt \\ & \le C_p \|g\|_{0,p}^p \int_0^{T_\lambda} e^{p(\mathrm{Re}\lambda)t} |h(\varepsilon_\lambda t)|^p dt,\end{aligned}$$

so that

$$(5) \qquad (1 + |\lambda|^p) \|v\|_{0,p}^p \le C_p \|g\|_{0,p}^p / (1 - \delta), \qquad p \neq 3/2.$$

Thus the proof is completed in the case $p \neq 3/2$. The case $p = 3/2$ is treated by the use of (5) with $p = 3$ and a simple duality argument.

PROOF OF LEMMA 2.2. From our choice of $h(t)$ it follows that

$$(6)\qquad C_p\varepsilon^p\int_0^T e^{p(\mathrm{Re}\lambda)t}|h'(\varepsilon t)|^p dt \le C_p\varepsilon^p M^p\int_{1/2\varepsilon}^{1/\varepsilon} e^{p(\mathrm{Re}\lambda)t}dt$$
$$= C_p\varepsilon^p M^p\{e^{p(\mathrm{Re}\lambda)/\varepsilon}-e^{p(\mathrm{Re}\lambda)/2\varepsilon}\}/p\cdot\mathrm{Re}\,\lambda,$$

where $M=\max|h'(t)|$. On the other hand we have

$$(7)\qquad (1+|\lambda|^p)\int_0^T e^{p(\mathrm{Re}\lambda)t}|h(\varepsilon t)|^p dt \ge (1+|\lambda|^p)\int_{1/\varepsilon}^T e^{p(\mathrm{Re}\lambda)t}dt$$
$$=(1+|\lambda|^p)\{e^{p(\mathrm{Re}\lambda)T}-e^{p(\mathrm{Re}\lambda)/\varepsilon}\}/p\cdot\mathrm{Re}\,\lambda.$$

Now, put $\varepsilon_\lambda=\mathrm{Re}\,\lambda$. Then, since the right hand sides of (6) and (7) are equal to $C_p(\mathrm{Re}\,\lambda)^{p-1}M^p\{e^p-e^{p/2}\}/p$ and $(1+|\lambda|^p)\{e^{p(\mathrm{Re}\lambda)T}-e^p\}/p\cdot\mathrm{Re}\,\lambda$ respectively, it suffices to prove the inequality

$$(8)\qquad C_pM^p(\mathrm{Re}\,\lambda)^p\{e^p-e^{p/2}\}\le\delta(1+|\lambda|^p)\{e^{p(\mathrm{Re}\lambda)T}-e^p\},$$

by choosing $T>0$ appropriately. But the validity of (8) for a large $T>0$ is obvious, and so the proof is completed.

REMARK 2.3. The above proof of Theorem 2.1 is motivated by the work of Sobolevskii ([13]). The proof in [13] seems to be incorrect.

Theorem 2.1 enables us to define the fractional powers of A. The proposition below plays an important role in constructing a solution of (I) for an arbitrary initial value.

PROPOSITION 2.4. *Suppose that* $n<p<\infty$.
(i) *There exists for each* $\varepsilon>0$ *a constant* $M_1>0$ *such that*

$$\|P(u,\mathrm{grad})v\|_{0,p}\le M_1\|A^{\varepsilon+1/2}u\|_{0,p}\|A^{\varepsilon+1/2}v\|_{0,p}$$

for any u *and* v *in* $D(A^{\varepsilon+1/2})$.
(ii) *There exists for each* $\varepsilon_j>0$, $j=1,2$, *a constant* $M_2>0$ *such that*

$$\|A^{-\varepsilon_1-1/4}P(u,\mathrm{grad})v\|_{0,p}\le M_2\|u\|_{0,p}\|A^{\alpha+\varepsilon_2}v\|_{0,p}$$

for each $u\in X_p$ *and each* $v\in D(A^{\alpha+\varepsilon_2})$. *Here* $\alpha=1/2$, *if* $p\ge 2n$, *and* $\alpha=1/4+n/(2p)$, *if* $n<p\le 2n$.

This proposition is an immediate consequence of two lemmas stated below, and so the proof is omitted. In what follows we denote by $B=B_p$ the Laplacian, $-\Delta$, on $(L^p(D))^n$, $1<p<\infty$, with the Dirichlet boundary condition. It is well known that B is a closed operator defined on $D(B)=(W_0^{1,p}(D))^n\cap(W^{2,p}(D))^n$, and that $B_p^*=B_{p'}$, $p'=p/(p-1)$ (see [10]). By the apriori estimate for A it is easy to see that $D(A)\subset D(B)$ and $\|Bu\|_{0,p}\leq C\|Au\|_{0,p}$ with a constant $C>0$ independent of $u\in D(A)$.

LEMMA 2.5. *Let $n<p<\infty$ be fixed. Then there exists a constant $M>0$ such that*
(i) $\|(u, \mathrm{grad})v\|_{0,p} \leq M\|B^{1/2}u\|_{0,p}\|B^{1/2}v\|_{0,p}$ *for any* u, v *in* $D(B^{1/2})$,
(ii) $\|B^{-1/4}(u, \mathrm{grad})v\|_{0,p} \leq M\|u\|_{0,p}\|B^{\alpha}v\|_{0,p}$ *for each* $u\in(L^p(D))^n$ *and each $v\in D(B^{\alpha})$, where α is the number in* (ii) *of Proposition* 2.4.

LEMMA 2.6. *Let $0<\alpha<1$ and $1<p<\infty$ be fixed. Then there exists for each $\beta>\alpha$ a constant $C_{\alpha\beta}>0$ such that $\|B^{\alpha}u\|_{0,p}\leq C_{\alpha\beta}\|A^{\beta}u\|_{0,p}$ for each $u\in D(A^{\beta})$. Thus $B^{\alpha}A^{-\beta}$ is a bounded operator from X_p into $(L^p(D))^n$, and hence, by duality, $A^{-\beta}PB^{\alpha}$ is extended uniquely to a bounded operator from $(L^p(D))^n$ into X_p.*

PROOF OF LEMMA 2.5. (i) From a result of Fujiwara [3] it follows that $D(B^{1/2})\subset(W_0^{1,p}(D))^n$ with a continuous injection. Therefore, by the Sobolev imbedding theorem and the Poincaré inequality, we have

$$\begin{aligned}\|(u, \mathrm{grad})v\|_{0,p} &\leq C\sup|u(x)|\,\|\mathrm{grad}\, v\|_{0,p} \leq C\|\mathrm{grad}\, u\|_{0,p}\|\mathrm{grad}\, v\|_{0,p}\\ &\leq C\|B^{1/2}u\|_{0,p}\|B^{1/2}v\|_{0,p}.\end{aligned}$$

This completes the proof of (i).

(ii) Let $B^{-1/4}(x, y)$ be the kernel function of $B^{-1/4}$. Then we have (see [2])

$$|B^{-1/4}(x, y)| \leq C/|x-y|^{n-1/2} \qquad \text{for any} \quad x, y\in\bar{D},\ x\neq y. \tag{9}$$

Therefore we see, for each $\psi\in(L^{p'}(D))^n$, that

$$\begin{aligned}|(B^{-1/4}(u, \mathrm{grad})v, \psi)| &\leq |((u, \mathrm{grad})v, B_{p'}^{-1/4}\psi)|\\ &\leq C\iint_{D\times D}|u(x)|\cdot|\nabla v(x)|\cdot|\psi(y)|\cdot|x-y|^{1/2-n}dx\,dy.\end{aligned} \tag{10}$$

Put $w(x)=\int_D\psi(y)|x-y|^{1/2-n}dy$. By the Sobolev inequality we have

$$w\in(L^q(D))^n,\ q^{-1} = 1-(2n)^{-1}-p^{-1}, \quad \text{and} \quad \|w\|_{0,q}\leq C\|\psi\|_{0,p'},$$

so that, by Hölder's inequality,

$$\begin{aligned}|(B^{-1/4}(u, \mathrm{grad})v, \psi)| &\leq C\|\,|u|\cdot|\nabla v|\,\|_{0,q'}\|w\|_{0,q}\\ &\leq C\|\,|u|\cdot|\nabla v|\,\|_{0,q'}\|\psi\|_{0,p'} \leq C\|u\|_{0,p}\|v\|_{1,2n}\|\psi\|_{0,p'},\end{aligned} \tag{11}$$

since $q'^{-1}=1-q^{-1}=(2n)^{-1}+p^{-1}$. This implies that when $p\geq 2n$ we have

$$\|B^{-1/4}(u, \mathrm{grad})v\|_{0,p} \leq C\|u\|_{0,p}\|B^{1/2}u\|_{0,p}. \tag{12}$$

When $n<p<2n$, we proceed as follows: A result of Fujiwara [3] tells us that $D(B^\alpha)\subset(H^{2\alpha,p}(D))^n$ with a continuous injection, where $H^{s,p}(D)$ denotes the space of the Bessel potentials (see [1], [10]). Since $H^{2\alpha,p}\subset H^{1,2n}$ if $2\alpha-n/p=1/2$, $n<p<2n$ (see [1]), the assertion follows from (11) by taking $\alpha=1/4+n/(2p)$.

PROOF OF LEMMA 2.6. By the moment inequality, we have

$$\|B^\alpha u\|_{0,p}\le C_\alpha\|u\|_{0,p}^{1-\alpha}\|Bu\|_{0,p}^{\alpha}\le C_\alpha\|u\|_{0,p}^{1-\alpha}\|Au\|_{0,p}^{\alpha} \tag{13}$$

for each $u\in D(A)$. Using this, we see, for each $\beta>\alpha$ and $u\in D(A)$,

$$\begin{aligned}(14)\quad \|B^\alpha u\|_{0,p} &= \|B^\alpha A^\beta A^{-\beta}u\|_{0,p}\\ &\le \frac{\sin\beta\pi}{\pi(1-\beta)}\int_0^\infty s^{1-\beta}\|B^\alpha(s+A)^{-1}A^\beta(s+A)^{-1}u\|_{0,p}ds\\ &\le \frac{\sin\beta\pi}{\pi(1-\beta)}C_\alpha\int_0^\infty s^{1-\beta}\|(s+A)^{-1}A^\beta(s+A)^{-1}u\|_{0,p}^{1-\alpha}\|A(s+A)^{-1}A^\beta(s+A)^{-1}u\|_{0,p}^{\alpha}ds\\ &\le \frac{\sin\beta\pi}{\pi(1-\beta)}C_{\alpha\beta}\left\{\int_0^N\frac{s^{1-\beta}ds}{(1+s)^{1-\alpha}(1+s)^{1-\beta}}\|u\|_{0,p}+\int_N^\infty\frac{s^{1-\beta}ds}{(1+s)^{2-\alpha}}\|A^\beta u\|_{0,p}\right\}\\ &\le C_{\alpha\beta}\{N^\alpha\|u\|_{0,p}/\alpha+N^{\alpha-\beta}\|A^\beta u\|_{0,p}/(\beta-\alpha)\}\,.\end{aligned}$$

Taking the minimum in N we obtain

$$\|B^\alpha u\|_{0,p}\le C_{\alpha\beta}\|u\|_{0,p}^{1-\alpha/\beta}\|A^\beta u\|_{0,p}^{\alpha/\beta}\le C_{\alpha\beta}\|A^{-\beta}\|^{1-\alpha/\beta}\|A^\beta u\|_{0,p}\,.$$

Note that here we have used the formula

$$A^{-\beta}=\frac{\sin\beta\pi}{\pi(1-\beta)}\int_0^\infty s^{1-\beta}\{(s+A)^{-1}\}^2ds,$$

which is easily verified by an integration by parts and by shifting the path of integration. This completes the proof.

Choose now $\varepsilon_j>0$, $j=1, 2$, so that $\varepsilon_1+\varepsilon_2+\alpha<3/4$, and put $\beta=3/4-(\alpha+\varepsilon_1+\varepsilon_2)$. Then, by (ii) of Proposition 2.4, we get

$$\|A^{-\varepsilon_1-1/4}P(u,\,\mathrm{grad})v\|_{0,p}\le M_2\|A^\beta u\|_{0,p}\|A^{\alpha+\varepsilon_2}v\|_{0,p}\,. \tag{15}$$

Note that $(\varepsilon_1+1/4)+\beta+(\alpha+\varepsilon_2)=1$.

Let us now construct a solution of the integral equation:

$$u(t)=e^{-tA}a+\int_0^t e^{-(t-s)A}\{Fu(s)+Pf(s)\}ds, \tag{II}$$

by means of the following iteration scheme:

$$(16)\qquad \begin{cases} u_0(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}Pf(s)ds, \\ u_{m+1}(t) = u_0(t) + \int_0^t e^{-(t-s)A}Fu_m(s)ds, \qquad m \geq 0. \end{cases}$$

Using the estimate (15) one can prove the proposition below in just the same way as in [11] (see also [7]). In what follows the L^p-norm will be denoted simply by $\|\cdot\|$.

PROPOSITION 2.7. *For each $Pf \in C([0, T]; X_p)$ and each $a \in X_p$, $n<p<\infty$, the functions $u_m(t)$ in* (16) *are well-defined and belong to $C([0, T]; X_p) \cap C((0, T]; D(A^\gamma))$ for each γ, $0<\gamma<3/4-\varepsilon_1$. Moreover, we have the estimates*

$$(17)\qquad \|A^\gamma u_m(t)\| \leq K_{\gamma m} t^{-\gamma}, \qquad m \geq 0,$$

where $K_{\gamma m}$ are defined inductively by

$$(18)\qquad \begin{aligned} K_{\gamma 0} &= \sup_t t^\gamma \|A^\gamma e^{-tA}a\| + C_\gamma N B(3/4-\varepsilon_1-\gamma,\ 1/4+\varepsilon_1) \\ K_{\gamma,m+1} &= K_{\gamma 0} + C_\gamma M_2 B(3/4-\varepsilon_1-\gamma,\ 1/4+\varepsilon_1) K_{\beta m} K_{\alpha+\varepsilon_2,m} \qquad m \geq 0, \end{aligned}$$

with a constant $C_\gamma>0$ depending on γ and ε_1. Here $B(a, b)$ is the beta function and

$$N = \sup_t t^{3/4-\varepsilon_1} \|A^{-\varepsilon_1-1/4} Pf(t)\|.$$

Repeating the arguments given in [7] and [11, Sec. 4] one can now prove

THEOREM 2.8. *For each $a \in X_p$ and each $Pf \in C([0, \infty); X_p)$, $n<p<\infty$, there exist a $T>0$ and a solution $u(t)$ of* (II) *belonging to $C([0, T]; X_p) \cap C((0, T]; D(A^\gamma))$ for any γ, $0<\gamma<3/4$. If the data are sufficiently small, then we can take $T=\infty$. Further, $u(t)$ is unique within the class of functions $v(t)$ in $C([0, T]; X_p)$ such that $A^\theta v(t) \in C((0, T]; X_p)$ and $\|A^\theta v(t)\| = o(t^{-\theta})$ as $t \to 0$, for some θ, $1/2<\theta<3/4$.*

THEOREM 2.9. *If, in addition to the assumptions of Theorem* 2.8, *Pf is Hölder continuous on each compact subset of* $(0, T]$, *then the solution $u(t)$ of* (II) *satisfies the evolution equation* (I).

3. An application

In this section we consider the equation (N.S) with $f=0$. Let us recall the definition of weak solutions (see [9]).

DEFINITION 3.1. Let $u(t)$ be a function defined on $(0, T)$ with values in X_2 and a be an arbitrary element in X_2. We call $u(t)$ a *weak solution* of (N.S) with

$f=0$ and with the initial value a if it satisfies the following conditions.

(i) $u(t)$ lies in $L^2(0, T; X_2 \cap (W_0^{1,2}(D))^n) \cap L^\infty(0, T; X_2)$.

(ii) The identity

$$- \int_0^T (u(t), v)h'(t)dt + \int_0^T (\nabla u(t), \nabla v)h(t)dt + \int_0^T ((u(t), \mathrm{grad})u(t), v)h(t)dt = (a, v)h(0)$$

is valid for each $v \in X_2 \cap (W_0^{1,2}(D))^n \cap (L^n(D))^n$ and each $h(t) \in C^1([0, T]; R)$ with $h(T)=0$.

(iii) $u(t)$ satisfies the energy inequality

$$\|u(t)\|_{0,2}^2 + 2\int_0^t \|\nabla u(s)\|_{0,2}^2 ds \leq \|a\|_{0,2}^2, \qquad \text{for a.e.} \quad t \in (0, T). \tag{19}$$

From the above definition it follows that every weak solution is weakly continuous on $[0, T]$ with values in X_2. Although the existence of a global weak solution for an arbitrary a is established e.g. by Hopf [5], its uniqueness is still an open problem. We note that the solution of (I) constructed in the preceding section is in fact a weak solution in the above sense which, moreover, satisfies the energy equality

$$\|u(t)\|_{0,2}^2 + 2\int_0^t \|\nabla u(s)\|_{0,2}^2 ds = \|a\|_{0,2}^2, \qquad \text{for each} \quad t \in (0, T). \tag{20}$$

The existence result in the preceding section enables us to prove the following theorem. Note that we do not impose any regularity assumption on the initial value a.

THEOREM 3.2. *For each $a \in X_p$, $p>n$, the weak solutions with the initial value a coincide in a neighbourhood of $t=0$. Furthermore, there exists on* $(0, \infty)$ *a unique weak solution with the initial value a if a is sufficiently small in X_p.*

REMARK 3.3. The above theorem is a generalization of a result of Serrin [12], where a uniqueness criterion is given when $n \leq 4$ (see [12, Th. 6]). By our results in the preceding section, we can remove the restriction $n \leq 4$, (see Lemma 3.4 below).

To prove the above result we begin with the following

LEMMA 3.4. *Let $v(t)$ be an arbitrary weak solution on* $(0, T)$ *with the initial value $a \in X_p$, $p>n$, and suppose that the solution $u(t)$ of* (I) *exists on* $[0, T]$ *with* $u(0)=a$. *Then we have*

(21) $$(v(t), u(t)) + 2\int_0^t (\nabla v(s), \nabla u(s))ds$$
$$= \|a\|_0^2 - \int_0^t \{b(v(s), v(s), u(s)) + b(u(s), u(s), v(s))\}ds \qquad \text{for each } t\in[0, T],$$

where $b(u, v, w) = ((u, \mathrm{grad})v, w)$.

PROOF. First we note that Hölder's inequality and the Sobolev imbedding theorem imply

$$|b(u, v, w)| \leq C\|u\|_{0,2n/(n-2)}\|v\|_{1,2}\|w\|_{0,n} \leq C\|u\|_{1,2}\|v\|_{1,2}\|w\|_{0,n},$$

for any $u, v\in X_2 \cap (W_0^{1,2}(D))^n$ and any $w\in X_n$. Thus the integrand on the right hand side of (21) makes sense. (Note that the roles of u and w can be interchanged in the above inequality.) Let B be the nonlinear operator from $X_2 \cap (W_0^{1,2}(D))^n$ to the dual of $X_2 \cap (W_0^{1,2}(D))^n \cap (L^n(D))^n$ defined by $b(u, u, w) = \langle Bu, w\rangle$. Then we see from Definition 3.1 that the weak solution $v(t)$ satisfies

(22) $$(v'(t), w) + (\nabla v(t), \nabla w) + \langle Bv(t), w\rangle = 0$$
$$w\in X_2 \cap (W_0^{1,2}(D))^n \cap (L^n(D))^n,$$

as a distribution on $(0, T)$. Now, take a non-negative even function $\rho\in C_0^\infty(R)$ such that $\int_{-\infty}^{\infty}\rho(t)dt = 1$, $\mathrm{supp}\,\rho = [-1, 1]$, and set $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$. Defining $v(t) = u(t) = 0$ outside $(0, T)$ and putting $v_\varepsilon = v*\rho_\varepsilon$ we see from (22) that

(23) $$(d/dt)(v_\varepsilon(t), u(t)) = (v_\varepsilon'(t), u(t)) + (v_\varepsilon(t), u'(t))$$
$$= -2(\nabla v_\varepsilon(t), \nabla u(t)) - \langle (Bv)*\rho_\varepsilon, u(t)\rangle - b(u(t), u(t), v_\varepsilon(t))$$

on $(\varepsilon, T-\varepsilon)$. Choose $t, s\in(0, T)$ and $\varepsilon>0$ such that $0<s-2\varepsilon<s<t<t+2\varepsilon<T$. Then, integrating (23) over (s, t) we obtain

(24) $$(v_\varepsilon(t), u(t)) - (v_\varepsilon(s), u(s))$$
$$= -2\int_s^t (\nabla v_\varepsilon(\sigma), \nabla u(\sigma))d\sigma$$
$$+ \left\{\int_{s-\varepsilon}^{s+\varepsilon} d\tau \int_{s-2\varepsilon}^{s} d\sigma + \int_{t-\varepsilon}^{t+\varepsilon} d\tau \int_t^{t+2\varepsilon} d\sigma\right\}\cdot\rho_\varepsilon(\tau-\sigma)b(v(\tau), v(\tau), u(\sigma))$$
$$-\int_s^t \{b(v(\sigma), v(\sigma), u_\varepsilon(\sigma)) + b(u(\sigma), u(\sigma), v_\varepsilon(\sigma))\}d\sigma$$
$$-\left\{\int_{s-\varepsilon}^{s} + \int_t^{t+\varepsilon}\right\} b(v(\sigma), v(\sigma), u_\varepsilon(\sigma))d\sigma.$$

Here we have used the fact that $\rho(t)$ is an even function. As $\varepsilon\to 0$, u_ε and v_ε tend respectively to u and v in $L^2_{\mathrm{loc}}(0, T; X_2 \cap (W_0^{1,2}(D))^n)$. Therefore we may assume, by passing to a subsequence, that $v_\varepsilon(\sigma)$ tends to $v(\sigma)$ in $(W_0^{1,2}(D))^n$ almost everywhere in $(0, T)$. Moreover, since $u \in C([0, T]; X_p)$, we see that $u_\varepsilon(\sigma)$ tends to $u(\sigma)$ in X_p uniformly on each compact subset of $(0, T)$. These observations together with (24) imply that for almost all t and s,

$$(25) \qquad \begin{aligned}&(v(t), u(t)) - (v(s), u(s)) + 2\int_s^t (\nabla v(\sigma), \nabla u(\sigma))d\sigma \\ &= -\int_s^t \{b(v(\sigma), v(\sigma), u(\sigma)) + b(u(\sigma), u(\sigma), v(\sigma))\}d\sigma.\end{aligned}$$

But, since $v(t)$ is weakly continuous and $u(t)$ is strongly continuous on $[0, T]$ with values in X_2, one concludes that (25) is valid everywhere on $[0, T]$. Letting $s \to 0$ we obtain (21). Thus the proof of Lemma 3.4 is completed.

PROOF OF THEOREM 3.2. Let $v(t)$ be a weak solution and $u(t)$ the solution of (I) with the same initial function a. As is noticed before, $v(t)$ satisfies the energy inequality

$$\|v(t)\|_{0,2}^2 + 2\int_0^t \|\nabla v(s)\|_{0,2}^2 ds \le \|a\|_{0,2}^2 \qquad \text{for a.e.} \quad t \in (0, T),$$

and $u(t)$ satisfies the energy equality

$$\|u(t)\|_{0,2}^2 + 2\int_0^t \|\nabla u(s)\|_{0,2}^2 ds = \|a\|_{0,2}^2 \qquad \text{for each} \quad t \in [0, T].$$

Adding these and subtracting twice the equality (21) we see, with $w(t) = v(t) - u(t)$, that

$$(26) \qquad \begin{aligned}&\|w(t)\|_{0,2}^2 + 2\int_0^t \|\nabla w(s)\|_{0,2}^2 ds \\ &\le 2\int_0^t \{b(v(s), v(s), u(s)) + b(u(s), u(s), v(s))\}ds \\ &= 2\int_0^t \{b(v(s), v(s), u(s)) - b(u(s), v(s), u(s))\}ds \\ &= 2\int_0^t b(w(s), w(s), u(s))ds.\end{aligned}$$

By the Sobolev imbedding theorem we get

$$\begin{aligned}|b(w, w, u)| &\le C\| \,|w| \cdot |u|\, \|_{0,2} \|\nabla w\|_{0,2} \le C\|w\|_{0,2p/(p-2)} \|u\|_{0,p} \|w\|_{1,2} \\ &\le C\|u\|_{0,p} \|w\|_{1,2}^{1+n/p} \|w\|_{0,2}^{1-n/p} \\ &\le C\|u\|_{0,p} \|\nabla w\|_{0,2}^{1+n/p} \|w\|_{0,2}^{1-n/p}.\end{aligned}$$

Since $(1+n/p)/2 + (1-n/p)/2 = 1$, we see from Young's inequality that

$$2|b(w, w, u)| \le 2\|\nabla w\|_{0,2}^2 + C\|u\|_{0,p}^{2p/(p-n)} \|w\|_{0,2}^2,$$

with some constant $C>0$. Thus it follows from (26) that

$$\|w(t)\|_{0,2}^2 \leq C\int_0^t \|u(s)\|_{0,p}^{2p/(p-n)}\|w(s)\|_{0,2}^2 ds.$$

Since $u(t)$ is strongly continuous with values in X_p the assertion follows from the Gronwall inequality. This completes the proof.

Finally we state a regularity theorem for the 4-dimensional problem.

THEOREM 3.5. *Suppose $n=4$. Then for each $\varepsilon>0$ and each $a\in X_2\cap (W^{1+\varepsilon,2}(D))^4\cap(W_0^{1,2}(D))^4$ there exist a $T>0$ and a unique weak solution on $(0, T)$, which belongs to $L^2(0, T; (W^{2,2}(D))^4)\cap W^{1,2}(0, T; X_2)$. Further, $T>0$ can be chosen arbitrarily if a is sufficiently small in $(W^{1+\varepsilon,2}(D))^4$.*

REMARK 3.6. In [12] Serrin proved that for each $a\in X_2\cap(W_0^{1,2}(D))^4\cap (W^{2,2}(D))^4$ there exist a $T>0$ and a weak solution u such that $u'\in L^2(0, T; X_2\cap (W_0^{1,2}(D))^4)\cap L^\infty(0, T; X_2)$. But he could not prove its uniqueness except when a is sufficiently small in $(W^{2,2}(D))^4$. Then, von Wahl [15] proved that the (unique) weak solution obtained by Serrin belongs to $L^p(0, T; (W^{2,p}(D))^4)\cap W^{1,p}(0, T; X_p)$ if a is in $X_p\cap(W_0^{1,p}(D))^4\cap(W^{2,p}(D))^4$ for some $p>5$ with its $W^{2,2}$-norm sufficiently small. Thus Theorem 3.5 is, in a way, an improvement of the results of [12] and [15]. The unique existence result for the initial data in $(W^{1+\varepsilon,2}(D))^4$ seems to be new.

As for the proof of Theorem 3.5, the existence and uniqueness part follows from Theorem 3.2 since the initial value a belongs to $X_{4/(1-\varepsilon)}$ by the Sobolev imbedding theorem. The regularity result follows from a result of Ladyzhenskaya [8] concerning the uniqueness and regularity of solutions for Oseen's linearized equation.

REMARK 3.7. In [6], Inoue and Wakimoto extended the results of Kato and Fujita to the case $n=4$, 5. They proved the existence of strong solutions in X_2 for the initial data in $X_2\cap(W_0^{1,2}(D))^4$, if $n=4$, and in $X_2\cap(W_0^{1,2}(D))^5\cap (W^{3/2,2}(D))^5$, if $n=5$. Since the Sobolev imbedding theorem tells us that $W^{1,2}\subset L^4$ $(n=4)$ and $W^{3/2,2}\subset L^5$ $(n=5)$, it would be possible to include the results of [6] if we could solve the equation (I) in X_n. It is not known whether the solutions constructed in [6] are unique or not in the class of weak solutions.

REMARK 3.8. In [16], von Wahl proved the existence of solutions of the integral equation (II) with $Pf=0$ and $p>n$ by a method similar to ours. In addition, he showed that if $u\in C([0, T]; X_p)$ is a solution of (II) then $\partial u/\partial t$, $\partial u/\partial x_j$ and $\partial^2 u/\partial x_j\partial x_k$ all belong to $C((0, T]; C^{1/n+1}(\bar{D}))$, where $C^\alpha(\bar{D})$ is the space of functions which are Hölder continuous on $\bar{D}$ with exponent α. This extends the interior regularity result of Serrin [12].

References

[1] J. Bergh and J. Löfström, Interpolation Spaces, an introduction, Springer-Verlag, Berlin, 1976.
[2] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, Arch. Rational Mech. Anal., **16** (1964), 269–315.
[3] D. Fujiwara, On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of some second order operators, J. Math. Soc. Japan, **21** (1969), 481–521.
[4] D. Fujiwara and H. Morimoto, An L_r-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo, Sec. IA, **24** (1977), 685–700.
[5] E. Hopf, Uber die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr., **4** (1950–51), 213–231.
[6] A. Inoue and M. Wakimoto, On existence of solutions of the Navier-Stokes equation in a time dependent domain, J. Fac. Sci. Univ. Tokyo, Sec. IA, **24** (1977), 303–320.
[7] T. Kato and H. Fujita, On the nonstationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova, **32** (1962), 243–260.
[8] O. A. Ladyzhenskaya, Uniqueness and smoothness of generalized solutions of the Navier-Stokes equations, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov, **5** (1967), 169–185.
[9] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Gauthier-Villars, Paris, 1969.
[10] J. L. Lions and E. Magenes, Problemi ai limiti non omogenei (III), Ann. Scuola Norm. Sup. Pisa **15** (1961), 39–101.
[11] T. Miyakawa, The L^p approach to the Navier-Stokes equations with the Neumann boundary condition, Hiroshima Math. J., **10** (1980), 517–537.
[12] J. Serrin, The initial value problem for the Navier-Stokes equations, in "Nonlinear problems", R. E. Langer ed., University of Wisconsin Press, Madison, 1963.
[13] P. E. Sobolevskii, Study of Navier-Stokes equations by the methods of the theory of parabolic equations in Banach spaces, Soviet Math. Dokl., **5** (1964), 720–723.
[14] V. A. Solonnikov, Estimates of the solutions of a nonstationary linearized system of Navier-Stokes eauations, Amer. Math. Soc. Translations, Ser. 2, **75** (1968), 1–116.
[15] W. von Wahl, Regularitätsfragen für die instationären Navier-Stokesschen Gleichungen in höheren Dimensionen, J. Math. Soc. Japan, **32** (1980), 263–284.
[16] W. von Wahl, Regularity questions for the Navier-Stokes equations, Lecture Notes in Math. 771, 538–542, Springer-Verlag, Berlin, 1980.

Department of Mathematics,
Faculty of Science,
Hiroshima University