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Ascendancy in locally solvable, ideally finite Lie algebras

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In a recent work, N. Kawamoto has obtained conditions which are sufficient for a subalgebra A to be ascendant in a generalized solvable Lie algebra L. One such condition is that for each $a \in L$, there exists $k=k(a)$ such that $(a)\,\mathrm{ad}^k x \in A$ for all $x \in A$. The results are obtained when the scalars come from a field of characteristic 0, a condition which is shown to be necessary for certain of the results. It seems to be of interest to obtain similar results without restrictions on the characteristic. Such a result is shown here and some consequences are derived.

The Lie algebras considered here are assumed to be over a field. The algebras are assumed locally solvable and ideally finite (see [3]); that is, each element of the algebra is contained in a finite dimensional ideal. Let $\mathfrak{J}$ denote the class of locally solvable, ideally finite Lie algebras. Let $L \in \mathfrak{J}$ and A be a subalgebra of L. For each $a \in A$, let $L_0(a)=\{x \in L; (x)\,\mathrm{ad}^k a=0$ for some $k=1, 2,\dots\}$ and $L_1(a)=\cap_{k=1}^{\infty}$ range $(\mathrm{ad}^k a)$. Since $L \in \mathfrak{J}$, clearly $L=L_0(a)+L_1(a)$ for each $a \in L$. In the conclusion of the main result, a condition which is apparently stronger than ascendancy is obtained. A is ω-ascendant in L if there exists a chain $A=A(0)\lhd A(1)\lhd \cdots A(\omega)=L$ where $A(\omega)=\cup_{k=0}^{\infty} A(k)$. The conditions which are sufficient for ω-ascendancy are also necessary. This is the context of the following main result.

THEOREM 1. *Let $L \in \mathfrak{J}$ and let A be a subalgebra of L. Then the following are equivalent*:

1. *A is ω-ascendant in L.*
2. *$A+L_0(a)=L$ for all $a \in A$.*
3. *$L_1(a)\subseteq A$ for all $a \in A$.*

PROOF. That 1 implies 3 is clear. Assume that 3 holds. Let $a \in A$, $x \in L$ and B be a finite dimensional ideal which contains x. Then $B=B_0(a)+B_1(a)$ and $x \in A+B=A+B_0(a)\subseteq A+L_0(a)$. Hence $L=A+L_0(a)$ and 2 follows. Now assume that 2 holds. L is the union of finite dimensional ideals $\{H(\lambda)\}$. Hence each $H(\lambda)$ contains a chain $0=H(\lambda, 0)\subset\cdots\subset H(\lambda, n(\lambda))=H(\lambda)$ where each $H(\lambda, i)$ is an ideal in L and $H(\lambda, i)/H(\lambda, i-1)$ is an irreducible L-module. Since L is locally solvable and $H(\lambda, 1)$, $H(\beta, 1)$ are minimal, $H(\lambda, 1)H(\beta, 1)=0$. Consider $H(\lambda, j)$, $H(\beta, k)$. Then $T=H(\lambda, j-1)+H(\beta, k-1)$ is an ideal in L and

$H(\lambda, j)+T/T$, $H(\beta, k)+T/T$ are either minimal ideals or 0 in L/T. Hence $(H(\lambda, j)+T)(H(\beta, k)+T)\subseteq T$ and

$$(*) \qquad H(\lambda, j)H(\beta, k) \subseteq H(\lambda, j-1) + H(\beta, k-1).$$

We construct a chain of subspaces $\{\Omega(i)\}$ such that $\{A+\Omega(i)\}$ is of the desired type. Let $\Omega(0)=0$ and $\Omega(1)$ is constructed as follows. If $H(\lambda)\not\subseteq A$, then let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda))\not\subseteq A$. Let $W(\lambda, 1)=\{z\in H(\lambda, r(\lambda));\ zA\subseteq A\}$. Now $H(\lambda, r(\lambda))+A/A$ is finite dimensional and since $L_0(a)+A=L$ for all $a\in A$, each $a\in A$ induces a nilpotent operator on this space. Hence $W(\lambda, 1)\neq 0$. If $H(\lambda)\subseteq A$, then let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda))=H(\lambda)$ and define $W(\lambda, 1)$ as above. In this case $W(\lambda, 1)$ may be 0. Since $W(\lambda, 1)\subseteq H(\lambda, r(\lambda))$ and $H(\lambda, r(\lambda)-1)\subseteq A$, $W(\lambda, 1)W(\beta, 1)\subseteq H(\lambda, r(\lambda))H(\beta, r(\beta))\subseteq A$ by $(*)$. Also $W(\lambda, 1)A\subseteq A$. Let $\Omega(1)$ be the subalgebra of L generated by all $W(\lambda, 1)$. Then $(A+\Omega(1))A\subseteq A$. Now suppose that $\Omega(k)$ has been constructed and we construct $\Omega(k+1)$. If $H(\lambda)\not\subseteq A+\Omega(k)$, let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda))\not\subseteq\Omega(k)+A$. Then $H(\lambda, r(\lambda))+A+\Omega(k)/A+\Omega(k)$ is finite dimensional and each element of A induces a nilpotent linear transformation on this space. Hence $W(\lambda, k+1)=\{z\in H(\lambda, r(\lambda));\ zA\subseteq A+\Omega(k)\}$ is not zero. If $H(\lambda)\subseteq A+\Omega(k)$, let $r(\lambda)$ be the smallest positive integer such that $H(\lambda, r(\lambda))=H(\lambda)$ and define $W(\lambda, k+1)$ as before. Let $\Omega(k+1)$ be the algebra generated by all $W(\lambda, k+1)$. Then $\Omega(k+1)A\subseteq A+\Omega(k)$. Also $W(\lambda, k+1)\subseteq H(\lambda, r(\lambda))$, $H(\lambda, r(\lambda)-1)\subseteq A+\Omega(k)$. Hence $W(\lambda, k+1)W(\beta, k+1)\subseteq H(\lambda, r(\lambda))H(\beta, r(\beta))\subseteq A+\Omega(k)$ by $(*)$. Therefore $\Omega(k+1)\Omega(k+1)\subseteq\Omega(k)+A$ and $(\Omega(k+1)+A)(\Omega(k+1)+A)\subseteq\Omega(k)+A$. Since $W(\lambda, 1)\subset W(\lambda, 2)\subset\cdots$ until $H(\lambda)$ is reached, $H(\lambda)\subseteq\Omega(\dim H(\lambda))$ and $L\subseteq\bigcup_{i=1}^{\infty}\Omega(i)$. Hence $\{A+\Omega(i)\}$ shows that A is ω-ascendant in L.

Let $A\subseteq L\in\mathfrak{I}$. We investigate the subalgebras of L in which A is ω-ascendant. First a well-known lemma on subinvariant subalgebras is obtained in the present setting. Let $L\supseteq L^2\supseteq\cdots$ be the lower central series of L and $L^\omega=\bigcap L^k$.

LEMMA 1. *Let L be a Lie algebra over a field and A be ω-ascendant in L. Then $A^\omega\lhd L$.*

PROOF. Let $A=A(0)\lhd A(1)\lhd\cdots A(\omega)=L$ where $A(\omega)=\bigcup A(i)$. Let $x\in L$. Then $x\in A(n)$ for some n and A is subinvariant in $A(n)$. Since $A^\omega\lhd A(n)$, $xA^\omega\subseteq A^\omega$ and $A^\omega\lhd L$.

LEMMA 2. *Let $L\in\mathfrak{I}$ and let A, H, K be subalgebras of L such that $A\subseteq H\cap K$. If A is ω-ascendant in H and K, then A is ω-ascendant in $\langle H, K\rangle$.*

PROOF. We may assume that $L=\langle H, K\rangle$. Then $A^\omega\lhd H, K$, hence $A^\omega\lhd L$

and we may assume that $A^{\omega}=0$. Let $a \in A$. Then $H_1(a) \subseteq A$ by Theorem 1. Let $x \in H$. Then there exists k such that $x \operatorname{ad}^k a \in A$ and then, since $A^{\omega}=0$, $x \in L_0(a)$. Hence $L_0(a) \supseteq H$ and similarly $L_0(a) \supseteq K$. Therefore $L_0(a)=\langle H, K\rangle =L$ for each $a \in A$. Hence A is ω-ascendant in L by Theorem 1.

THEOREM 2. *Let $A \subseteq L \in \mathfrak{I}$. Then there exists a unique maximal subalgebra of L in which A is ω-ascendant.*

PROOF. Let $\mathscr{S}$ be the collection of all subalgebras of L in which A is ω-ascendant. Clearly $\mathscr{S} \neq \emptyset$. Order $\mathscr{S}$ by inclusion and let $T(1) \subseteq T(2) \subseteq \cdots$ be an increasing chain in $\mathscr{S}$. Let $R=\bigcup T(i)$. If $x \in R$, then $x \in T(n)$ for some n. Let $a \in A$. Then there exists k such that $x \operatorname{ad}^k a \in A$, hence $R_1(a) \subseteq A$ for each $a \in A$ and A is ω-ascendant in R. Hence $\mathscr{S}$ has a maximal element S by Zorn's lemma. Let $U \in \mathscr{S}$. Then $U+S \in \mathscr{S}$ by Lemma 2. Hence $U \subseteq S$ and S contains all elements of $\mathscr{S}$.

The maximal subalgebra of L in which A is ω-ascendant will be denoted by $\alpha(A, L)$. Note that $\alpha(A, L)$ is the maximal subalgebra B of L such that $B_1(a) \subseteq A$ for all $a \in A$. We use $\alpha(A, L)$ to prove the following

THEOREM 3. *Let $A \subseteq L \in \mathfrak{I}$. Then the following are equivalent:*

1. *A is ω-ascendant in L.*
2. *A is ω-ascendant in every subalgebra B of L such that $A \subseteq B$ and $\dim B/A$ is finite.*
3. *A is ω-ascendant in $\langle A, x\rangle$ for every $x \in L$.*
4. *A is ω-ascendant in $\langle A, A \operatorname{ad} x\rangle$ for every $x \in L$.*

PROOF. Since each of the conditions clearly implies the next, it remains to show that 4 implies 1. Suppose that A is ω-ascendant in $\langle A, A \operatorname{ad} x\rangle$ for every $x \in L$ but A is not ω-ascendant in L. Then $\alpha(A, L) \neq L$. Let $x \in L$, $x \notin \alpha(A, L)$ and let N be a finite dimensional ideal of L containing x. Then $\alpha(A, L)$ has finite codimension in $\alpha(A, L)+N$, hence there exists a minimal subalgebra of $\alpha(A, L)+N$ which contains $\alpha(A, L)$ properly. Hence we may assume that there exists $A \subseteq L \in \mathfrak{I}$ such that A is ω-ascendant in $\langle A, A \operatorname{ad} x\rangle$ for all $x \in L$, A is not ω-ascendant in L but A is ω-ascendant in a maximal subalgebra M of L. We break off the next piece of the proof as

LEMMA 3. *There exists $x \in A$ such that $A+L_0(x)=M$.*

PROOF. Since M is maximal in L and $L \in \mathfrak{I}$, M is of finite codimension in L. Let K be the maximal ideal of L contained in M. Then K has finite codimension, for let N be a finite dimensional ideal of L which supplements M and let $C_L(N)$ be the centralizer of N in L. Then $C_L(N) \cap M=J \lhd L$ and since $C_L(N)$ has finite codimension in L, J does also. Furthermore since A is ω-ascendant in M, $M \subseteq A + L_0(x)$ for all $x \in A$, hence we need only to find $x \in A$ such that $L_0(x) \subseteq M$. Now $A + K/K = \bar{A}$ is subinvariant in $M/K = \bar{M}$ and $\bar{M}$ is self-normalizing in $L/K = \bar{L}$. Now $\bar{L}$ is a primitive algebra and by the same proof as [1, Theorem 4, Case II], there exists $x \in A$ such that $\bar{M} = \bar{A} + \bar{L}_0(\bar{x})$. (Note that [1, Theorem 4] is under stronger assumptions that the $\bar{L}$ in this situation but the Case II still carries through.) Then $L_0(x) \subseteq M$ since if $y \in L$ and $y \operatorname{ad}^n x = 0$, then $(y + K) \operatorname{ad}^n (x + K) = (y \operatorname{ad}^n x) + K = K$ and $\bar{y} \in \bar{M}$, hence $y \in M$ and $L_0(x) \subseteq M$.

We now complete the proof of Theorem 3. We show that if $y \notin M$, then $\langle A, A \operatorname{ad} y \rangle = L$. Hence A is ω-ascendant in L, a contradiction. Let $x \in A$ such that $M = A + L_0(x)$. Suppose that $\langle A, A \operatorname{ad} y \rangle \neq L$. Then $\langle A, A \operatorname{ad} y \rangle \subseteq \alpha(A, L) = M$ by Theorem 2 and $xy \in M$. Let B be a finite dimensional ideal such that $y \in B$. Then $y \in B_0(x) + B_1(x) \subseteq L_0(x) + L_1(x)$. Let $y = s + t$ where $s \in L_0(x)$, $t \in L_1(x)$. Now $t \notin M$ since $s \in M$. Then $xt = xy - xs \in M$. However, $\operatorname{ad} x$ is non-singular on the finite dimensional space $L_1(x)$ and $M \cap L_1(x)$ is $\operatorname{ad} x$ invariant. Hence if $z \in L_1(x)$ and $xz \in M \cap L_1(x)$, then $z \in M \cap L_1(x)$. In particular this holds for $z = t$, a contradiction. Hence $\langle A, A \operatorname{ad} y \rangle = L$ and $\alpha(A, L) = L$. Therefore, 4 implies 1 and Theorem 3 is shown.

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