HIROSHIMA MATH. J.
11 (1981), 379–413

Balanced fractional $r^m \times s^n$ factorial designs and their analysis

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(Received January 17, 1981)

1. Introduction and summary

The theory of a fractional factorial design was originated by Fisher [18], who treated the development of confounding systems for factorial designs (cf. [17, 40]), and further Finney [16] gave the first definitive approach. This theory takes aim at the search of "good" fractional factorial designs (cf. [14, 19]). There are many criteria of *goodness*, some of which are:

A. *Save the number of assemblies (treatment combinations).*
B. *Estimate the unknown effects independently.*
C. *Minimize the value of some function $f(T)$ on a class of designs T having the same size (the number of assemblies) N, where $f(T)$ evaluates a sort of the loss of the information.*

As $f(T)$, the following types are used commonly:

$$\det(V_T),\ \mathrm{tr}(V_T) \text{ and the } \textit{maximum characteristic root of } V_T,$$

where $\sigma^2 V_T$ is the variance-covariance matrix of the estimates of the effects based on a design T. These optimality criteria are called the determinant, trace and maximum root criteria, respectively. They aim to minimize the volume of a confidence region for the effects of interest, the average variance, and the largest variance of the estimates of all normalized linear combinations of the effects, respectively (cf. [33]).

The complete design satisfies the *criteria B* and *C*, but it needs a large number of assemblies, which imply that the complete design is unreasonable in the sense of the *criterion A*. An *orthogonal design*, defined by Rao [27] in s^m factorials in which each of m factors has s levels, satisfies the *criteria* B and C (cf. [1, 4, 6, 15, 20, 26]). This design can reduce the number of assemblies in comparison with the complete design. However, an orthogonal design exists only for special values of the size, and the use of such a design may be, in general, uneconomic in the sense that it involves more than the desirable size. For an example of 2^7 factorials of resolution V (the term *resolution* was defined by Box and Hunter [2]), an orthogonal design needs $2^6=64$ or $2^7=128$ (the complete design) assemblies since there exists no orthogonal design of size $2^5=32$ and of resolution V (see Chopra [9]). On the other hand, the number of unknown effects is $1+7+\binom{7}{2}=29$. In an attempt to remedy this defect, Chakravarti [5] proposed a *balanced array* (BA) by relaxing certain conditions to be an orthogonal array. A fractional factorial design derivable from a balanced array has a *goodness* such that

D. *The variance-covariance matrix of the estimates is invariant under any permutation of factors' symbols.*

A design satisfying the *criterion D* is called a *balanced fractional factorial* (BFF) design, and it asserts some invariant test (see Section 5 in detail). The equivalence between a BFF design and a balanced array was proved by Yamamoto, Shirakura and Kuwada [38] in 2^m factorials of resolution $2\ell+1$. Furthermore, Kuwada [22], and Kuwada and Nishii [24] gave the similar equivalence in 3^m factorials of resolution V and in s^m factorials of resolution $2\ell+1$, respectively.

The analysis of a BFF design is not so easy since the estimates of the effects of interest have some correlation. Srivastava and Chopra [34] gave the characteristic polynomial of the information matrix of a balanced fractional 2^m factorial (2^m-BFF) design of resolution V by the direct computation. They further obtained trace optimal designs (cf. [7, 8, 10–13, 30, 32, 35, 36]). It is natural to consider the class of BFF designs since they reflect the relation inherent to the structure of the effects. The algebra generated by relation matrices can be expressed as a direct sum of two-sided ideals. This fact enables to make the analysis of a BFF design relatively easy. Yamamoto, Shirakura and Kuwada [39] succeeded to give the characteristic polynomials of the information matrix of a BFF design of resolution $2\ell+1$. Optimal 2^m-BFF designs of resolution VII are given by Shirakura [28, 29]. These results are derived by using the property of the triangular multidimensional partially balanced association scheme defined in the set of the effects up to ℓ-factor interactions. The algebraic structure of the multidimensional relationship enabled Kuwada and/or Nishii [23, 25] to get the characteristic polynomial of the information matrix of 3^m- and of s^m-BFF designs of resolution V, respectively. Kuwada [21] further obtained optimal designs in 3^m-factorials of resolution V.

On another viewpoint of the development of a fractional factorial design, a *fold-over design* in 2^m factorials was introduced by Box and Wilson [3], who showed that a fold-over design has a *good* property such that no two-factor interactions appear as aliases of the main effects. This property turned out to be useful to construct 2^m-FF designs of resolution IV (cf. [37]). A generalization of the concept of a fold-over design will be proposed in Section 9.

This paper consists of ten sections. Section 2 provides the preliminary results on an $r^m\times s^n$-FF design. In Section 3, asymmetrical orthogonal arrays are introduced, and the equivalence between orthogonal arrays and orthogonal designs in $r^m \times s^n$ factorials is proved. Section 4 is devoted to propose asymmetrical balanced arrays and balanced designs in $r^m \times s^n$ factorials. Section 5 provides the definition of a multidimensional relationship. In particular we define a multidimensional relationship in the set of unknown effects to show the equivalence between balanced arrays and balanced designs in $r^m \times s^n$ factorials. In Section 6, some methods of constructing asymmetrical balanced arrays are described. Sections 7 and 8 deal with the derivation of the characteristic polynomial of the information matrix of balanced designs in $r^m \times s^n$ factorials. This approach is based on the the structure of the algebra containing the information matrix. In Section 9, level-symmetric designs are proposed and their *goodness* is newly shown. Section 10 deals with some structural properties of balanced level-symmetric designs in 2^m factorials.

For convenience, the notations and symbols below are used throughout this paper. Their meanings are as follows:

$\mathfrak{m}$: The set $\{1, 2,..., m\}$.

$\mathfrak{n}$: The set $\{1, 2,..., n\}$.

Z_k : The set $\{0, 1,..., k-1\}$ for any natural number k.

$|S|$: The cardinality of a set S.

I_k : The unit matrix of order k.

$G_{k,l}$: The $k \times l$ matrix whose elements are unity everywhere, and $G_{k,1}$ is denoted by $\boldsymbol{j}_k$.

A' : The transposed matrix of A.

$w(\boldsymbol{a})$: The number of non-zero elements contained in a vector $\boldsymbol{a}=(a_1,..., a_k)$.

$w_\psi(\boldsymbol{a})$: The number of occurrence of ψ among elements of a vector $\boldsymbol{a}$.

$\boldsymbol{w}_1(\boldsymbol{a})$: The r-rowed vector $(w_0(\boldsymbol{a}), w_1(\boldsymbol{a}),..., w_{r-1}(\boldsymbol{a}))$.

$\boldsymbol{w}_2(\boldsymbol{a})$: The s-rowed vector $(w_0(\boldsymbol{a}), w_1(\boldsymbol{a}),..., w_{s-1}(\boldsymbol{a}))$.

$\delta_{\boldsymbol{a},\boldsymbol{b}}$: Kronecker's delta.

$\mathfrak{S}_k$: The symmetric group of k objects.

$A_{(k)}$: The k-times Kronecker product of a matrix A, $\underbrace{A \otimes \cdots \otimes A}_{k}$, for $k \geqq 1$ and $A_{(0)}$ is defined to be 1.

$\boldsymbol{R}(S_1, S_2)$: The set of matrices of size $|S_1| \times |S_2|$ over the real field, where S_1 and S_2 are nonempty finite sets, and the rows and columns of matrices are numbered by the elements of S_1 and S_2, respectively.

$\mathrm{diag}[K_1,..., K_k]$: A matrix of size $\sum_{i=1}^{k} n_i \times \sum_{i=1}^{k} n_i$ whose diagonal positions are given by K_i $(i=1,..., k)$ and the remaining positions are given by zero matrices, where K_i is a matrix of size $n_i \times n_i$.

2. Fractional designs in $r^m \times s^n$ factorials

Consider an $r^m \times s^n$ factorial design with $m+n$ factors $F_1,\dots, F_m, G_1,\dots, G_n$, where F_i $(1 \leqq i \leqq m)$ has r levels in Z_r and G_j $(1 \leqq j \leqq n)$ has s levels in Z_s. An assembly (treatment combination) will be represented by a row vector $\boldsymbol{t}=(f_1\cdots f_m, g_1\cdots g_n)$, where f_i $(\in Z_r)$ and g_j $(\in Z_s)$ denote levels of the factors F_i and G_j, respectively. Let $y(\boldsymbol{t})$ be the observed value based on $\boldsymbol{t}$, and its expectation will be denoted by $\eta(\boldsymbol{t})$ for any assembly $\boldsymbol{t}$. Let $\boldsymbol{\eta}$ be an $r^m s^n$-columned vector of all $\eta(\boldsymbol{t})$ which are arranged in the lexicographic order of $\boldsymbol{t} \in Z_r{}^m \times Z_s{}^n$, i.e.,

$$\boldsymbol{\eta}' = (\eta(0\cdots 0,\ 0\cdots 0),\ \eta(0\cdots 0,\ 0\cdots 01),\dots,\ \eta(0\cdots 0,\ 0\cdots 0s-1),\dots,$$
$$\eta(0\cdots 0,\ s-1\cdots s-1),\dots,\ \eta(r-1\cdots r-1,\ s-1\cdots s-1)).$$

We consider a linear model that $\boldsymbol{\eta}$ can be decomposed as

$$\boldsymbol{\eta} = D_{(m)} \otimes E_{(n)}\boldsymbol{\theta},$$

where $\boldsymbol{\theta}$ is an $r^m s^n$-columned vector composed of effects $\theta(\boldsymbol{\varepsilon})$ arranged in the lexicographic order of $\boldsymbol{\varepsilon}=(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n) \in Z_r{}^m \times Z_s{}^n$, and

$$D = [d(f, \xi)]\ \ (f, \xi \in Z_r), \qquad E = [e(g, \zeta)]\ \ (g, \zeta \in Z_s)$$

are, respectively, $r \times r$, $s \times s$ non-singular matrices whose first columns are composed of 1's and whose all column vectors are mutually orthogonal. The above equality is equivalent to

$$(2.1) \qquad \eta(\boldsymbol{t}) = \textstyle\sum_{\xi_i \in Z_r, \zeta_j \in Z_s} \prod_{i=1}^m d(f_i, \xi_i) \prod_{j=1}^n e(g_j, \zeta_j)\theta(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n)$$

for any assembly $\boldsymbol{t}=(f_1\cdots f_m, g_1\cdots g_n)$.

The effects $\theta(0\cdots 0,\ 0\cdots 0)$, $\theta(0\cdots 0\xi_i 0\cdots 0,\ 0\cdots 0)$ $(1 \leqq \xi_i \leqq r-1)$ and $\theta(0\cdots 0,\ 0\cdots 0\zeta_j 0\cdots 0)$ $(1 \leqq \zeta_j \leqq s-1)$ are called the general mean, the main effects of the factor F_i and those of the factor G_j, respectively. In general, $\theta(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n)$ is called a k-factor interaction if precisely k elements among ξ_i and ζ_j are non-zero.

Note that in the quantitative equi-spaced case, $d(f, \xi)$ and $e(g, \zeta)$ are often defined to be $\Phi_\xi(f)$ and $\Psi_\zeta(g)$ where Φ_ξ and Ψ_ζ are orthogonal polynomials on Z_r and Z_s of degree ξ and ζ, respectively. For example, D and E are defined by $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}$, respectively, when $r=2$ and $s=3$.

Throughout this paper, we shall consider the situation that the set of unknown effects is given by the following Θ_1 or Θ_2 and the remaining effects are assumed to be negligible:

Case 1. $\Theta_1 = \{\theta(\boldsymbol{\varepsilon}) \mid \boldsymbol{\varepsilon} = (\boldsymbol{\xi}, \boldsymbol{\zeta}),\ w(\boldsymbol{\varepsilon}) \leqq \ell\}$ for $\ell(\leqq m+n)$.

Case 2. $\Theta_2 = \{\theta(\boldsymbol{\varepsilon}) \mid \boldsymbol{\varepsilon} = (\boldsymbol{\xi}, \boldsymbol{\zeta}), w(\boldsymbol{\xi}) \leqq \ell_r, w(\boldsymbol{\zeta}) \leqq \ell_s\}$ for $\ell_r(\leqq m)$ and $\ell_s(\leqq n)$. Put

$$\nu_1 = \sum_{p+q \leqq \ell} \binom{m}{p}\binom{n}{p}(r-1)^p(s-1)^q,$$

$$\nu_2 = \left\{\sum_{p=0}^{\ell_r}\binom{m}{p}(r-1)^p\right\}\left\{\sum_{q=0}^{\ell_s}\binom{n}{q}(s-1)^q\right\}.$$

Let $\boldsymbol{\theta}_i$ be a ν_i-columned vector composed of all effects in Θ_i $(i=1, 2)$.

Let T be a fractional $r^m \times s^n$ factorial ($r^m \times s^n$-FF) design with N assemblies $\boldsymbol{t}^{(\alpha)} = (f_1^{(\alpha)} \cdots f_m^{(\alpha)}, g_1^{(\alpha)} \cdots g_n^{(\alpha)}) = (\boldsymbol{f}^{(\alpha)}, \boldsymbol{g}^{(\alpha)})$ for $\alpha = 1,\dots, N$. Then T can be partitioned into two submatrices F of size $N \times m$ and G of size $N \times n$, which is denoted by $T=[F: G]$. Let $\boldsymbol{y}(T)=[y(\boldsymbol{t}^{(\alpha)})]$ $(\alpha=1,\dots, N)$ be the N-columned vector composed of the observed values $y(\boldsymbol{t}^{(\alpha)})$. From (2.1), it can be expressed by

$$\boldsymbol{y}(T) = E_T\boldsymbol{\theta}_i + \boldsymbol{e}(T),$$

where E_T is the design matrix (of size $N \times \nu_i$) of T, and $\boldsymbol{e}(T)$ is the error vector (of size $N \times 1$) whose components are assumed to be uncorrelated and each has mean zero and the same variance σ^2. The normal equation for estimating $\boldsymbol{\theta}_i$ can be written as

$$M_T\hat{\boldsymbol{\theta}}_i = E'_T\boldsymbol{y}(T),$$

where $M_T = E'_TE_T$ is called the information matrix (of size $\nu_i \times \nu_i$). If M_T is non-singular, the best linear unbiased estimate of $\boldsymbol{\theta}_i$ is given by $M_T^{-1}E'_T\boldsymbol{y}(T)$ and its variance-covariance matrix is $\sigma^2M_T^{-1}$. In this case, the resolution of a design T is defined to be $2\ell+1$ or $(2\ell_r+1, 2\ell_s+1)$ according as $i=1$ or 2.

The rows and columns of E_T are numbered by the elements of $\boldsymbol{y}(T)$ and $\boldsymbol{\theta}_i$, respectively. The $(y(\boldsymbol{t}^{(\alpha)}), \theta(\boldsymbol{\varepsilon}))$-entry of E_T is given by

$$d(f_1^{(\alpha)}, \xi_1)\cdots d(f_m^{(\alpha)}, \xi_m)e(g_1^{(\alpha)}, \zeta_1)\cdots e(g_n^{(\alpha)}, \zeta_n) \quad (= d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon},) \text{ say}),$$

where $\boldsymbol{t}^{(\alpha)} = (f_1^{(\alpha)} \cdots f_m^{(\alpha)}, g_1^{(\alpha)} \cdots g_n^{(\alpha)})$ and $\boldsymbol{\varepsilon} = (\xi_1 \cdots \xi_m, \zeta_1 \cdots \zeta_n)$. Thus a $(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$-entry of M_T, denoted by $m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$, can be expressed by

$$m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*)) = \sum_{\alpha=1}^{N} d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon})d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}^*), \tag{2.2}$$

where $\boldsymbol{\varepsilon}^* = (\xi_1^* \cdots \xi_m^*, \zeta_1^* \cdots \zeta_n^*)$, $\xi_i^* \in Z_r$ and $\zeta_j^* \in Z_s$. This relation implies that $m_T(\theta(\boldsymbol{\delta}), \theta(\boldsymbol{\delta}^*)) = m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$ if the k-th element of $\boldsymbol{\delta}$ and $\boldsymbol{\delta}^*$ are, respectively, given by those of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^*$ or those of $\boldsymbol{\varepsilon}^*$ and $\boldsymbol{\varepsilon}$ for $k=1, 2,\dots, m+n$.

Note that if $\boldsymbol{\varepsilon}^* = \boldsymbol{0} = (0\cdots0, 0\cdots0)$ in (2.2), then

$$m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{0})) = \sum_{\alpha=1}^{N} d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}),$$

since $d(*, 0) = e(*, 0) = 1$.

Now some symbols describing an $r^m \times s^n$-FF design $T=[F: G]$ are introduced, where F and G are matrices composed of elements in Z_r and Z_s, respectively.

For sequences $u=(u_1 \cdots u_p)$ with $1 \leqq u_1 < \cdots < u_p \leqq m$ and $v=(v_1 \cdots v_q)$ with $1 \leqq v_1 < \cdots < v_q \leqq n$, let $T_{u,v}=[F_u: G_v]$ be the $N \times (p+q)$ submatrix of $T=[F: G]$, where F_u is the $N \times p$ submatrix of F generated by u_i-th columns $(1 \leqq i \leqq p)$ of F and G_v is the $N \times q$ submatrix of G given in the same way. In the special case $p=0$ or $q=0$, $T_{u,v}$ is defined to be G_v or F_u. For a $(p+q)$-rowed vector $(\boldsymbol{f}, \boldsymbol{g})=(f_1 \cdots f_p, g_1 \cdots g_q)$ in $Z_r^p \times Z_s^q$, let $\mu_{u,v}^{\boldsymbol{f};\boldsymbol{g}}$ be the number of times that $(\boldsymbol{f}, \boldsymbol{g})$ occurs in $T_{u,v}$ as row vectors. Let $\gamma_{u,v}^{\boldsymbol{\xi};\boldsymbol{\zeta}} = \sum_{\alpha=1}^N d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon})$ for any $(\boldsymbol{\xi}, \boldsymbol{\zeta})=(\xi_1 \cdots \xi_p, \zeta_1 \cdots \zeta_q) \in Z_r^p \times Z_s^q$, where

$$\boldsymbol{\varepsilon} = (0 \cdots 0 \underset{(u_1)}{\xi_1} 0 \cdots 0 \underset{(u_p)}{\xi_p} 0 \cdots 0, 0 \cdots 0 \underset{(v_1)}{\zeta_1} 0 \cdots 0 \underset{(v_q)}{\zeta_q} 0 \cdots 0). \tag{2.3}$$

$\{\mu\}$ and $\{\gamma\}$ are arranged in the lexicographic order of upper indices $(\boldsymbol{f}, \boldsymbol{g}) \in Z_r^p \times Z_s^q$ and $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in Z_r^p \times Z_s^q$ as

$$\boldsymbol{\mu}_{u,v} = [\mu_{u,v}^{\boldsymbol{f};\boldsymbol{g}}] \quad \text{and} \quad \boldsymbol{\gamma}_{u,v} = [\gamma_{u,v}^{\boldsymbol{\xi};\boldsymbol{\zeta}}].$$

Then we have the following

Lemma 2.1. *For* $(u, v)=(u_1 \cdots u_p, v_1 \cdots v_q)$ $(1 \leqq u_1 < \cdots < u_p \leqq m;\ 1 \leqq v_1 < \cdots < v_q \leqq n)$, *it holds that*

$$\boldsymbol{\gamma}_{u,v} = (D'_{(p)} \otimes E'_{(q)}) \boldsymbol{\mu}_{u,v}. \tag{2.4}$$

Proof. From the definitions of $\gamma_{u,v}^{\boldsymbol{\xi};\boldsymbol{\zeta}}$ and $\mu_{u,v}^{\boldsymbol{f};\boldsymbol{g}}$, and $d(*, 0)=e(*, 0)=1$, we have

$$\begin{aligned} \gamma_{u,v}^{\boldsymbol{\xi};\boldsymbol{\zeta}} &= \textstyle\sum_{\alpha=1}^N d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}) = \sum_{\alpha=1}^N \{\prod_{i=1}^p d(f_{u_i}^{(\alpha)}, \xi_i)\} \{\prod_{j=1}^q e(g_{v_j}^{(\alpha)}, \zeta_j)\} \\ &= \textstyle\sum_{f_i \in Z_r, g_j \in Z_s} \{\prod_{i=1}^p d(f_i, \xi_i)\} \{\prod_{j=1}^q e(g_j, \zeta_j)\} \mu_{u,v}^{\boldsymbol{f};\boldsymbol{g}} \end{aligned} \tag{2.5}$$

for any $(\boldsymbol{\xi}, \boldsymbol{\zeta})=(\xi_1 \cdots \xi_p, \zeta_1 \cdots \zeta_q) \in Z_r^p \times Z_s^q$, which yields the required relation. Here $(\boldsymbol{f}, \boldsymbol{g})=(f_1 \cdots f_p, g_1 \cdots g_q)$ and $\boldsymbol{\varepsilon}$ is given in (2.3).

3. Equivalence between orthogonal arrays and orthogonal designs

An orthogonal array OA$[N, m, r, d]$ was defined by Rao [27] as an $N \times m$ matrix with entries in Z_r whose any $N \times d$ submatrix contains all possible d-rowed vectors in the same frequency $\lambda\ (=N/r^d)$. An OA is an interesting subject in combinatorics, and many works have been done. Now we shall extend the concept of an orthogonal array.

Consider an $N \times (m+n)$ matrix $T=[F: G]$, where F and G are $N \times m$ and $N \times n$ matrices with entries in Z_r and Z_s, respectively. We present following definitions of an OA according to unknown effects Θ_1 or Θ_2.

DEFINITION 3.1. An $N \times (m+n)$ matrix T is called an *asymmetrical orthogonal array of type* 1 of strength t, size N, (m, n) constraints, (r, s) levels and index set $\{\lambda_{p,q}\}$ (for brevity, AOA1$[N, (m, n), (r, s), t]$), if for arbitrary non-negative integers p and q satisfying $p+q=t$, $0 \leqq p \leqq m$ and $0 \leqq q \leqq n$, $\mu_{u,v}^{f;g}=\lambda_{p,q}$ for any $(u, v)=(u_1 \cdots u_p, v_1 \cdots v_q)$ $(1 \leqq u_1 < \cdots < u_p \leqq m;\ 1 \leqq v_1 < \cdots < v_q \leqq n)$ and any $(\boldsymbol{f}, \boldsymbol{g})= (f_1 \cdots f_p, g_1 \cdots g_q) \in Z_r^p \times Z_s^q$.

REMARK. (i) It is unnecessary to assume that $t \leqq m$ and $t \leqq n$. (ii) $N= r^p s^q \lambda_{p,q}$. (iii) F and G are orthogonal arrays of levels r and s, respectively.

DEFINITION 3.2. T is called an *asymmetrical orthogonal array of type* 2 of strength (d, e), size N, (m, n) constraints, (r, s) levels and index λ (for brevity, AOA2$[N, (m, n), (r, s), (d, e)]$), if $\mu_{u,v}^{f;g}=\lambda$ for any $(u, v)=(u_1 \cdots u_d, v_1 \cdots v_e)$ $(1 \leqq u_1 < \cdots < u_d \leqq m;\ 1 \leqq v_1 < \cdots < v_e \leqq n)$ and any $(\boldsymbol{f}, \boldsymbol{g})=(f_1 \cdots f_d, g_1 \cdots g_e) \in Z_r^d \times Z_s^e$.

REMARK. (i) $N=r^d s^e \lambda$. (ii) F and G are an OA$[N, m, r, d]$ and an OA$[N, n, s, e]$, respectively.

DEFINITION 3.3. An $r^m \times s^n$-FF design T is called an *orthogonal design* of resolution $2\ell+1$ or $(2\ell_r+1, 2\ell_s+1)$ if its information matrix M_T, with unknown effects Θ_1 or Θ_2, is diagonal.

Let Θ_1 be the set of effects given in Section 2 satisfying $2l \leqq m+n$, and $T= [F\colon G]=[\boldsymbol{t}^{(\alpha)}]_{\alpha=1,\ldots,N}$ be an $r^m \times s^n$-FF design of resolution $2\ell+1$, where $\boldsymbol{t}^{(\alpha)}= (f_1^{(\alpha)} \cdots f_m^{(\alpha)}, g_1^{(\alpha)} \cdots g_n^{(\alpha)}) \in Z_r^m \times Z_s^n$. In this case, we have

THEOREM 3.1. *An $r^m \times s^n$-FF design T is an orthogonal design of resolution $2\ell+1$ if and only if T is an AOA1$[N, (m, n), (r, s), 2\ell]$.*

PROOF (*Sufficiency*). Let $\theta(\boldsymbol{\varepsilon})$ and $\theta(\boldsymbol{\varepsilon}^*)$ be elements in Θ_1. Then the sum of the number of non-zero elements of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^*$ is at most 2ℓ. We can assume that $\{i \mid \xi_i \neq 0 \text{ or } \xi_i^* \neq 0\} \subset \{u_1,\ldots, u_p\} \subset \mathfrak{m}$ and $\{j \mid \zeta_j \neq 0 \text{ or } \zeta_j^* \neq 0\} \subset \{v_1,\ldots, v_q\} \subset \mathfrak{n}$ for $p+q=2\ell$ $(0 \leqq p \leqq m,\ 0 \leqq q \leqq n)$, where $\boldsymbol{\varepsilon}$ is given by (2.3) and $\boldsymbol{\varepsilon}^*$ is defined by changing ξ_i into ξ_i^*, and ζ_j into ζ_j^* in the elements of $\boldsymbol{\varepsilon}$. Here ξ_i, ξ_i^* $(\in Z_r)$ and ζ_j, ζ_j^* $(\in Z_s)$ may be equal to zero. From the assumption, the $N \times 2\ell$ submatrix $T_{u,v}$ contains all possible 2ℓ-rowed vectors in the same frequency $\lambda_{p,q}=N/(r^p s^q)$, where $(u, v)=(u_1 \cdots u_p, v_1 \cdots v_q)$. Since D and E are non-singular and their column vectors are mutually orthogonal respectively, the relation (2.2) can be reduced to

$$
\begin{aligned}
m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*)) &= \textstyle\sum_{\alpha=1}^N d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}) d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}^*) \\
&= \textstyle\sum_{\alpha=1}^N \{\prod_{i=1}^p d(f_{u_i}^{(\alpha)}, \xi_i) d(f_{u_i}^{(\alpha)}, \xi_i^*)\} \{\prod_{j=1}^q e(g_{v_j}^{(\alpha)}, \zeta_j) e(g_{v_j}^{(\alpha)}, \zeta_j^*)\} \\
&= \lambda_{p,q} \textstyle\sum_{f_i \in Z_r, g_j \in Z_s} \{\prod_{i=1}^p d(f_i, \xi_i) d(f_i, \xi_i^*)\} \{\prod_{j=1}^q e(g_j, \zeta_j) e(g_j, \zeta_j^*)\}
\end{aligned}
$$

$$\begin{cases} = 0 & \text{if} \quad \boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}^*, \\ > 0 & \text{if} \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*. \end{cases}$$

This shows that M_T is a diagonal matrix.

(*Necessity*). Let T be an $r^m \times s^n$-FF design whose information matrix M_T is diagonal. Any off-diagonal entry, $m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$, of M_T is equal to zero for $\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*) \in \Theta_1$. This fact implies

$$\boldsymbol{\gamma}_{u,v} = (N, 0, \dots, 0)'$$

for any $(u, v) = (u_1 \cdots u_p, v_1 \cdots v_q)$ $(1 \leqq u_1 < \cdots < u_p \leqq m;\ 1 \leqq v_1 < \cdots < v_q \leqq n)$, where $p + q = 2\ell$ $(0 \leqq p \leqq m;\ 0 \leqq q \leqq n)$. Solving (2.4) with respect to $\boldsymbol{\mu}_{u,v}$, we have

$$\begin{aligned} \boldsymbol{\mu}_{u,v} &= (D'_{(p)} \otimes E'_{(q)})^{-1}(N, 0, \dots, 0)' \\ &= (D(D'D)^{-1})_{(p)} \otimes (E(E'E)^{-1})_{(q)}(N, 0, \dots, 0)' = N/(r^p s^q)\boldsymbol{j}_{r^p s^q} \end{aligned}$$

since $D'D$ and $E'E$ are diagonal and the first columns of D and E are $\boldsymbol{j}_r$ and $\boldsymbol{j}_s$, respectively. Thus T is an AOA1 with index set $\{\lambda_{p,q} = N/(r^p s^q) \mid p + q = 2\ell\}$.

For T being an AOA1, the non-singularity of the information matrix M_T yields

$$N \geqq \operatorname{rank} E_T = \operatorname{rank} E'_T E_T = |\Theta_1| = \nu_1 .$$

Therefore, we have the following

COROLLARY 3.2. *For an AOA1*$[N, (m, n), (r, s), 2\ell]$ *satisfying* $2\ell \leqq m + n$, *it holds that* $N \geqq \sum_{i+j \leqq \ell} \binom{m}{i}\binom{n}{j}(r-1)^i(s-1)^j$.

COROLLARY 3.3. *For an AOA1*$[N, (m, n), (r, s), 2\ell + 1]$ *satisfying* $2\ell + 1 \leqq m + n$, *it holds that*

$$\begin{aligned} N \geqq \sum_{i+j \leqq \ell} \binom{m}{i}\binom{n}{j}(r-1)^i(s-1)^j + \max\Big\{ &\sum_{i+j=\ell} \binom{m-1}{i}\binom{n}{j} \\ &\cdot (r-1)^{i+1}(s-1)^j,\ \sum_{i+j=\ell} \binom{m}{i}\binom{n-1}{j}(r-1)^i(s-1)^{j+1} \Big\}. \end{aligned}$$

PROOF. Let $\Theta^* = \Theta_1 \cup \{\theta(\boldsymbol{\varepsilon}) \mid \boldsymbol{\varepsilon} = (\xi_1 \cdots \xi_m, \zeta_1 \cdots \zeta_n),\ \xi_1 \neq 0,\ w(\boldsymbol{\varepsilon}) = \ell + 1\}$ and let $\Theta^{**} = \Theta_1 \cup \{\theta(\boldsymbol{\varepsilon}) \mid \zeta_1 \neq 0,\ w(\boldsymbol{\varepsilon}) = \ell + 1\}$. The information matrix M_T^* of T given by unknown effects Θ^* is diagonal, since

$$\boldsymbol{\mu}_{u,v} = \lambda_{p+1,q}\boldsymbol{j}_{r^{p+1}s^q},$$

where $(u, v) = (u_1 \cdots u_{p+1}, v_1 \cdots v_q)$ $(1 = u_1 < u_2 < \cdots < u_{p+1} \leqq m;\ 1 \leqq v_1 < \cdots < v_q \leqq n)$ and $p + q + 1 = 2\ell + 1$. Similarly, the information matrix M_T^{**} given by unknown effects Θ^{**} can be shown to be diagonal. Therefore $N \geqq \max\{|\Theta^*|, |\Theta^{**}|\}$.

Let Θ_2 be the set of effects given in Section 2 satisfying $2\ell_r \leqq m$ and $2\ell_s \leqq n$, and let T be an $r^m \times s^n$-FF design of resolution $(2\ell_r+1, 2\ell_s+1)$. An argument similar to Theorem 3.1 and Corollaries 3.2–3 shows the following theorem and corollary.

THEOREM 3.4. *T is an orthogonal design of resolution $(2\ell_r+1, 2\ell_s+1)$ if and only if T is an $AOA2[N, (m, n), (r, s), (2\ell_r, 2\ell_s)]$.*

COROLLARY 3.5. *For an $AOA2[N, (m, n), (r, s), (d, e)]$, it holds that*

$$N \geqq L_r(d) \cdot L_s(e),$$

where

$$L_r(d) = \begin{cases} \sum_{i=0}^{d^*} \binom{m}{i}(r-1)^i & \text{if } d = 2d^* \quad (even), \\ \sum_{i=0}^{d^*} \binom{m}{i}(r-1)^i + \binom{m-1}{d^*}(r-1)^{d^*+1} & \text{if } d = 2d^*+1 \ (odd), \end{cases}$$

$$L_s(e) = \begin{cases} \sum_{j=0}^{e^*} \binom{n}{j}(s-1)^j & \text{if } e = 2e^* \quad (even), \\ \sum_{j=0}^{e^*} \binom{n}{j}(s-1)^j + \binom{n-1}{e^*}(s-1)^{e^*+1} & \text{if } e = 2e^*+1 \ (odd). \end{cases}$$

4. Asymmetrical balanced arrays and balanced designs

Orthogonal designs are desirable in the sense that all unknown effects can be estimated uncorrelatedly. However, since the existence conditions of an orthogonal design are severe, such a design exists only in restricted cases. Next we consider the *criterion D* in Section 1.

As an illustration of *goodness*, we consider a $2^m \times 3^n$-FF design T of resolution III, in which unknown effects are the general mean and all main effects. Let $\boldsymbol{\theta}_*$ be a $(m+n)$-columned vector composed of some main effects

$$\boldsymbol{\theta}'_* = (\theta(10\cdots0, 0\cdots0), \ldots, \theta(0\cdots01, 0\cdots0), \theta(0\cdots0, 10\cdots0), \ldots, \theta(0\cdots0, 0\cdots01)).$$

For testing hypothesis $H: \boldsymbol{\theta}_* = c\boldsymbol{j}_{m+n}$ for a given constant c against alternative that $\boldsymbol{\theta}_* \neq c\boldsymbol{j}_{m+n}$, we give the statistic F defined by

$$F = \{\boldsymbol{u}'V_*^{-1}\boldsymbol{u}/(m+n)\}/\{S^2/(N-\nu_*)\},$$

where

$$\boldsymbol{u} = [\boldsymbol{0}: I_{m+n}: 0]V_T E'_T \boldsymbol{y}(T) - c\boldsymbol{j}_{m+n},$$
$$S^2 = \boldsymbol{y}(T)'(I_N - E_T V_T E'_T)\boldsymbol{y}(T),$$

$v_* = 1+m+2n$, $\sigma^2 V_*$ is the variance-covariance matrix of $\hat{\boldsymbol{\theta}}_*$ and $V_T = M_T^{-1}$. Here F is distributed according to a *noncentral F-distribution* with $m+n$ and $N-v_*$ degrees of freedom and the noncentrality parameter $(\boldsymbol{\theta}_* - c\boldsymbol{j}_{m+n})' V_*^{-1} \cdot (\boldsymbol{\theta}_* - c\boldsymbol{j}_{m+n})$ if the distribution of the error vector $\boldsymbol{e}(T)$ is $N(\mathbf{0}, \sigma^2 I_N)$. This test is desirable to be symmetric in $F_1, \ldots, F_m$ and in $G_1, \ldots, G_n$. This requirement means that V_*^{-1} should belong to the matrix algebra $\mathscr{B}_*$ which is generated by $(m+n) \times (m+n)$ matrices

$$\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} G_{m,m} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & G_{m,n} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ G_{n,m} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & G_{n,n} \end{pmatrix}.$$

In this case, $V_*^{-1} \in \mathscr{B}_*$ holds if M_T is invariant under any permutation of $F_1, \ldots, F_m$ and of $G_1, \ldots, G_n$, i.e., $M_T = M_{T^{\tau,\rho}}$ for any $\tau \in \mathfrak{S}_m$ and $\rho \in \mathfrak{S}_n$, where $T = [F: G]$, $T^{\tau,\rho} = [F^\tau: G^\rho]$ and F^τ denotes the matrix whose i-th column is given by $\tau(i)$-th column of F ($i \in \mathfrak{m}$) and G^ρ is defined in the same way. Thus it is reasonable to consider designs having the *good* property that M_T is invariant under any permutation of the factors $F_1, \ldots, F_m$ and $G_1, \ldots, G_n$, respectively.

Let $T = [F: G]$ be an $r^m \times s^n$-FF design with unknown effects Θ, where $\Theta = \Theta_1$ or Θ_2.

DEFINITION 4.1. The information matrix M_T is said to be *balanced* with respect to Θ if $M_T = M_{T^{\tau,\rho}}$ for any $(\tau, \rho) \in \mathfrak{S}_m \times \mathfrak{S}_n$.

DEFINITION 4.2. A fraction T is called a *balanced design* if M_T is nonsingular and $M_T^{-1} = M_{T^{\tau,\rho}}^{-1}$ for any $(\tau, \rho) \in \mathfrak{S}_m \times \mathfrak{S}_n$.

The following definitions of an asymmetrical balanced array (ABA) are given by relaxing the some condition of the asymmetrical orthogonal array of type 1 or of type 2.

DEFINITION 4.3. An $N \times (m+n)$ matrix is called an *asymmetrical balanced array of type* 1 of strength t, size N, (m, n) constraints, (r, s) levels and index set $\{\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1}) \mid \sum_{i=0}^{r-1} p_i + \sum_{j=0}^{s-1} q_j = t, \ \Sigma p_i \leqq m, \ \Sigma q_j \leqq n\}$ (for brevity, ABA1$[N, (m, n), (r, s), t]\{\lambda(\boldsymbol{p}, \boldsymbol{q})\}$), if $\mu_{u,v}^{f;g} = \lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1})$ for any $(\boldsymbol{f}, \boldsymbol{g}) = (f_1 \cdots f_p, g_1 \cdots g_q)$ satisfying $\boldsymbol{w}_1(\boldsymbol{f}) = (p_0 \cdots p_{r-1})$ and $\boldsymbol{w}_2(\boldsymbol{g}) = (q_0 \cdots q_{s-1})$, and for any $(u, v) = (u_1 \cdots u_p, v_1 \cdots v_q)$ $(1 \leqq u_1 < \cdots < u_p \leqq m; \ 1 \leqq v_1 < \cdots < v_q \leqq n)$, where p_i and q_j are non-negative integers such that $\Sigma p_i = p$ and $\Sigma q_j = q$. Here $p+q=t$.

DEFINITION 4.4. An $N \times (m+n)$ matrix is called an *asymmetrical balanced array of type* 2 of strength (d, e), size N, (m, n) constraints, (r, s) levels and index set $\{\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1}) \mid \sum_{i=0}^{r-1} p_i = d, \ \sum_{j=0}^{s-1} q_j = e\}$ (for brevity, ABA2$[N, (m, n), (r, s), (d, e)]\{\lambda(\boldsymbol{p}, \boldsymbol{q})\}$), if $\mu_{u,v}^{f;g} = \lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1})$ for any $(\boldsymbol{f}, \boldsymbol{g}) = (f_1 \cdots f_d, g_1 \cdots g_e)$ satisfying $\boldsymbol{w}_1(\boldsymbol{f}) = (p_0 \cdots p_{r-1})$ and $\boldsymbol{w}_2(\boldsymbol{g}) = (q_0 \cdots q_{s-1})$, and for any

$(u, v)=(u_1 \cdots u_d, v_1 \cdots v_d)$ $(1 \leqq u_1 < \cdots < u_d \leqq m;\ 1 \leqq v_1 < \cdots < v_e \leqq n)$, where p_i and q_j are nonnegative integers such that $\Sigma p_i = d$ and $\Sigma q_j = e$.

Then we have the following

THEOREM 4.1. *Let T be an $r^m \times s^n$-FF design with unknown effects Θ_1 satisfying $2\ell \leqq m+n$. Then M_T is balanced with respect to Θ_1 if and only if T is an ABA1$[N, (m, n), (r, s), 2\ell]\{\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1}) \mid \Sigma p_i + \Sigma q_j = 2\ell\}$.*

PROOF (*Sufficiency*). Suppose T be an ABA1$[N, (m, n), (r, s), 2\ell]$. Then any entry, $m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$, of M_T can be expressed as

$$m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*)) = \sum_{\alpha=1}^{N} \{\prod_{i=1}^{p} d(f_{u_i}^{(\alpha)}, \xi_{u_i}) d(f_{u_i}^{(\alpha)}, \xi_{u_i}^*)\} \{\prod_{j=1}^{q} e(g_{v_j}^{(\alpha)}, \zeta_{v_j}) e(g_{v_j}^{(\alpha)}, \zeta_{v_j}^*)\},$$

where $\xi_i = \xi_i^* = 0$ for any $i \in \mathfrak{m} - \{u_1, \ldots, u_p\}$ and $\zeta_j = \zeta_j^* = 0$ for any $j \in \mathfrak{n} - \{v_1, \ldots, v_q\}$ and $p+q=2\ell$ $(0 \leqq p \leqq m,\ 0 \leqq q \leqq n)$. From the assumption of T,

$$m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*)) = \sum_{f_i \in Z_r, g_j \in Z_s} \{\prod_{i=1}^{p} d(f_i, \xi_{u_i}) d(f_i, \xi_{u_i}^*)\} \cdot \{\prod_{j=1}^{q} e(g_j, \zeta_{v_j}) e(g_j, \zeta_{v_j}^*)\} \lambda(\boldsymbol{w}_1(f_1 \cdots f_p), \boldsymbol{w}_2(g_1 \cdots g_q)).$$

This relation shows that $m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*)) = m_T(\theta(\boldsymbol{\varepsilon}^\omega), \theta(\boldsymbol{\varepsilon}^{*\omega}))$ for any $\omega = (\tau, \rho) \in \mathfrak{S}_m \times \mathfrak{S}_n$, where $\boldsymbol{\varepsilon}^\omega = (\xi_{\tau(1)} \cdots \xi_{\tau(m)}, \zeta_{\rho(1)} \cdots \zeta_{\rho(n)})$ and $\boldsymbol{\varepsilon}^{*\omega}$ is defined similarly. Therefore M_T is balanced with respect to Θ_1.

(*Necessity*). The assumption that M_T is balanced implies that all $\gamma_{u,v}^{\xi;\zeta}$ depend only on $\boldsymbol{w}_1(\boldsymbol{\xi})$ and $\boldsymbol{w}_2(\boldsymbol{\zeta})$ for any $(\boldsymbol{\xi}, \boldsymbol{\zeta}) = (\xi_1 \cdots \xi_p, \zeta_1 \cdots \zeta_q)$, and for any $(u, v) = (u_1 \cdots u_p, v_1 \cdots v_q)$ $(1 \leqq u_1 < \cdots < u_p \leqq m;\ 1 \leqq v_1 < \cdots < v_q \leqq n)$, where $p+q=2\ell$ $(0 \leqq p \leqq m,\ 0 \leqq q \leqq n)$. Solving (2.4) with respect to $\boldsymbol{\mu}_{u,v}$, we have

$$\boldsymbol{\mu}_{u,v} = (D_{(p)}^{-1} \otimes E_{(q)}^{-1})' \boldsymbol{\gamma}_{u,v}.$$

Therefore, $\boldsymbol{\mu}_{u,v}$ does not depend on (u, v) since $\boldsymbol{\gamma}_{u,v}$ depends only on p and q. We can define $\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1})$ by $\mu_{u,v}^{f;g}$ if $\boldsymbol{w}_1(\boldsymbol{g}) = (p_0 \cdots p_{r-1})$ and $\boldsymbol{w}_2(\boldsymbol{f}) = (q_0 \cdots q_{s-1})$ for any p_i and q_j satisfying $\Sigma p_i + \Sigma q_j = 2\ell$, since $\mu_{u,v}^{f;g}$ depends only on $\boldsymbol{w}_1(\boldsymbol{f})$ and $\boldsymbol{w}_2(\boldsymbol{g})$. Thus T is shown to be an ABA1$[N, (m, n), (r, s), 2\ell]$ $\{\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1})\}$.

An argument similar to Theorem 4.1 shows the following

THEOREM 4.2. *Let T be an $r^m \times s^n$-FF design with unknown effects Θ_2 satisfying $2\ell_r \leqq m$ and $2\ell_s \leqq n$. Then M_T is balanced with respect to Θ_2 if and only if T is an ABA2$[N, (m, n), (r, s), (2\ell_r, 2\ell_s)]\{\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1})\}$.*

5. Multidimensional relationships

As a generalization of an association scheme, a multidimensional partially balanced association scheme was introduced. By omitting the condition that the relation of a multidimensional partially balanced association scheme is symmetrical, we have a multidimensional relationship. The following definition is due to Kuwada [21].

Consider p mutually disjoint nonempty finite sets $S_1,\dots, S_p$ with $|S_i|=n_i$ each. Suppose that an association is defined for each ordered pair (x_{ia}, x_{jb}), where $x_{ia}\in S_i$ and $x_{jb}\in S_j$. Let $\Pi^{i,j}$ be a set of associations defined on the set $S_i\times S_j$. We denote

$$\mathscr{S} = \{S_1,\dots, S_p\} \quad \text{and} \quad \mathscr{R} = \{\Pi^{1,1}, \Pi^{1,2}, \Pi^{2,1},\dots, \Pi^{p,p}\}.$$

DEFINITION 5.1. The pair $(\mathscr{S}, \mathscr{R})$ is called a *multidimensional relationship* if the following two conditions are satisfied.

C1. With respect to any $x_{ia}\in S_i$, the objects of S_j can be divided into $n^{i,j}$ disjoint classes and the number of objects in the set $\{x_{jb}\in S_j \mid$ the association of (x_{ia}, x_{jb}) is $\alpha\}$ is $n_\alpha^{i,j}$ for $\alpha\in\Pi^{i,j}$. The numbers $n^{i,j}$ and $n_\alpha^{i,j}$ are independent of the particular object x_{ia} chosen in S_i.

C2. Let S_i, S_j and S_k be any three sets, where they are not necessarily distinct. Let the association of $(x_{ia}, x_{jb})\in S_i\times S_j$ be α, where $\alpha\in\Pi^{i,j}$. Then the number of objects x_{kc} $(\in S_k)$, which satisfies that the associations of (x_{ia}, x_{kc}) and of (x_{kc}, x_{jb}) are respectively β and γ, is $q(i, j, \alpha; k, \beta, \gamma)$ which is dependent only on i, j, α, k, β and γ, where $\beta\in\Pi^{i,k}$ and $\gamma\in\Pi^{k,j}$.

Consider an association $\alpha\in\Pi^{i,j}$. Let $A_\alpha^{i,j}\in\boldsymbol{R}(S_i, S_j)$ be the adjacency matrix defined by

$$A_\alpha^{i,j}(x_{ia}, x_{jb}) = \begin{cases} 1 & \text{if the association of } (x_{ia}, x_{jb}) \text{ is } \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $A_\alpha^{i,j}=[A_\alpha^{i,j}(x_{ia}, x_{jb})]$. Let $D_\alpha^{i,j}=[D_\alpha^{i,j}(x, x^*)]\in\boldsymbol{R}(\cup_{i=1}^p S_i, \cup_{i=1}^p S_i)$ be the relation matrix defined by

$$D_\alpha^{i,j}(x, x^*) = \begin{cases} 1 & \text{if } (x, x^*)\in S_i\times S_j \text{ and the association of } (x, x^*) \text{ is } \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

LEMMA 5.1. *The matrices $A_\alpha^{i,j}$ and $D_\alpha^{i,j}$ satisfy the following:*

(I) $A_\alpha^{i,j}\boldsymbol{j}_{n_j} = n_\alpha^{i,j}\boldsymbol{j}_{n_i}$ *for* $\alpha\in\Pi^{i,j}$.

(II) $\sum_{\alpha\in\Pi^{i,j}} A_\alpha^{i,j} = G_{n_i,n_j}$.
(III) $A_\beta^{i,k} A_\gamma^{k,j} = \sum_{\alpha\in\Pi^{i,j}} q(i,j,\alpha;k,\beta,\gamma) A_\alpha^{i,j}$ *for* $\beta\in\Pi^{i,k}$ *and* $\gamma\in\Pi^{k,j}$.
(IV) $\sum_{i,j}\sum_{\alpha\in\Pi^{i,j}} D_\alpha^{i,j} = G_{a,a}$ *where* $a=\sum n_i$.
(V) $D_\beta^{i,k} D_\gamma^{k^*,j} = \delta_{k,k^*}\sum_{\alpha\in\Pi^{i,j}} q(i,j,\alpha;k,\beta,\gamma) D_\alpha^{i,j}$ *for* $\beta\in\Pi^{i,k}$ *and* $\gamma\in\Pi^{k^*,j}$.

LEMMA 5.2. *The linear closure $\mathscr{B}$ of $D_\alpha^{i,j}$ ($\alpha\in\Pi^{i,j}$; $i, j=1,\dots, p$) is a matrix algebra.*

PROOF. Lemma 5.1 (V) shows that $AB\in\mathscr{B}$ if A and B are contained in $\mathscr{B}$. Therefore $\mathscr{B}$ is a matrix algebra.

Consider an $r^m\times s^n$-FF design with unknown effects Θ, where $\Theta=\Theta_1$ or Θ_2. A multidimensional relationship is defined in Θ as follows:
The set of all effects $\{\theta(\boldsymbol{\varepsilon})\mid\boldsymbol{\varepsilon}=(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n), \xi_i\in Z_r, \zeta_j\in Z_s\}$ is partitioned into $\cup S_{\boldsymbol{p},\boldsymbol{q}}$, where

$$S_{\boldsymbol{p},\boldsymbol{q}} = \{\theta(\boldsymbol{\varepsilon})\mid \boldsymbol{\varepsilon}=(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n)=(\boldsymbol{\xi},\boldsymbol{\zeta}), \boldsymbol{w}_1(\boldsymbol{\xi})=\boldsymbol{p}, \boldsymbol{w}_2(\boldsymbol{\zeta})=\boldsymbol{q}\}$$

and $\boldsymbol{p}=(p_0\cdots p_{r-1})$, $\boldsymbol{q}=(q_0\cdots q_{s-1})$. Here p_i and q_j are non-negative integers satisfying $\sum p_i=m$ and $\sum q_j=n$. The set $S_{\boldsymbol{p},\boldsymbol{q}}$ has $m!n!/(p_0!\cdots p_{r-1}!q_0!\cdots q_{s-1}!)$ ($=n_{\boldsymbol{p},\boldsymbol{q}}$, say) elements. Let $S_{\boldsymbol{r}}\times S_{\boldsymbol{r}^*}$ be a subset of $\Theta\times\Theta$, where $\boldsymbol{r}=(\boldsymbol{p},\boldsymbol{q})$ and $\boldsymbol{r}^*=(\boldsymbol{p}^*,\boldsymbol{q}^*)$. Let

$$\Pi^{\boldsymbol{r},\boldsymbol{r}^*} = \left\{W=(U,V)\,\middle|\,\begin{array}{l}U: r\times r, j_r'U'=\boldsymbol{p}, j_r'U=\boldsymbol{p}^*\\ V: s\times s, j_s'V'=\boldsymbol{q}, j_s'V=\boldsymbol{q}^*\end{array}\right\},$$

where all entries of $U=[u(i,i^*)]$ and $V=[v(j,j^*)]$ are non-negative integers for $i, i^*\in Z_r$ and $j, j^*\in Z_s$. An association of $(\theta(\boldsymbol{\varepsilon}),\theta(\boldsymbol{\varepsilon}^*))\in S_{\boldsymbol{r}}\times S_{\boldsymbol{r}^*}$ is defined by $W=(U,V)\in\Pi^{\boldsymbol{r},\boldsymbol{r}^*}$ if $u(i,i^*)=|\{u\in\mathfrak{m}\mid\xi_u=i, \xi_u^*=i^*\}|$ and $v(j,j^*)=|\{v\in\mathfrak{n}\mid\zeta_v=j, \zeta_v^*=j^*\}|$ for any $i, i^*\in Z_r$ and $j, j^*\in Z_s$, where $\boldsymbol{\varepsilon}=(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n)$ and $\boldsymbol{\varepsilon}^*=(\xi_1^*\cdots\xi_m^*, \zeta_1^*\cdots\zeta_n^*)$. It can be shown that the associations, defined in the set Θ, satisfy C1 and C2. Put $\mathscr{R}=\{\Pi^{\boldsymbol{r},\boldsymbol{r}^*}\}$.

THEOREM 5.3. *The pair $(\Theta,\mathscr{R})$ is a multidimensional relationship and the algebra $\mathscr{B}$ generated by all relation matrices contains the unit matrix I.*

PROOF. It follows that $A_W^{\boldsymbol{r},\boldsymbol{r}}=I_{n_{\boldsymbol{r}}}$ if $W=(\mathrm{diag}(\boldsymbol{p}), \mathrm{diag}(\boldsymbol{q}))$, where $\boldsymbol{r}=(\boldsymbol{p},\boldsymbol{q})$. Therefore $\mathscr{B}$ contains I.

The parameters of the associations are given below:

$$n_{\boldsymbol{r}} = |S_{\boldsymbol{r}}| = m!n!/(p_0!\cdots p_{r-1}!q_0!\cdots q_{s-1}!),\quad n^{\boldsymbol{r},\boldsymbol{r}^*}=|\Pi^{\boldsymbol{r},\boldsymbol{r}^*}|,$$

$$n_W^{\boldsymbol{r},\boldsymbol{r}^*} = \{\textstyle\prod_{i=0}^{r-1} p_i!/(u(i,0)!\cdots u(i,r-1)!)\}\{\prod_{j=0}^{s-1} q_j!/(v(j,0)!\cdots v(j,s-1)!)\},$$

$$q(\boldsymbol{r},\boldsymbol{r}^*,W_1;\boldsymbol{r}^{**},W_2,W_3)$$

$$= \{\sum_x \prod_{i,i^*=0}^{r-1} u_1(i, i^*)!/(x_{i0i^*}!\cdots x_{ir-1i^*}!)\}$$
$$\cdot\{\sum_y \prod_{j,j^*=0}^{s-1} v_1(j, j^*)!/(y_{j0j^*}!\cdots y_{js-1j^*}!)\},$$

where $\sum_x$ extends over all non-negative integers x_{ijk} $(0\leqq i, j, k\leqq r-1)$ such that $\sum_i x_{ijk}=u_3(j, k)$, $\sum_j x_{ijk}=u_1(i, k)$ and $\sum_k x_{ijk}=u_2(i, j)$, and $\sum_y$ extends over all non-negative integers y_{ijk} $(0\leqq i, j, k\leqq s-1)$ such that $\sum_i y_{ijk}=v_3(j, k)$, $\sum_j y_{ijk}=v_1(i, k)$ and $\sum_k y_{ijk}=v_2(i, j)$. Here $\boldsymbol{r}=(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$, $\boldsymbol{r}^*=(p_0^*\cdots p_{r-1}^*, q_0^*\cdots q_{s-1}^*)$, $\boldsymbol{r}^{**}=(p_0^{**}\cdots p_{r-1}^{**}, q_0^{**}\cdots q_{s-1}^{**})$, and $W_l=([u_l(i, i^*)], [v_l(j, j^*)])$ $(0\leqq i, i^*\leqq r-1, 0\leqq j, j^*\leqq s-1; l=1, 2, 3)$.

Note that M_T (or M_T^{-1}) is balanced, i.e., $M_T=M_{T^{\tau,\rho}}$ for any $(\tau, \rho)\in\mathfrak{S}_m\times\mathfrak{S}_n$, if and only if M_T (or M_T^{-1}) is contained in $\mathscr{B}$, since a maximal invariant with respect to $\mathfrak{S}_m\times\mathfrak{S}_n$ is $W=(U, V)$.

In this case, we have the following

THEOREM 5.4. *Let T be an $r^m\times s^n$-FF design of resolution $2\ell+1$ or $(2\ell_r+1, 2\ell_s+1)$. Then the following conditions are equivalent each other*:

(i) *T is an asymmetrical balanced array.*

(ii) *M_T is balanced.*

(iii) *M_T^{-1} is balanced.*

PROOF. Theorems 4.1 and 4.2 show that conditions (i) and (ii) are equivalent. Since $\mathscr{B}$ is a matrix algebra with the unit matrix I, it follows that $M_T\in\mathscr{B}$ is equivalent to $M_T^{-1}\in\mathscr{B}$.

6. Constructions of asymmetrical balanced arrays

Srivastava [31] gave a necessary and sufficient condition for the existence of a balanced array $[N, m, 2, t]$ by solving some linear program when $m=t+1$ and $t+2$. His method can be extended to the general case $m=t+l$ $(l\geqq 3)$. But it is difficult to solve its linear program when l is large. For practical use, we may only consider a simple array, named by Shirakura [29] in 2^m factorials. We now construct an asymmetrical balanced array derivable from a balanced array.

M1. Simple array method.

Let $\Omega(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$ be a matrix of size $\{m!n!/(p_0!\cdots p_{r-1}!q_0!\cdots q_{s-1}!)\}\times(m+n)$ whose all row vectors are different each other and each row vector $(f_1\cdots f_m, g_1\cdots g_n)$ has the same weights $\boldsymbol{w}_1(f_1\cdots f_m)=(p_0\cdots p_{r-1})$ and $\boldsymbol{w}_2(g_1\cdots g_n)=(q_0\cdots q_{s-1})$, where p_i and q_j are non-negative integers satisfying $\sum p_i=m$ and $\sum q_j=n$. The row vectors of $\Omega(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$ are considered as assemblies of an $r^m\times s^n$-FF design. T is called a simple array with index set $\{\lambda(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})\mid p_i\geqq 0, q_j\geqq 0, \sum p_i=m, \sum q_j=n\}$ if T is composed of

$\Omega(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})\,\lambda(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$-times for any $(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$. Then T is an ABA1$[N, (m, n), (r, s), m+n]$ with index set $\{\lambda(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})\}$.

M2. Direct concatenation method

Let F be a balanced array $[N, m, r, t]$ and G be a balanced array $[N^*, n, s, t^*]$. Then T, defined by the direct concatenation of F and G, is an ABA2$[NN^*, (m, n), (r, s), (t, t^*)]$. Indices of T, $\lambda(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$, are given by $\lambda_F(p_0\cdots p_{r-1})\lambda_G(q_0\cdots q_{s-1})$, where $\lambda_F(\cdot)$ and $\lambda_G(\cdot)$ are indices of F and G, respectively. Now the direct concatenation T of $F=[f_{ij}]$ and $G=[g_{kl}]$ is defined by the matrix of size $NN^*\times(m+n)$ whose rows are given by $(f_{i1},\dots,f_{im}, g_{k1},\dots, g_{kn})$ $(1\leqq i\leqq N, 1\leqq k\leqq N^*)$.

M3. Reduction method

Let ϕ and ψ be mappings from Z_{s^*} into Z_r and from Z_{s^*} into Z_s, respectively. Let T^* be a BA$[N, m^*, s^*, t]$ for $m^*=m+n$. Partitioning T^* as $[F^*: G^*]$ ($F^*=[f_{ij}]$ and $G^*=[g_{kl}]$ are matrices of size $N\times m$ and $N\times n$, respectively), T is defined by $[\phi(F^*): \psi(G^*)]$, where $\phi(F^*)$ and $\psi(G^*)$ are derived from ϕ and ψ, i.e., $\phi(F^*)$ is a matrix of size $N\times m$ whose i-th row is given by $(\phi(f_{i1}),\dots, \phi(f_{im}))$ and $\psi(G^*)$ is a matrix of size $N\times m$ whose i-th row is given by $(\psi(g_{i1}),\dots, \psi(g_{in}))$. Then T is an ABA1$[N, (m, n), (r, s), t]$.

M4. Fold-over method

Let $T^*=[F^*: G^*]$ be an ABA1$[N, (m, n), (r, s), t]$. Let mappings $\phi: Z_r\to Z_r$ and $\psi: Z_s\to Z_s$ such that $\phi(i)=r-1-i$ and $\psi(j)=s-1-j$ for any $i\in Z_r$ and $j\in Z_s$. Then the $2N\times(m+n)$ matrix $\begin{bmatrix} F^* & : & G^* \\ \phi(F^*) & : & \psi(G^*)\end{bmatrix}$ is also an ABA1$[2N, (m, n), (r, s), t]$, which, further, is a fold-over design. Note that the definition of a fold-over design was given by Box and Wilson [3].

7. Notations of the associations

We consider the multidimensional relationship algebra defined in the set of effects $\Theta_1^*=\{\theta(\boldsymbol{\varepsilon})\mid \boldsymbol{\varepsilon}=(\xi_1\cdots\xi_m, \zeta_1\cdots\zeta_n), w(\boldsymbol{\varepsilon})\leqq 2\}$ i.e., Θ_1^* is the set of effects up to two-factor interactions.

Let $W=(U, V)$ be an association defined in the set $S_{\boldsymbol{r}}\times S_{\boldsymbol{r}^*}\subset\Theta_1^*\times\Theta_1^*$, where $\boldsymbol{r}=(\boldsymbol{p}, \boldsymbol{q})=(p_0\cdots p_{r-1}, q_0\cdots q_{s-1})$, $\boldsymbol{r}^*=(\boldsymbol{p}^*, \boldsymbol{q}^*)=(p_0^*\cdots p_{r-1}^*, q_0^*\cdots q_{s-1}^*)$, $U=[u_{ij}]$ $(i, j\in Z_r)$ and $V=[v_{kl}]$ $(k, l\in Z_s)$. Here $p_i, p_i^*, u_{ij}, q_k, q_k^*, v_{kl}$ are non-negative integers satisfying

$$\begin{aligned}(7.1)\quad &\textstyle\sum_{i=0}^{r-1}p_i = \sum_{i=0}^{r-1}p_i^* = m,\quad \sum_{k=0}^{s-1}q_k = \sum_{k=0}^{s-1}q_k^* = n,\quad \sum_{i=1}^{r-1}p_i + \sum_{k=1}^{s-1}q_k \\ &\leqq 2,\quad \textstyle\sum_{i=1}^{r-1}p_i^* + \sum_{k=1}^{s-1}q_k^* \leqq 2,\quad \sum_{j=0}^{r-1}u_{ij} = p_i,\ \sum_{i=0}^{r-1}u_{ij} = p_j^*, \\ &\textstyle\sum_{l=0}^{s-1}v_{kl} = q_k \quad\text{and}\quad \sum_{k=0}^{s-1}v_{kl} = q_l^*\,.\end{aligned}$$

Hence the number of non-zero elements among p_i $(1 \leqq i \leqq r-1)$ or p_i^* $(1 \leqq i \leqq r-1)$ is at most two. According to the non-zero elements of $\boldsymbol{p}, \boldsymbol{p}^*$ and U, the matrix U will be denoted as follows:

$$U = \begin{cases} u_0 & \text{if } p_0 = p_0^* = u_{00} = m, \\ u(0, k; 0) & \text{if } p_0 = m,\ p_0^* = m-1,\ p_k^* = 1,\ u_{00} = m-1,\ u_{0k} = 1, \\ u(0, kk; 0) & \text{if } p_0 = m,\ p_0^* = m-2,\ p_k^* = 2,\ u_{00} = m-2,\ u_{0k} = 2, \\ u(0, kl; 0) & \text{if } p_0 = m,\ p_0^* = m-2,\ p_k^* = p_l^* = 1,\ u_{00} = m-2,\ u_{0k} = u_{0l} = 1, \\ u(i, k; 0) & \text{if } p_0 = p_0^* = m-1,\ p_i = p_k^* = 1,\ u_{00} = m-1,\ u_{ik} = 1, \\ u(i, k; 1) & \text{if } p_0 = p_0^* = m-1,\ p_i = p_k^* = 1,\ u_{00} = m-2,\ u_{i0} = u_{0k} = 1, \\ u(i, kk; 0) & \text{if } p_0 = m-1,\ p_i = 1,\ p_0^* = m-2,\ p_k^* = 2,\ u_{00} = m-2, \\ & \qquad u_{0k} = u_{ik} = 1, \\ u(i, kk; 1) & \text{if } p_0 = m-1,\ p_i = 1,\ p_0^* = m-2,\ p_k^* = 2,\ u_{00} = m-2, \\ & \qquad u_{0k} = 2,\ u_{i0} = 1, \\ u(i, kl; 0) & \text{if } p_0 = m-1,\ p_0^* = m-2,\ p_i = p_k^* = p_l^* = 1,\ u_{00} = m-2, \\ & \qquad u_{0l} = u_{ik} = 1, \\ u(i, kl; 1) & \text{if } p_0 = m-1,\ p_0^* = m-2,\ p_i = p_k^* = p_l^* = 1,\ u_{00} = m-2, \\ & \qquad u_{0k} = u_{il} = 1, \\ u(i, kl; 2) & \text{if } p_0 = m-1,\ p_0^* = m-2,\ p_i = p_k^* = p_l^* = 1,\ u_{00} = m-3, \\ & \qquad u_{0k} = u_{0l} = u_{i0} = 1, \\ u(ii, kk; 0) & \text{if } p_0 = p_0^* = m-2,\ p_i = p_k^* = 2,\ u_{00} = m-2,\ u_{ik} = 2, \\ u(ii, kk; 1) & \text{if } p_0 = p_0^* = m-2,\ p_i = p_k^* = 2,\ u_{00} = m-3, \\ & \qquad u_{i0} = u_{0k} = u_{ik} = 1, \\ u(ii, kk; 2) & \text{if } p_0 = p_0^* = m-2,\ p_i = p_k^* = 2,\ u_{00} = m-2,\ u_{i0} = u_{0k} = 2, \\ u(ii, kl; 0) & \text{if } p_0 = p_0^* = m-2,\ p_i = 2,\ p_k^* = p_l^* = 1,\ u_{00} = m-2, \\ & \qquad u_{ik} = u_{il} = 1, \\ u(ii, kl; 1) & \text{if } p_0 = p_0^* = m-2,\ p_i = 2,\ p_k^* = p_l^* = 1,\ u_{00} = m-3, \\ & \qquad u_{0l} = u_{i0} = u_{ik} = 1, \\ u(ii, kl; 2) & \text{if } p_0 = p_0^* = m-2,\ p_i = 2,\ p_k^* = p_l^* = 1,\ u_{00} = m-3, \\ & \qquad u_{0k} = u_{i0} = u_{il} = 1, \\ u(ii, kl; 3) & \text{if } p_0 = p_0^* = m-2,\ p_i = 2,\ p_k^* = p_l^* = 1,\ u_{00} = m-4, \\ & \qquad u_{i0} = 2,\ u_{0k} = u_{0l} = 1, \end{cases}$$

$$
\begin{cases}
u(ij, kl; 0) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-4, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{ik} = u_{0l} = 1, \\
u(ij, kl; 1) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-2, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{ik} = u_{jl} = 1, \\
u(ij, kl; 2) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-3, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{0l} = u_{ik} = u_{j0} = 1, \\
u(ij, kl; 3) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-3, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{0k} = u_{i0} = u_{jl} = 1, \\
u(ij, kl; 4) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-3, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{0k} = u_{il} = u_{j0} = 1, \\
u(ij, kl; 5) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-3, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{0l} = u_{i0} = u_{jk} = 1, \\
u(ij, kl; 6) \text{ if } p_0 = p_0^* = m-2,\ p_i = p_j = p_k^* = p_l^* = 1,\ u_{00} = m-4, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{0k} = u_{0l} = u_{i0} = u_{j0} = 1,
\end{cases}
$$

where $1 \leqq i, j \leqq r-1$, $i<j$, $1 \leqq k$, $l \leqq r-1$, $k<l$.
Furthermore, the transposed matrix of $u(x, y; \cdot)$ in the above will be denoted by $u(y, x; \cdot)$ for $(x, y)=(0, k)$, $(0, kk)$, $(0, kl)$, (i, kk), (i, kl) and (ii, kl).

The notation on V is defined by changing U, u, p, m and r into V, v, q, n and s, respectively. These matrix notations on U and V will be used from now on. (We notice that these notations are different from those used in Kuwada and Nishii [25].)

8. Irreducible representations of M_T with effects Θ_1^*

We consider an $r^m \times s^n$-BFF design with unknown effects

$$\Theta_1^* = \{\theta(\boldsymbol{\varepsilon}) \mid w(\boldsymbol{\varepsilon}) \leqq 2\}.$$

THEOREM 8.1. *The algebra $\mathscr{B}$ generated by the relation matrices $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ of size $v_1^* \times v_1^*$ is a semi-simple, completely reducible matrix algebra, where* $v_1^* = \sum_{i+j \leqq 2} \binom{m}{i} \binom{n}{j} (r-1)^i (s-1)^j$.

PROOF. Let $B(\boldsymbol{r}, \boldsymbol{r}^*; W)$ be a symmetric matrix of size $v_1^* \times v_1^*$ defined as follows:

$$B(\boldsymbol{r}, \boldsymbol{r}^*; W) = \begin{cases} D(\boldsymbol{r}, \boldsymbol{r}^*; W) & \text{if } U' = U \text{ and } V' = V, \\ D(\boldsymbol{r}, \boldsymbol{r}^*; W) + D(\boldsymbol{r}, \boldsymbol{r}^*; W') & \text{otherwise,} \end{cases}$$

where $W=(U, V)$ is the ordered pair of matrices U and V, and $W'=(U', V')$. Then $\mathscr{B}$ is generated by symmetric matrices $B(\boldsymbol{r}, \boldsymbol{r}^*; W)$. This completes the proof.

We can represent $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ by the linear combination of $D^\sharp_\alpha(\boldsymbol{r}, \boldsymbol{r}^*)$ which are the basis of two-sided ideals of $\mathscr{B}$ (see Kuwada [21], Kuwada and Nishii [25]). In fact we have the following relations between $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ and $D^\sharp_\cdot(\boldsymbol{r}, \boldsymbol{r}^*)$, where $W=(U, V)$, $\boldsymbol{r}=(\boldsymbol{p}, \boldsymbol{q})$ and $\boldsymbol{r}^*=(\boldsymbol{p}^*, \boldsymbol{q}^*)$. (We use the notation $D^\sharp_\cdot$ instead of $D^\sharp_\cdot(\boldsymbol{r}, \boldsymbol{r}^*)$, for brevity.)

In the case $V=v_0$ (i.e., $\boldsymbol{q}=\boldsymbol{q}^*=(n, 0,\dots, 0)$),

$D(\boldsymbol{r}, \boldsymbol{r}^*; W)$

$$= \begin{cases} D^\sharp_0 & \text{if } U=u_0, \\ m^{1/2}D^\sharp_0 & \text{if } U=u(0, k; 0), \\ \binom{m}{2}^{1/2}D^\sharp_0 & \text{if } U=u(0, kk; 0), \\ \left\{2\binom{m}{2}\right\}^{1/2}D^\sharp_0 & \text{if } U=u(0, kl; 0), \\ (8.1)\begin{cases} D^\sharp_0+D^\sharp_{f_{11}} & \text{if } U=u(i, k; 0), \\ (m-1)D^\sharp_0-D^\sharp_{f_{11}} & \text{if } U=u(i, k; 1), \\ \{2(m-1)\}^{1/2}D^\sharp_0+(m-2)^{1/2}D^\sharp_{f_{12}} & \text{if } U=u(i, kk; 0), \\ (m-2)\{(m-1)/2\}^{1/2}D^\sharp_0-(m-2)^{1/2}D^\sharp_{f_{12}} & \text{if } U=u(i, kk; 1), \\ (m-1)^{1/2}D^\sharp_0+(m/2)^{1/2}D^\sharp_{f_{13}}+\{(m-2)/2\}^{1/2}D^\sharp_{f_{14}} & \\ & \text{if } U=u(i, kl; 0), \\ (m-1)^{1/2}D^\sharp_0-(m/2)^{1/2}D^\sharp_{f_{13}}+\{(m-2)/2\}^{1/2}D^\sharp_{f_{14}} & \\ & \text{if } U=u(i, kl; 1), \\ (m-2)(m-1)^{1/2}D^\sharp_0-\{2(m-2)\}^{1/2}D^\sharp_{f_{14}} & \text{if } U=u(i, kl; 2), \end{cases} \\ D^\sharp_0+D^\sharp_1+D^\sharp_{f_{22}} & \text{if } U=u(ii, kk; 0), \\ 2(m-2)D^\sharp_0-2D^\sharp_1+(m-4)D^\sharp_{f_{22}} & \text{if } U=u(ii, kk; 1), \\ \binom{m-2}{2}D^\sharp_0+D^\sharp_1-(m-3)D^\sharp_{f_{22}} & \text{if } U=u(ii, kk; 2), \\ 2^{1/2}[D^\sharp_0+D^\sharp_1+D^\sharp_{f_{24}}] & \text{if } U=u(ii, kl; 0), \end{cases}$$

$$\begin{cases}
(1/2)^{1/2}[2(m-2)D_0^\# - 2D_1^\# + \{m(m-2)\}^{1/2}D_{f_{23}}^\# + (m-4)D_{f_{24}}^\#] & \text{if } U = u(ii,\, kl;\, 1), \\
(1/2)^{1/2}[2(m-2)D_0^\# - 2D_1^\# - \{m(m-2)\}^{1/2}D_{f_{23}}^\# + (m-4)D_{f_{24}}^\#] & \text{if } U = u(ii,\, kl;\, 2), \\
2^{1/2}\left[\binom{m-2}{2}D_0^\# + D_1^\# - (m-3)D_{f_{24}}^\#\right] & \text{if } U = u(ii,\, kl;\, 3), \\
D_0^\# + D_1^\# + D_2^\# + D_{f_{33}}^\# + D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 0), \\
D_0^\# + D_1^\# - D_2^\# - D_{f_{33}}^\# + D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 1), \\
(m-2)D_0^\# - D_1^\# - D_2^\# + (m-2)/2D_{f_{33}}^\# + (1/2)\{m(m-2)\}^{1/2}[D_{f_{34}}^\# + D_{f_{43}}^\#] & \\
\qquad + (m-4)/2D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 2), \\
(m-2)D_0^\# - D_1^\# - D_2^\# + (m-2)/2D_{f_{22}}^\# - (1/2)\{m(m-2)\}^{1/2}[D_{f_{34}}^\# + D_{f_{43}}^\#] & \\
\qquad + (m-4)/2D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 3), \\
(m-2)D_0^\# - D_1^\# + D_2^\# - (m-2)/2D_{f_{33}}^\# + (1/2)\{m(m-2)\}^{1/2}[D_{f_{34}}^\# - D_{f_{43}}^\#] & \\
\qquad + (m-4)/2D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 4), \\
(m-2)D_0^\# - D_1^\# + D_2^\# - (m-2)/2D_{f_{33}}^\# - (1/2)\{m(m-2)\}^{1/2}[D_{f_{34}}^\# - D_{f_{43}}^\#] & \\
\qquad + (m-4)/2D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 5), \\
2\binom{m-2}{2}D_0^\# + 2D_1^\# - 2(m-3)D_{f_{44}}^\# & \text{if } U = u(ij,\, kl;\, 6).
\end{cases}$$

In the case $U=u_0$ (i.e., $\boldsymbol{p}=\boldsymbol{p}^*=(m, 0,\dots, 0)$), $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ is expressed by the linear combination of $D_\cdot^\#(\boldsymbol{r}, \boldsymbol{r}^*)$ by changing U, u, p, f, r, m, D_1 and D_2 into V, v, q, g, s, n, D_3 and D_4, respectively. For example,

$$D(\boldsymbol{r}, \boldsymbol{r}^*; W) = 2(n-2)D_0^\# - 2D_3^\# + (n-4)D_{g_{22}}^\# \text{ if } U = u_0 \text{ and } V = v(ii,\, kk;\, 1).$$

In the case $U=u(*,\, 0;\, 0)$ and $V=v(0,\, *;\, 0)$,

$$D(\boldsymbol{r}, \boldsymbol{r}^*; W)$$

$$\begin{cases}
(mn)^{1/2}D_0^\# & \text{if } U = u(i,\, 0;\, 0),\ V = v(0,\, k;\, 0), \\
\left\{\binom{m}{2}n\right\}^{1/2}D_0^\# & \text{if } U = u(ii,\, 0;\, 0),\ V = v(0,\, k;\, 0), \\
\left\{2\binom{m}{2}n\right\}^{1/2}D_0^\# & \text{if } U = u(ij,\, 0;\, 0),\ V = v(0,\, k;\, 0), \\
\left\{m\binom{n}{2}\right\}^{1/2}D_0^\# & \text{if } U = u(i,\, 0;\, 0),\ V = v(0,\, kk;\, 0),
\end{cases}$$

$$= \begin{cases} \left\{\binom{m}{2}\binom{n}{2}\right\}^{1/2} D_0^\# & \text{if} \quad U = u(ii, 0; 0), \ V = v(0, kk; 0), \\ \left\{2\binom{m}{2}\binom{n}{2}\right\}^{1/2} D_0^\# & \text{if} \quad U = u(ij, 0; 0), \ V = v(0, kk; 0), \\ \left\{2m\binom{n}{2}\right\}^{1/2} D_0^\# & \text{if} \quad U = u(i, 0; 0), \ V = v(0, kl; 0), \\ \left\{2\binom{m}{2}\binom{n}{2}\right\}^{1/2} D_0^\# & \text{if} \quad U = u(ii, 0; 0), \ V = v(0, kl; 0), \\ \left\{4\binom{m}{2}\binom{n}{2}\right\}^{1/2} D_0^\# & \text{if} \quad U = u(ij, 0; 0), \ V = v(0, kl; 0). \end{cases}$$

In the case $V=v(j, 0; 0)$, we get $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ by multiplying $n^{1/2}$ to (8.1). For example,

$$D(\boldsymbol{r}, \boldsymbol{r}^*; W) = n^{1/2}[D_0^\# + D_{f_{11}}^\#] \quad \text{if} \quad U = u(i, k; 0) \quad \text{and} \quad V = v(j, 0; 0).$$

In the case $U=u(j, 0; 0)$, we get $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ by changing m, n, p, U and u, which are contained in the terms given by multiplying $n^{1/2}$ to (8.1), into n, m, q, V and v, respectively. For example,

$$D(\boldsymbol{r}, \boldsymbol{r}^*; W) = \{2m(n-1)\}^{1/2} D_0^\# + \{m(n-2)\}^{1/2} D_{g_{12}}^\# \quad \text{if} \quad U = u(j, 0; 0) \quad \text{and} \quad V = v(i, kk; 0).$$

In the case $U=u(i, j; \delta_u)$ and $V=v(k, l; \delta_v)$ $(\delta_u, \delta_v = 0, 1)$,

$$D(\boldsymbol{r}, \boldsymbol{r}^*; W)$$
$$= \begin{cases} D_0^\# + D_{f_{11}}^\# + D_{g_{11}}^\# + D_5^\# & \text{if} \quad \delta_u = \delta_v = 0, \\ (m-1)[D_0^\# + D_{g_{11}}^\#] - [D_{f_{11}}^\# + D_5^\#] & \text{if} \quad \delta_u = 1 \quad \text{and} \quad \delta_v = 0, \\ (n-1)[D_0^\# + D_{f_{11}}^\#] - [D_{g_{11}}^\# + D_5^\#] & \text{if} \quad \delta_u = 0 \quad \text{and} \quad \delta_v = 1, \\ (m-1)(n-1)D_0^\# - (m-1)D_{g_{11}}^\# - (n-1)D_{f_{11}}^\# + D_5^\# & \text{if} \quad \delta_u = \delta_v = 1. \end{cases}$$

In the case $U=u(0, j; 0)$ and $V=v(0, l; 0)$, it holds that $D(\boldsymbol{r}, \boldsymbol{r}^*; W) = (mn)^{1/2} D_0^\#$, where $1 \leqq j \leqq r-1$ and $1 \leqq l \leqq s-1$.

Let $D_\alpha^\#(\boldsymbol{r}^*, \boldsymbol{r}) = D_\alpha^\#(\boldsymbol{r}, \boldsymbol{r}^*)'$ $(\alpha = 0, 1, \ldots, 5)$, $D_{f_{ij}}^\#(\boldsymbol{r}^*, \boldsymbol{r}) = D_{f_{ji}}^\#(\boldsymbol{r}, \boldsymbol{r}^*)'$ and $D_{g_{ij}}^\#(\boldsymbol{r}, \boldsymbol{r}^*) = D_{g_{ji}}^\#(\boldsymbol{r}, \boldsymbol{r}^*)'$, where $D_\cdot^\#(\boldsymbol{r}, \boldsymbol{r}^*)$ are matrices appear in the above relation. Note that $D(\boldsymbol{r}, \boldsymbol{r}^*; W)' = D(\boldsymbol{r}^*, \boldsymbol{r}; W')$. Then any of the relation matrices $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ is expressed by a linear combination of $D_\cdot^\#(\boldsymbol{r}, \boldsymbol{r}^*)$.

Let $\mathscr{B}_\alpha$, $\mathscr{B}_f$ and $\mathscr{B}_g$ be the linear closures $[D_\alpha^\#(\boldsymbol{r}, \boldsymbol{r}^*)]$, $[D_{f_{ij}}^\#(\boldsymbol{r}, \boldsymbol{r}^*)]$ and $[D_{g_{ij}}^\#(\boldsymbol{r}, \boldsymbol{r}^*)]$, respectively, for $\alpha = 0, 1, \ldots, 5$. These ideals satisfy the following theorem and we omit its proof.

THEOREM 8.2. (i) $\mathscr{B}_\alpha \mathscr{B}_\beta = \delta_{\alpha,\beta} \mathscr{B}_\alpha$ for $\alpha, \beta = 0, 1, \ldots, 5, f, g$. (ii) *The multi-dimensional relationship algebra is decomposed into the direct sum of eight two-sided ideals* $\mathscr{B}_\alpha$ ($\alpha = 0, 1, \ldots, 5, f, g$), *i.e.*,

$$\mathscr{B} = \mathscr{B}_0 \oplus \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_5 \oplus \mathscr{B}_f \oplus \mathscr{B}_g .$$

(iii) $\mathscr{B}_\alpha$ *is isomorphic to the complete* $\tau_\alpha \times \tau_\alpha$ *matrix algebra for* $\alpha = 0, 1, \ldots, 5, f, g$, *where* $\tau_0 = (r+s)(r+s-1)/2$, $\tau_1 = r(r-1)/2$, $\tau_2 = (r-1)(r-2)/2$, $\tau_3 = s(s-1)/2$, $\tau_4 = (s-1)(s-2)/2$, $\tau_5 = (r-1)(s-1)$, $\tau_f = (r-1)(r+s-1)$ *and* $\tau_g = (s-1)\cdot(r+s-1)$.

(iv) *The multiplicity of the irreducible representation of* M_T *with respect to* $\mathscr{B}_\alpha$ *is* ϕ_α ($\alpha = 0, 1, \ldots, 5, f, g$), *where* $\phi_0 = 1$, $\phi_1 = m(m-3)/2$, $\phi_2 = \binom{m-1}{2}$, $\phi_3 = n(n-3)/2$, $\phi_4 = \binom{n-1}{2}$, $\phi_5 = (m-1)(n-1)$, $\phi_f = m-1$ *and* $\phi_g = n-1$.

Let T be an ABA1$[N, (m, n), (r, s), 4]$ with index set $\{\lambda(p_0 \cdots p_{r-1}, q_0 \cdots q_{s-1})\}$. Let $p(W)$ be the entry, $m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$, of M_T if an association of $(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$ is $W = (U, V)$, where $\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*) \in \Theta_1$. All $p(W)$ can be expressed by linear combinations of $\{\gamma\}$ (see Lemma 9.3 described shortly), where $\boldsymbol{\gamma}$ is given by the linear combination of $\{\lambda\}$ (see (2.4)).

The information matrix M_T is represented by $D(\boldsymbol{r}, \boldsymbol{r}^*; W)$ (see Theorem 5.4). Therefore M_T is also represented by $D^{\#}_{\cdot}(\boldsymbol{r}, \boldsymbol{r}^*)$ as

$$\begin{aligned} M_T &= \textstyle\sum_W p(W) D(\boldsymbol{r}, \boldsymbol{r}^*; W) \\ &= \textstyle\sum_{\boldsymbol{r},\boldsymbol{r}^*} \sum_{\alpha=0}^{5} \kappa_\alpha(\boldsymbol{r}, \boldsymbol{r}^*) D^{\#}_\alpha(\boldsymbol{r}, \boldsymbol{r}^*) + \sum_{\boldsymbol{r},\boldsymbol{r}^*} \sum_{i,j=1}^{4} \{\kappa_{f_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*) D^{\#}_{f_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*) \\ &\qquad + \kappa_{g_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*) D^{\#}_{g_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*)\} . \end{aligned}$$

Here we recall the fact that $\boldsymbol{r}$ and $\boldsymbol{r}^*$ are represented by U and V (see (7.1)). Then $\kappa_\alpha(\boldsymbol{r}, \boldsymbol{r}^*)$ are given as follows:

In the case $V = v_0$,

$\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*)$

$$(8.2)\quad \left\{ \begin{array}{l} p(u_0, v_0), \\ m^{1/2} p(u(0, k; 0), v_0), \\ \binom{m}{2}^{1/2} p(u(0, kk; 0), v_0), \\ \left\{2\binom{m}{2}\right\}^{1/2} p(u(0, kl; 0), v_0), \\ p(u(i, k; 0), v_0) + (m-1) p(u(i, k; 1), v_0), \\ \{(m-1)/2\}^{1/2} \{2p(u(i, kk; 0), v_0) + (m-2) p(u(i, kk; 1), v_0)\}, \end{array} \right.$$

$$
= \begin{cases} \quad \Big\{ (m-1)^{1/2}\{p(u(i, kl; 0), v_0) + p(u(i, kl; 1), v_0)\} \\ \qquad\qquad + (m-2)p(u(i, kl; 2), v_0), \\ p(u(ii, kk; 0), v_0) + 2(m-2)p(u(ii, kk; 1), v_0) + \binom{m-2}{2}p(u(ii, kk; 2), v_0), \\ 2^{1/2}\Big[p(u(ii, kl; 0), v_0) + (m-2)\{p(u(ii, kl; 1), v_0) + p(u(ii, kl; 2), v_0)\} \\ \qquad\qquad + \binom{m-2}{2}p(u(ii, kl; 3), v_0)\Big] \\ p(u(ij, kl; 4), v_0) + p(u(ij, kl; 5), v_0) + 2\binom{m-2}{2}p(u(ij, kl; 6), v_0). \end{cases}
$$

In the case $U=u_0$, $\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*)$ is expressed by $p(U, V)$ as above by changing U, u, p, r and m into V, v, q, s and n, respectively. For example,

$$
\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*) = \left\{2\binom{n}{2}\right\}^{1/2} p(u_0, v(0, kl; 0)).
$$

In the case $U=u(*, 0; 0)$ and $V=v(0, *; 0)$,

$$
\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*) = \begin{cases} (mn)^{1/2}p(u(i, 0; 0), v(0, k; 0)), \\ \left\{\binom{m}{2}n\right\}^{1/2} p(u(ii, 0; 0), v(0, k; 0)), \\ \left\{2\binom{m}{2}n\right\}^{1/2} p(u(ij, 0; 0), v(0, k; 0)), \\ \left\{\binom{m}{2}\binom{n}{2}\right\}^{1/2} p(u(ii, 0; 0), v(0, kk; 0)), \\ \left\{2\binom{m}{2}\binom{n}{2}\right\}^{1/2} p(u(ij, 0; 0), v(0, kk; 0)), \\ \left\{2m\binom{n}{2}\right\}^{1/2} p(u(i, 0; 0), v(0, kl; 0)), \\ \left\{2\binom{m}{2}\binom{n}{2}\right\}^{1/2} p(u(ii, 0; 0), v(0, kl; 0)), \\ \left\{4\binom{m}{2}\binom{n}{2}\right\}^{1/2} p(u(ij, 0; 0), v(0, kl; 0)). \end{cases}
$$

In the case $V=v(j, 0; 0)$, we get $\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*)$ by multiplying $n^{1/2}$ to (8.2). For example,

$$
\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*) = n^{1/2}\{p(u(i, k; 0), v(j, 0; 0)) + (m-1)p(u(i, k; 1), v(j, 0; 0))\}.
$$

In the case $U=u(0, j, 0)$, we get $\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*)$ by changing m, n, p, U and u, which are contained in the terms given by multiplying $n^{1/2}$ to (8.2), into n, m, q, V and v, respectively.

$$\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*) = \{m(n-1)/2\}^{1/2}\{2p(u(0, j;\, 0), v(i, kk;\, 0)) + (n-2)p(u(0, j;\, 0), v(i, kk;\, 1))\}.$$

In the case $U=u(i, j;\, *)$ and $V=v(k, l;\, *)$, it holds that

$$\kappa_0(\boldsymbol{r}, \boldsymbol{r}^*) = p(u(i, j;\, 0), v(k, l;\, 0)) + (m-1)p(u(i, j;\, 1), v(k, l;\, 0)) + (n-1)p(u(i, j;\, 0), v(k, l;\, 1)) + (m-1)(n-1)p(u(i, j;\, 1), v(k, l;\, 1)).$$

$$\kappa_1(\boldsymbol{r}, \boldsymbol{r}^*) = \begin{cases} p(u(ii, kk;\, 0), v_0) - 2p(u(ii, kk;\, 1), v_0) + p(u(ii, kk;\, 2), v_0), \\ 2^{1/2}\{p(u(ii, kl;\, 0), v_0) - p(u(ii, kl;\, 1), v_0) - p(u(ii, kl;\, 2), v_0) \\ \quad + p(u(ii, kl;\, 3), v_0)\}, \\ p(u(ij, kl;\, 0), v_0) + p(u(ij, kl;\, 1), v_0) - p(u(ij, kl;\, 2), v_0) \\ \quad - p(u(ij, kl;\, 3), v_0) - p(u(ij, kl;\, 4), v_0) - p(u(ij, kl;\, 5), v_0) \\ \quad + 2p(u(ij, kl;\, 6), v_0). \end{cases}$$

$\kappa_3(\boldsymbol{r}, \boldsymbol{r}^*)$ is given in the same way as $\kappa_1(\boldsymbol{r}, \boldsymbol{r}^*)$. For example,

$$\kappa_3(\boldsymbol{r}, \boldsymbol{r}^*) = p(u_0, v(ii, kk;\, 0)) - 2p(u_0, v(ii, kk;\, 1)) + p(u_0, v(ii, kk;\, 2)).$$

$$\kappa_2(\boldsymbol{r}, \boldsymbol{r}^*) = p(u(ij, kl;\, 0), v_0) - p(u(ij, kl;\, 1), v_0) - p(u(ij, kl;\, 2), v_0) - p(u(ij, kl;\, 3), v_0) + p(u(ij, kl;\, 4), v_0) + p(u(ij, kl;\, 5), v_0).$$

$$\kappa_4(\boldsymbol{r}, {}^*\boldsymbol{r}) = p(u_0, v(ij, kl;\, 0)) - p(u_0, v(ij, kl;\, 1)) - p(u_0, v(ij, kl;\, 2)) - p(u_0, v(ij, kl;\, 3)) + p(u_0, v(ij, kl;\, 4)) + p(u_0, v(ij, kl;\, 5)).$$

$$\kappa_5(\boldsymbol{r}, \boldsymbol{r}^*) = p(u(i, j;\, 0), v(k, l;\, 0)) - p(u(i, j;\, 1), v(k, l;\, 0)) - p(u(i, j;\, 0), v(k, l;\, 1)) + p(u(i, j;\, 1), v(k, l;\, 1)).$$

$$\kappa_{f_{11}}(\boldsymbol{r}, \boldsymbol{r}^*) = \begin{cases} p(u(i, j;\, 0), v_0) - p(u(i, j;\, 1), v_0), \\ m^{1/2}\{p(u(i, j;\, 0), v(0, k;\, 0)) - p(u(i, j;\, 1), v(0, k;\, 0))\}. \end{cases}$$

$$\kappa_{f_{12}}(\boldsymbol{r}, \boldsymbol{r}^*) = (m-2)^{1/2}\{p(u(i, jj;\, 0), v_0) - p(u(i, jj;\, 1), v_0)\}.$$

$$\kappa_{f_{13}}(\boldsymbol{r}, \boldsymbol{r}^*) = (m/2)^{1/2}\{p(u(i, kl;\, 0), v_0) - p(u(i, kl;\, 1), v_0)\}.$$

$$\kappa_{f_{14}}(\boldsymbol{r}, \boldsymbol{r}^*) = \{(m-2)/2\}^{1/2}\{p(u(i, kl;\, 0), v_0) + p(u(i, kl;\, 1), v_0) - 2p(u(i, kl;\, 2), v_0)\}.$$

$$\kappa_{f_{22}}(\boldsymbol{r}, \boldsymbol{r}^*) = p(u(ii, kk;\, 0), v_0) + (m-4)p(u(ii, kk;\, 0), v_0) - (m-3)p(u(ii, kk;\, 0), v_0).$$

$$\kappa_{f_{23}}(\boldsymbol{r}, \boldsymbol{r}^*) = \{m(m-2)/2\}^{1/2}\{p(u(ii, kl;\, 1), v_0) - p(u(ii, kl;\, 2), v_0)\}.$$

$$\kappa_{f_{24}}(\boldsymbol{r}, \boldsymbol{r}^*) = (1/2)^{1/2}[2p(u(ii, kl; 0), v_0) + (m-4)\{p(u(ii, kl; 1), v_0) + p(u(ii, kl; 2), v_0)\} - 2(m-3)p(u(ii, kl; 3), v_0)].$$

$$\kappa_{f_{33}}(\boldsymbol{r}, \boldsymbol{r}^*) = (1/2)[2\{p(u(ij, kl; 0), v_0) - p(u(ij, kl; 1), v_0)\} + (m-2)\{p(u(ij, kl; 2), v_0) + p(u(ij, kl; 3), v_0) - p(u(ij, kl; 4), v_0) - p(u(ij, kl; 5), v_0)\}].$$

$$\kappa_{f_{34}}(\boldsymbol{r}, \boldsymbol{r}^*) = (1/2)\{m(m-2)\}^{1/2}\{p(u(ij, kl; 2), v_0) - p(u(ij, kl; 3), v_0)\}.$$

$$\kappa_{f_{44}}(\boldsymbol{r}, \boldsymbol{r}^*) = (1/2)[2\{p(u(ij, kl; 0), v_0) + p(u(ij, kl; 1), v_0)\} + (m-4)\{p(u(ij, kl; 2), v_0) + p(u(ij, kl; 3), v_0) + p(u(ij, kl; 4), v_0) + p(u(ij, kl; 5), v_0)\} - 4(m-3)p(u(ij, kl; 6), v_0)].$$

$\kappa_{g_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*)$ $(1 \leqq i \leqq j \leqq 4)$ are given in the same way as $\kappa_{f_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*)$. For example,

$$\kappa_{g_{12}}(\boldsymbol{r}, \boldsymbol{r}^*) = (n-2)^{1/2}\{p(u_0, v(i, jj; 0)) - p(u_0, v(i, jj; 1))\}.$$

Here $\kappa_\alpha(\boldsymbol{r}^*, \boldsymbol{r})$, $\kappa_{f_{ji}}(\boldsymbol{r}^*, \boldsymbol{r})$ and $\kappa_{g_{ji}}(\boldsymbol{r}^*, \boldsymbol{r})$ are defined by $\kappa_\alpha(\boldsymbol{r}, \boldsymbol{r}^*)$, $\kappa_{f_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*)$ and $\kappa_{g_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*)$, respectively, for $0 \leqq \alpha \leqq 5$ and $1 \leqq i \leqq j \leqq 4$.

Let $K_\alpha = [\kappa_\alpha(\boldsymbol{r}, \boldsymbol{r}^*)]$ (of size $\tau_\alpha \times \tau_\alpha$) for $\alpha = 0, 1, \ldots, 5$. Let

$$K_f = \begin{bmatrix} \overbrace{K_{f_{11}}(1,1)}^{r-1} & \overbrace{K_{f_{12}}(1,2)}^{r-1} & \overbrace{K_{f_{13}}(1,3)}^{\binom{r-1}{2}} & \overbrace{K_{f_{14}}(1,4)}^{\binom{r-1}{2}} & \overbrace{K_{f_{11}}(1,5)}^{(r-1)(s-1)} \\ K_{f_{21}}(2,1) & K_{f_{22}} & K_{f_{23}} & K_{f_{24}} & K_{f_{21}}(2,5) \\ K_{f_{31}}(3,1) & K_{f_{32}} & K_{f_{33}} & K_{f_{34}} & K_{f_{31}}(3,5) \\ K_{f_{41}}(4,1) & K_{f_{42}} & K_{f_{43}} & K_{f_{44}} & K_{f_{41}}(4,5) \\ K_{f_{11}}(5,1) & K_{f_{12}}(5,2) & K_{f_{13}}(5,3) & K_{f_{14}}(5,4) & K_{f_{11}}(5,5) \end{bmatrix}$$

of size $\tau_f \times \tau_f$, where $K_{f_{ij}} = K'_{f_{ji}} = [\kappa_{f_{ij}}(\boldsymbol{r}, \boldsymbol{r}^*)]$ $(2 \leqq i, j \leqq 4)$, $K_{f_{1i}}(1, i) = K_{f_{i1}}(i, 1)' = [\kappa_{f_{1i}}((\boldsymbol{p}, (n0\cdots0)), (\boldsymbol{p}^*, \boldsymbol{q}^*))]$ $(i = 1, 2, \ldots, 4)$,

$$K_{f_{1i}}(5, 1) = K_{f_{i1}}(i, 5)' = [\kappa_{f_{1i}}((\boldsymbol{p}_\alpha, \boldsymbol{q}), (\boldsymbol{p}^*, (n0\cdots0))]_{\gamma=1,\ldots,s-1}^{\alpha=1,\ldots,r-1}$$

(the range of $\boldsymbol{p}^*$ is dependent on $i = 1, 2, 3, 4$), $K_{f_{11}}(5, 5) = [\kappa_{f_{11}}((\boldsymbol{p}_\alpha, \boldsymbol{q}_\gamma), (\boldsymbol{p}_\beta, \boldsymbol{q}_\delta))]$ $(\alpha, \beta = 1, \ldots, r-1;\ \gamma, \delta = 1, \ldots, s-1)$. Here $\boldsymbol{p}_\alpha = (\underset{(o)}{m-1}, 0, \ldots, 0, \underset{(\alpha)}{1}, 0, \ldots, \underset{(r-1)}{0})$ and $\boldsymbol{q}_\gamma = (\underset{(o)}{n-1}, 0, \ldots, 0, \underset{(\gamma)}{1}, 0, \ldots, \underset{(s-1)}{0})$. We define the matrix K_g of size $\tau_g \times \tau_g$ in the same way as K_f.

From Theorem 8.2, there exists an orthogonal matrix P of order v_1^* such that

$$P' M_T P = \mathrm{diag}[K_0, \overbrace{K_1, \ldots, K_1}^{\phi_1}, \ldots, \overbrace{K_5, \ldots, K_5}^{\phi_5}, \overbrace{K_f, \ldots, K_f}^{\phi_f}, \overbrace{K_g, \ldots, K_g}^{\phi_g}].$$

Now M_T is the information matrix of size $v_1^* \times v_1^*$, and v_1^* is dependent on constraints m and n. As shown above, however, M_T can be expressed by the matrices K_α of size $\tau_\alpha \times \tau_\alpha$ for $\alpha = 0, 1, \ldots, 5, f, g$. Note that τ_α is independent of m and n.

Thus we have established the followings:

THEOREM 8.3. *Let T be an $ABA1[N, (m, n), (r, s), 4]$. Then T is a balanced design of resolution* V *if and only if all K_α ($\alpha = 0, 1, \ldots, 5, f, g$) are positive definite.*

THEOREM 8.4. *The characteristic polynomial of M_T is given by*

$$\det(M_T - xI_{v_1^*}) = [\prod_{\alpha=0}^{5} \{\det(K_\alpha - xI_{\tau_y})\}^{\phi_\alpha}]\{\det(K_f - xI_{\tau_f})\}^{\phi_f}\{\det(K_g - xI_{\tau_g})\}^{\phi_g},$$

if T is an $ABA1[N, (m, n), (r, s), 4]$.

COROLLARY 8.5. *For T being a balanced $r^m \times s^n$-FF design of resolution* V, *the inverse matrix of M_T is expressed as*

$$M_T^{-1} = P\operatorname{diag}[K_0^{-1}, K_1^{-1}, \ldots, K_1^{-1}, \ldots, K_5^{-1}, \ldots, K_5^{-1}, K_f^{-1}, \ldots, K_f^{-1}, K_g^{-1}, \ldots, K_g^{-1}]P'.$$

The trace and determinant of M_T^{-1} are given by

$$\operatorname{tr}(M_T^{-1}) = \sum_{\alpha=0}^{5} \phi_\alpha \operatorname{tr}(K_0^{-1}) + \phi_f \operatorname{tr}(K_f^{-1}) + \phi_g \operatorname{tr}(K_g^{-1}),$$
$$\det(M_T^{-1}) = [\sum_{\alpha=0}^{5} \{\det(K_\alpha^{-1})\}^{\phi_\alpha}]\{\det(K_f^{-1})\}^{\phi_f}\{\det(K_g^{-1})\}^{\phi_g}.$$

There are, in general, a large number of possible balanced $r^m \times s^n$-FF designs of resolution V with each number of assemblies N ($\geqq v_1^*$). Out of these designs, one must choose a design which allows us to estimate all v_1^* effects and, further, which minimizes the loss of the information in some sense. The functions, which evaluate the loss of information, are mostly defined in terms of characteristic roots of the information matrix M_T as shown in Section 1. Thus it is very useful to obtain the characteristic polynomial of M_T (or M_T^{-1}).

Consider a $2^2 \times 3^2$-FF design of resolution V derived from an ABA1[N, (2, 2), (2, 3), 4] with index set $\{\lambda(p_0p_1, q_0q_1q_2) \mid p_0 + p_1 = 2, q_0 + q_1 + q_2 = 2\}$ for v_1^* ($=20$) $\leqq N \leqq 36$. In Table, optimal balanced designs with respect to the trace and determinant criteria are given with values of $\operatorname{tr}(M_T^{-1})$ and $\det(M_T^{-1})$, respectively, for each N in the above-mentioned range. Here matrices D and E are defined by $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$, respectively.

TABLE Optimal balanced $2^2 \times 3^2$-FF designs of resolution V

N	$\boldsymbol{\lambda}$	tr (V_T)	$\boldsymbol{\lambda}$	det (V_T)
20	011101100110110001	1.83025	100011011100110011	6.36818E–26
21	011101110010101101	1.45307	001100110011011101	9.43434E–27
22	010101111010101101	1.31019	010101101110111001	2.23445E–27
23	011101110011101101	1.14578	011100110011011101	5.89646E–28
24	010101101110111101	1.06156	110011011101110011	1.47412E–28
25	100111011101101111	0.99675	011101101110111101	5.70625E–29
26	010101101111111101	0.93256	111011010110111011	2.21117E–29
27	100111111101101111	0.87151	110011011101111111	8.57667E–30
28	011101101111110111	0.81967	011110111011011110	3.53788E–30
29	110111111101101111	0.76828	010110111011111111	1.41515E–30
30	110111101111110111	0.71995	111011110111111011	6.14215E–31
31	011101110111111111	0.69937	110011011111111111	2.50913E–31
32	101111110111111111	0.65344	110111011111110111	1.09681E–31
33	111111110111111102	0.63474	110111011111111111	4.83634E–32
34	111111110111111111	0.59375	110111111111110111	2.13269E–32
35	110111111111111111	0.57465	110111111111111111	9.47862E–33
36	111111111111111111	0.55556	111111111111111111	4.21272E–33

$\boldsymbol{\lambda} = (\lambda(02, 002), \lambda(02, 011), \lambda(02, 020), \lambda(02, 101), \lambda(02, 110), \lambda(02, 200),$
$\lambda(11, 002), \lambda(11, 011), \lambda(11, 020), \lambda(11, 101), \lambda(11, 110), \lambda(11, 200),$
$\lambda(20, 002), \lambda(20, 011), \lambda(20, 020), \lambda(20, 101), \lambda(20, 110), \lambda(20, 200))$.

9. Optimality of level-symmetric designs in $s_1 \cdots s_m$ factorials

We consider an $s_1 \cdots s_m$ factorial design with m factors $F_1, \ldots, F_m$, where F_i has levels $0, 1, \ldots, s_i - 1$ for $i = 1, \ldots, m$. We use notations similar to Section 2. The assembly $\boldsymbol{t} = (t_1, \ldots, t_m)$ is represented as an element of $Z_{s_1} \times \cdots \times Z_{s_m}$. Let $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ be the expected values of all observations and all factorial effects, respectively. Then we assume that $\boldsymbol{\eta}$ can be expressed by the effects $\boldsymbol{\theta}$ as

$$\boldsymbol{\eta} = D_1 \otimes \cdots \otimes D_m \boldsymbol{\theta}.$$

Here $D_i = [d_i(t, \varepsilon)]_{0 \leqq t, \varepsilon \leqq s_i - 1} = [\boldsymbol{d}_i(0), \boldsymbol{d}_i(1), \ldots, \boldsymbol{d}_i(s_i - 1)]$ is an $s_i \times s_i$ non-singular matrix whose first column $\boldsymbol{d}_i(0)$ is composed of 1's, whose all column vectors are mutually orthogonal, and whose entries $d_i(t, \varepsilon)$ satisfy $d_i(s_i - 1 - t, \varepsilon) = (-1)^{\varepsilon} d_i(t, \varepsilon)$ for any $t, \varepsilon \in Z_{s_i}$ $(i = 1, \ldots, m)$. Note that the matrix D_i, defined by orthogonal polynomials, satisfies these restrictions.

We assume that $(\ell + 1)$-factor and more interactions are negligible, i.e., all unknown effects are elements of

$$\Theta_{\ell} = \{\theta(\varepsilon_1, \ldots, \varepsilon_m) \mid \varepsilon_i \in Z_{s_i}, \ w(\varepsilon_1, \ldots, \varepsilon_m) \leqq \ell\}.$$

Let $\boldsymbol{\theta}_\ell$ be a ν_ℓ-columned vector composed of all effects in Θ_ℓ, where $\nu_\ell = |\Theta_\ell|$. Let T be a fractional $s_1 \cdots s_m$ factorial ($s_1 \cdots s_m$-FF) design with N assemblies $\boldsymbol{t}^{(\alpha)} = (t_1^{(\alpha)}, \dots, t_m^{(\alpha)})$. Then T can be identified with an $N \times m$ matrix whose α-th row is $\boldsymbol{t}^{(\alpha)}$. Let $y(\boldsymbol{t}^{(\alpha)})$ be the observation based on an assembly $\boldsymbol{t}^{(\alpha)}$ and $\boldsymbol{y}(T)$ be an N-columned vector $[y(\boldsymbol{t}^{(\alpha)})]$ expressed by

$$\boldsymbol{y}(T) = E_T \boldsymbol{\theta}_\ell + \boldsymbol{e}(T),$$

where $\boldsymbol{e}(T)$ is the error vector whose components are assumed to be uncorrelated and each has mean zero and the same variance σ^2. The $(y(\boldsymbol{t}^{(\alpha)}), \theta(\boldsymbol{\varepsilon}))$-entry of the design matrix E_T is given by

$$d_1(t_1^{(\alpha)}, \varepsilon_1) \cdots d_m(t_m^{(\alpha)}, \varepsilon_m) \ (= d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}), \text{ say}).$$

The normal equation for estimating $\boldsymbol{\theta}_\ell$ can be written as

$$M_T \hat{\boldsymbol{\theta}}_\ell = E'_T \boldsymbol{y}(T),$$

where $M_T = E'_T E_T$ is the information matrix whose $(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*))$-entry is given by

$$\textstyle\sum_{\alpha=1}^N d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}) d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}^*) \ (= m_T(\theta(\boldsymbol{\varepsilon}), \theta(\boldsymbol{\varepsilon}^*)), \text{ say})$$

for $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$ and $\boldsymbol{\varepsilon}^* = (\varepsilon_1^*, \dots, \varepsilon_m^*)$. An $N \times m$ matrix T is called the fractional $s_1 \cdots s_m$ factorial design of resolution $2\ell + 1$ if M_T is non-singular. For the design of resolution $2\ell + 1$, the best linear unbiased estimate of $\boldsymbol{\theta}_\ell$ can be obtained by

$$\hat{\boldsymbol{\theta}}_\ell = V_T E'_T \boldsymbol{y}(T),$$

where $V_T = M_T^{-1}$. The variance-covariance matrix of $\hat{\boldsymbol{\theta}}_\ell$ can be shown to be $\mathrm{Var}(\hat{\boldsymbol{\theta}}_\ell) = \sigma^2 V_T$.

Let $\gamma(\boldsymbol{\varepsilon}) = \sum_{\alpha=1}^N d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon})$ for any $\boldsymbol{\varepsilon} \in Z_{s_1} \times \cdots \times Z_{s_m}$. Let $\lambda(\boldsymbol{t})$ be the multiplicity of the assembly $\boldsymbol{t}$ in T for any $\boldsymbol{t} = (t_1, \dots, t_m)$. Using $\lambda(\boldsymbol{t})$, we have

$$\textstyle\sum_{\alpha=1}^N d(\boldsymbol{t}^{(\alpha)}, \boldsymbol{\varepsilon}) = \sum_{\boldsymbol{t}} d(\boldsymbol{t}, \boldsymbol{\varepsilon}) \lambda(\boldsymbol{t}).$$

Therefore we can get the following

LEMMA 9.1. $$\boldsymbol{\gamma} = D'_1 \otimes \cdots \otimes D'_m \boldsymbol{\lambda},$$

where $\boldsymbol{\gamma}$ *and* $\boldsymbol{\lambda}$ *are the* $s_1 \cdots s_m$*-columned vectors*

$$\boldsymbol{\gamma} = [\gamma(\varepsilon_1, \dots, \varepsilon_m)] \quad \textit{and} \quad \boldsymbol{\lambda} = [\lambda(t_1, \dots, t_m)] \quad (\varepsilon_i, t_i \in Z_{s_i}).$$

DEFINITION 9.1. An $N \times m$ matrix, T, is called an *orthogonal array* of strength d if any $N \times d$ submatrix T_i contains all possible d-rowed vectors in the same frequency λ_i for any sequence $i = (i_1 \cdots i_d)$ with $1 \leqq i_1 < \cdots < i_d \leqq m$, where T_i

is given in the same way as Section 2. Here λ_i is equal to $N/(s_{i_1}\cdots s_{i_d})$.

We have the following by an argument similar to Theorem 3.1.

THEOREM 9.2. *Let T be an $s_1\cdots s_m$-FF design of resolution $2\ell+1$, where $2\ell\leqq m$. Then $\hat{\boldsymbol{\theta}}_\ell$ can be estimated uncorrelatedly, i.e., M_T^{-1} becomes a diagonal matrix, if and only if T is an orthogonal array of strength 2ℓ.*

REMARK. One of the assumptions on D_i, $d_i(s_i-1-t, \varepsilon)=(-1)^\varepsilon d_i(t, \varepsilon)$ $(1\leqq i\leqq m)$, is not necessary to prove Lemma 9.1 and Theorem 9.2.

The following definition of a level-symmetric design is a generalization of the concept of a fold-over design.

DEFINITION 9.2. For T being an $s_1\cdots s_m$-FF design, T is called a *d-level-symmetric design* if the following relation holds:
$\sum^*\lambda(t_1,\dots, t_{i_1},\dots, t_{i_d},\dots, t_m) = \sum^*\lambda(t_1,\dots, s_{i_1}-1-t_{i_1},\dots, s_{i_d}-1-t_{i_d},\dots, t_m)$
for any $1\leqq i_1<\cdots<i_d\leqq m$ and any $(t_{i_1},\dots, t_{i_d})$ $(t_{i_k}=0, 1,\dots, s_{i_k}-1)$, where two summations $\sum^*$ extend over $t_j=0, 1,\dots, s_j-1$ for any $j\in \mathfrak{m}-\{i_1,\dots, i_d\}$.

Note that if T is a d-level-symmetric design, then T is also a d^*-level-symmetric design for any $d^*=1,\dots, d-1$.

An effect $\theta(\varepsilon_1,\dots, \varepsilon_m)$ is called an odd or even effect according as $\sum\varepsilon_i$ is odd or even. The set of unknown effects Θ_ℓ can be partitioned into the two sets $\Theta_{\ell,o}$ and $\Theta_{\ell,e}$ composed of odd and even effects, respectively. Corresponding to this partition, the vector $\boldsymbol{\theta}_\ell$ can be decomposed into

$$\boldsymbol{\theta}_\ell = \begin{pmatrix} \boldsymbol{\theta}_{\ell,o} \\ \boldsymbol{\theta}_{\ell,e} \end{pmatrix}.$$

LEMMA 9.3. *Let $m_T(\theta(\boldsymbol{\alpha}), \theta(\boldsymbol{\beta}))$ be an entry of M_T such that $\sum(\alpha_i+\beta_i)$ is an odd integer, where $\boldsymbol{\alpha}=(\alpha_1,\dots, \alpha_m)$ and $\boldsymbol{\beta}=(\beta_1,\dots, \beta_m)$ are elements in $Z_{s_1}\times\cdots\times Z_{s_m}$. Then $m_T(\theta(\boldsymbol{\alpha}), \theta(\boldsymbol{\beta}))$ can be represented by a linear combination of elements of $\{\gamma(\varepsilon_1,\dots, \varepsilon_m)\mid \sum\varepsilon_i$ is odd$\}$.*

PROOF. An entry $m_T(\theta(\boldsymbol{\alpha}), \theta(\boldsymbol{\beta}))$ is given by $\sum_{\boldsymbol{t}} d_1(t_1, \alpha_1)d_1(t_1, \beta_1)\cdots d_m(t_m, \alpha_m)d(t_m, \beta_m)\lambda(t_1,\dots, t_m)$. The column vector $(d_i(0, \alpha)d_i(0, \beta),\dots, d_i(s_i-1, \alpha)\cdot d_i(s_i-1, \beta))'$ $(=\boldsymbol{d}_i(\alpha)*\boldsymbol{d}_i(\beta)$, say) can be expressed as $\boldsymbol{d}_i(\alpha)*\boldsymbol{d}_i(\beta)=\sum_{\varepsilon=0}^{s_i-1}\cdot c_i(\varepsilon; \alpha, \beta)\boldsymbol{d}_i(\varepsilon)=D_i\boldsymbol{c}_i(\alpha, \beta)$ where $\boldsymbol{c}_i(\alpha, \beta)=(c_i(0; \alpha, \beta),\dots, c_i(s_i-1; \alpha, \beta))'$ is given by $(D_i'D_i)^{-1}D_i'\boldsymbol{d}_i(\alpha)*\boldsymbol{d}_i(\beta)$. Therefore we have $c_i(\varepsilon; \alpha, \beta)=c_i(\varepsilon)\boldsymbol{d}_i(\varepsilon)'\boldsymbol{d}_i(\alpha)*\boldsymbol{d}_i(\beta)$, where $c_i(\varepsilon)$ is the $(\varepsilon, \varepsilon)$-entry of $(D_i'D_i)^{-1}$, since $D_i'D_i$ is a diagonal matrix. By the condition $d_i(s_i-1-t, \varepsilon)=(-1)^\varepsilon d_i(t, \varepsilon)$, it holds that

$$c_i(\varepsilon; \alpha, \beta) = 0 \qquad \text{if} \quad \varepsilon+\alpha+\beta \text{ is odd.}$$

Any entry of M_T can be expressed as a linear combination of elements of $\{\gamma(\varepsilon_1,\ldots, \varepsilon_m)\}$ as follows:

$$m_T(\theta(\boldsymbol{\alpha}), \theta(\boldsymbol{\beta})) = \sum_{\boldsymbol{t}} [\prod_{i=1}^m \{\sum_{\varepsilon_i=0}^{s_i-1} c_i(\varepsilon_i; \alpha_i, \beta_i) d_i(t_i, \varepsilon_i)\}] \lambda(t_1,\ldots, t_m)$$
$$= \sum_{\boldsymbol{\varepsilon}} \{\sum_{i=1}^m c_i(\varepsilon_i; \alpha_i, \beta_i)\} \gamma(\varepsilon_1, \varepsilon_2,\ldots, \varepsilon_m).$$

Here $c_1(\varepsilon_1; \alpha_1, \beta_1)\cdots c_m(\varepsilon_m; \alpha_m, \beta_m)=0$ if $\sum(\alpha_i+\beta_i)$ is an odd integer and $\sum \varepsilon_i$ is an even integer. This completes the proof.

THEOREM 9.4. *Let T be an $s_1\cdots s_m$-FF design of resolution $2\ell+1$, where $2\ell \leqq m$ and $s_i \geqq 3$ $(i=1,\ldots, m)$. The best linear unbiased estimate* $\hat{\boldsymbol{\theta}}_\ell = \begin{pmatrix} \hat{\boldsymbol{\theta}}_{\ell,o} \\ \hat{\boldsymbol{\theta}}_{\ell,e} \end{pmatrix}$ *satisfies* $\mathrm{Cov}(\hat{\boldsymbol{\theta}}_{\ell,o}, \hat{\boldsymbol{\theta}}_{\ell,e})=0$ *if and only if T is a 2ℓ-level-symmetric design, where* $\mathrm{Cov}(X, Y)$ *denotes the covariance matrix between random variables X and Y.*

PROOF (*Sufficiency*). Consider $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots, \varepsilon_m) \in Z_{s_1} \times \cdots \times Z_{s_m}$ such that the sum of ε_{i_k} $(1 \leqq k \leqq n)$ is odd and the remaining elements are equal to zero where $1 \leqq n \leqq 2\ell$. Since T is an n-level-symmetric design and $d(\boldsymbol{t}, \boldsymbol{\varepsilon})=(-1)^{\Sigma \varepsilon_i} d(\boldsymbol{t}^*, \boldsymbol{\varepsilon})$, the following relation holds:

$$\gamma(\boldsymbol{\varepsilon}) = \sum_{\boldsymbol{t}} d_1(t_1, \varepsilon_1)\cdots d_m(t_m, \varepsilon_m)\lambda(\boldsymbol{t}) = \sum{}^* \{\prod_{k=1}^n d_{i_k}(t_{i_k}, \varepsilon_{i_k})\} \sum{}^{**} \lambda(\boldsymbol{t})$$
$$= (1/2)[\sum{}^* [\{\prod_{k=1}^n d_{i_k}(t_{i_k}, \varepsilon_{i_k})\} \sum{}^{**} \lambda(\boldsymbol{t})]$$
$$+ \sum{}^* [\{\prod_{k=1}^n d_{i_k}(t^*_{i_k}, \varepsilon_{i_k})\} \sum{}^* \lambda^*(\boldsymbol{t}^*)]]$$
$$= (1/2) \sum{}^* [(1+(-1)^\delta) \{\prod_{k=1}^n d_{i_k}(t_{i_k}, \varepsilon_{i_k})\} \sum{}^{**} \lambda(\boldsymbol{t})] = 0,$$

where $\boldsymbol{t}^*=(t_1^*,\ldots, t_m^*)$ is defined by $(s_1-1-t_1,\ldots, s_m-1-t_m)$ for any $\boldsymbol{t}=(t_1,\ldots, t_m)$, the summations $\sum^*$ and $\sum^{**}$ extend over all $t_{i_1},\ldots, t_{i_n}$ and the remaining t_j, respectively, and $\delta=\sum \varepsilon_i$ (odd). Therefore Lemma 9.3 leads to $m_T(\theta(\boldsymbol{\alpha}), \theta(\boldsymbol{\beta}))=0$ for any $\theta(\boldsymbol{\alpha}) \in \Theta_{\ell,o}$ and any $\theta(\boldsymbol{\beta}) \in \Theta_{\ell,e}$. Thus we have $\mathrm{Cov}(\hat{\boldsymbol{\theta}}_{\ell,o}, \hat{\boldsymbol{\theta}}_{\ell,e})=0$.

(*Necessity*). The submatrix of M_T corresponding to $\boldsymbol{\theta}_{\ell,o}$-row and $\boldsymbol{\theta}_{\ell,e}$-column is 0 since $\mathrm{Cov}(\hat{\boldsymbol{\theta}}_{\ell,o}, \hat{\boldsymbol{\theta}}_{\ell,e})=0$, i.e., $m_T(\theta(\boldsymbol{\alpha}), \beta(\boldsymbol{\beta}))=0$ for all $\theta(\boldsymbol{\alpha}) \in \Theta_{\ell,o}$ and $\theta(\boldsymbol{\beta}) \in \Theta_{\ell,e}$. These relations and the assumption that $s_i \geqq 3$ imply $\gamma(\varepsilon_1,\ldots, \varepsilon_m)=0$ for any $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots, \varepsilon_m)$ $(\in Z_{s_1} \times \cdots \times Z_{s_m})$ such that $w(\boldsymbol{\varepsilon}) \leqq 2\ell$ and $\sum \varepsilon_i$ is odd. Solving the relation in Lemma 9.1 with respect to $\boldsymbol{\lambda}$, we get

$$\boldsymbol{\lambda} = E_1 \otimes \cdots \otimes E_m \boldsymbol{\gamma}, \tag{9.1}$$

where $E_i=D_i(D_i'D_i)^{-1}$. The entries of the $s_i \times s_i$ matrix $E_i=[e_i(t, \varepsilon)]$ $(t, \varepsilon \in Z_{s_i})$ satisfy $e_i(s_i-1-t, \varepsilon)=(-1)^\varepsilon e_i(t, \varepsilon)$ since $D_i'D_i$ is diagonal. Put $e_1(t_1, \varepsilon_1)\cdots e_m(t_m, \varepsilon_m)=e(\boldsymbol{t}, \boldsymbol{\varepsilon})$. Then it holds that

$$\lambda(\boldsymbol{t}) = \sum_{\boldsymbol{\varepsilon}} e(\boldsymbol{t}, \boldsymbol{\varepsilon})\gamma(\boldsymbol{\varepsilon}).$$

Furthermore, we have $e(\boldsymbol{t}^*, \boldsymbol{\varepsilon})=(-1)^{\sum\varepsilon_i}e(\boldsymbol{t}, \boldsymbol{\varepsilon})$. Let $\{i_1,\dots, i_{2\ell}\}$ be any subset of $\mathfrak{m}$ and let $\{j_1,\dots, j_{m-2\ell}\}$ be $\mathfrak{m}-\{i_1,\dots, i_{2\ell}\}$. Let $X(i_1,\dots, i_{2\ell})=I_1^{x_1}\otimes\cdots\otimes I_m^{x_m}$, where $x_{i_k}=1$ $(k=1,\dots, 2\ell)$, $x_{j_s}=0$ $(s=1,\dots, m-2\ell)$ and

$$I_\alpha^{x_\alpha}=\begin{cases} I_{s_\alpha} & \text{if } x_\alpha=1,\\ j'_{s_\alpha} & \text{if } x_\alpha=0.\end{cases}$$

From (9.1) we have

$$X(i_1,\dots, i_{2\ell})\boldsymbol{\lambda}=(I_1^{x_1}E_1)\otimes\cdots\otimes(I_m^{x_m}E_m)\boldsymbol{\gamma}. \tag{9.2}$$

If $x_\alpha=0$, then $I_\alpha^{x_\alpha}E_\alpha=j'_{s_\alpha}E_\alpha=(1, 0,\dots, 0)$. The relation (9.2) yields

$$\textstyle\sum_{t_j}\lambda(t_1,\dots, t_m)=\sum_{\varepsilon_i}\{\prod_{k=1}^{2\ell}e_{i_k}(t_{i_k}, \varepsilon_{i_k})\}\gamma(0\cdots0\varepsilon_{i_1}0\cdots0\varepsilon_{i_{2\ell}}0\cdots0)$$

where the summations $\sum_{t_j}$ and $\sum_{\varepsilon_i}$ extend over all t_{j_s} $(1\leqq s\leqq m-2\ell)$ and ε_{i_k} $(1\leqq k\leqq 2\ell)$, respectively. Now $\gamma(0\cdots0\varepsilon_{i_1}0\cdots0\varepsilon_{i_{2\ell}}0\cdots0)=0$ when $\sum\varepsilon_{i_k}$ is odd. Hence the range of the last summation can be restricted to ε_{i_k} satisfying $\sum\varepsilon_{i_k}$ is even. Recall the relation $\prod_{k=1}^{2\ell}e_{i_k}(t_{i_k}, \varepsilon_{i_k})=\prod_{k=1}^{2\ell}e_{i_k}(s_{i_k}-1-t_{i_k}, \varepsilon_{i_k})$ for ε_{i_k} satisfying $\sum\varepsilon_{i_k}$ is even. Thus we have

$$\begin{aligned}&\textstyle\sum_{t_j}\lambda(t_1,\dots, s_{i_1}-1-t_{i_1},\dots, s_{i_{2\ell}}-1-t_{i_{2\ell}},\dots, t_m)\\ &\textstyle\quad=\sum_{t_j}\lambda(t_1,\dots, t_{i_1},\dots, t_{i_{2\ell}},\dots, t_m),\end{aligned}$$

which implies that T is a 2ℓ-level-symmetric design.

In the case $s_1=\cdots=s_m=2$, we have the following

Theorem 9.5. *Let T be a 2^m-FF design of resolution $2\ell+1$, where $2\ell-1\leqq m$. The best linear unbiased estimate of $\boldsymbol{\theta}_\ell$ satisfies* $\mathrm{Cov}(\hat{\boldsymbol{\theta}}_{\ell,o}, \hat{\boldsymbol{\theta}}_{\ell,e})=0$ *if and only if T is a $(2\ell-1)$-level-symmetric design.*

10. Structural properties of 2^m-BFF designs

Throughout this section, we consider a balanced fractional 2^m factorial (2^m-BFF) design T of resolution $2\ell+1$ with N assemblies, where D is defined by $\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$. For simplicity, we use symbols θ_ϕ and $\theta_{i_1\cdots i_k}$ instead of $\theta(0, 0,\dots, 0)$ and $\theta(\varepsilon_1,\dots, \varepsilon_m)$, respectively, where $\varepsilon_{i_1}=\cdots=\varepsilon_{i_k}=1$ and the remaining elements are all equal to zero. Let γ_0 and γ_k be $\gamma(0, 0,\dots, 0)$ $(=N)$ and $\gamma(\overbrace{1,\dots, 1}^{k}, 0,\dots, 0)$. Since T is a 2^m-BFF design, $\gamma_k=\gamma(\varepsilon_1,\dots, \varepsilon_m)$ for any $(\varepsilon_1,\dots, \varepsilon_m)\in Z_2{}^m$ satisfying $w(\varepsilon_1,\dots, \varepsilon_m)=k$, where $1\leqq k\leqq 2\ell$ (cf. [38]).

We use the method of the analysis of a 2^m-BFF design in Yamamoto, Shirakura and Kuwada [39] to derive the following two theorems.

THEOREM 10.1. *Let T be a 2^m-BFF design of resolution V derived from a $BA[N, m, 2, 4]$ with index set $\{\mu_0, \mu_1, \ldots, \mu_4\}$. The covariance matrix between the estimates of main effects and those of two-factor interactions is zero if and only if the indices satisfy $\mu_0 = \mu_4$ and $\mu_1 = \mu_3$.*

PROOF (*Sufficiency*). The relations $\mu_0 = \mu_4$ and $\mu_1 = \mu_3$ suggest that T is a 4-level-symmetric design. This fact implies that T is also a 3-level-symmetric design. Therefore Theorem 9.5 yields $\mathrm{Cov}(\hat{\boldsymbol{\theta}}_{2,o}, \hat{\boldsymbol{\theta}}_{2,e}) = 0$, where $\boldsymbol{\theta}'_{2,o} = (\theta_1, \ldots, \theta_m)$ and $\boldsymbol{\theta}'_{2,e} = (\theta_\phi, \theta_{12}, \theta_{13}, \ldots, \theta_{m-1,m})$.

(*Necessity*). The information matrix, M_T, of a balanced design T can be decomposed by the orthogonal matrix P_2 of order v_2^* as

$$M_T = P_2' \mathrm{diag}\,[K_0, \overbrace{K_1, \ldots, K_1}^{m-1}, \overbrace{K_2, \ldots, K_2}^{\binom{m}{2}-m}] P_2,$$

where $v_2^* = 1 + m + \binom{m}{2}$ and K_0, K_1 and K_2 are matrices of size 3×3, 2×2, 1×1, respectively. By changing K_i for K_i^{-1} $(i = 0, 1, 2)$, M_T^{-1} can also be represented in the same way. Since the submatrix corresponding to $\boldsymbol{\theta}_{2,o}$-row and $\boldsymbol{\theta}_{2,e}$-column is 0, K_1^{-1} is diagonal and the (2, 3)-entry of K_1^{-1} is zero, while K_1 is given by

$$K_1 = \begin{bmatrix} \gamma_0 - \gamma_2 & (m-2)^{1/2}(\gamma_1 - \gamma_3) \\ (m-2)^{1/2}(\gamma_1 - \gamma_3) & \gamma_0 + (m-4)\gamma_2 - (m-3)\gamma_4 \end{bmatrix}.$$

Therefore $\gamma_1 = \gamma_3$, and further from the form of

$$K_0 = \begin{bmatrix} \gamma_0 & m^{1/2}\gamma_1 & \binom{m}{2}^{1/2}\gamma_2 \\ & \gamma_0 + (m-1)\gamma_2 & m\{m-1)/2\}^{1/2}\gamma_1 \\ (sym.) & & \gamma_0 + 2(m-2)\gamma_2 + \binom{m-2}{2}\gamma_4 \end{bmatrix},$$

we get $\gamma_1(\gamma_0 - \gamma_2) = 0$. Since K_1 is positive definite, $\gamma_0 - \gamma_2 > 0$. Hence $\gamma_1 = \gamma_3 = 0$. The relation between μ_i and γ_j is given by

$$\begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = (1/16) \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}.$$

Thus we have $\mu_0 = \mu_4 = (1/16)(\gamma_0 + 6\gamma_2 + \gamma_4)$ and $\mu_1 = \mu_3 = (1/16)(\gamma_0 - \gamma_4)$.

Theorem 10.2. *Let T be a 2^m-BFF design of resolution $2\ell+1$ derivable from a $BA[N, m, 2, 2\ell]$ with index set $\{\mu_0, \mu_1, \dots, \mu_{2\ell}\}$, where $6 \leqq 2\ell \leqq m$. The covariance matrix between the estimates of p-factor interactions and those of q-factor interactions is 0 for any $1 \leqq p < q \leqq \ell$ if and only if T is an orthogonal array of strength $2\ell-1$*, i.e.,

$$\mu_0 + \mu_1 = \mu_1 + \mu_2 = \cdots = \mu_{2\ell-1} + \mu_{2\ell}, \quad \text{or}$$

$$\mu_0 = \mu_2 = \cdots = \mu_{2\ell} \text{ and } \quad \mu_1 = \mu_3 = \cdots = \mu_{2\ell-1}\,.$$

Proof (*Sufficiency*). Let T be an orthogonal array of strength $2\ell-1$. Then $m_T(\theta_{i_1\cdots i_p}, \theta_{j_1\cdots j_q})=0$ for all p-factor and q-factor interactions $(1 \leqq p < q \leqq \ell)$. Therefore the estimates of p-factor interactions and those of q-factor interactions have no correlations.

(*Necessity*). There exists an orthogonal matrix P_ℓ of order v_ℓ^* such that

$$M_T = P'_\ell \operatorname{diag}[K_0, \overbrace{K_1, \dots, K_1}^{m-1}, \overbrace{K_2, \dots, K_2}^{\binom{m}{2}-\binom{m}{1}}, \dots, \overbrace{K_\ell, \dots, K_\ell}^{\binom{m}{\ell}-\binom{m}{\ell-1}}]P_\ell\,,$$

where K_i is a $(\ell-i+1)\times(\ell-i+1)$ matrix $(i=0, 1, \dots, \ell)$ and $v_\ell^* = 1+\binom{m}{1}+\cdots+\binom{m}{\ell}$ because T is a balanced design. From the assumption on M_T, $M_T^{-1}(p, q)$ is the zero matrix for $1 \leqq p < q \leqq \ell$, where $M_T^{-1}(p, q)$ (resp. $M_T(p, q)$) denotes a submatrix of M_T^{-1} (resp. M_T) corresponding to (p-factor interactions)-rows and (q-factor interactions)-columns. Therefore K_i and K_i^{-1} are diagonal for $1 \leqq i < \ell$. Here all entries of $M_T(\ell-1, \ell)$ equal either $\gamma_1, \gamma_3, \dots, \gamma_{2\ell-3}$ or $\gamma_{2\ell-1}$, and $(\ell-i, \ell-i+1)$-entry of K_i is given by some contrast of these elements $(1 \leqq i \leqq \ell-1)$. These contrasts are linearly independent. Therefore $\gamma_1=\gamma_3=\cdots=\gamma_{2\ell-1}$ since K_i $(1 \leqq i \leqq \ell-1)$ is diagonal. Considering $M_T^{-1}(\ell-2, \ell)$, we can also prove that $\gamma_2=\gamma_4=\cdots=\gamma_{2\ell-2}$. Now our assumption on M_T^{-1} implies that K_0^{-1} can be expressed by

$$K_0^{-1} = \begin{bmatrix} a_0 & a_1 & \cdots & a_\ell \\ a_1 & b_1 & & 0 \\ \vdots & & \ddots & \\ a_\ell & 0 & & b_\ell \end{bmatrix}.$$

On the other hand, K_0 is given by

$$K_0 = \begin{bmatrix} \gamma_0 & m^{1/2}\gamma_1 & \cdots \\ m^{1/2}\gamma_1 & \gamma_0 + (m-1)\gamma_2 & \cdots \\ \binom{m}{2}^{1/2}\gamma_2 & \left\{m\binom{m}{2}\right\}^{1/2}\gamma_1 & \cdots \\ \binom{m}{3}^{1/2}\gamma_1 & \left\{m\binom{m}{3}\right\}^{1/2}\gamma_2 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

The (1, 2)-, (3, 2)-, and (4, 2)-entries of $K_0 K_0^{-1}$ $(=I_{\ell+1})$ imply that $\gamma_0 a_1 + m^{1/2}\gamma_1 b_1 = 0$, $\gamma_2 a_1 + m^{1/2}\gamma_1 b_1 = 0$, and $\gamma_1 a_1 + m^{1/2}\gamma_2 b_1 = 0$, respectively. These relations must hold for some (a_1, b_1). Here $b_1 > 0$ since K_0^{-1} is positive definite. Therefore we have $(\gamma_2)^2 - (\gamma_1)^2 = 0$ and $\gamma_0\gamma_2 - (\gamma_1)^2 = 0$. Thus $\gamma_2(\gamma_2 - \gamma_0) = 0$, i.e., $\gamma_2 = 0$ or $\gamma_2 = \gamma_0$. If $\gamma_2 \neq 0$, the relation $\gamma_2 = \gamma_0$ $(=N)$ must hold. This contradicts the assumption that M_T is non-singular. Therefore $\gamma_1 = \gamma_2 = \cdots = \gamma_{2\ell-1} = 0$, i.e., T is an orthogonal array of strength $2\ell - 1$. Here T is a balanced array of strength 2ℓ and with index set $\{\mu_0, \mu_1, \ldots, \mu_{2\ell}\}$. It can be easily proved that T is also a balanced array of strength $2\ell - 1$ and with index set $\{\mu_0 + \mu_1, \mu_1 + \mu_2, \ldots, \mu_{2\ell-1} + \mu_{2\ell}\}$. For T being an orthogonal array of strength $2\ell - 1$, the relation $\mu_0 + \mu_1 = \mu_1 + \mu_2 = \cdots = \mu_{2\ell-1} + \mu_{2\ell}$ must hold, i.e., $\mu_0 = \mu_2 = \cdots = \mu_{2\ell}$ and $\mu_1 = \mu_3 = \cdots = \mu_{2\ell-1}$. This completes the proof.

Acknowledgments

The author would like to express his deepest gratitude to Professor Sumiyasu Yamamoto, Hiroshima University, for his valuable advice and encouragements. Many thanks are also due to Dr. Sanpei Kageyama, Hiroshima University, and Dr. Masahide Kuwada, Maritime Safety Academy, who read the preliminary draft and offered helpful comments.

References

[1] R. C. Bose, Mathematical theory of the symmetrical factorial design, Sankhyā **8** (1947), 107–166.

[2] G. E. P. Box and J. S. Hunter, The 2^{k-p} fractional factorial designs, Part I, II, Technometrics **3** (1961), 311–351; 449–458.

[3] G. E. P. Box and K. B. Wilson, On the experimental attainment of optimum conditions (with discussion), J. Roy Statist. Soc. (B) **13** (1951), 1–45.

[4] P. J. Cameron and J. H. Van Lint, *Graph theory, coding theory and block designs,* Cambridge Univ. Press, 1975.

[5] I. M. Chakravarti, Fractional replication in asymmetrical factorial designs and partially balanced arrays, Sankhyā **17** (1956), 143–164.

[6] C. S. Cheng, Optimality of certain asymmetrical experimental designs, Ann. Statist. **6** (1978), 1239–1261.

[7] D. V. Chopra, Balanced optimal 2^8 fractional factorial designs of resolution V, $52 \leqq N \leqq 59$, "*A Survey of Statistical Design and Linear Models*" (Ed., J. N. Srivastava et al.), North-Holland Publishing Co., Amsterdam (1975), 91–100.

[8] D. V. Chopra, Optimal balanced 2^8 fractional factorial designs of resolution V, with 60 to 65 runs, Bull. Intern. Statist. Inst. **46** (1975), 161–166.

[9] D. V. Chopra, On the maximum number of constraints for some orthogonal arrays of strength four, Proc. 5-th British Combinatorial Conf. (1975), 113–123.

[10] D. V. Chopra and J. N. Srivastava, Optimal balanced 2^7 fractional factorial designs of resolution V, with $N \leqq 42$, Ann. Inst. Statist. Math. **25** (1973), 587–604.

[11] D. V. Chopra and J. N. Srivastava, Optimal balanced 2^7 fractional factorial designs of resolution V, $49 \leqq N \leqq 55$, Commun. Statist. **2** (1973), 59–84.
[12] D. V. Chopra and J. N. Srivastava, Optimal balanced 2^8 fractional factorial designs of resolution V, $37 \leqq N \leqq 51$, Sankhyā (A) **36** (1974), 41–52.
[13] D. V. Chopra and J. N. Srivastava, Optimal balanced 2^7 fractional factorial designs of resolution V, $43 \leqq N \leqq 48$. Sankhyā (B) **37** (1975), 429–447.
[14] W. G. Cochran and G. M. Cox, *Experimental designs,* John Wiley and Sons, Inc., New York, 1968.
[15] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Repts. Suppl., 1973, No. 10.
[16] D. J. Finney, The fractional replication of factorial arrangements, Ann. Eugen. **12** (1945), 291–301.
[17] R. A. Fisher, *The design of experiments,* Oliver and Boyd, Edinburgh, 1935.
[18] R. A. Fisher, The theory of confounding in factorial experiments in relation to the theory of groups, Ann. Eugen. **11** (1942), 341–353.
[19] O. Kempthorne, *Design and analysis of experiments,* John Wiley and Sons, Inc., New York, 1952.
[20] J. Kiefer, Optimal experimental designs, J. Roy. Statist. Soc. (B) **21** (1959), 272–319.
[21] M. Kuwada, Optimal balanced fractional 3^m factorial designs of resolution V and balanced third-order designs, Hiroshima Math. J. **9** (1979), 347–450.
[22] M. Kuwada, Balanced arrays of strength 4 and balanced fractional 3^m factorial designs, J. Statist. Planning Inf. **3** (1979), 347–360.
[23] M. Kuwada, Characteristic polynomials of the information matrices of balanced fractional 3^m factorial designs of resolution V, (to appear in) J. Statist. Planning Inf.
[24] M. Kuwada and R. Nishii, On a connection between balanced arrays and balanced fractional s^m factorial designs, J. Japan Statist. Soc. **9** (1979), 93–101.
[25] M. Kuwada and R. Nishii, On the characteristic polynomial of the information matrix of balanced fractional s^m factorial designs of resolution $V_{p,q}$, submitted for publication.
[26] D. Raghavarao, *Constructions and combinatorial problems in design of experiments,* John Wiley and Sons, Inc., New York, 1971.
[27] C. R. Rao, Hypercubes of strength "d" leading to confounded designs in factorial experiments, Bull. Cal. Math. Soc., **38** (1946), 67–78.
[28] T. Shirakura, Optimal balanced fractional 2^m factorial designs of resolution VII, $6 \leqq m \leqq 8$, Ann. Statist. **4** (1976), 515–531.
[29] T. Shirakura, Contributions to balanced fractional 2^m factorial designs derived from balanced arrays of strength 2ℓ, Hiroshima Math. J. **7** (1977), 217–285.
[30] J. N. Srivastava, Optimal balanced 2^m fractional factorial designs, S. N. Roy Memorial Volume, Univ. of North Carolina, and Indian Statist. Institute (1970), 689–706.
[31] J. N. Srivastava, Some general existence conditions for balanced arrays of strength t and 2 symbols, J. Combinatorial Theory (A) **13** (1972), 198–206.
[32] J. N. Srivastava and D. A. Anderson, Optimal fractional factorial plans for main effects orthogonal to two-factor interactions: 2^m series, J. Amer. Statist. Soc. **65** (1970), 828–843.
[33] J. N. Srivastava and D. A. Anderson, A comparison of the determinant, maximum root, and trace optimality criteria, Commun. Statist. **3** (1974), 933–940.
[34] J. N. Srivastava and D. V. Chopra, On the characteristic roots of the information matrix of 2^m balanced factorial designs of resolution V, with applications, Ann. Math. Statist. **42** (1971), 722–734.

[35] J. N. Srivastava and D. V. Chopra, Balanced optimal 2^m fractional factorial designs of resolution V, $m \leqq 6$, Technometrics **13** (1971), 257–269.

[36] J. N. Srivastava and D. V. Chopra, Balanced trace-optimal 2^7 fractional factorial designs of resolution V, with 56 to 68 runs, Utilitas Mathematica **5** (1974), 263–279.

[37] S. Webb, Non-orthogonal designs of even resolution, Technometrics **10** (1968), 291–299.

[38] S. Yamamoto, T. Shirakura and M. Kuwada, Balanced arrays of strength 2ℓ and balanced fractional 2^m factorial designs, Ann. Inst. Statist. Math. **27** (1975), 143–157.

[39] S. Yamamoto, T. Shirakura and M. Kuwada, Characteristic polynomials of the information matrices of balanced fractional 2^m factorial designs of higher $(2\ell+1)$ resolution, "*Essays in Probability and Statistics*" (S. Ikeda et al., Eds.), birthday volume in honor of Professor J. Ogawa, Shinko Tsusho Co. Ltd., Tokyo (1976), 73–94.

[40] F. Yates, Complex experiments, J. Roy. Statist. Soc. **2** (1935), 181–247.

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